Spectral Points of Type π_+ of Symmetric Operators in Indefinite Inner Product Spaces

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joint work with C. Trunk

Let G be a bounded self-adjoint operator in a Hilbert space $(\mathcal{H}, (\cdot, \cdot))$. We define an inner product by $[\cdot\,,\cdot] = (G\cdot, \cdot)$. Let A be a closed operator in $\mathcal{H}$ satisfying the symmetry condition

$$[Ax, y] = [x, Ay] \quad \text{for all} \quad x, y \in \mathcal{D}(A).$$

We call such an operator *G-symmetric*. An approximative eigenvalue $\lambda \in \sigma_{\mathrm{ap}}(A)$ is called a *spectral point of type π_+ of A*, if there exists a linear manifold $\mathcal{H}_\lambda$ in $\mathcal{H}$ with finite codimension, such that for every approximative eigensequence (x_n) (i.e. $\|x_n\| = 1$ and $(A-\lambda)x_n \to 0$ as $n \to \infty$) in $\mathcal{H}_\lambda \cap \mathcal{D}(A)$ the accumulation points of the sequence $([x_n, x_n])$ are positive. We denote the set of all those spectral points by $\sigma_{\pi_+}(A)$.

For $\lambda \in \sigma_{\pi_+}(A)$ we describe all the possible linear manifolds $\mathcal{H}_\lambda$ with the property mentioned above and we determine among these a special manifold $\mathcal{H}_\lambda$ with minimal codimension. Let A_1 be another G-symmetric operator such that $\rho(A) \cap \rho(A_1) \neq \emptyset$ and $(A - \mu)^{-1} - (A_1 - \mu)^{-1}$ is compact. Then we show that λ either is contained in $\sigma_{\pi_+}(A_1)$ or in $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(A_1)$.

For real intervals $[a, b]$ with $[a, b] \cap \sigma_{\mathrm{ap}}(A) \subset \sigma_{\pi_+}(A)$ we prove, that there is an open neighbourhood U of $[a, b]$ in $\mathbb{C}$, such that either all points in U are eigenvalues of A or $(U \setminus \mathbb{R}) \cap \sigma_{\mathrm{ap}}(A)$ is empty. If even $U \setminus \mathbb{R} \subset \rho(A)$ holds, then the norm of the resolvent $(A - \mu)^{-1}$ can be estimated by some power of $|\mathrm{Im}\mu|^{-1}$ in a neighbourhood of $[a, b]$.

Moreover, we show the existence of a local spectral function E of A on (a, b). The spectral subspace $E(\Delta)\mathcal{H}$ is always the direct G-orthogonal sum of a Pontryagin space and a finite dimensional neutral subspace.