Approximation of High Order Singular Perturbations

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Let $\mathcal{H}$, $\langle\cdot,\cdot\rangle$ be a Hilbert space, L be a positive self-adjoint operator in $\mathcal{H}$ and φ be an element of the space $\mathcal{H}_{-k-1}\setminus\mathcal{H}_{-k}$, $k\geq 2$. Here $\mathcal{H}_l$'s are the Hilbert scale spaces with the inner products $\langle (L+1)^l\cdot,\cdot\rangle$.

For a given set $\widehat{g}=\{g_s\}_{s=2}^k$ of real numbers m-model (realization) for singular perturbation of L generated by φ is described by the triple $\Pi(\widehat{g}), S(\widehat{g}), H^g(\widehat{g})$, where $\Pi(\widehat{g})$ is a π_m–space with $m=[\frac{k}{2}]$, $S(\widehat{g})$ is a symmetric operator with defect indices $(1,1)$ in $\Pi(\widehat{g})$ and $H^g(\widehat{g})$, $g\in\mathbb{R}\cup\infty$, is one parameter family of canonical s.a. extensions of $S(\widehat{g})$. With $\mu<0$ the function

$$Q_k(z)=\langle\frac{(z-\mu)^k}{(L-z)(L-\mu)^k}\varphi,\varphi\rangle_0+\sum_{l=0}^{k-1}g_{j+1}(z-\mu)^l,\quad g_1\equiv g,$$

belongs to the class $\mathcal{N}_m$ and it is Q-function for $S(\widehat{g})$ and $H^\infty(\widehat{g})$.

For an "approximant" we take a smother $(1k)$–model , which is determined by L, an element $\psi\in\mathcal{H}_{-2}\setminus\mathcal{H}_0$ and real parameters from $\widehat{\gamma}=\{\gamma_s\}_{s=2}^k$ and is described by the triple $\mathcal{K}(\widehat{\gamma}), S(\widehat{\gamma}), A^\gamma(\widehat{\gamma})$, where $\Pi(\widehat{g})$ is a π_κ–space, $S(\widehat{\gamma})$ is a symmetry with defect indices $(1,1)$ in $\Pi(\widehat{g})$ and $A^\gamma(\widehat{\gamma})$, $\gamma\in\mathbb{R}\cup\infty$, is one parameter family of canonical s.a. extensions of $S(\widehat{\gamma})$. The function

$$Q_{1k}(z)=\langle\frac{(z-\mu)}{(L-z)(L-\mu)}\psi,\psi\rangle_0+\sum_{l=0}^{k-1}\gamma_{l+1}(z-\mu)^k,\quad \gamma_1\equiv\gamma,$$

belongs to the class $\mathcal{N}_\kappa$ and it is Q-function for $S(\widehat{\gamma})$ and $A^\infty(\widehat{\gamma})$.

Let sequences of numbers $\gamma_s^{(n)}$, $s=\overline{2,k}$, and elements $\psi^{(n)}$ from $\mathcal{H}_{-2}$ are taken in such way that $\psi^{(n)}\xrightarrow{n\to\infty}\varphi$ in $\mathcal{H}_{-k-1}$ and

$$\gamma_s^{(n)}+\langle (L-\mu)^{-s}\psi^{(n)},\psi^{(n)}\rangle_0\xrightarrow{n\to\infty}g_s,$$

then the spaces $\mathcal{K}(\widehat{\gamma}^{(n)})$ strongly approximates the space $\Pi(\widehat{g})$ and for $z \in \rho(H^g(\hat{g}))$ the sequence $(A^g(\hat{\gamma}^{(n)}) - z)^{-1}$ strongly approximates the resolvent $(H^g(\widehat{q})-z)^{-1}$ in the sense of approximation with variable spaces. An example of approximation by a boundary condition will be discussed.