Non-negative perturbations of non-negative selfadjoint operators

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We consider

- Laplace operators $-\Delta$ in $L^2(R_3)$ and $L^2(R_2)$;
- the restriction $-\Delta^0$ of $-\Delta$ onto the Sobolev subspaces $H^2_0(R_i \setminus \{0\});$
- self-adjoint extensions $-\Delta_\alpha$, $\alpha \in \mathbb{R}$ of $-\Delta^0$ in $L^2(R_i)$, $i = 3, 2.$
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Problem motivation and definition

Domains of $-\Delta_\alpha$:

\[ D^{(3)}_\alpha := \left\{ f : f \in H^2_2(R^3), \lim_{|x| \downarrow 0} \left[ \frac{d}{d|x|} (|x| f(x)) - \alpha |x| f(x) \right] = 0 \right\}, \]

\[ D^{(2)}_\alpha := \left\{ f : f \in H^2_2(R^2), \lim_{|x| \downarrow 0} \left[ \left( \frac{2\pi \alpha}{\ln |x|} + 1 \right) f(x) - \lim_{|x'| \downarrow 0} \frac{\ln |x|}{\ln |x'|} f(x') \right] = 0. \right\} \]
Domains of $-\Delta_\alpha$:

$$D_{\alpha}^{(3)} := \left\{ f : f \in H^2_2(\mathbb{R}^3), \lim_{|x| \downarrow 0} \left[ \frac{d}{d|x|} (|x|f(x)) - \alpha |x|f(x) \right] = 0 \right\},$$

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Problem motivation and definition

Resolvent kernels (Green functions):

\[
G_{\alpha,Z}^{(3)}(x, x') = \begin{cases} 
G^0_Z(x, x') + \frac{1}{\alpha - i \sqrt{z}/4\pi} G^0_Z(x, 0) G^0_Z(0, x') \\
G^0_Z(x, x') = \frac{\exp i \sqrt{z}|x - x'|}{4\pi |x - x'|}.
\end{cases}
\]

\[
G_{\alpha,Z}^{(2)}(x, x') = \begin{cases} 
G^0_Z(x, x') + \frac{2\pi}{2\pi \alpha - \psi(1) + \ln \left( \frac{\sqrt{z}}{2i} \right)} G^0_Z(x, 0) G^0_Z(0, x') \\
G^0_Z(x, x') = \left( \frac{i}{4} \right) H_0^{(1)}(i \sqrt{z}|x - x'|).
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Resolvent kernels (Green functions):

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\]
All singular perturbations $-\Delta_\alpha$ of the Laplace operator in two dimensions have one negative eigenvalue or the standardly defined Laplace operator $-\Delta$ is the unique non-negative self-adjoint extension in $L_2(\mathbb{R}^2)$ of the symmetric operator $-\Delta^0$. 
Why in some cases the Friedrichs extension is the unique non-negative extension of given non-negative symmetric operator?
Problem motivation and definition: Question

**Question**

Why in some cases the Friedrichs extension is the unique non-negative extension of given non-negative symmetric operator?
Let $A \geq 0$ - self-adjoint operator in the Hilbert space $\mathcal{H}$

- $A^{(0)}$ be a densely defined closed restriction of $A$ onto $\mathcal{D}(A^{(0)}) \subset \mathcal{D}(A)$ of $A$.

Put

$$M := (I + A^{(0)})\mathcal{D}(A^{(0)}) \neq \mathcal{H},$$

$$N := \mathcal{H} \ominus M.$$

We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than $A$ singular perturbations of $A$ (associated with $A_0$).
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We call all self-adjoint extensions of $A^{(0)}$ in $\mathcal{H}$ other than $A$ **singular perturbations** of $A$ (associated with $A_0$).
Let us consider $K_0 : \mathcal{M} \to \mathcal{H}$:

$$\begin{cases} 
  f = (I + A^{(0)}) x, \\
  K_0 f = A^{(0)} x, \ x \in \mathcal{D}(A^{(0)}).
\end{cases}$$

$A_1$ is a non-negative self-adjoint extension of $A_0$ in $\mathcal{H}$ iff $K_1 := A_1 (A_1 + I)^{-1}$ is a non-negative contractive extension of $K_0$ from $\mathcal{M}$ onto $\mathcal{H}$, $K_1 f = K_0 f$, $f \in \mathcal{M}$, $1 \in \sigma(K_1)$.

$A_0$ has unique non-negative self-adjoint extension in $\mathcal{H}$ if and only if $K_0$ admits only one non-negative contractive extension onto the whole $\mathcal{H}$, no eigenvalue of which $= 1$, that is $K := A(I + A)^{-1}$.

The uniqueness of $A$ as non-negative extension of $A_0$ is equivalent to uniqueness of $K$ as non-negative contractive extension of $K_0$. 
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Notation

- $\mathcal{G}$ - the set consisting of $A$ and all its non-negative singular perturbations;
- $\mathcal{C}$ denote the set of non-negative contractions obtained from $\mathcal{G}$ by transformation $A_1 \to A_1 (A_1 + I)^{-1}$, $A_1 \in \mathcal{G}$;
- $P_M$ the orthogonal projector onto $\mathcal{M}$ in $\mathcal{H}$;
- $P_N = I - P_M$. 
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- $P_M$ the orthogonal projector onto $M$ in $\mathcal{H}$
- $P_N = I - P_M$. 
With respect to the representation $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ each $K_X \in \mathcal{C}$ can be represented as

$$K_X = \begin{pmatrix} T & \Gamma^* \\ \Gamma & X \end{pmatrix}$$

Here

$$T = P_M K_0|_M,$$
$$\Gamma = P_M K_0|_M.$$ 

$X$ is a non-negative contraction in $\mathcal{N}$ distinguishing elements from $\mathcal{C}$. 
Since each $K_X \in \mathbf{C}$ is non-negative and contractive then

$$T \geq 0; \quad I \geq T^2 + \Gamma^* \Gamma$$

$K_X \in \mathbf{C}$ is equivalent to

$$K_X + \varepsilon I \geq 0;$$

$$(1 + \varepsilon)I - K_X \geq 0 \quad \varepsilon > 0.$$
By the Schur-Frobenius factorization formula:

\[
\begin{pmatrix}
I \\
\Gamma(T + \varepsilon)^{-1} I
\end{pmatrix} 
\times 
\begin{pmatrix}
T + \varepsilon \\
0
\end{pmatrix} 
\begin{pmatrix}
0 \\
X + \varepsilon - \Gamma(T + \varepsilon)^{-1} \Gamma^*
\end{pmatrix} 
\times 
\begin{pmatrix}
I \\
0
\end{pmatrix} 
\begin{pmatrix}
(T + \varepsilon)^{-1} \Gamma^* \\
I
\end{pmatrix} \geq 0
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\begin{pmatrix}
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-\Gamma(I + \epsilon - T)^{-1} & I
\end{pmatrix} \times 
\begin{pmatrix}
1 + \epsilon - T & 0 \\
0 & 1 + \epsilon - X - \Gamma(1 + \epsilon - T)^{-1}\Gamma^*
\end{pmatrix} \times 
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I & -(1 + \epsilon - T)^{-1}\Gamma^* \\
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\end{pmatrix} \geq 0
\]
Since $T \geq 0$ and $I - T \geq 0$ the above inequalities are reduced to
\[
\begin{cases}
X + \varepsilon I - \Gamma(T + \varepsilon I)^{-1}\Gamma^* \geq 0, \\
(1 + \varepsilon)I - X - \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \geq 0, \quad \varepsilon > 0.
\end{cases}
\]

Setting
\[
Y := X - \lim_{\varepsilon \downarrow 0} \Gamma(T + \varepsilon I)^{-1}\Gamma^*
\]
we conclude that $K_X \in C$ if and only if
\[
0 \leq Y \leq I - \lim_{\varepsilon \downarrow 0} \left( \Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \right).
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Hence

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I - \lim_{\varepsilon \downarrow 0} \left( \Gamma(T + \varepsilon I)^{-1} \Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1} \Gamma^* \right) = 0
\]

is the criterium that there are no contractive non-negative extension of \( K_0 \) in \( \mathcal{H} \) other than \( K \).

To express this criterium in terms of given \( K \) and \( A \) we use the following proposition.
Hence

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To express this criterium in terms of given \( K \) and \( A \) we use the following proposition.
Proposition.

Let $L$ be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as $2 \times 2$ block operator matrix,

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where $R$ and $S$ are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and $U, V$ act between $\mathcal{M}$ and $\mathcal{N}$.

If $R$ is invertible operator in $\mathcal{M}$, then

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_\mathcal{N} \Lambda^{-1} P_\mathcal{N} L^{-1},$$

$$\Lambda = P_\mathcal{N} L^{-1} |_\mathcal{N}.$$
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where $R$ and $S$ are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and $U$, $V$ act between $\mathcal{M}$ and $\mathcal{N}$.

If $R$ is invertible operator in $\mathcal{M}$, then

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_N \Lambda^{-1} P_N L^{-1},$$

$$\Lambda = P_N L^{-1} |_N.$$
Proposition.

Let $L$ be a bounded invertible operator in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ given as $2 \times 2$ block operator matrix,

$$L = \begin{pmatrix} R & U \\ V & S \end{pmatrix},$$

where $R$ and $S$ are invertible operators in $\mathcal{M}$ and $\mathcal{N}$, respectively, and $U$, $V$ act between $\mathcal{M}$ and $\mathcal{N}$.

If $R$ is invertible operator in $\mathcal{M}$, then

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_{\mathcal{N}} \Lambda^{-1} P_{\mathcal{N}} L^{-1},$$

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If $R$ is invertible operator in $\mathcal{M}$, then

$$
\begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} = L^{-1} - L^{-1} P_\mathcal{N} \Lambda^{-1} P_\mathcal{N} L^{-1},
$$

$$
\Lambda = P_\mathcal{N} L^{-1} |_\mathcal{N}.
$$
Set

\[ \Lambda_{1,\varepsilon} = \mathcal{P}_\mathcal{N}(K + \varepsilon I)^{-1}|_{\mathcal{N}} \]
\[ \Lambda_{2,\varepsilon} = \mathcal{P}_\mathcal{N}[(1 + \varepsilon)I - K]^{-1}|_{\mathcal{N}}. \]

Applying the above Proposition with \( L = K + \varepsilon I \) and \( R = T + \varepsilon I \),
\[ R = T + \varepsilon I, \]
\[ U = \Gamma^* = P_\mathcal{M}K|_{\mathcal{N}} = P_\mathcal{M}(K + \varepsilon I)|_{\mathcal{N}}, \]
\[ V = \Gamma = \mathcal{P}_\mathcal{N}K|_{\mathcal{M}} = \mathcal{P}_\mathcal{N}(K + \varepsilon I)|_{\mathcal{M}}, \]
\[ S = \mathcal{P}_\mathcal{N}K|_{\mathcal{N}} + \varepsilon I \]
yields
\[ \Gamma(T + \varepsilon I)^{-1}\Gamma^* = \mathcal{P}_\mathcal{N}K|_{\mathcal{N}} + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}. \]
Set

\[ \Lambda_{1,\varepsilon} = P_{\mathcal{N}}(K + \varepsilon I)^{-1}|_{\mathcal{N}} \]
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\[ R = T + \varepsilon I, \]
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Set

\[ \Lambda_{1,\varepsilon} = P_N(K + \varepsilon I)^{-1}|_N \]
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Applying the above Proposition with \( L = K + \varepsilon I \) and \( R = T + \varepsilon I \),

\[ U = \Gamma^* = P_M K|_N = P_M[K + \varepsilon I]|_N, \]
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\[ S = P_N K|_N + \varepsilon I \]

yields

\[ \Gamma(T + \varepsilon I)^{-1}\Gamma^* = P_N K|_N + \varepsilon I - \Lambda_{1,\varepsilon}^{-1}. \]
In the same fashion we get

$$\Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* = P_N[I - K]|_N + \varepsilon I - \Lambda^{-1}_{2,\varepsilon}.$$ 

Hence

$$I - \lim_{\varepsilon \downarrow 0} \left( \Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \right) = \lim_{\varepsilon \downarrow 0} \Lambda_{1,\varepsilon}^{-1} + \lim_{\varepsilon \downarrow 0} \Lambda_{2,\varepsilon}^{-1}.$$
In the same fashion we get

$$
\Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* = P_N[I - K]|_N + \varepsilon I - \Lambda_{2,\varepsilon}^{-1}.
$$

Hence

$$
\lim_{\varepsilon \downarrow 0} \left( \Gamma(T + \varepsilon I)^{-1}\Gamma^* + \Gamma[(1 + \varepsilon)I - T]^{-1}\Gamma^* \right) = \lim_{\varepsilon \downarrow 0} \Lambda_{1,\varepsilon}^{-1} + \lim_{\varepsilon \downarrow 0} \Lambda_{2,\varepsilon}^{-1}.
$$
Theorem.

Let $K$ be a non-negative contraction in the Hilbert space $\mathcal{H} = M \oplus N$, $K_0$ is the restriction of $K$ onto the subspace $M (= M \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} (P_N[K + \varepsilon I]_N)^{-1}$$

$$G_2 = \lim_{\varepsilon \downarrow 0} (P_N[I - K + \varepsilon I]_N)^{-1}$$

Then the set $\mathcal{C}$ of all non-negative contractive extensions $K_X$ of $K_0$ in $\mathcal{H}$ is described by expression

$$K_X = \begin{pmatrix} P_M K_M & P_M K_N \\ P_M K_N & X \end{pmatrix}, \quad (1)$$

where $X$ runs the set of all non-negative contractions in $N$ satisfying inequalities

$$P_N K_N - G_1 \leq X \leq P_N K_N + G_2. \quad (2)$$

In particular, $K$ is the unique non-negative contractive extension of $K_0$ if and only if $G_1 = G_2 = 0$. 
Theorem.

Let $K$ be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, $K_0$ is the restriction of $K$ onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} (P_\mathcal{N}[K + \varepsilon I]|_\mathcal{N})^{-1}$$

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Then the set $\mathcal{C}$ of all non-negative contractive extensions $K_X$ of $K_0$ in $\mathcal{H}$ is described by expression

$$K_X = \begin{pmatrix} P_\mathcal{M} K|_\mathcal{M} & P_\mathcal{M} K|_\mathcal{N} \\ P_\mathcal{M} K|_\mathcal{N} & X \end{pmatrix},$$

where $X$ runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_\mathcal{N} K|_\mathcal{N} - G_1 \leq X \leq P_\mathcal{N} K|_\mathcal{N} + G_2.$$  

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Theorem.

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$$
G_1 = \lim_{\varepsilon \downarrow 0} (P_N[K + \varepsilon I]|_N)^{-1}
$$

$$
G_2 = \lim_{\varepsilon \downarrow 0} (P_N[I - K + \varepsilon I]|_N)^{-1}
$$

Then the set $\mathcal{C}$ of all non-negative contractive extensions $K_X$ of $K_0$ in $\mathcal{H}$ is described by expression

$$
K_X = \begin{pmatrix} P_MK|_M & P_MK|_N \\ P_MK|_N & X \end{pmatrix},
$$

(1)

where $X$ runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$
P_NK|_N - G_1 \leq X \leq P_NK|_N + G_2.
$$

(2)

In particular, $K$ is the unique non-negative contractive extension of $K_0$ if and only if $G_1 = G_2 = 0.$
Theorem.

Let $K$ be a non-negative contraction in the Hilbert space $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, $K_0$ is the restriction of $K$ onto the subspace $\mathcal{M}(= \mathcal{M} \oplus \{0\})$ and

$$G_1 = \lim_{\varepsilon \downarrow 0} (P_{\mathcal{N}}[K + \varepsilon I]|_{\mathcal{N}})^{-1}$$

$$G_2 = \lim_{\varepsilon \downarrow 0} (P_{\mathcal{N}}[I - K + \varepsilon I]|_{\mathcal{N}})^{-1}$$

Then the set $C$ of all non-negative contractive extensions $K_X$ of $K_0$ in $\mathcal{H}$ is described by expression

$$K_X = \begin{pmatrix} P_{\mathcal{M}}K|_{\mathcal{M}} & P_{\mathcal{M}}K|_{\mathcal{N}} \\ P_{\mathcal{M}}K|_{\mathcal{N}} & X \end{pmatrix},$$ \hspace{1cm} (1)

where $X$ runs the set of all non-negative contractions in $\mathcal{N}$ satisfying inequalities

$$P_{\mathcal{N}}K|_{\mathcal{N}} - G_1 \leq X \leq P_{\mathcal{N}}K|_{\mathcal{N}} + G_2.$$ \hspace{1cm} (2)

In particular, $K$ is the unique non-negative contractive extension of $K_0$ if and only if $G_1 = G_2 = 0$. 

Vadim Adamyan
Non-negative perturbations . . .
Remark.

The set of all non-negative singular perturbations of $A$ contains the minimal perturbation $A_\mu$ with and the maximal perturbation $A_M$ such that any non-negative perturbation $A_1$ satisfies inequalities $A_\mu \leq A_1 \leq A_M$. The corresponding values of parameters $X$ in the above theorem are

$$X_\mu = I|_\mathcal{N} + P_{\mathcal{N}}[I + A]^{-1}|_\mathcal{N} - G_1$$

$$X_M = I|_\mathcal{N} + P_{\mathcal{N}}[I + A]^{-1}|_\mathcal{N} + G_2$$

If $G_1 = 0$ ($G_2 = 0$), then the minimal (maximal) perturbation coincides with $A$. 

Vadim Adamyan
Non-negative perturbations...
Proposition.

The set of resolvents of all non-negative singular perturbations $A_Y$ of $A$ is described by the M.G. Krein formula

$$(A_Y - zI)^{-1} = (A - zI)^{-1} - (A + I)(A - zI)^{-1} P_{\mathcal{N}} Y \times \left[ I + (1 + z)P_{\mathcal{N}}(A + I)(A - zI)^{-1} Y \right]^{-1} \times P_{\mathcal{N}}(A + I)(A - zI)^{-1},$$

where $Y$ runs contractions in $\mathcal{N}$ satisfying inequalities $-G_1 \leq Y \leq G_2$. 
Let $A$ denote the multiplication operator in $L_2(\mathbb{R}_n)$ by the continuous function $\varphi(k)$, $k^2 = k_1^2 + \ldots + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

$$
\int_0^\infty \frac{k^{n-1}}{(1 + \varphi(k))^2} \, dk < \infty.
$$

$A$ is a non-negative self-adjoint operator,

$$
\mathcal{D}(A) = \left\{ f : \int_{\mathbb{R}_n} |1 + \varphi(k)|^2 |f(k)|^2 \, dk < \infty, \ f \in L_2(\mathbb{R}_n) \right\}.
$$
Let $A$ denote the multiplication operator in $L_2(\mathbb{R}^n)$ by the continuous function $\varphi(k)$, $k^2 = k_1^2 + \ldots + k_n^2$, such that $\varphi(k) > 0$ almost everywhere and

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$A$ is a non-negative self-adjoint operator,

$$D(A) = \left\{ f : \int_{\mathbb{R}^n} |1 + \varphi(k)|^2 |f(k)|^2 \, dk < \infty, \; f \in L_2(\mathbb{R}^n) \right\}.$$
Let $\hat{\delta}$ denote the unbounded linear functional in $L_2(\mathbb{R}_n)$:

$$\hat{\delta}(f) = \int_{\mathbb{R}_n} f(k)\,dk.$$ 

Note that $\mathcal{D}(\hat{\delta}) \subset \mathcal{D}(A)$.

Let us denote by $A_0$ the restriction of $A$ onto linear set

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A), \hat{\delta}(f) = 0 \right\}.$$

The closure of $A_0 \neq A$ and

$$\mathcal{N} = (L_2(\mathbb{R}_n) \ominus (I + A)\mathcal{D}_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)}, \xi \in \mathbb{C} \right\}.$$
Let $\hat{\delta}$ denote the unbounded linear functional in $L_2(\mathbb{R}_n)$:

$$
\hat{\delta}(f) = \int_{\mathbb{R}_n} f(k) dk.
$$

Note that $D(\hat{\delta}) \subset D(A)$.

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\[
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\[
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\]

The closure of \( A_0 \neq A \) and

\[
\mathcal{N} = (L_2(\mathbb{R}_n) \ominus (I + A)D_0(A)) = \left\{ \xi \cdot \frac{1}{1 + \varphi(k)}, \, \xi \in \mathbb{C} \right\}.
\]
Proposition.

A is the unique non-negative self-adjoint extension of $A_0$ that is $A$ has no non-negative singular perturbations if and only if

\[ \int_0^\infty \frac{k^{n-1}}{\varphi(k)(1 + \varphi(k))} \, dk = \infty \]

and

\[ \int_0^\infty \frac{k^{n-1}}{(1 + \varphi(k))} \, dk = \infty. \]
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and

$$\int_0^\infty \frac{k^{n-1}}{(1 + \varphi(k))} \, dk = \infty.$$
Put $\varphi(k) = k^2$ and let $n = 2$.

**Corollary.**

*The self-adjoint Laplace operator in $L_2(\mathbb{R}^2)$ has no non-negative singular perturbations with support at one point of $\mathbb{R}^2$.***
The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbb{R}_2)$ with support at two or more points do exist. Let us consider the restriction $A_0$ of the multiplication operator operator by $k^2$, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

$$e_0(k) = \frac{1 - \exp(-i(k \cdot x_0))}{1 + k^2}, \quad x_0 \in \mathbb{R}_2.$$ 

In this case

$$\|e_0\|^2 = \int_{\mathbb{R}_2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{(1 + k^2)^2} \, dk < \infty,$$

$$((I + A)A^{-1}e_0, e_0) = \int_{\mathbb{R}_2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{k^2(1 + k^2)} \, dk < \infty,$$

$$((I + A)e_0, e_0) = \int_{\mathbb{R}_2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{1 + k^2} \, dk = \infty.$$

Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$. 

Vadim Adamyan
Non-negative perturbations . . .
The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbb{R}_2)$ with support at two or more points do exist. Let us consider the restriction $A_0$ of the multiplication operator operator by $k^2$, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

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Vadim Adamyan
Non-negative perturbations ...
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Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$. 

Vadim Adamyan  
Non-negative perturbations...
The non-negative singular perturbations of $-\Delta$ in $L_2(\mathbb{R}_2)$ with support at two or more points do exist. Let us consider the restriction $A_0$ of the multiplication operator operator by $k^2$, for which the defect subspace $\mathcal{N}$ consists of functions collinear to

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Hence $G_1 = \|e_0\|^2 \cdot ((I + A)e_0, e_0)^{-1} > 0$, but $G_2 = 0$. 
The non-negative singular perturbations of \(-\Delta\) in \(L_2(\mathbb{R}^2)\) with support at two or more points do exist. Let us consider the restriction \(A_0\) of the multiplication operator operator by \(k^2\), for which the defect subspace \(\mathcal{N}\) consists of functions collinear to

\[
e_0(k) = \frac{1 - \exp(-i(k \cdot x_0))}{1 + k^2}, \quad x_0 \in \mathbb{R}^2.
\]

In this case

\[
\|e_0\|^2 = \int_{\mathbb{R}^2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{(1 + k^2)^2} \, dk < \infty,
\]

\[
((I+A)A^{-1}e_0, e_0) = \int_{\mathbb{R}^2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{k^2(1 + k^2)} \, dk < \infty,
\]

\[
((I+A)e_0, e_0) = \int_{\mathbb{R}^2} \frac{4 \sin^2 \frac{1}{2}(k \cdot x_0)}{1 + k^2} \, dk = \infty.
\]

Hence \(G_1 = \|e_0\|^2 \cdot ((I+A)e_0, e_0)^{-1} > 0\), but \(G_2 = 0\).
As follows, the concerned restriction $A_0$ of the multiplication operator $A$ by $k^2$ has non-negative self-adjoint extensions in $L_2(\mathbb{R}^2)$ others then $A$ and $A$ is the maximal element in the set of these extensions.
It remains to note that $A$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $L_2(\mathbb{R}_2)$ and $A_0$ is isomorphic to the restriction of this $-\Delta$ on the subset of function $f(x)$ from $\mathcal{D}(-\Delta)$ satisfying conditions:

\[
\lim_{|x| \to 0} (\ln |x|)^{-1} f(x) - \lim_{|x-x_0| \to 0} (\ln |x-x_0|)^{-1} f(x) = 0,
\]

\[
\lim_{|x| \to 0} \left[ f(x) - \ln |x| \lim_{|x'| \to 0} (\ln |x'|)^{-1} f(x') \right] - \lim_{|x-x_0| \to 0} \left[ f(x) - \ln |x-x_0| \lim_{|x'-x_0| \to 0} (\ln |x'-x_0|)^{-1} f(x') \right] = 0.
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\]

\[
\lim_{|x| \to 0} \left[ f(x) - \ln |x| \lim_{|x'| \to 0} (\ln |x'|)^{-1} f(x') \right] - \lim_{|x-x_0| \to 0} \left[ f(x) - \ln |x-x_0| \lim_{|x'-x_0| \to 0} (\ln |x'-x_0|)^{-1} f(x') \right] = 0.
\]
It remains to note that $A$ is isomorphic to the self-adjoint Laplace operator $-\Delta$ in $L^2(\mathbb{R}^2)$ and $A_0$ is isomorphic to the restriction of this $-\Delta$ on the subset of function $f(x)$ from $\mathcal{D}(-\Delta)$ satisfying conditions:

$$
\lim_{|x| \to 0} (\ln |x|)^{-1} f(x) - \lim_{|x-x_0| \to 0} (\ln |x-x_0|)^{-1} f(x) = 0,
$$

$$
\lim_{|x| \to 0} \left[ f(x) - \ln |x| \lim_{|x'| \to 0} (\ln |x'|)^{-1} f(x') \right] - \lim_{|x-x_0| \to 0} \left[ f(x) - \ln |x-x_0| \lim_{|x'-x_0| \to 0} (\ln |x'-x_0|)^{-1} f(x') \right] = 0.
$$
The self-adjoint Laplace operator in $L_2(\mathbb{R}_3)$ has infinitely many non-negative singular perturbations with support at one point of $\mathbb{R}_3$ and the standardly defined Laplace the maximal element in the set of this perturbation.
Consider the multiplication operator $A$ by $k^{2l}$ in $L_2(\mathbb{R}^n)$ assuming that $4l \leq n + 1$. $A$ is isomorphic to the polyharmonic operator $(-\Delta)^l$ in $L_2(\mathbb{R}^n)$.

Let us consider the restriction $A_0$ of $A$ with the domain

$$\mathcal{D}(A_0) := \left\{ f : f \in \mathcal{D}(A), \hat{\delta}(f) = 0 \right\}.$$

that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $H^{2l}_2(\mathbb{R}^n \setminus \{0\})$. 
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that is non-negative symmetric operator which is isomorphic to the restriction of the polyharmonic operator $(-\Delta)^l$ onto the Sobolev subspace $H^{2l}_{2l}(\mathbb{R}^n \setminus \{0\})$. 
Proposition.

If $n < 2l$ then there are infinitely many non-negative singular perturbations of $(-\Delta)^l$ associated with the one-point symmetric restriction $A_0$ and $(-\Delta)^l$ is the minimal element in the set of the non-negative extensions of $A_0$ in $H^2_{2l}(\mathbb{R}_n \setminus \{0\})$.

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If $n = 2l$ then $(-\Delta)^l$ has no such perturbations in $H^2_{2l}(\mathbb{R}_n \setminus \{0\})$.

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