A Completeness Theorem for a Non-Standard Two-Parameter Eigenvalue Problem

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We consider two simultaneous Sturm-Liouville systems involving two spectral parameters,

$$\lambda y''(s) + \mu a(s) y(s) + b(s) y(s) = 0, \quad 0 \le s \le 1, \quad {}' = d/ds, \tag{1}$$
$$y(0) = y(1) = 0, \tag{2}$$
$$-\lambda c(t) z(t) + \mu z''(t) + d(t) z(t) = 0, \quad 0 \le t \le 1, \quad {}' = d/dt, \tag{3}$$
$$z(0) = z(1) = 0, \tag{4}$$

where a, b, c, and d are real-valued and continuous, and in addition, a and c are positive, and at least one of the functions b and d has no zeros, and if $b \neq 0$ (resp. $d \neq 0$), then b (resp. d) is not a scalar multiple of a (resp. c).

Eliminating the eigenvalue parameter μ, the boundary problem

$$\lambda(D_1^2 D_2^2 u + acu) + (bD_2^2 u - adu) = 0 \quad \text{in} \quad \Omega, \tag{5}$$
$$u = 0 \quad \text{on} \quad \Gamma, \tag{6}$$

arises, where $D_1 = \partial/\partial s$, $D_2 = \partial/\partial t$, $\Omega = (0,1) \times (0,1)$, and $\Gamma = \partial\Omega$.

Since the differential equation (5) is not elliptic, the usual methods used in standard multiparameter spectral theory to obtain the required regularity results for solutions of (5–6) no longer apply. However, we are able to show that the domain of the self-adjoint operator associated with (5–6) contains the Sobolev space $H^2(\Omega)$, which enables us to show that the eigenvectors of (1–4) are complete in $L^2(\Omega)$ in case $b = 0$ or $d = 0$.

In case $b \neq 0$, the operator A has essential spectrum in $\mathbb{R} \setminus \{0\}$, and the completeness of the eigenvectors is still an open problem.