Summerschool 2010 – TU Berlin
Infinite Dimensional Operator Matrices
Theory and Applications

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Exercise A

Let $(\mathcal{L}, [\cdot,\cdot])$ be a linear space with a sesquilinear form.

1) If $[\cdot,\cdot]$ is indefinite then there exist neutral elements.

2) If $\mathcal{M}$ is a maximal non-negative subspace of $\mathcal{L}$, then $\mathcal{M}^{[\perp]}$ is a non-positive subspace.

3) Let $[\cdot,\cdot]$ be indefinite and $[\cdot,\cdot]_1$ be another sesquilinear form on $\mathcal{L}$. If

$$[x,x] = 0 \quad \Longrightarrow \quad [x,x]_1 = 0$$

then

a) $\mu := \inf\{[x,x]_1 \big| [x,x] = 1\} > -\infty$,

b) $[x,x]_1 \geq \mu[x,x]$ for all $x \in \mathcal{L}$.

(Hint: Show that $y,z \in \mathcal{L}$ with $[y,y] < 0$ and $[z,z] > 0$ implies:

$$\frac{[y,y]_1}{[y,y]} \leq \frac{[z,z]_1}{[z,z]}.$$

Otherwise there would be y,z with $[y,y] = -1$, $[z,z] = 1$ and $-[y,y]_1 > [z,z]_1$. Let $x = \epsilon y + z$ with $|\epsilon| = 1$. Then $[x,x] = 2\Re\{\epsilon[y,z]\}$, $[x,x]_1 < 2\Re\{\epsilon[y,z]_1\}$. Choose ϵ such that $[x,x] = 0$ and $[x,x]_1 < 0$.)

4) Let $[\cdot,\cdot]$ be non-degenerate. Then all Banach space norms on $\mathcal{L}$ are equivalent.
(Hint: With two Banach space norms $\|\cdot\|, \|\cdot\|'$ consider the norm $\|\cdot\|'' := \|\cdot\| + \|\cdot\|'$.)

5) Let $(\mathcal{H}, (\cdot,\cdot))$ be a Hilbert space, G be a bounded self-adjoint operator in $\mathcal{H}$, $[x,y] := (Gx,y)$, $x,y \in \mathcal{H}$. Then:

$$(\mathcal{H}, [\cdot,\cdot]) \text{ is a Krein space} \quad \Longleftrightarrow \quad 0 \in \rho(G).$$

6) Let $(\mathcal{K}, [\cdot,\cdot])$ be a Pontryagin space. Then the sequence $(x_n) \subset \mathcal{K}$ converges with respect to the norm of $\mathcal{K}$ to x_0 if and only if the following two relations hold:

(1) $[x_n, x_n] \to [x_0, x_0], \quad n \to \infty$,

(2) $[x_n, y] \to [x_0, y], \quad n \to \infty$, for all elements y of a total subset of $\mathcal{K}$.

The sequence $(x_n) \subset \mathcal{K}$ is a Cauchy sequence with respect to the norm of $\mathcal{K}$ if and only if the following two relations hold:

(1) $[x_n - x_m, x_n - x_m] \to 0, \quad m,n \to \infty$,

(2) $[x_n - x_m, y] \to 0, \quad m, n \to \infty$, for all elements y of a total subset of $\mathcal{K}$.

7) If $\mathcal{L}$ is a closed subspace of a Krein space $\mathcal{K}$, then

$$\mathcal{L} \cap \mathcal{L}^{[\perp]} = \{0\} \quad \Longleftrightarrow \quad \overline{\mathcal{L} \, [\dot{+}] \, \mathcal{L}^{[\perp]}} = \mathcal{K}.$$

8) A closed subspace $\mathcal{L}$ of a Krein space $\mathcal{K}$ is called *projection complete* if for each $x \in \mathcal{K}$ there exists an $x_{\mathcal{L}} \in \mathcal{L}$ such that $x - x_{\mathcal{L}} [\perp] \mathcal{L}$. Equivalent are:

(1) $\mathcal{L}$ is projection complete.

(2) $\mathcal{K} = \mathcal{L} [\dot{+}] \mathcal{L}^{[\perp]}$.

(3) $\mathcal{L}$ is the range of a $[\cdot, \cdot]$-self-adjoint projection E ($E = E^2 = E^+$).

(4) The inner product $[\cdot, \cdot]$ is nondegenerate on $\mathcal{L}$ and $\mathcal{L}$ is sequentially complete in its $\sigma_{[\cdot,\cdot]}$-topology.

Let $(\mathcal{K}, [\cdot, \cdot])$ be a Krein space. For a closed operator T in $\mathcal{K}$, its adjoint is denoted by T^+:

$$[Tx, y] = [x, T^+ y], \quad x \in \operatorname{dom} T, \; y \in \operatorname{dom} T^+.$$

9) If T is a closed operator in $\mathcal{K}$, σ a bounded spectral set of T and $E(T, \sigma)$ the corresponding Riesz projection, then

$$E(T; \sigma)^+ = E(T^+; \sigma^*).$$

10) If E is a self-adjoint projection in $\mathcal{K}$ then $\mathcal{L}_1 := \operatorname{ran} E$ and $\mathcal{L}_2 := \operatorname{ran} E^+$ are in duality, that is

$$\forall\, x \in \mathcal{L}_1, \; x \neq 0, \; \exists\, y \in \mathcal{L}_2 : [x, y] \neq 0$$

and

$$\forall\, y \in \mathcal{L}_2, \; y \neq 0, \; \exists\, x \in \mathcal{L}_1 : [x, y] \neq 0.$$