Summerschool 2010 – TU Berlin
Infinite Dimensional Operator Matrices
Theory and Applications

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Exercises on variational principles

1. Let A be a closed operator in a Hilbert space $\mathcal{H}$. Show that the form
$$\mathfrak{a}[x,y] := (Ax, Ay), \qquad x,y \in \operatorname{dom}(A),$$
is closed and non-negative.

2. Let $\mathcal{H} = \ell^2$ and consider the operator function T:
$$T(\lambda) = \operatorname{diag}\big(T_n(\lambda)\big), \qquad \lambda \in (-1,3),$$
with
$$T_n(\lambda) := \begin{cases} 1, & \lambda \in (-1,0], \\ 1-n\lambda, & \lambda \in \big(0,\frac{2}{n}\big), \\ -1, & \lambda \in \big[\frac{2}{n},3\big) \end{cases}$$
for $n \in \mathbb{N}$. Calculate the spectrum of the operator function T. Can the variational principle from the lecture be applied?

3. Let $\mathcal{H} = L^2(0,1) \oplus \mathbb{C}$ and A the operator of multiplication by the independent variable in $L^2(0,1)$. Show that the operator function
$$T(\lambda) = \begin{pmatrix} A+\lambda^2 I & 0 \\ 0 & 1-\lambda \end{pmatrix}, \qquad \lambda \in (-1,2),$$
satisfies the assumptions of the variational principle. Calculate κ and
$$\min_{\substack{L\subset\mathcal{D}\\ \dim L=\kappa+n}} \max_{\substack{x\in L\\ x\neq 0}} p(x)$$
for $n \in \mathbb{N}$. Determine the spectrum of T.

4. Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ so that
$$A \geq -c \quad \text{for some } c>0, \qquad \sigma_{\text{ess}}(A) \cap (-\infty,0) = \emptyset.$$
Let B be a symmetric, non-negative operator which is A-bounded with A-bound 0. Apply the variational principle principle to the operator function
$$T(\lambda) = \lambda^2 I - \lambda B + A, \qquad \lambda \in \big(-\sqrt{c}-1, 0\big).$$

5. Prove the formula
$$\lambda_n = \max_{\substack{L\subset\mathcal{H}\\ \dim L=\kappa+n-1}} \inf_{\substack{x\in\mathcal{D},\, x\neq 0\\ x\perp L}} p(x)$$
from the theorem in the lecture about operator functions.