SPARSE FUSION FRAMES: EXISTENCE AND CONSTRUCTION

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Abstract. Fusion frame theory is an emerging mathematical theory that provides a natural framework for performing hierarchical data processing. A *fusion frame* can be regarded as a frame-like collection of subspaces in a Hilbert space, and thereby generalizes the concept of a frame for signal representation. However, when the signal and/or subspace dimensions are large, the decomposition of the signal into its fusion frame measurements through subspace projections typically requires a large number of additions and multiplications, and this makes the decomposition intractable in applications with limited computing budget. To address this problem, in this paper, we introduce the notion of a *sparse fusion frame*, that is, a fusion frame whose subspaces are generated by orthonormal basis vectors that are sparse in a 'uniform basis' over all subspaces, thereby enabling low-complexity fusion frame decompositions.

We study the existence and construction of sparse fusion frames, but our focus is on developing simple algorithmic constructions that can easily be adopted in practice to produce sparse fusion frames with desired (given) operators. By a desired (or given) operator we simply mean one that has a desired (or given) set of eigenvalues for the fusion frame operator. We start by presenting a complete characterization of Parseval fusion frames in terms of the existence of special isometries defined on an encompassing Hilbert space. We then introduce two general methodologies to generate new fusion frames from existing ones, namely the Spatial Complement Method and the Naimark Complement Method, and analyze the relationship between the parameters of the original and the new fusion frame. We proceed by establishing existence conditions for 2-sparse fusion frames for *any* given fusion frame operator, for which the eigenvalues are greater than or equal to two. We then provide an easily implementable algorithm for computing such 2-sparse fusion frames.

1. Introduction

Recent advances in hardware technology have enabled the economic production and deployment of sensing and computing networks consisting of a large number of low-cost components, which through collaboration enable reliable and efficient operation. Across different

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disciplines there is a fundamental shift from centralized information processing to distributed or network-wide information processing. Data communication is shifting from point-to-point communication to packet transport over wide area networks where network management is distributed and the reliability of individual links is less critical. Radar imaging is moving away from single platforms to multiple platforms that cooperate to achieve better performance. Wireless sensor networks are emerging as a new technology with the potential to enable cost-effective and reliable surveillance.

These applications typically involve a large number of data streams, which need to be integrated at a central processor. Low communication bandwidth and limited transmit/computing power at each single node in the network give rise to the need for decentralized data analysis, where data reduction/processing is performed in two steps: local processing at neighboring nodes followed by the integration of locally processed data streams at a central processor.

Fusion frames (or *frames of subspaces*) [21] are a recent development that provide a natural mathematical framework for two-stage (or, more generally, hierarchical) data processing. The need for robust designs against noise, data loss, and channel erasure effects impose certain constraints on the structure of the fusion frame operator [5, 20, 40, 43, 21]. At the same time, constraints on the available computing power and bandwidth for data processing motivates the design of fusion frames that enable signal decomposition with a minimal number of additions and multiplications. These two factors motivate the study of the existence and algorithmic constructions of fusion frames that not only have desired operators but also enjoy some degree of "sparsity" to reduce computational cost. This study is the focus of this paper. We will further clarify our terminology and objectives in Sections 1.3 and 1.4. Our main contributions are highlighted in Section 1.5.

1.1. **Fusion Frames.** The notion of *fusion frames* (or *frames of subspaces*) was introduced in [21] with the main ideas already contained in [18] (see also [7]). However, the concept of a frame-like collection of subspaces was exploited much earlier in relation to domain decomposition techniques in papers by Bjørstad and Mandel [4] and Oswald [42].

In contrast to frame theory, where a signal is represented by a collection of *scalars*, which measure the amplitudes of the projections of the signal onto the frame vectors, in fusion frame theory the signal is represented by a collection of *vectors*, more precisely, the projections of the signal onto the fusion frame subspaces. In a two-stage data processing setup, these projections serve as locally processed data, which can be combined to reconstruct the signal of interest.

Given a Hilbert space $\mathcal{H}$ and a family of closed subspaces $\{\mathcal{W}_i\}_{i\in I}$ with associated positive weights v_i, $i \in I$, a *fusion frame* for $\mathcal{H}$ is a collection of weighted subspaces $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|P_i f\|^2 \leq B\|f\|^2 \qquad \text{for any } f \in \mathcal{H},$$

where P_i is the orthogonal projection onto $\mathcal{W}_i$. The constants A and B are called *fusion frame bounds*. We refer to a fusion frame as being *tight*, if A and B can be chosen to be equal, and *Parseval*, if $A = B = 1$. If $v_i = 1$ for all $i \in I$, for the sake of brevity, we sometimes write $\{\mathcal{W}_i\}_{i\in I}$ instead of $\{(\mathcal{W}_i, 1)\}_{i\in I}$.

The decomposition of any signal $f \in \mathcal{H}$ according to a fusion frame $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ is given by the *fusion frame measurements* $\{v_i P_i f\}_{i\in I}$. These completely characterize the signal f, which can be reconstructed from those by performing

$$f = \sum_{i\in I} v_i S^{-1}(v_i P_i f), \tag{1.1}$$

where $S = \sum_{i\in I} v_i^2 P_i f$ is the *fusion frame operator* known to be positive and self-adjoint. We refer the interested reader to [21] for more details. If dimension reduction is what we seek, the sequence of vector-valued data $\{v_i U_i^* f\}_{i\in I}$ can be regarded as fusion frame measurements (cf. [40]), where U_i is a left-orthogonal basis for $\mathcal{W}_i$, i.e., $P_i = U_i U_i^*$ and $U_i^* U_i = I$. In this case, the reconstruction formula takes the form

$$f = \sum_{i\in I} v_i S^{-1} U_i (v_i U_i^* f).$$

1.2. **Applications of Fusion Frames.** Frame theory has been established as a powerful mathematical framework for robust and stable representation of signals by introducing redundancy.[1] It has found numerous applications in sampling theory [33], data quantization [8], quantum measurements [34], coding [2, 46], image processing [11, 26], wireless communications [35, 37, 45], time-frequency analysis [30, 31, 48], speech recognition [1], and bioimaging [27]. The reader is referred to survey papers [38, 39] and the references therein for more examples.

Since fusion frame theory is a generalization of frame theory that is more suited for applications where two-stage (local and global) signal/data analysis is required, its main applications are situated in areas which require distributed processing. To highlight this, we give three signal processing applications wherein fusion frames arise naturally. We also discuss the connection between fusion frames and two pressing questions in pure mathematics.

Distributed Sensing. Consider a large number of small and inexpensive sensors that are deployed in an area of interest to measure various physical quantities or to keep the area under surveillance. Due to practical and economical factors, such as low communication bandwidth, limited signal processing power, limited battery life, or the topography of the surveillance area, the sensors are typically deployed in clusters, where each cluster includes a unit with higher computational and transmission power for local data processing. A typical large sensor network can thus be viewed as a redundant collection of subnetworks forming a set of subspaces (e.g., see [22, 40, 43]). The local subspace information are passed to a central processing station for joint processing. A similar local-global signal processing principle is applicable to modeling of human visual cortex as discussed in [44].

Parallel Processing. If a frame system is simply too large to handle effectively (from either computational complexity or numerical stability standpoints), we can divide it into multiple small subsystems for simple and perhaps parallelizable processing. Fusion frames provide

[1]Traditionally, frame redundancy has been measured by the ratio of the number of frame elements to the dimension of the Hilbert space it spans. Recently, a quantitative notion of frame redundancy was introduced in [6]. This notion brings valuable insight into the nature of redundancy and the ability of the frame to represent different signals from the Hilbert space. It includes the traditional notion of redundancy as a special case.

a natural framework for splitting a large frame system into smaller subsystems and then recombining the subsystems. We wish to mention that splitting of a large frame system into smaller subsystems for parallel processing was first considered in [4, 42] and predates the introduction of fusion frames.

Packet Encoding. Information bearing symbols are typically encoded into a number of packets and then transmitted over a communication network, e.g., the internet. The transmitted packet may be corrupted during the transmission or completely lost due to buffer overflows. By introducing redundancy in encoding the symbols, we can increase the reliability of the communication scheme. Fusion frames, as redundant collections of subspaces, can be used to produce a redundant representation of a source symbol. In the simplest form, each fusion frame projection can be viewed as a packet that carries some new information about the symbol. The packets can be decoded jointly at the destination to recover the transmitted symbol. The use of fusion frames for packet encoding is considered in [5].

The Kadison-Singer Problem and Optimal Packings. The Kadison-Singer Problem [25] has been among the most famous unsolved problems in analysis since 1959. It turns out that this problem is, roughly speaking, equivalent to the following question (cf. [25]): Can a frame be partitioned such that the spans of the partitions as a fusion frame lead to a 'good' lower fusion frame bound? The reader is referred to [25] for details. Therefore, advances in the design of fusion frames will have direct impact in providing new angles for a renewed attack to the Kadison-Singer Problem. In addition, there is a close connection between Parseval fusion frames and Grassmannian packings. In fact, as shown in [40], Parseval fusion frames consisting of equi-distance and equi-dimensional subspaces are optimal Grassmannian packings. Therefore, new methods for constructing such fusion frames also provide ways to construct optimal packings. We note that the frame counterpart of this connection also exists (cf. [46]).

1.3. **Fusion Frames and Sparsity.** Distributed data processing applications are typically characterized by low on-board computing power, small bandwidth budget, and/or short battery life. When the signal dimension is large, the decomposition of the signal into its fusion frame measurements through subspace projections requires a large of number additions and multiplications, which may be infeasible for on-board data processing. This is unless the fusion frame is designed to have a rather "sparse" structure that reduces the computation of the signal coefficient vectors.

Over the past few years, sparsity has become a key concept in various areas of applied mathematics, computer science, and electrical engineering. Sparse signal processing methodologies explore the fundamental fact that many types of signals can be represented by only a few non-vanishing coefficients when choosing a suitable basis or, more generally, a frame. A signal representable by only k, say, basis or frame elements is called *k-sparse*. If signals possess such a sparse representation, they can in general be recovered from few measurements using ℓ_1 minimization techniques (see, e.g., [10, 12, 29] and the references therein). A natural question to ask is whether sparse representations in fusion frames enjoy similar properties as sparse representation in frames, that is, whether or not they provide a possibility for precise signal reconstruction using only an underdetermined set of equations. And, in fact, this questions was positively answered in [9].

However, we pose a different question concerning sparsity in this paper, by viewing sparsity from a very different standpoint. As mentioned earlier, fusion frame processing suffers from the fact that the computational complexity of determining fusion frame measurements may be high due to the possibly large signal and/or subspace dimensions. It would hence be a significant improvement, if each fusion frame subspace would be spanned by an orthonormal basis, which ensures low-complexity processing. This is precisely the case, if each vector of such a basis is k-sparse with small k.

A generalization of this consideration is made precise in the following definition.

Definition 1.1. *A fusion frame $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ is k-sparse with respect to an orthonormal basis $\{e_j\}_{j=1}^M$ for $\mathbb{R}^M$ if each subspace $\mathcal{W}_i$ is spanned by an orthonormal basis $\{e_{ij}\}_{j=1}^{m_i}$ so that for each $j = 1, 2, \ldots, m_i$, $e_{ij} \in \text{span } \{e_\ell\}_{\ell\in J}$ and $|J| \leq k$.*

This view of sparsity is fundamentally new in fusion frame theory, and even in classical frame theory, and it constitutes a main focus of our paper.

1.4. **Fusion Frames With Desired Operators.** The value of fusion frames for signal processing is that the interplay between local-global processing and redundant representation provides resilience to noise and erasures due to, for instance, sensor failures or buffer overflows [5, 20, 40, 43]. It also provides robustness to subspace perturbations [21], which may be due to imprecise knowledge of sensor network topology. In most cases, extra structure on fusion frames is required to manage distortion in the presence of noise and erasures.

Our recent work [40, 43] shows that in order to minimize the mean-squared error in the linear minimum mean-squared error estimation of a random vector from its fusion frame measurements in white noise the fusion frame needs to be Parseval or tight. The Parseval property is also desirable for managing signal processing complexity, since in this situation the fusion frame operator S is equal to the identity operator, and hence the operator inversion required for signal reconstruction as in (1.1) is trivial. To allow additional flexibility, we might however also aim to design a fusion frame not only with S equaling the identity operator, but also equaling a different operator of our choice. Furthermore, to provide maximal robustness against erasures of one fusion frame subspace the fusion frame subspaces must also be equi-dimensional. If maximal robustness with respect to two or more subspace erasures is desired then the fusion frame subspaces must all have the same pairwise chordal distance as well. Other examples of optimality of structured fusion frames for signal reconstruction can be found in [5, 20, 40, 43, 21].

The need to manage both distortion and computational complexity motivates a fundamental question, that is, *how can one construct sparse fusion frames with desired properties?* More specifically, how can one construct sparse fusion frames for which a set of parameters such as

- eigenvalues of the fusion frame operator,
- dimensions of the subspaces,
- chordal distances between subspaces, and/or
- weights assigned to the subspaces

can be prescribed?

1.5. **Main Contributions.** In this paper, we consider the construction of fusion frames with desired properties, with emphasis on sparsity. Our main contributions are highlighted below.

Characterization of Parseval Fusion Frames. We present a complete characterization of Parseval fusion frames, as an important class of fusion frames, in terms of the existence of special isometries defined on an encompassing Hilbert space. This characterization is expressed in Theorem 2.1.

New Fusion Frames from Existing Ones. We introduce two general methodologies to generate new fusion frames from existing ones, namely the Spatial Complement Method described in Theorem 3.3 as well as the Naimark Complement Method detailed in Theorem 3.6. The former approach applies to general fusion frames, whereas the latter is only applicable to Parseval fusion frames. Both methods are carefully analyzed concerning the relationship between the parameters of the two fusion frames. In particular, we show how the weights, subspace dimensions, fusion frame bounds, eigenvalues of the fusion frame operator, and the chordal distance between the subspaces for the new fusion frame can be determined from those of the original fusion frame prior to construction. This provides further insights into the analysis and construction of fusion frames with desired properties, including Parseval fusion frames.

Existence and Algorithmic Constructions of Sparse Fusion Frames. We establish existence conditions for 2-sparse fusion frames for *any* given fusion frame operator[2], for which the eigenvalues are greater than or equal to two, and provide an easily implementable algorithm for computing such 2-sparse fusion frames. The key feature of our construction is its simplicity. The mild restriction on the eigenvalues (being greater than or equal to 2) may seem artificial at first glance, however we conjecture that this is indeed necessary for achieving 2-sparsity; but we do not have a rigorous proof for this claim. We believe that as the eigenvalues of the fusion frame operator decrease from two to one, the sparsity deteriorates rapidly. For instance, presumably, the sparsest tight frames with $M+1$ unit vectors in $\mathbb{R}^M$ are those constructed by Tremain in [47], which have sparsity on the order of $M^2/2$. Finally, we note that, since fusion frame theory contains frame theory as a special case, our construction includes the construction of sparse frames with desired frame operator.

1.6. **Related Results.** The construction of frames with arbitrary frame operators has already been studied by several authors and we refer to [23, 3, 14, 24, 19, 32] and references therein. Establishing existence conditions for fusion frames is an even deeper and much more involved problem.

In 2004, Dykema *et al.* [32] gave fusion frame constructions (not under the name of fusion frames) in the operator theoretic setting using a degeneracy condition on the eigenvalues of the fusion frame operator. Their paper establishes an intriguing and deep set of results and one can argue that it even gives an "algorithm" for constructing fusion frames. However, such construction involves an induction proof on the dimension of the Hilbert space, and hence

[2]Throughout this paper whenever we say a fusion frame with a desired (or given) fusion frame operator we mean a fusion frame for which the fusion frame operator has a desired set of eigenvalues. A similar language is used to refer to a frame for which the frame operator has a desired set of eigenvalues.

is not easily implementable in practice. Also, their procedure is not designed to address the sparsity question, which is one of our main focusses in this paper.

Frame potentials, which were introduced in [3], have proven to be a valuable tool in asserting the existence of tight frames. The recent paper [15] introduced and studied fusion frame potentials, showing that its minimization is equivalent to the minimization of the usual frame potential over a particular domain. The problem with this approach is however that minimizers of the fusion frame potential are not necessarily tight fusion frames.

A groundbreaking and comprehensive study of fusion frames with weights and non-constant dimensional subspaces was later carried out by Massey, Ruiz and Stojanoff [41]. Their paper presents a general characterization of the eigenvalues of a fusion frame operator in the form of Horn-Klyachko compatibility conditions. Although this fundamental piece of work classifies when fusion frames exist, it does not provide an algorithmic construction method for finding them. It does not consider sparsity either.

A significant advance for the construction of equi-dimensional tight fusion frames was presented in [16]. The authors have provided a complete characterization of triples (M, N, m) for which tight fusion frames exist, where M denotes the total dimension of the Hilbert space, N the number of subspaces, and m the dimension of the fusion frame subspaces. They have also developed an elegant and simple algorithm which can produce a tight fusion frame for most (M, N, m) triples. It turns out that this construction actually results in some degree of sparsity but sparsity was not pursued as a construction principle in [16]. In comparison, our paper is concerned with a more general question, more precisely, the construction of fusion frames for which the fusion frame operator can possess any desired set of eigenvalues greater than or equal to two.

1.7. **Outline.** In Section 2, we provide a complete characterization of the class of Parseval fusion frames. Section 3 presents two general methodologies – the Spatial Complement Method and the Naimark Complement Method – to construct a new fusion frame from an existing one with control on particular properties of the generated fusion frame. In Section 4, we focus on the algorithmic construction of sparse fusion frames with prescribed fusion frame operators and present simple algorithms for this complex problem under mild assumptions. Extensions and related problems are discussed in Section 5.

2. Characterization of Parseval Fusion Frames

In this section, we provide a characterization of *Parseval* fusion frames in terms of the existence of special isometries defined on an encompassing Hilbert space. This characterization may be viewed as the fusion frame counterpart to *Naimark's theorem* [13, 17, 28, 36], where Parseval frames are characterized as frame systems generated by an orthogonal projection of an orthonormal basis from a larger Hilbert space. However, these characterizations cannot be easily exploited for constructing Parseval frames or Parseval fusion frames. The difficulty arises from the uncontrollable nature of the projection of the larger Hilbert space. For fusion frames, the construction of appropriate isometries are particularly difficult. In fact, these problems are equivalent to serious unsolved problems in operator theory concerning the construction of projections which sum to a given operator. Nonetheless, these isometries are illuminating for understanding Parseval fusion frames.

The following theorem states the main result of this section, which can be regarded as a quantitative version of [21, Thm. 3.1].

Theorem 2.1. *For a complete family of subspaces*[3] $\{\mathcal{W}_i\}_{i\in I}$ *of* $\mathcal{H}$ *and positive weights* $\{v_i\}_{i\in I}$*, the following conditions are equivalent.*

(i) $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ *is a Parseval fusion frame for* $\mathcal{H}$*.*

(ii) *There exists a Hilbert space* $\mathcal{K} \supset \mathcal{H}$*, an orthonormal basis* $\{e_j\}_{j\in J}$ *for* $\mathcal{K}$*, a partition* $\{J_i\}_{i\in I}$ *of* J*, and isometries* $L_i : \mathcal{E}_i := span\{e_j\}_{j\in J_i} \to \mathcal{W}_i$*,* $i \in I$*, such that*

$$P = \sum_{i\in I} v_i L_i$$

is an orthogonal projection of $\mathcal{K}$ *onto* $\mathcal{H}$*.*

Proof. (i) $\Rightarrow$ (ii). For every $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{W}_i$. Since $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$, by [21, Thm. 2.3], the family $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathcal{H}$. This implies (cf. [13, 28, 36]) that there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ with an orthonormal basis $\{\tilde{e}_{ij}\}_{i\in I, j\in J_i}$ so that the orthogonal projection P of $\mathcal{K}$ onto $\mathcal{H}$ satisfies

$$P(\tilde{e}_{ij}) = v_i e_{ij}, \qquad i \in I,\ j \in J_i.$$

Setting $\mathcal{E}_i = \mathrm{span}\{\tilde{e}_{ij}\}_{j\in J_i}$, the map

$$L_i := \frac{1}{v_i} P|_{\mathcal{E}_i} : \mathcal{E}_i \to \mathcal{W}_i$$

is an isometry for all $i \in I$, and

$$P = \sum_{i\in I} v_i L_i$$

is an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

(ii) $\Rightarrow$ (i). Since $P = \sum_{i\in I} v_i L_i$ is an orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, $\{Pe_j\}_{j\in J}$ is a Parseval frame for $\mathcal{H}$. Further, since $L_i := 1/v_i \cdot P|_{\mathcal{E}_i} : \mathcal{E}_i \to \mathcal{W}_i$ is an isometry, it follows that $\{1/v_i \cdot Pe_j\}_{j\in J_i}$ is an orthonormal basis for $\mathcal{W}_i$, $i \in I$. Applying these observations and denoting by P_i the orthogonal projection onto $\mathcal{W}_i$, for all $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in I} v_i^2 \|P_i f\|^2 &= \sum_{i\in I} v_i^2 \Big\| \sum_{j\in J_i} \Big\langle f, \frac{1}{v_i} Pe_j \Big\rangle \frac{1}{v_i} Pe_j \Big\|^2 \\
&= \sum_{i\in I} v_i^2 \sum_{j\in J_i} |\Big\langle f, \frac{1}{v_i} Pe_j \Big\rangle|^2 \\
&= \sum_{i\in I} \sum_{j\in J_i} |\langle f, Pe_j\rangle|^2 \\
&= \|f\|^2.
\end{aligned}$$

Thus $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ is a Parseval fusion frame as claimed. □

[3] A family of subspaces is called *complete* in $\mathcal{H}$, if their span equals $\mathcal{H}$.

Considering this theorem and its proof, we can derive an interesting corollary which links the construction of Parseval fusion frames to the construction of special Parseval frames. In fact, the question of existence of Parseval fusion frames is equivalent to the question of existence of Parseval frames for which certain subsets of frame vectors are orthonormal. The answer to this question is not known, but the connection between the two problems may provide insights into the construction of Parseval fusion frames.

Corollary 2.2. *For a family of subspaces $\{\mathcal{W}_i\}_{i\in I}$ of $\mathcal{H}$ and positive weights $\{v_i\}_{i\in I}$, the following conditions are equivalent.*

(i) *$\{(W_i, v_i)\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{H}$.*
(ii) *There exists a Parseval frame $\{e_{ij}\}_{i\in I, j\in J_i}$ for $\mathcal{H}$ such that $\{1/v_i \cdot e_{ij}\}_{j\in J_i}$ is an orthonormal basis for $\mathcal{W}_i$ for all $i \in I$.*

3. Construction of New Fusion Frames from Existing Ones

In this section, we present two general ways, namely the Spatial Complement Method and the Naimark Complement Method, for constructing a new fusion frame from a given fusion frame and establish the relationship between the parameters of the two fusion frames. These ideas were first developed in [16] for constructing tight fusion frames with given parameters. A special case of the construction methods presented here is reported in [16]. The result of [16] deals only with the construction of Parseval fusion frames in a finite dimensional Hilbert space and does not investigate the relation between the new and the original fusion frame parameters.

3.1. The Spatial Complement Method.

Taking the spatial complement appears to be a natural way for generating a new fusion frame from a given fusion frame. We begin by defining the notion of an *orthogonal fusion frame to a given fusion frame*, which is central to our discussion.

Definition 3.1. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. If the family $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$, where $W_i^\perp$ is the orthogonal complement of W_i, is also a fusion frame, then we call $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$ the orthogonal fusion frame to $\{(\mathcal{W}_i, v_i)\}_{i\in I}$.*

Theorem 3.2. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with optimal fusion frame bounds $0 < A \le B < \infty$ such that $\sum_{i\in I} v_i^2 < \infty$. Then the following conditions are equivalent.*

(i) $\bigcap_{i\in I} \mathcal{W}_i = \{0\}$.
(ii) $B < \sum_{i\in I} v_i^2$.
(iii) *The family $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with optimal fusion frame bounds $\sum_{i\in I} v_i^2 - B$ and $\sum_{i\in I} v_i^2 - A$.*

Proof. (iii) $\Rightarrow$ (i): Suppose that (i) is false. Then there exists a vector $0 \ne f \in \cap_{i\in I}\mathcal{W}_i$. This implies $f \perp \mathcal{W}_i^\perp$ for all $i \in I$, hence $\{\mathcal{W}_i^\perp\}_{i\in I}$ does not span $\mathcal{H}$. This is a contradiction to (iii).

(i) $\Rightarrow$ (ii): Since B is optimal, by using the fusion frame property, it follows that there exists some $f \in \mathcal{H}$ so that

$$B\|f\|^2 = \Big\langle \sum_{i\in I} v_i^2 P_i f, f \Big\rangle = \sum_{i\in I} v_i^2 \|P_i f\|^2 \le \sum_{i\in I} v_i^2 \|f\|^2.$$

Hence

$$B \le \sum_{i\in I} v_i^2. \tag{3.2}$$

It now suffices to observe that we have equality in (3.2) if and only if

$$f \in \bigcap_{i\in I} \mathcal{W}_i \neq \{0\}.$$

(ii) $\Rightarrow$ (iii): Since $AI \le \sum_{i\in I} v_i^2 P_i \le BI$, we have

$$\left(\sum_{i\in I} v_i^2 - B\right) I \le \sum_{i\in I} v_i^2 (I - P_i) \le \left(\sum_{i\in I} v_i^2 - A\right) I. \tag{3.3}$$

From (ii), we have $\sum_{i\in I} v_i^2 - B > 0$ and hence

$$\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I} = \{((I - P_i)\mathcal{H}, v_i)\}_{i\in I},$$

is a fusion frame. The fusion frame bounds from (3.3) are optimal. □

The following theorem shows that all the parameters of the new fusion frame can be determined from those of the generating fusion frame prior to the construction.

Theorem 3.3. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$ be its associated orthogonal fusion frame. Then the following conditions hold.*

(i) *Let S denote the frame operator for $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ with eigenvectors $\{e_j\}_{j\in J}$ and respective eigenvalues $\{\lambda_j\}_{j\in J}$. Then the fusion frame operator for $\{(\mathcal{W}_i^\perp, v_i)\}_{i=1}^N$ possesses the same eigenvectors $\{e_j\}_{j\in J}$ and respective eigenvalues $\{\sum_{i\in I} v_i^2 - \lambda_j\}_{j\in J}$.*

(ii) *Assume that $\dim \mathcal{H} < \infty$ and $m := \dim \mathcal{W}_i$ for all $i \in I$. Then,*

$$d_c^2(\mathcal{W}_i^\perp, \mathcal{W}_j^\perp) = d_c^2(\mathcal{W}_i, \mathcal{W}_j) + 2m - \dim \mathcal{H} \quad \text{for all } i, j \in \{1, \dots, N\},\ i \neq j.$$

where $d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ denotes the squared chordal distance between subspaces $\mathcal{W}_i$ and $\mathcal{W}_j$ and is given by

$$d_c^2(\mathcal{W}_i, \mathcal{W}_j) = \dim \mathcal{H} - \mathrm{tr}[P_i P_j].$$

Proof. (i). For each $j \in J$, we have

$$\sum_{i\in I} v_i^2 P_i e_j = \lambda_j e_j.$$

Hence,

$$\sum_{i\in I} v_i^2 (I - P_i) e_j = \left(\sum_{i\in I} v_i^2 - \lambda_j\right) e_j,$$

which implies the claimed properties for the fusion frame operator $S^{\perp}$.
(ii). The orthogonal projection onto $\mathcal{W}_i^{\perp}$ is given by $I - P_i$. Hence,

$$d_c^2(\mathcal{W}_i^{\perp}, \mathcal{W}_j^{\perp}) = \dim \mathcal{H} - \operatorname{tr}[(I - P_i)(I - P_j)].$$

The claim follows from

$$\operatorname{tr}[(I - P_i)(I - P_j)] = \operatorname{tr}[I - P_i - P_j + P_i P_j] = \dim \mathcal{H} - 2m + \operatorname{tr}[P_i P_j]$$

and the definition of $d_c^2(\mathcal{W}_i, \mathcal{W}_j)$. □

Corollary 3.4. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be an A-tight fusion frame for $\mathbb{R}^M$ such that $\mathcal{W}_k \neq \mathcal{H}$ for some $k \in \{1, \ldots, N\}$. Then $\{W_i^{\perp}\}_{i=1}^N$ is an $(N - A)$-tight fusion frame for $\mathbb{R}^M$. If $m := \dim \mathcal{W}_i$ for all $i \in \{1, \ldots, N\}$ and $d^2 := d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ for all $i, j \in \{1, \ldots, N\}$, $i \neq j$, then*

$$d_c^2(\mathcal{W}_i^{\perp}, \mathcal{W}_j^{\perp}) = d^2 + 2m - M \quad \text{for all } i, j \in \{1, \ldots, N\},\ i \neq j.$$

Proof. Assume that $W_k \neq \mathbb{R}^M$. Then by choosing some $0 \neq f \in W_k^{\perp}$, we obtain

$$A\|f\|^2 = \sum_{i=1}^N v_i^2 \|P_i f\|^2 = \sum_{i \neq k} v_i^2 \|P_i f\|^2 < \Big(\sum_{i=1}^N v_i^2\Big) \|f\|^2.$$

Thus we have $A < \sum_{i=1}^N v_i^2$, and the application of Theorem 3.2 proves the first part of the claim. The second part follows immediately from Theorem 3.3 (ii). □

A straightforward application of Corollary 3.4 provides a way of constructing tight fusion frames with equi-dimensional subspaces. This construction starts with a given set of equi-dimensional subspaces that do not form a tight fusion frames and fills up the Hilbert space by adding a new set of subspaces, with the same dimension, to produce a tight fusion frame.

Corollary 3.5. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a family of m-dimensional subspaces of $\mathbb{R}^M$. Then there exist $N(M-1)$ m-dimensional subspaces $\{\mathcal{V}_i\}_{i=1}^{N(M-1)}$ of $\mathbb{R}^M$ so that $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N(M-1)}$ is a tight fusion frame. Moreover, if $N = 1$ and $\dim \mathcal{W}_1 = M - 1$ then the construction is minimal in the sense that it identifies the smallest number of m-dimensional subspaces which need to be added to obtain a tight fusion frame.*

Proof. For each $i = 1, \ldots, N$, we choose an orthonormal basis $\{e_j^i\}_{j=1}^M$ for $\mathbb{R}^M$ in such a way that $\{e_j^i\}_{j=1}^m$ is an orthonormal basis for $\mathcal{W}_i$. Let T_i, $i = 1, \ldots, N$, denote the circular shift operator on the orthonormal basis $\{e_j^i\}_{j=1}^M$. Then

$$\{T_i^k \mathcal{W}_i\}_{i=1,k=0}^{N\ ,M-1},$$

is a tight fusion frame for $\mathbb{R}^M$ of m-dimensional subspaces which contains $\{\mathcal{W}_i\}_{i=1}^N$.

Now consider the case where $N = 1$ and $\dim \mathcal{W}_1 = M - 1$. Let $\{\mathcal{V}_i\}_{i=1}^{N_1}$ be any collection of $(M-1)$-dimensional subspaces so that $\{\mathcal{W}_1\} \cup \{\mathcal{V}_i\}_{i=1}^{N_1}$ is a tight fusion frame. By Theorem 3.2, we have $1 + N_1 = M$, hence $N_1 = M - 1$, which equals $N(M-1)$. □

3.2. **The Naimark Complement Method.** Another approach to constructing a new fusion frame from an existing one is to use the notion of *Naimark complement*. This approach however applies to Parseval fusion frames only, as stated in the following theorem.

Theorem 3.6. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$ with $0 < v_i < 1$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Parseval fusion frame $\{(\mathcal{W}_i', \sqrt{1-v_i^2})\}_{i\in I}$ for $\mathcal{K}\ominus\mathcal{H}$ with the following properties.*

(i) $\dim \mathcal{W}_i' = \dim \mathcal{W}_i$ *for all* $i \in I$.

(ii) *If* $\dim \mathcal{H} < \infty$ *and* $\dim \mathcal{W}_i = \dim \mathcal{W}_j$ *for all* $i, j \in I$, $i \neq j$, *then*

$$d_c^2(\mathcal{W}_i', \mathcal{W}_j') = d_c^2(\mathcal{W}_i, \mathcal{W}_j) \quad \text{for all } i, j \in \{1, \ldots, N\},\ i \neq j.$$

Proof. For each $i \in I$, let $\{f_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\mathcal{W}_i$. Then the family

$$\{v_i f_{ij}\}_{i\in I, j\in J_i}$$

is a Parseval frame for $\mathcal{H}$. By [13, 28, 36], there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, an orthogonal projection $P : \mathcal{K} \to \mathcal{H}$, and an orthonormal basis $\{e_{ij}\}_{i\in I, j\in J_i}$ for $\mathcal{K}$ so that

$$Pe_{ij} = v_i f_{ij}, \qquad i \in I,\ j \in J_i. \tag{3.4}$$

This implies that $\{(I - P)e_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathcal{K} \ominus \mathcal{H}$. Further,

$$\|(I - P)e_{ij}\| = \sqrt{1 - v_i^2}, \qquad i \in I,\ j \in J_i,$$

and, for $j, j' \in J_i$, $j \neq j'$, we have

$$\langle (I - P)e_{ij}, (I - P)e_{ij'}\rangle = -\langle Pe_{ij}, e_{ij'}\rangle = -\langle v_i f_{ij}, v_i f_{ij'}\rangle = 0.$$

Defining

$$\mathcal{W}_i' = \operatorname{span}\{(I - P)e_{ij} : j \in J_i\},$$

we conclude that $\{(\mathcal{W}_i', \sqrt{1 - v_i^2})\}_{i\in I}$ is a Parseval fusion frame for $\mathcal{K} \ominus \mathcal{H}$. Also, since $v_i, \sqrt{1 - v_i^2} \neq 0$ we have that dim $\mathcal{W}_i'$ = dim $\mathcal{W}_i$.

(i). By construction,

$$\dim \mathcal{W}_i' = |J_i| = \dim \mathcal{W}_i \qquad \text{for all } i \in I.$$

(ii). Set $M := \dim \mathcal{H}$, $L := \dim \mathcal{K}$, $I := \{1, \ldots, N\}$, and $m := \dim \mathcal{W}_i$ for all $i \in \{1, \ldots, N\}$. For the sake of brevity, we define $E_i := ((I - P)e_{i1}, \ldots, (I - P)e_{im}) \in \mathbb{R}^{M\times m}$ and $F_i := (v_i f_{i1}, \ldots, v_i f_{im}) \in \mathbb{R}^{M\times m}$. Then, for every $i, i' \in \{1, \ldots, N\}$, $i \neq i'$, we obtain

$$\operatorname{tr}[P_i P_{i'}] = \operatorname{tr}[F_i F_i^T F_{i'} F_{i'}^T] = \operatorname{tr}[(F_{i'}^T F_i)(F_i^T F_{i'})] = \operatorname{tr}[(\langle v_i f_{i'j}, v_i f_{ik}\rangle)_{j,k}(\langle v_i f_{ij}, v_i f_{i'k}\rangle)_{j,k}].$$

By employing (3.4),

$$\operatorname{tr}[P_i P_{i'}] = \operatorname{tr}[(\langle Pe_{i'j}, Pe_{ik}\rangle)_{j,k}(\langle Pe_{ij}, Pe_{i'k}\rangle)_{j,k}]. \tag{3.5}$$

Now letting P_i' denote the orthogonal projection onto $\mathcal{W}_i'$, for each $i, i' \in \{1, \ldots, N\}$, $i \neq i'$, the definition of $\mathcal{W}_i'$ implies

$$\operatorname{tr}[P_i' P_{i'}'] = \operatorname{tr}[E_i E_i^T E_{i'} E_{i'}^T] = \operatorname{tr}[(E_{i'}^T E_i)(E_i^T E_{i'})]$$

and

$$(E_{i'}^T E_i) = (\langle (I - P)e_{i'j}, (I - P)e_{ik}\rangle)_{j,k}.$$

Utilizing the choice of $\{e_{ij}\}$ and careful dealing with the inner products on $\mathcal{K}, \mathcal{H}$, and $\mathcal{K} \ominus \mathcal{H}$, for each j, k,

$$\langle (I-P)e_{i'j}, (I-P)e_{ik}\rangle = \langle e_{i'j}, e_{ik}\rangle - \langle Pe_{i'j}, Pe_{ik}\rangle = -\langle Pe_{i'j}, Pe_{ik}\rangle.$$

Combining the above three equations,

$$\mathrm{tr}[P_i' P_{i'}'] = \mathrm{tr}[(\langle Pe_{i'j}, Pe_{ik}\rangle)_{j,k}(\langle Pe_{ij}, Pe_{i'k}\rangle)_{j,k}].$$

Comparison with (3.5) completes the proof. □

Definition 3.7. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a tight fusion frame for $\mathcal{H}$. We refer to the tight fusion frame $\{(\mathcal{W}_i', \sqrt{1-v_i^2})\}_{i\in I}$ for $\mathcal{K}\ominus\mathcal{H}$ from Theorem 3.6 as the Naimark fusion frame associated with $\{(\mathcal{W}_i, v_i)\}_{i\in I}$. The rationale for this terminology is that this is the fusion frame version of the Naimark theorem* [13, 28, 36].

Corollary 3.8. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be an A-tight fusion frame for $\mathbb{R}^M$. Then there exists some $L \geq M$ and a $\sqrt{1-1/A^2}$-tight fusion frame for $\mathbb{R}^{L-M}$ which satisfies $\dim \mathcal{W}_i' = \dim \mathcal{W}_i$ for all $i \in \{1,\ldots,N\}$. If, in addition, $d^2 := d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ for all $i,j \in \{1,\ldots,N\}$, $i \neq j$, then*

$$d_c^2(\mathcal{W}_i', \mathcal{W}_j') = d^2 \quad \text{for all } i,j \in \{1,\ldots,N\},\ i\neq j.$$

Proof. This follows immediately from Theorem 3.6. □

We note that Theorem 3.6 is not always constructive, since it requires the knowledge of a larger Hilbert space from which the given Parseval frame is derived by an orthogonal projection of an orthonormal basis.

4. Construction of Sparse Fusion Frames with A Desired Fusion Frame Operator

We now focus on the existence and construction of sparse fusion frames whose fusion frame operators possess a desired set of eigenvalues. We answer the following questions: (1) Given a set of eigenvalues, does there exist a sparse fusion frame whose fusion frame operator possesses those eigenvalues? (2) If such a fusion frame exists how can it be constructed?

Let $\lambda_1 \geq \ldots \geq \lambda_M > 0$, $M \in \mathbb{N}$, be real positive values satisfying a factorization as

$$\text{(FAC)} \qquad \sum_{j=1}^{M} \lambda_j = mN \in \mathbb{N}.$$

We wish to construct a 2-sparse fusion frame $\{\mathcal{W}_i\}_{i=1}^N$, $\mathcal{W}_i \subseteq \mathbb{R}^M$, such that

(FF1) $\dim \mathcal{W}_i = m$ for all $i = 1\ldots,N$, and

(FF2) the associated fusion frame operator has $\{\lambda_j\}_{j=1}^M$ as its eigenvalues.

In [32], Dykema at al. proved that an operator A with discrete spectrum having kn strictly positive eigenvalues, each repeated a multiple of k times, is the sum of r rank-k projections provided that $\mathrm{tr}[A] = rk$. Interestingly, the trace condition coincides with (FAC) if we restrict our situation to the hypotheses put on the eigenvalues in [32]. Our goal is different here. We will give an implementable algorithm for computing sparse fusion frames with a desired fusion frame operator. This will require a restriction on the eigenvalues of the fusion frame operator to bring us to the case where sparse fusion frames exist.

4.1. **The Integer Case.** We first consider the simple case where $\lambda_j \in \mathbb{N}$ for all $i = 1, \ldots, M$. This case is central to developing intuition about the construction algorithms to be developed.

Proposition 4.1. *If the positive integers $N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M > 0$, $N \in \mathbb{N}$, and $m \in \mathbb{N}$ satisfy* (Fac)*, then the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ constructed via the* (SFFI) *algorithm outlined in Figure 1 satisfies both* (FF1) *and* (FF2) *and the fusion frame is 1-sparse.*

SFFI: Sparse Fusion Frame Construction for Integer Eigenvalues

Parameters:
- Dimension $M \in \mathbb{N}$.
- Integer eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M > 0$, number of subspaces N, and dimension of subspaces m satisfying (Fac).

Algorithm:

1) Set $k := 1$.
2) For $j = 1, \ldots, M$ do
3)   Repeat
4) $w_k := e_j$.
5) $k := k + 1$.
6) $\lambda_j := \lambda_j - 1$.
7)   until $\lambda_j = 0$.
8) end.

Output:
- Fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with $\mathcal{W}_i := \text{span}\{w_{i+kN} : k = 0, \ldots, m-1\}$.

Figure 1. The SFFI Algorithm for constructing a 1-sparse fusion frame with a fusion frame operator with prescribed integer eigenvalues.

Proof. If the set of vectors
$$\{w_{i+kN} : k = 0, \ldots, m-1\}$$
is pairwise orthogonal for each $i = 1, \ldots, N$, then (FF1) and (FF2) follow automatically. Now fix $i \in \{1, \ldots, N\}$. By construction, it is sufficient to show that, for each $0 \leq k \leq m-2$, the vectors w_{i+kN} and $w_{i+(k+1)N}$ are orthogonal. Again by construction, the only possibility for this to fail is that there exists some $j_0 \in \{1, \ldots, M\}$ satisfying $\lambda_{j_0} > N$. But this was excluded by the hypothesis.

The fact that the fusion frame is 1-sparse follows immediately from the construction and Definition 1.1. □

The algorithm outlined in Figure 1 shuffles the intended eigenvalues in terms of associated unit vectors $e_1, \ldots, e_M \in \mathbb{R}^M$ as basis vectors into the subspaces of the 1-sparse fusion frame to be constructed. Considering a matrix $W \in \mathbb{R}^{mN \times M}$ with the vectors $w_1, \ldots, w_{mN}$ as rows, intuitively (SFFI) fills this matrix up from top to bottom, row by row in such a way that the ℓ_2 norm of the rows is 1, the ℓ_2 norm of column j is λ_j, $j = 1, \ldots, M$, and the columns are orthogonal. The vectors w_k are then assigned to subspaces in such a way that the vectors assigned to each subspace forms an orthonormal system. We note that the generated vectors w_k, $k = 1 \ldots, mN$ are as sparse as possible, providing fast computation abilities.

We wish to note that the condition $N \geq \lambda_1$ is always necessary since each fusion frame subspace can contribute at most one to the largest eigenvalue.

4.2. **The General Case.** We now discuss the general case where the desired eigenvalues for the fusion frame operator are real positive values that satisfy (FAC).

4.2.1. *The Algorithm.* As a first step we generalize (SFFI) (see Figure 1) by introducing Lines 4) – 9), which deal with the non-integer parts. The construction algorithm for real eigenvalues, called (SFFR), is outlined in Figure 2.

SFFR: SPARSE FUSION FRAME CONSTRUCTION FOR REAL EIGENVALUES

Parameters:

- Dimension $M \in \mathbb{N}$.
- Eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M > 0$, number of subspaces N, and dimension of subspaces m satisfying (FAC).

Algorithm:

1) Set $k := 1$.
2) For $j = 1, \ldots, M$ do
3)   Repeat
4)   If $\lambda_j < 1$ then
5) $w_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
6) $w_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
7) $k := k + 2$.
8) $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
9) $\lambda_j := 0$.
10)   else
11) $w_k := e_j$.
12) $k := k + 1$.
13) $\lambda_j := \lambda_j - 1$.
14)   end;
15)   until $\lambda_j = 0$.
16) end;

Output:

- Fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with $\mathcal{W}_i := \text{span}\{w_{i+kN} : k = 0, \ldots, m-1\}$.

FIGURE 2. The SFFR algorithm for constructing a 2-sparse fusion frame with a desired fusion frame operator.

The principle for constructing the row vectors w_k which generate the subspaces $\mathcal{W}_i$ of the fusion frame is similar to that in (SFFI), that is, again the matrix W which contains the vectors w_k, $k = 1\dots, mN$ as rows is filled up from top to bottom, row by row in such a way that the ℓ_2 norm of the rows is 1, the ℓ_2 norm of column j is λ_j, $j = 1,\dots,M$, and the columns are orthogonal. The vectors w_k are then grouped in such a way that the vectors assigned to each subspace form an orthonormal system. However, here the task is more delicate since the λ_j's are not all integers. This forces the introduction of (2×2)-submatrices of the type

$$\begin{pmatrix} \sqrt{\frac{\lambda_j}{2}} & \sqrt{1-\frac{\lambda_j}{2}} \\ \sqrt{\frac{\lambda_j}{2}} & -\sqrt{1-\frac{\lambda_j}{2}} \end{pmatrix}.$$

These submatrices have orthogonal columns and unit norm (ℓ_2 norm) rows and allow us to handle non-integer eigenvalues. This construction was originally introduced in [16] for constructing tight fusion frames.

Before we prove that (SFFR) indeed produces 2-sparse fusion frames with desired operators we consider a special case, in which the construction coincides with the construction of frames with desired frame operators. Our intention is to highlight the applicability of (SFFR) to the construction of frames with arbitrary frame operators and to present a simple example that demonstrates how the algorithm works. A detailed analysis of the algorithm and the proof of its correctness are provided in Subsection 4.2.5.

4.2.2. *A Special Case and An Example.* In the special case where $m = 1$ a fusion frame reduces to a frame and (SFFR) simplifies to an algorithm for constructing 2-sparse frames with desired fusion frame operators. This algorithm, which we refer to as (SFR), is outlined in Figure 3.

We now present an example to demonstrate the application of (SFR) as a special case of (SFFR).

Example 4.2. *Let $M = 3$, $m = 1$ (special case of frame construction), $N = 8$, and $\lambda_1 = \frac{11}{4}$, $\lambda_2 = \frac{11}{4}$, $\lambda_3 = \frac{10}{4}$. Then, the algorithm constructs the following matrix W. Notice that indeed the ℓ_2 norm of the rows is 1, the ℓ_2 norm of the column j is λ_j, $j = 1,\dots,M$, and the columns are orthogonal.*

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \sqrt{3/8} & \sqrt{5/8} & 0 \\ \sqrt{3/8} & -\sqrt{5/8} & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{1/4} & \sqrt{3/4} \\ 0 & \sqrt{1/4} & -\sqrt{3/4} \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of the frame operator of the constructed frame $\{w_{k,\cdot}\}_{k=1}^{8}$ are indeed $\frac{11}{4}$, $\frac{11}{4}$, and $\frac{10}{4}$ as a simple computation shows. This also follows from Theorem 4.8 or Corollary 4.9 presented later in this subsection.

SFR: SPARSE FRAME CONSTRUCTION FOR REAL EIGENVALUES

Parameters:

- Dimension $M \in \mathbb{N}$.
- Eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M > 0$, number of frame vectors N satisfying (FAC) with $m = 1$.

Algorithm:

1) Set $k := 1$.
2) For $j = 1, \ldots, M$ do
3) Repeat
4) If $\lambda_j < 1$ then
5) $w_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
6) $w_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
7) $k := k + 2$.
8) $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
9) $\lambda_j := 0$.
10) else
11) $w_k := e_j$.
12) $k := k + 1$.
13) $\lambda_j := \lambda_j - 1$.
14) end.
15) until $\lambda_j = 0$.
16) end.

Output:

- Frame $\{w_k\}_{k=1}^N$.

FIGURE 3. The SFR algorithm for constructing a 2-sparse frame with a desired frame operator.

From now on we concentrate on the analysis of (SFFR), keeping in mind that our analysis also applies to (SFR) as a special case.

4.2.3. *Feasibility Checks.* Before proving that (SFFR) indeed produces a 2-sparse fusion frame satisfying (FF1) and (FF2), we investigate the feasibility of the solution furnished by the algorithm.

Lemma 4.3. *For all $k = 1, \ldots, mN$,*

$$\|w_k\|_2^2 = 1.$$

Proof. This follows immediately from Lines 5), 6), and 11) of (SFFR). □

Denoting the λ_j's in Lines 4) – 6) of (SFFR) by $\tilde{\lambda}_j$'s to distinguish them from the eigenvalues λ_j, $j = 1\ldots, M$, the only two problems which could occur while running (SFFR) are:

(P1) $\lambda_{j+1} - (2 - \tilde{\lambda}_j) < 0$ in Line 9) for some $j = 1, \ldots, M-1$,
(P2) using e_{M+1} in Lines 5) – 9) when performing the step for $j = M$.

The following result shows that these cannot happen.

Proposition 4.4. *If $\lambda_j \geq 2$ for all $j = 1\ldots, M$, then* (P1) *and* (P2) *cannot happen.*

Proof. (P1). Since $\lambda_j \geq 2$ for all $j = 1\ldots, M$, we have

$$\lambda_{j+1} \geq 2 \geq 2 - \tilde{\lambda}_j \quad \text{for all } j = 1, \ldots, M-1.$$

(P2). Suppose the algorithm is executed until Line 16) with $j = M-1$. Let $K+1$ denote the value which k has reached at this point, and denote the coefficients of the vectors w_k by $w_k = (w_{k1}, \ldots, w_{kM})$. This means that so far we have constructed w_{kj} for $k = 1, \ldots, K$, $j = 1\ldots, M-1$. Then, by construction,

$$\sum_{k=1}^{K} w_{kj}^2 = \lambda_j \quad \text{for all } 1 \leq j \leq M-1. \tag{4.6}$$

We have to distinguish between two cases:

Case 1. $w_{K-2,M} = 0$ and $w_{K-1,M} = 0$. Then, by (4.6) and Lemma 4.3,

$$\sum_{j=1}^{M-1} \lambda_j = \sum_{j=1}^{M-1} \sum_{k=1}^{K} w_{kj}^2 = \sum_{k=1}^{K} \sum_{j=1}^{M-1} w_{kj}^2 = \sum_{k=1}^{K} 1 = K.$$

Since

$$\sum_{j=1}^{M} \lambda_j = \lambda_M + \sum_{j=1}^{M-1} \lambda_j = \lambda_M + K$$

is an integer, it follows that λ_M is an integer as well. Hence during the step $j = M$ only the Block 11) – 14) as opposed to the Block 5) – 9) will be executed. Thus (P2) does not happen.

Case 2. $w_{K-2,M} = \sqrt{1 - \frac{\tilde{\lambda}_{M-1}}{2}}$ and $w_{K-1,M} = -\sqrt{1 - \frac{\tilde{\lambda}_{M-1}}{2}}$. In this case,

$$\sum_{j=1}^{M-1} \lambda_j + (2 - \tilde{\lambda}_{M-1}) = \sum_{j=1}^{M-1} \sum_{k=1}^{K} w_{kj}^2 = \sum_{k=1}^{K} \sum_{j=1}^{M-1} w_{kj}^2 = \sum_{k=1}^{K} 1 = K,$$

an integer. Since $\sum_{j=1}^{M} \lambda_j$ is an integer as well, so is

$$\sum_{j=1}^{M} \lambda_j - \left(\sum_{j=1}^{M-1} \lambda_j + (2 - \tilde{\lambda}_{M-1}) \right) = \lambda_M - (2 - \tilde{\lambda}_{M-1}).$$

Hence, as before, in the step $j = M$ only the Block 11) – 14) as opposed to the Block 5) – 9) will be executed; here $\lambda_M - (2 - \tilde{\lambda}_{M-1})$ times. Thus, in this situation, (P2) does not occur. □

4.2.4. *Terminology and Lemmata.* In preparation for a detailed analysis of (SFFR), which is presented in Subsection 4.2.5, we need to establish some terminology and a few results.

Definition 4.5. *An entry of a vector w_k, $k \in \{1, \ldots, mN\}$ of the form $\pm\sqrt{1 - \frac{\tilde{\lambda}_j}{2}}$ (entered in Line 5) or 6) of* (SFFR)*) will be termed a* terminal point. *An* initial point *will be an entry of the form $\pm\sqrt{\tilde{\lambda}_j/2}$ (entered in Line 5) or 6)).*

Considering the matrix $W \in \mathbb{R}^{mN \times M}$ with the vectors $w_1, \ldots, w_{mN}$ as rows, the initial points start non-zero entries in a row with more than one non-zero entry, whereas the terminal points end such non-zero entries. It is obvious from algorithm (SFFR) that column n of W has no initial points if and only if

$$\sum_{j=1}^{n} \lambda_j \text{ is an integer,}$$

and it has no terminal points if and only if

$$\sum_{j=1}^{n-1} \lambda_j \text{ is an integer.}$$

Let $N(j)$ denote the number of non-zero terms in each column j, $j = 1, \ldots, M$ of the matrix W, that is, let $N(j)$ denote the number of non-zero entries of the vector $w_{\cdot,j}$. The following proposition determines exactly the value of $N(j)$ depending on the occurrence of initial and/or terminal points. We remind the reader of the definition of $\tilde{\lambda}_j$ right before Proposition 4.4.

Lemma 4.6. *The following conditions hold for the previously defined values $N(j)$, $j = 1, \ldots, M$.*

(i) *$N(j) = \lambda_j$, if $w_{\cdot,j}$ contains no initial or terminal points.*
(ii) *$N(j) = \lfloor \lambda_j \rfloor + 1$, if $w_{\cdot,j}$ contains terminal, but no initial points.*
(iii) *$N(j) = \lfloor \lambda_j \rfloor + 2$, if $w_{\cdot,j}$ contains initial, but no terminal points.*
(iv) *If $w_{\cdot,j}$ contains both initial and terminal points, then*
  (a) *if $\tilde{\lambda}_j \geq \tilde{\lambda}_{j-1}$, then $N(j) = \lfloor \lambda_j \rfloor + 2$,*
  (b) *if $\tilde{\lambda}_j < \tilde{\lambda}_{j-1}$ then $N(j) = \lfloor \lambda_j \rfloor + 3$.*
(v) *If λ_{j_0} is the first non-integer value, then $N(j_0) = \lfloor \lambda_{j_0} \rfloor + 2$.*
(vi) *If λ_{j_1} is the last non-integer value, then $N(j_1) = \lfloor \lambda_{j_1} \rfloor + 1$.*

Proof. (i). This is obvious, since in this case only the Block 11) – 14) is performed as opposed to the Block 5) – 9).
(ii). Letting n_j denote the number of ones in the vector $w_{\cdot,j}$, it follows that $N(j) = n_j + 2$. Since the entries of $w_{\cdot,j}$ are n_j times a 1 as well as the values $\pm\sqrt{1 - \frac{\tilde{\lambda}_{j-1}}{2}}$,

$$\lambda_j = n_j + (2 - \tilde{\lambda}_{j-1}) = n_j + 1 + (1 - \tilde{\lambda}_{j-1}) \quad \text{with } 0 < 1 - \tilde{\lambda}_{j-1} < 1.$$

This implies $\lfloor \lambda_j \rfloor = n_j + 1$, and thus

$$N(j) = n_j + 2 = \lfloor \lambda_j \rfloor + 1.$$

(iii). Now the non-zero entries of the vector $w_{\cdot,j}$ are $\pm\sqrt{\frac{\tilde{\lambda}_j}{2}}$ as well as n_j, say, entries 1. Hence $N(j) = n_j + 2$, and

$$\lambda_j = n_j + \tilde{\lambda}_j \quad \text{with } 0 < \tilde{\lambda}_j < 1.$$

This implies $\lfloor \lambda_j \rfloor = n_j$, and thus

$$N(j) = n_j + 2 = \lfloor \lambda_j \rfloor + 2.$$

(iv). The vector $w_{\cdot,j}$ contains as non-zero entries, the initial points $\pm\sqrt{\frac{\tilde{\lambda}_j}{2}}$ and the terminal points $\pm\sqrt{1 - \frac{\tilde{\lambda}_{j-1}}{2}}$ as well as, say, n_j entries 1, hence $N(j) = n_j + 4$. Thus

$$\lambda_j = n_j + \tilde{\lambda}_j + (2 - \tilde{\lambda}_{j-1}) = n_j + 2 + (\tilde{\lambda}_j - \tilde{\lambda}_{j-1}).$$

If $\tilde{\lambda}_j - \tilde{\lambda}_{j-1} \geq 0$, then $\lfloor \lambda_j \rfloor = n_j + 2$, which implies

$$N(j) = n_j + 4 = \lfloor \lambda_j \rfloor + 2.$$

If $\tilde{\lambda}_j - \tilde{\lambda}_{j-1} < 0$, then $\lfloor \lambda_j \rfloor = n_j + 1$, which implies

$$N(j) = n_j + 4 = \lfloor \lambda_j \rfloor + 3.$$

(v) and (vi). These are direct consequences from the previous conditions. □

The following lemma shows an interesting relation between consecutive values of $N(j)$ as j progresses. However, we note that only the previous lemma is required for the proofs of the main theorems that will be presented in Subsection 4.2.5.

Lemma 4.7. *For any $j \in \{1, \ldots, M-1\}$, the following conditions hold for the previously defined values $N(j)$ and $N(j+1)$ supposing that they are not integers.*

(i) *If $w_{\cdot,j}$ contains no initial or terminal points, then $N(j) \geq N(j+1) - 1$.*
(ii) *If $w_{\cdot,j}$ contains initial, but no terminal points, then*
   (a) *if $\lambda_j + \lambda_{j+1}$ is an integer, then $N(j) \geq N(j+1) + 1$,*
   (b) *if $\lambda_j + \lambda_{j-1}$ is not an integer, then $N(j) \geq N(j+1) - 1$.*
(iii) *If $w_{\cdot,j}$ contains both initial and terminal points, then $N(j) \geq N(j+1) - 1$.*

Proof. Recall that we have $\lambda_j \geq \lambda_{j+1}$.
(i). Since $w_{\cdot,j}$ contains no initial points and λ_{j+1} is not an integer, the vector $w_{\cdot,j+1}$ contains initial, but no terminal points. Thus, by Lemma 4.6,

$$N(j) = \lfloor \lambda_j \rfloor + 1 \geq \lfloor \lambda_{j+1} \rfloor + 2 - 1 = N(j+1) - 1.$$

(ii). By Lemma 4.6, $N(j) = \lfloor \lambda_j \rfloor + 2$. Also $w_{\cdot,j}$ contains initial points, hence the vector $w_{\cdot,j+1}$ contains terminal points.
(a). Since $\lambda_j + \lambda_{j+1}$ is an integer and $w_{\cdot,j}$ does not contain any terminal points, the vector $w_{\cdot,j+1}$ does not contain initial points. This implies $N(j+1) = \lfloor \lambda_{j+1} \rfloor + 1$.
(b). Since $\lambda_j + \lambda_{j+1}$ is not an integer and $w_{\cdot,j}$ does not contain any terminal points, the vector $w_{\cdot,j+1}$ does contain initial points. This implies $N(j+1) \leq \lfloor \lambda_{j+1} \rfloor + 3$.
(iii). By Lemma 4.6, $N(j) \geq \lfloor \lambda_j \rfloor + 2$ and the vector $w_{\cdot,j+1}$ can not contain more than $\lfloor \lambda_{j+1} \rfloor + 3$ non-zero entries. □

4.2.5. *Main Results: Analysis of* (SFFR). We now present the main results concerning (SFFR). We first show that the algorithm indeed delivers the correct 2-sparse fusion frame, i.e., a 2-sparse fusion frame with the prescribed fusion frame operator. From this result, we deduce that in certain cases a fusion frame can be turned into a tight fusion frame by careful adding of new subsets (compare also with Corollary 3.5). Also, here we will need to assume that the eigenvalues are greater than or equal to two since otherwise k-sparse frames (or fusion frames) do not exist for small values of k (see the discussion in Section 1.5).

Theorem 4.8. *Suppose the real values $N \geq \lambda_1 \geq \cdots \geq \lambda_M$, $N \in \mathbb{N}$, and $m \in \mathbb{N}$ satisfy* (FAC) *as well as the following conditions.*

(i) $\lambda_M \geq 2$.
(ii) *If j_0 is the first integer in $\{1, \ldots, M\}$, for which λ_{j_0} is not an integer, then $\lfloor \lambda_{j_0} \rfloor \leq N - 3$.*

Then the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ constructed by (SFFR) *fulfills* (FF1) *and* (FF2) *and the fusion frame is 2-sparse.*

Proof. If the set of vectors
$$\{w_{i+kN} : k = 0, \ldots, m-1\}$$
is pairwise orthogonal for each $i = 1, \ldots, N$, then (FF1) and (FF2) follow automatically. Fix $i \in \{1, \ldots, N\}$. By construction, it is sufficient to show that, for each $0 \leq k \leq m-2$, the vectors w_{i+kN} and $w_{i+(k+1)N}$ are disjointly supported. We distinguish between the following two cases:

Case 1. The vector w_{i+kN} is a unit vector, e_n, say. By (ii) and Lemma 4.6, $w_{\cdot,n}$ does not have more than N non-zero elements. When defining the vector $w_{i+(k+1)N}$, already $N-1$ vectors w_ℓ have been defined before its construction. Therefore this definition takes place in a different step of the loop in Line 1). Hence $w_{i+(k+1)N,j} = 0$ for all $j = 1, \ldots, n$. This proves the claim in this case.

Case 2. The vector w_{i+kN} has two non-zero entries, namely an initial and a terminal point, where the terminal point is at the nth position, say. Again, by (ii) and Lemma 4.6, $w_{\cdot,n}$ does not have more than N non-zero elements. Hence, concluding as before, $w_{i+(k+1)N,j} = 0$ for all $j = 1, \ldots, n$. This proves the claim also in this case.

The fact that the fusion frame is 2-sparse follows immediately from the construction and Definition 1.1. □

Certainly, this theorem also holds in the special case of frames, i.e., 1-dimensional subspaces.

Corollary 4.9. *Suppose the real values $\lambda_1 \geq \cdots \geq \lambda_M$ and $N \in \mathbb{N}$ satisfy*
$$\sum_{j=1}^{M} \lambda_j = N$$
as well as the following conditions.

(i) $\lambda_M \geq 2$.
(ii) *If j_0 is the first integer in $\{1, \ldots, M\}$, for which λ_{j_0} is not an integer, then $\lfloor \lambda_{j_0} \rfloor \leq N - 3$.*

Then the eigenvalues of the frame operator of the frame $\{w_k\}_{k=1}^N$ *constructed by* (SFFR) *are* $\{\lambda_j\}_{j=1}^M$ *and the frame is* 2*-sparse.*

Proof. This result follows directly from Theorem 4.8 by choosing $m = 1$. □

Theorem 4.8 is now applied to generate a tight fusion frame from a given fusion frame, satisfying some mild conditions.

Theorem 4.10. *Let* $\{\mathcal{W}_i\}_{i=1}^N$ *be a fusion frame for* $\mathbb{R}^M$ *with* $\dim \mathcal{W}_i = m < M$ *for all* $i = 1, \ldots, N$*, and let* S *be the associated fusion frame operator with eigenvalues* $\lambda_1 \geq \ldots \geq \lambda_M$ *and eigenvectors* $\{e_j\}_{j=1}^M$*. Further, let* A *be the smallest positive integer, which satisfies the following conditions.*

(i) $\lambda_1 + 2 \leq A$.
(ii) $AM = mN_0$ *for some* $N_0 \in \mathbb{N}$.
(iii) $A \leq \lambda_M + N_0 - (N+3)$.

Then there exists a fusion frame $\{\mathcal{V}_i\}_{i=1}^{N_0-N}$ *for* $\mathbb{R}^M$ *with dim* $\mathcal{V}_i = m$ *for all* $i \in \{1, \ldots, N_0 - N\}$ *so that*

$$\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_0-N}$$

is an A*-tight fusion frame.*

Proof. The first task is to check whether such a positive integer A exists at all. We use the ansatz $A = nm$ for some $n \in \mathbb{N}$. This immediately satisfies (ii). Now choose n as the smallest positive integer still satisfying

$$\lambda_1 + 2 \leq A.$$

Thus (i) and (ii) are fulfilled (and they will still be fulfilled for all larger $n \in \mathbb{N}$.) For inequality (iii), we require

$$mn \leq \lambda_1 + nM - (N+3),$$

which we can reformulate as

$$\frac{m}{M} \leq \frac{\lambda_1}{Mn} + 1 - \frac{N+3}{Mn}.$$

Since $\frac{m}{M} < 1$ by assumption, n can be chosen large enough for this inequality to be satisfied.

Next, we set

$$\mu_j = A - \lambda_j \quad \text{for all } j = 1, \ldots, M.$$

In particular, we have $\mu_1 \leq \ldots \leq \mu_M$. We claim that the hypotheses of Theorem 4.8 are satisfied by the sequence $\{\mu_j\}_{j=1}^M$. For the proof, we refer to the assumption of the present theorem as (i'), (ii'), and (iii').

(i). By (i'),

$$\mu_1 = A - \lambda_1 \geq 2.$$

Letting $N_1 = N_0 - N$,

$$\sum_{j=1}^M \mu_j = \sum_{j=1}^M (A - \lambda_j) = AM - \sum_{j=1}^M \lambda_j = AM - mN = mN_0 - mN = mN_1.$$

(ii). By (iii'),

$$\mu_M = A - \lambda_M \leq (N_0 - N) - 3 = N_1 - 3.$$

From Theorem 4.8 it follows that there exists a fusion frame $\{\mathcal{V}_i\}_{i=1}^{N_1}$ for $\mathbb{R}^M$ whose fusion frame operator S_1, say, has eigenvectors $\{e_j\}_{j=1}^M$ and respective eigenvalues $\{\mu_j\}_{j=1}^M$. The fusion frame operator for $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_1}$ is $S + S_1$, which then possesses as eigenvectors the sequence $\{e_j\}_{j=1}^M$ with associated eigenvalues

$$\lambda_j + \mu_j = \lambda_j + (A - \lambda_j) = A.$$

Hence $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_0-N}$ constitutes an A-tight fusion frame. □

The number of m-dimensional subspaces added in Theorem 4.10 to force a fusion frame to become tight is in fact the smallest number that can be added in general. For this, let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for $\mathbb{R}^M$ with fusion frame operator S having eigenvalues $\{\lambda_j\}_{j=1}^M$. Suppose $\{\mathcal{V}_i\}_{i=1}^{N_1}$ is any family of m-dimensional subspaces with fusion frame operator S_1, say, and so that the union of these two families is an A-tight fusion frame for $\mathbb{R}^M$. Thus

$$S + S_1 = AI,$$

which implies that the eigenvalues $\{\mu_j\}_{j=1}^M$ of S_1 satisfy

$$\mu_j = A - \lambda_j \quad \text{for all } j = 1, \ldots, M,$$

and

$$\sum_{j=1}^M \mu_j = \sum_{j=1}^M (A - \lambda_j) = AM - mN = mN_1.$$

In particular,

$$AM = m(N_1 - N) = mN_0.$$

Thus, we have examples to show that – in general – fusion frames with the above properties of S_1 cannot be constructed unless the hypotheses of Theorem 4.10 are satisfied. This shows that the smallest constant satisfying this theorem is in general the smallest number of subspaces we can add to obtain a tight fusion frame.

5. Extensions and Related Problems

Finally, we would like to discuss several extensions and related problems.

Weights. The handling of weights is particularly delicate. When turning a frame $\{f_i\}_{i=1}^N$ into the fusion frame $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i=1}^N$ consisting of 1-dimensional subspaces and having the same (fusion) frame operator as well as the same (fusion) frame bounds [21, Prop. 2.14], we notice that the subspaces are generated by the frame vectors and the weights have to be chosen equal to the norms of the frame vectors. Thus choosing weights is in a sense comparable to choosing the lengths of frame vectors. However the design is more delicate due to the necessary compensation of the dimensions of the subspaces. Generalizing, for instance, Theorem 4.8 to weighted sparse fusion frames requires careful handling and a thorough understanding of the interplay between subspace dimensions and weights. This is currently under investigation.

Chordal Distances. It was shown in [40] that maximal resilience of fusion frames to noise and erasures is closely related to the chordal distances between pairs of subspaces forming the fusion frame. Hence it would be desirable to be able to control the set of chordal distances in construction procedures for fusion frames. The results in Section 3 already allow this control. However, for instance, for Theorem 4.8 this control is more difficult.

Equivalence Classes. Our results normally produce one fusion frame satisfying a desired property. However, from a scholarly point of view, it would be desirable to be able to generate each such fusion frame in the sense of the whole "equivalence class" of fusion frames satisfying a special property. This is beyond our reach at this point, since even the following apparently simple problem is still unsolved: Construct one Parseval frame in each equivalence class choosing unitary equivalence as the equivalence relation.

Eigenvalues Less Than 2. At this time, we are not able to give an algorithm for producing fusion frames with fusion frame operator having eigenvalues less than 2. It was pointed out by one of the reviewers that reformulating the SFFR algorithm in the Horn-Klyachko language (see [41]) might allow one to combine the results of [32] with (SFFR) to get an algorithm which works for this case. Although we have not managed to carry out this program, it seems to be a good direction for future research - although *conventional wisdom* indicates that these fusion frames will be less sparse than those obtained for $\lambda_i \geq 2$.

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