Interpolation of Sobolev spaces and indefinite elliptic eigenvalue problems

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Let Ω be a bounded domain with a Lipschitz boundary Γ and let the symbol $W_p^m(\Omega)$ stand for the Sobolev space. By $\overset{\circ}{W}{}_p^m(\Omega)$ we mean the closure of the class $C_0^\infty(\Omega)$ in the norm of $W_p^m(\Omega)$. The main our results are connected with the property:

$$\exists s \in (0,1): \quad (W_p^m(\Omega), L_{p,g}(\Omega))_{1-s,p} = (\overset{\circ}{W}{}_p^m(\Omega), L_{p,g}(\Omega))_{1-s,p}. \qquad (1)$$

By definition of a Lipschitz domain, for any $x_0 \in \Gamma$ there exists a neighborhood U about x_0 and a local coordinate system y obtained by rotation and translation of the origin from the initial one in which

$$U \cap \Omega = \{y \in \mathbb{R}^n : y' \in B_r, \omega(y') < y_n < \omega(y') + \delta\},$$

$$y' = (y_1, y_2, \ldots, y_{n-1}),\ B_r = \{y' : |y'| < r\},$$

where the function ω meets the Lipschitz condition in B_r. Given $y \in U \cap \Omega$, put $K_y(a) = \{\eta \in \Omega : \ |\eta' - y'| < a(y_n - \eta_n)\}, \quad a > 0$. Our conditions on the weight g are connected with some integral inequalities. The simplest of them is the following analog of the A_1-condition.

(A) There exist a finite covering U_i $(i = 1, 2, \ldots, N)$ of Γ (the domains U_i possess the properties from the definition of a Lipschitz domain) and the corresponding local coordinate systems such that for some $a > 0, c > 0$ and almost all $y \in U_i \cap \Omega$ $(i = 1, 2, \ldots, N)$

$$\int_{K_y(a) \cap U_i} g(\eta)\, d\eta \le c\mu(K_y(a))g(y)$$

(here the nonnegative function $g(y)$ is written in the local coordinate system y). We have the following theorem.

Theorem. *Under the condition* (A) (1) *holds.*

We also present applications to the elliptic eigenvalue problems with indefinite weight function of the form

$$Lu = \lambda Bu \quad (x \in G \subset R^n), \quad B_j u|_\Gamma = 0 \quad (j = \overline{1, m}), \tag{2}$$

where L is an elliptic differential operator of order $2m$ defined in a domain $G \subset R^n$ with boundary Γ, the B_j's are differential operators defined on Γ, and $Bu = g(x)u$ with $g(x)$ a measurable function changing a sign in G. We assume that there exist open subsets G^+ and G^- of G such that $\mu(\overline{G^\pm} \setminus G^\pm) = 0$ (μ is the Lebesgue measure), $g(x) > 0$ almost everywhere in G^+, $g(x) < 0$ almost everywhere in G^-, and $g(x) = 0$ almost everywhere in $G^0 = G \setminus (\overline{G^+} \cup \overline{G^-})$. Let the symbol $L_{2,g}(G \setminus G^0)$ stand for the space of functions $u(x)$ measurable in $G^+ \cup G^-$ and such that $u|g|^{1/2} \in L_2(G \setminus G^0)$. We study the Riesz basis property of eigenfunctions and associated functions of problem (2) in the weighted space $L_{2,g}(G \setminus G^0)$.