

Shifted Hermite-Biehler Functions

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We investigate a certain subclass \mathcal{SHB} of indefinite Hermite-Biehler functions. Our aim is to characterize the belonging of a function to this subclass by means of the distribution of its zeros. Functions of the class \mathcal{SHB} appeared in various contexts. For example in the theory of generalized strings, the Regge problem, or the investigation of the vibrations of a damped string, where also the distribution of their zeros, in particular the purely imaginary zeros, proved to be of interest.

Let us describe our result in more detail. If E is an entire function, we say $E \in \mathcal{HB}_{<\infty}$ (E is an *indefinite Hermite-Biehler function*), if $E(z)$ and $E^\#(z) := \overline{E(\overline{z})}$ do not have common nonreal zeros and if the kernel

$$S(z, w) := \frac{i}{z - \overline{w}} \left[1 - \frac{E^\#(z)}{E(z)} \overline{\left(\frac{E^\#(w)}{E(w)} \right)} \right]$$

has a finite number of negative squares. We write $E \in \mathcal{HB}_{<\infty}^{sym}$ (E is *symmetric*), if in addition to $E \in \mathcal{HB}_{<\infty}$ the functional equation $E^\#(z) = E(-z)$ is satisfied. We write $E \in \mathcal{HB}_{<\infty}^{sb}$ (E is *semibounded*), if in addition to $E \in \mathcal{HB}_{<\infty}$ the function $E + E^\#$ has only finitely many zeros in $(-\infty, 0)$. The transformation

$$\mathfrak{T} : \{E \in \mathcal{HB}_{<\infty}^{sb} : E(t) \neq 0, t \in (-\infty, 0)\} \rightarrow \mathcal{HB}_{<\infty}^{sym}$$

defined as $(\mathfrak{T}E)(z) := A(z^2) - izB(z^2)$, where $A := \frac{1}{2}(E + E^\#)$ and $B := \frac{i}{2}(E - E^\#)$, is a bijection. The subclass \mathcal{SHB} under consideration is now

$$\mathcal{SHB} := \mathfrak{T} \left(\left\{ E \in \mathcal{HB}_{<\infty}^{sb} : \begin{array}{l} E(t) \neq 0, t \in (-\infty, 0), \\ S(z, w) \text{ is positive semidefinite} \end{array} \right\} \right)$$

The result we present can be roughly formulated as follows: A function $E \in \mathcal{HB}_{<\infty}^{sym}$ actually belongs to \mathcal{SHB} if and only if all zeros of F in the upper half plane are simple, lie on the imaginary axis, and if their location governs the distribution of zeros on the negative imaginary axis in a specific way.