Shift operators as fundamental symmetries of a Pontryagin spaces

M. Wojtylak
joint work with F.H. Szafraniec

Let S be a commutative $*$-semigroup with 0. We say that a function $\phi : S \to \mathbb{C}$ is *positive definite* (we write $\phi \in \mathcal{P}(S)$) if for every $N \in \mathbb{N}$ we have

$$\sum_{i,j=1}^{N} \xi_i \bar{\xi}_j \phi(s_j^* + s_i) \geq 0, \qquad s_1, \ldots, s_N \in S,\ \xi_1, \ldots, \xi_N \in \mathbb{C}.$$

Each function $\phi \in \mathcal{P}(S)$ generates a positive definite kernel K^ϕ on S by

$$K^\phi(s,t) := \phi(t^* + s), \quad s,t \in S.$$

Furthermore, with each K^ϕ there is linked the reproducing kernel Hilbert space $\mathcal{H}^\phi$ (consisting of complex functions on S). We set

$$K_s^\phi := K^\phi(\cdot, s) : S \to \mathbb{C}, \quad s \in S,$$

it is known that the linear span $\operatorname{lin}\{K_s^\phi : s \in S\}$ is contained and dense in $\mathcal{H}^\phi$.

For an element $u \in S$ and a function $\phi \in \mathcal{P}(S)$ we define *the shift operator*, by

$$A(u,\phi)K_s^\phi = K_{s+u}^\phi.$$

It can be shown that $A(u,\phi)$ is well defined and extends uniquely to a linear mapping on $\operatorname{lin}\{K_s^\phi : s \in S\}$. Moreover, as an operator in $\mathcal{H}^\phi$, it is densely defined and closable. The aim of this talk is the following: *Provide necessary and sufficient conditions on the element $u \in S$ for the operator $A := \overline{A(u,\phi)}$ to be a fundamental symmetry of a Pontryagin space, i.e. to satisfy*

$$A = A^*, \quad A^2 = I_{\mathcal{H}^\phi}, \quad \dim\ker(A + I_{\mathcal{H}^\phi}) < \infty$$

for every $\phi \in \mathcal{P}(S)$. To solve the problem we will use the theory of the structure of a $*$-semigroup, developed by T.M. Bisgaard and results on RKHS by F.H. Szafraniec.

References

[1] T.M. Bisgaard, Separation by characters or positive definite functions, *Semigroup Forum* 53 (1996), 317-320

[2] T.M. Bisgaard, Extensions of Hamburger's Theorem, *Semigroup Forum* 57 (1998), 397-429

[3] T.M. Bisgaard, Semiperfect countable $\mathbb{C}$-separative C-finite semigroups, *Collect. Math.* 52 (2001), 55-73

[4] F.H. Szafraniec, Boundness of the shift operator related to positive definite forms: an application to moment problems, *Ark. Mat.* 19 (1981), 251-259.

[5] F.H. Szafraniec, *Przestrzenie Hilberta z jądrem reprodukującym*, WUJ, Kraków 2004.