The Spectra of normal, equinormal and pseudonormal closed linear relations

J.-P. Labrousse

Let $\mathcal{H}$ be a complex Hilbert space and let $LR(\mathcal{H})$ denote the set of all closed linear relations on $\mathcal{H}$ (which includes all closed linear operators on $\mathcal{H}$).

Denote by $\mathbf{\Gamma_1}$ the sphere in $\mathbf{R}^3$ described by the equation: $x^2 + y^2 + z^2 = 1$ and let Φ be the mapping of $\overline{\mathbf{C}}$ (the one point compactification of the complex plane) onto $\mathbf{\Gamma_1}$ given by:

$$\text{If} \quad \lambda \in \mathbf{C}, \quad \Phi(\lambda) = \{ \frac{2Re\{\lambda\}}{|\lambda|^2 + 1}, \frac{2Im\{\lambda\}}{|\lambda|^2 + 1}, \frac{|\lambda|^2 - 1}{|\lambda|^2 + 1} \}$$

$$\Phi(\infty) = \{0, 0, 1\}.$$

$\Phi(\lambda)$ is the intersection in $\mathbf{R}^3$ of $\mathbf{\Gamma_1}$ with the straight line going from $\{0, 0, 1\}$ to the point $\{a, b, 0\}$ where $\lambda = a + ib$.

Let $\Psi : E \mapsto \Psi(E) = \{u_0, u_1, u_2, u_3\}$ denote a certain linear mapping of $LR(\mathcal{H})$ into the set of the 4-tuples of self-ajoint operators in $L(\mathcal{H})$ (the precise definition of Ψ is too long to include in the abstract).

The following results are proved:

• If $E \in LR(\mathcal{H})$ and $\Psi(E) = \{u_0, u_1, u_2, u_3\}$ then E is normal if and only if $u_0 = 0$ and the other three components of $\Psi(E)$ commute

• If $E \in LR(\mathcal{H})$ is normal and $\sigma(E)$ denotes its spectrum then $\Phi(\sigma(E))$ is the joint spectrum of $\Psi(E)$

Finally using Ψ and Φ two cathegories of closed linear relations are defined: the equinormal, which are normal with an additional property, and the pseudonormal, which generalize the normal but retain some good properties of the normal.