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Exercise B

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and T a closed operator.

11)
$$\begin{array}{llll} \lambda \in \rho(T) \iff \lambda^* \in \rho(T^+), & & \lambda \in \sigma_r(T) \implies \lambda^* \in \sigma_p(T^+), \\ \lambda \in \sigma_p(T) \implies \lambda^* \in \sigma_r(T^+) \cup \sigma_p(T^+), & & \lambda \in \sigma_c(T) \implies \lambda^* \in \sigma_c(T^+). \end{array}$$

12) If $A = A^+$ then $\sigma(A)$ is symmetric with respect to the real axis and if $\lambda, \mu \in \sigma_p(A)$, then for the algebraic eigenspaces $\mathcal{S}_\lambda$, $\mathcal{S}_\mu$ it holds

$$\mathcal{S}_\lambda [\perp] \mathcal{S}_\mu \quad \text{if} \;\; \lambda \neq \mu^*;$$

moreover, if λ and λ^* are isolated eigenvalues, then the Jordan structures of $A\big|_{\mathcal{S}_\lambda}$ and $A\big|_{\mathcal{S}_{\lambda^*}}$ coincide.

13) If $A = A^+$, $\rho(A) \neq \emptyset$ and $[Ax, x] \geq 0,\; x \in \operatorname{dom} A$, then $\sigma(A)$ is real.
(Hint: Show that $\sigma(A)$ cannot have a non-real boundary point.)

14) Let σ be a compactly supported bounded measure on $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Then the function

$$f(z) := \int_{\mathbb{R}} \frac{d\sigma(t)}{t - z} + \alpha - z \tag{1}$$

has either exactly one zero $z_0 \in \mathbb{C}^+$, or $f(z) \neq 0$, $z \in \mathbb{C}^+$, and there exists exactly one point $z_0 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{|t - z_0|^2} \leq 1 \quad \text{and} \quad \int_{\mathbb{R}} \frac{d\sigma(t)}{t - z_0} + \alpha - z_0 = 0.$$

(Hint: Consider the operator $A := \begin{pmatrix} t\cdot & \mathbf{1} \\ -(\cdot, \mathbf{1}) & \alpha \end{pmatrix}$ in the Hilbert space $L^2_\sigma \oplus \mathbb{C}$.)

15) Let f be as in (**??**) and set $g(z) := \displaystyle\int_{\mathbb{R}} \frac{d\sigma(t)}{t - z} + \alpha$. For arbitrary $z_1 \in \mathbb{C}^+$ the sequence $(z_n)_{n=1}^\infty$:

$$z_{n+1} := g(z_n), \quad n = 1, 2, \ldots,$$

belongs to $\mathbb{C}^+$ and converges to the point $z_0 \in \mathbb{C}^+ \cup \mathbb{R}$ from 14). (Exercises 14) and 15) are closely related to a result of Denjoy and Wolff (Compt. Rend. ...) on solutions of the equation $s(z) - z = 0$ in $\overline{\mathbb{D}}$ for a Schur function s.)

16) If U is a unitary operator in the Krein space $\mathcal{K}$ such that

$$\|U^n\| \leq C, \; n = 1, 2, \ldots,$$

for some $C > 0$, then there exist a maximal non-negative subspace $\mathcal{L}_+$ and a maximal non-positive subspace $\mathcal{L}_-$, which are invariant under U and such that $\mathcal{K} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-$.
(Hint: The assumptions imply $\|U^n\| \leq C$ for all $n = 0, \pm 1, \pm 2, \ldots$. According to a result of Sz.-Nagy, U is similar to a (Hilbert space) unitary operator V : $U = TVT^{-1}$. Consider $S := T^*JT$ for a fundamental symmetry J.)

17) Let T be expansive in the Krein space $(\mathcal{K}, [\cdot, \cdot])$, that is

$$[Tx, Tx] \geq [x, x], \; x \in \mathcal{K},$$

and suppose that $\mathbb{T} := \{z : \|z\| = 1\} \subset \rho(T)$. Then there exist a maximal non-negative subspace $\mathcal{L}_+$ and a maximal non-positive subspace $\mathcal{L}_-$, which are invariant under T and such that

$$\mathcal{K} = \mathcal{L}_+ \dot{+} \mathcal{L}_-.$$

(Hint: Consider the Riesz spectral subspaces of T corresponding to $\sigma(T) \cap \mathbb{D}$ and $\sigma(T) \cap (\mathbb{C} \setminus \overline{\mathbb{D}})$.)

18) Let $\widehat{J}$ be a $2n \times 2n$-matrix such that $\widehat{J}^* = -\widehat{J}$ and $\widehat{J}^2 = -I$, and let H be a locally summable, symmetric $2n \times 2n$-matrix function on $[0, \infty)$. Consider the *canonical* differential system

$$\widehat{J}U'(t) = H(t)U(t), \quad 0 \leq t < \infty; \qquad U(0) = I.$$

For $J := i\widehat{J}$, the solution $U(t)$ is a J-unitary matrix function: $U(t)^*JU(t) = J$. (Remark: If H is periodic with period $\tau > 0$: $H(t + \tau) = H(t)$, $t \geq 0$, then the stability behaviour of the solution $U(t)$ is determined by $U(\tau)$.)