Some Properties of Eigenfunctions and Associated Functions of Indefinite Sturm-Liouville Problems

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We consider the problem

$$Lu = \lambda g(x)u, \quad x \in (a,b), \tag{1}$$

where L is an ordinary differential operator of order $2m$ which is defined by the differential expression

$$Lu = \sum_{i,j=1}^{m} \frac{d^i}{dx^i} a_{ik} \frac{d^j u}{dx^j} \quad (x \in (a,b)) \tag{2}$$

and the boundary conditions

$$\begin{gathered}
B_k u = \sum_{i=0}^{2m-1} (\alpha_{ik} u^{(i)}(a) + \beta_{ik} u^{(i)}(b)) = 0 \ (k = 1, 2, \ldots, 2m), \\
B_k u = \sum_{i=0}^{2m-1} \alpha_{ik} u^{(i)}(a) = 0, \quad (k = 1, 2, \ldots, m), \quad \lim_{x \to +\infty} u^{(i)} = 0 \ (i = 0, 1 \ldots m-1), \\
B_k u = \sum_{i=0}^{2m-1} \beta_{ik} u^{(i)}(b) = 0 \ (k = 1, 2, \ldots, m), \quad \lim_{x \to -\infty} u^{(i)} = 0 \quad (i = 0, 1 \ldots m-1), \\
\lim_{x \to \infty} u^{(i)} = 0 \quad (i = 0, 1 \ldots m-1),
\end{gathered} \tag{3}$$

Here the first conditions are used in the case of a bounded interval (a,b), the second in the case of $b = +\infty$, the third when $a = -\infty$, and the last conditions in the case of $(a,b) = \mathbb{R}$. The operator L is assumed to be selfadjont in $L_2(a,b)$.

We study the Riesz basis property of the root functions of the problem (1) in the weighted space $L_{2,g}((a,b) \setminus G^0)$ ($G^0 = \{x \in (a,b) : g(x) = 0\}$)

endowed with the norm

$$\|u\|^2_{L_{2,g}((a,b)\setminus G^0)} = \int\limits_{(a,b)\setminus G^0} |g||u|^2(x)\ dx.$$

Assume that there exist an open subsets G^+ and G^- of $G=(a,b)$ such that $\mu(\overline{G^\pm}\setminus G^\pm)=0$, $g(x)>0$ a.e. (almost everywhere) in G^+, $g(x)<0$ a.e. in G^-, and $g(x)=0$ a.e. in $G^0=G\setminus\overline{G^+}\cup\overline{G^-})$. Here μ is the Lebesgue measure. A point $x_0\in\partial G^+\cap\partial G^-$ is called a turning point. We show that in almost all cases the Riesz basis property is independent of the boundary conditions (3). Next, we demonstrate that the Riesz basis property depends only on the behavior of the function g at the turning points and thus it does not matter how this function looks like beyond some neighborhood about the set of turning points. At last, with the use of new sufficient conditions of regularity of a turning point obtained by Parfenov A.I. we refine some previous results and present some sufficient conditions of the Riesz basis property.