A max-min-principle for pairs of Hermitian matrices

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Let $\lambda_1(H) \geq \lambda_2(H) \geq \ldots \geq \lambda_n(H)$ denote the eigenvalues of the Hermitian matrix $H \in \mathbb{F}^{n\times n}$ in decreasing order, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let $\mathcal{G}_k(\mathbb{F})$ denote the Grassmann manifold of k-dimensional subspaces of $\mathbb{F}^n$. The Courant-Fischer max-min-principle states that

$$\lambda_k(H) = \max_{\mathcal{S}\in\mathcal{G}_k(\mathbb{F})} \min_{\substack{x \in \mathcal{S} \\ \|x\| = 1}} x^* H x.$$

We show that the following formula holds for any pair of Hermitian matrices $H_0, H_1 \in \mathbb{F}^{n\times n}$. Let $H_t = (1-t)\, H_0 + t\, H_1$. Then

$$\min_{t\in[0,1]} \lambda_k(H_t) = \max_{\mathcal{S}\in\mathcal{G}_k(\mathbb{F})} \Big\{ \min_{\substack{x \in \mathcal{S} \\ \|x\| = 1}} x^* H_0 x, \quad \min_{\substack{x \in \mathcal{S} \\ \|x\| = 1}} x^* H_1 x \Big\}.$$

This formula is a corollary of the following theorem.

Theorem: Suppose that to each $t \in [0,1]$ there exists a k-dimensional subspace $\mathcal{S}_t$ on which the Hermitian form $x \mapsto x^* H_t x$ is positive definite. Then there exists a k-dimensional subspace $\mathcal{S}$ on which all of these forms are simultaneously positive definite.

The proof of the theorem uses the canonical form of Hermitian matrix pairs under congruence transformations.