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Minimizing Fusion Frame Potential

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Abstract Fusion frames are an emerging topic of frame theory, with applications to encoding and distributed sensing. However, little is known about the existence of tight fusion frames. In traditional frame theory, one method for showing that unit norm tight frames exist is to characterize them as the minimizers of an energy functional, known as the frame potential. We generalize the frame potential to the fusion frame setting. In particular, we introduce the fusion frame potential, and show how its minimization is equivalent to the minimization of the traditional frame potential over a particular domain. We then study this minimization problem in detail. Specifically, we show that if the desired number of fusion frame subspaces is large, and if the desired dimension of these subspaces is small compared to the dimension of the underlying space, then a tight fusion frame of those dimensions will necessarily exist, being a minimizer of the fusion frame potential.

Key words frames, fusion, potential, tight

1 Introduction

The *analysis* operator of some finite sequence of vectors $\{f_m\}_{m=1}^M$ in an N-dimensional Hilbert space $\mathbb{H}_N$ is $F:\mathbb{H}_N\rightarrow\mathbb{C}^M$, $(Ff)(m):=\langle f,f_m\rangle$.

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The corresponding *frame* operator is $F^*F:\mathbb{H}_N\rightarrow\mathbb{H}_N$,

$$F^*Ff=\sum_{m=1}^{M}\langle f,f_m\rangle f_m. \tag{1}$$

Generally speaking, *frame theory* is the study of how $\{f_m\}_{m=1}^M$ should be chosen in order to guarantee that F^*F is well-conditioned. In particular, $\{f_m\}_{m=1}^M$ is a *frame* for $\mathbb{H}_N$ if there exists *frame bounds* $0<A\leq B<\infty$ such that $A\mathrm{I}\leq F^*F\leq B\mathrm{I}$, and is a *tight frame* if $A=B$, that is, if $F^*F=A\mathrm{I}$. Of particular interest is the case of *unit norm tight frames*, that is, tight frames for which $\|f_m\|=1$ for all $m=1,\dots,M$; such frames, known to exist for any $M\geq N$, provide Parseval-like decompositions in terms of vectors of unit length, despite the nonorthogonality of these vectors.

Fusion frame theory generalizes these concepts. In particular, when each f_m is of unit norm, the summands of the frame operator (1), namely the operators $f\mapsto\langle f,f_m\rangle f_m$, are rank-one projections. Fusion frame theory is the study of sums of projections whose ranks are permitted to be greater than one. To be precise, a sequence of orthogonal projections $\{P_k\}_{k=1}^K$, is a *fusion frame* for $\mathbb{H}_N$ if:

$$A\mathrm{I}\leq\sum_{k=1}^{K}P_k\leq B\mathrm{I},$$

and is a *tight fusion frame* if $A=B$. As detailed below, tight fusion frames are a focus of many emerging applications. Despite their potential applicability, the question of the existence of tight fusion frames of a given size is mostly unresolved, being heretofore only addressed in special cases using particular constructions. The purpose of this article is to better address this question by adapting a recent concept of traditional frame theory, namely the minimization of frame potentials.

To be precise, the *frame potential* of a sequence $\{f_m\}_{m=1}^M$ is:

$$\mathrm{FP}(\{f_m\}_{m=1}^M):=\sum_{m,m'=1}^{M}|\langle f_m,f_{m'}\rangle|^2. \tag{2}$$

The frame potential quantifies the total orthogonality of a system of vectors by measuring the total potential energy stored within that system under a certain force which encourages orthogonality. Regarded as a functional over $E=\{\{f_m\}_{m=1}^M\in\mathbb{H}_N^M:\|f_m\|=1,\ m=1,\dots,M\}$, one may show that every local minimizer of the frame potential is necessarily a tight frame whenever $M\geq N$ [1]. In particular, as the frame potential is continuous and E is compact, one may conclude that unit norm tight frames for $\mathbb{H}_N$ of M elements must indeed exist for any $M\geq N$. We generalize these ideas to the fusion frame setting.

In particular, in the next section, we introduce the fusion frame potential (6), and show how the minimization of this potential is equivalent to minimizing the traditional frame potential (2) over a particular domain, as described in Theorem 2. As such, in the third section, we study the minimization of (2) in greater detail, strengthening and simplifying several of the main results of [1,5,10], as summarized in Theorem 3. In the final section, we then use these results to prove Theorem 4, which places a strong necessary structure on any local minimizer of the fusion frame potential. We then conclude with our main result, namely Theorem 5, which shows that tight fusion frames will always exist, provided the number of subspaces K is sufficiently large, and provided that the dimension of the whole space N is sufficiently large when compared to the dimension L of the fusion frame's subspaces. Indeed, having Theorem 5, it is straightforward to prove the following:

Theorem 1 *For any positive integer L, and any $\alpha > 1$, then for all large N and all positive integers $K \geq \alpha N$, there exists a tight fusion frame $\{P_k\}_{k=1}^K$ for $\mathbb{H}_N$ such that $\mathrm{Tr}(P_k) = L$ for all $k = 1, \dots, K$.*

Fusion frames were introduced in [6], and later refined in [8]. Applications of fusion frames include distributed sensing [9,12], the recovery of a signal from its frame coefficients even when some are unknown [3,4,7], and the modeling of the human visual cortex [15]. The frame potential was introduced in [1], with its domain of optimization being later generalized in [5]. It has been used to characterize tight filter bank frames [10,11]. Recently, generalized frame potentials have been a subject of interest [2,14].

Some of the theory below is a special case of the q-potential theory introduced in [13]. There, tight fusion frames are referred to as *uniformly weighted projective protocols*. In particular, our definition of the fusion frame potential is a special case of Definition 3.2 of [13], and our Proposition 1 is a special case of Theorem 3.4 of [13]. Most significantly, Corollary 5.3 of [13] characterizes the existence of tight fusion frames in a novel manner which is quite distinct from our main results.

2 Fusion frame potential

The *fusion frame operator* of a sequence of $\{P_k\}_{k=1}^K$ of orthogonal projections is their sum; the goal of this paper is to prove the existence of tight fusion frames, in which this frame operator is a positive scalar multiple of the identity. To be precise, our goal is to find sufficient conditions on N, K and a given sequence of positive integers $\{L_k\}_{k=1}^K$ so as to guarantee the existence of orthogonal projections $\{P_k\}_{k=1}^K$ over $\mathbb{H}_N$ such that $\mathrm{Tr}(P_k) = L_k$ for all $k = 1, \dots, K$ and such that:

$$A\mathrm{I} = \sum_{k=1}^{K} P_k, \tag{3}$$

for some $A > 0$. We begin by noting that for any orthogonal projection $P : \mathbb{H}_N \to \mathbb{H}_N$, letting $\{f_m\}_{m=1}^M$ be an orthonormal basis for the range

$\mathcal{R}(P)$ of P, we classically know that:

$$Pf = \sum_{m=1}^{M} \langle f, f_m \rangle f_m,$$

for all $f \in \mathbb{H}_N$, that is, that P is the frame operator for $\{f_m\}_{m=1}^M$. As such, given a sequence of projections $\{P_k\}_{k=1}^K$, and, for each $k = 1, \dots, K$, letting $\{f_{k,l}\}_{l=1}^{L_k}$ be an orthonormal basis for $\mathcal{R}(P_k)$, we have

$$\sum_{k=1}^{K} P_k f = \sum_{k=1}^{K} \sum_{l=1}^{L_k} \langle f, f_{k,l} \rangle f_{k,l}, \tag{4}$$

for all $f \in \mathbb{H}_N$, namely that the fusion frame operator of $\{P_k\}_{k=1}^K$ is equal to the traditional frame operator of $\{f_{k,l}\}_{k=1,l=1}^{K,L_k}$. As such, rather than regarding fusion frame theory as a generalization of traditional frame theory, one may instead regard it as a special case of traditional frame theory in which certain frame vectors are required to be orthogonal to others.

As we shall make repeated use of this equivalence (4) between traditional and fusion frames, we, to simplify notation, let $M = \sum_{k=1}^K L_k$ and consider the singly-indexed sequence $\{f_m\}_{m=1}^M$ obtained by concatenating each of the K sequences $\{f_{k,l}\}_{l=1}^{L_k}$ together. To be precise, we say that a sequence of vectors $\{f_m\}_{m=1}^M$ *generates* the projections $\{P_k\}_{k=1}^K$ if there exists a partition $\{\mathcal{I}_k\}_{k=1}^K$ of the indices $\{1, \dots, M\}$ such that each $\{f_m\}_{m \in \mathcal{I}_k}$ is an orthonormal basis for the range of P_k.

In light of (4), it is not surprising that many of the results that hold for traditional frames will also apply to fusion frames. For example, in a manner similar to Proposition 1 of [5], the constant A in (3) is uniquely determined by N and the dimensions $\{L_k\}_{k=1}^K$ of the projections' ranges; taking the trace of (3) yields:

$$AN = \mathrm{Tr}(A\mathrm{I}) = \mathrm{Tr}\Big(\sum_{k=1}^{K} P_k\Big) = \sum_{k=1}^{K} \mathrm{Tr}(P_k) = \sum_{k=1}^{K} L_k = M. \tag{5}$$

Moreover, as the traditional frame potential is the trace of the square of the frame operator [1], we define the *fusion frame potential* of $\{P_k\}_{k=1}^K$ as:

$$\mathrm{FFP}(\{P_k\}_{k=1}^K) := \mathrm{Tr}\Big(\sum_{k=1}^{K} P_k\Big)^2. \tag{6}$$

Indeed, in light of (4) and (6), the fusion frame potential of $\{P_k\}_{k=1}^K$ is equal to the traditional frame potential of any one of its generators $\{f_m\}_{m=1}^M$. We further note that by distributing the square and trace in (6), we may write the fusion frame potential in a manner more consistent with (2):

$$\mathrm{FFP}(\{P_k\}_{k=1}^K) = \sum_{k,k'=1}^{K} \mathrm{Tr}(P_k P_{k'}).$$

We shall accomplish our goal of proving the existence of tight fusion frames by characterizing them as local minimizers of (6). In particular, given any sequence of positive integers $\{L_k\}_{k=1}^K$, we take the domain of optimization of the fusion frame potential to be:

$$\mathcal{P}(\{L_k\}_{k=1}^K) := \Big\{\{P_k\}_{k=1}^K \;\Big|\; P_k : \mathbb{H}_N \to \mathbb{H}_N,\, P_k^* P_k = P_k,\, \mathrm{Tr}(P_k) = L_k\Big\}. \tag{7}$$

Fixing some partition $\{\mathcal{I}_k\}_{k=1}^K$ of $\{1, \ldots, M\}$ such that $|\mathcal{I}_k| = L_k$ for every $k = 1, \ldots, K$, note that every member of $\mathcal{P}(\{L_k\}_{k=1}^K)$ may be generated by a member of:

$$\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K) := \Big\{\{f_m\}_{m=1}^M : \{f_m\}_{m\in\mathcal{I}_k} \text{ is orthonormal, } \forall k = 1, \ldots, K\Big\}. \tag{8}$$

In the next result, which generalizes Proposition 4 of [5], we show that if a tight fusion frame of dimensions $\{L_k\}_{k=1}^K$ exists, then it is necessarily a global minimizer of the fusion frame potential FFP : $\mathcal{P}(\{L_k\}_{k=1}^K) \to \mathbb{R}$. Note that this result does not imply that such a frame actually exists.

Proposition 1 *For any sequence of positive integers* $\{L_k\}_{k=1}^K$,

$$\mathrm{FFP}(\{P_k\}_{k=1}^K) \geq \frac{1}{N}\Big(\sum_{k=1}^K L_k\Big)^2 \tag{9}$$

for any $\{P_k\}_{k=1}^K \in \mathcal{P}(\{L_k\}_{k=1}^K)$*, with equality holding in* (9) *if and only if* $\{P_k\}_{k=1}^K$ *is a tight fusion frame for* $\mathbb{H}_N$.

Proof For any $\{P_k\}_{k=1}^K \in \mathcal{P}(\{L_k\}_{k=1}^K)$, letting $\{\lambda_n\}_{n=1}^N$ be the eigenvalues of the corresponding self-adjoint, positive semi-definite fusion frame operator $\sum_{k=1}^K P_k$, we have:

$$\sum_{n=1}^N \lambda_n = \mathrm{Tr}\Big(\sum_{k=1}^K P_k\Big) = \sum_{k=1}^K \mathrm{Tr}(P_k) = \sum_{k=1}^K L_k. \tag{10}$$

Moreover, as the fact that $\mathrm{FFP}(\{P_k\}_{k=1}^K) = \sum_{n=1}^N \lambda_n^2$ immediately follows by definition (6), we, for any $\{P_k\}_{k=1}^K \in \mathcal{P}(\{L_k\}_{k=1}^K)$, have:

$$\mathrm{FFP}(\{P_k\}_{k=1}^K) \geq \min\Big\{\sum_{n=1}^N \lambda_n^2 : \sum_{n=1}^N \lambda_n = \sum_{k=1}^K L_k\Big\}. \tag{11}$$

The explicit value of the right-hand side of (11) occurs precisely when the λ_n's are constant, yielding (9), namely:

$$\mathrm{FFP}(\{P_k\}_{k=1}^K) \geq \sum_{n=1}^N \Big(\frac{1}{N}\sum_{k=1}^K L_k\Big)^2 = \frac{1}{N}\Big(\sum_{k=1}^K L_k\Big)^2. \tag{12}$$

Moreover, equality in (12) occurs precisely when the λ_n's are constant, that is, precisely when $\sum_{k=1}^K P_k$ is a constant multiple of the identity. □

Though Proposition 1 characterizes tight fusion frames, should they exist, as the global minimizers of $\mathrm{FFP}:\mathcal{P}(\{L_k\}_{k=1}^K)\rightarrow\mathbb{R}$, our approach, paralleling that of [1,5,10], is to study the *local* minimizers of this functional. Here, we take the distance between any $\{P_k\}_{k=1}^K,\{Q_k\}_{k=1}^K\in\mathcal{P}(\{L_k\}_{k=1}^K)$ to be:

$$d(\{P_k\}_{k=1}^K,\{Q_k\}_{k=1}^K):=\Big(\sum_{k=1}^K\|P_k-Q_k\|_{\mathrm{HS}}^2\Big)^{\frac{1}{2}}, \tag{13}$$

whereas the distance between any $\{f_m\}_{m=1}^M$, $\{g_m\}_{m=1}^M\in\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ is taken as:

$$\tilde{d}(\{f_m\}_{m=1}^M,\{g_m\}_{m=1}^M):=\Big(\sum_{m=1}^M\|f_m-g_m\|^2\Big)^{\frac{1}{2}}. \tag{14}$$

Below, we show that if two sequences of vectors are close with respect to (14), then the projections they generate are necessarily close with respect to (13). However, the appropriate converse statement is more complicated. In particular, as any single projection P_k may be generated by many distinct orthonormal bases $\{f_m\}_{m\in\mathcal{I}_k}$, the distance between the generating vectors (14) may be large even when the distance between the projections they generate (13) is zero. Nevertheless, as the next result shows, this may be remedied by choosing one's generators carefully.

Lemma 1 *For any $\{P_k\}_{k=1}^K,\{Q_k\}_{k=1}^K\in\mathcal{P}(\{L_k\}_{k=1}^K)$, we have:*

$$d(\{P_k\}_{k=1}^K,\{Q_k\}_{k=1}^K)\leq 2(\max_k\sqrt{L_k})\,\tilde{d}(\{f_m\}_{m=1}^M,\{g_m\}_{m=1}^M) \tag{15}$$

for any $\{f_m\}_{m=1}^M$, $\{g_m\}_{m=1}^M\in\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ which generate $\{P_k\}_{k=1}^K$ and $\{Q_k\}_{k=1}^K$, respectively. Conversely, for any generators $\{f_m\}_{m=1}^M$ of $\{P_k\}_{k=1}^K$ and any $\varepsilon>0$, there exists $\delta>0$ such that whenever $\{Q_k\}_{k=1}^K$ satisfies $d(\{P_k\}_{k=1}^K,\{Q_k\}_{k=1}^K)<\delta$, there necessarily exists $\{g_m\}_{m=1}^M$ which generates $\{Q_k\}_{k=1}^K$ and for which $\tilde{d}(\{f_m\}_{m=1}^M,\{g_m\}_{m=1}^M)<\varepsilon$.

Proof Throughout, we, for any $k=1,\dots,K$, let F_k and G_k be the analysis operators of the orthonormal sequences $\{f_m\}_{m\in\mathcal{I}_k}$ and $\{g_m\}_{m\in\mathcal{I}_k}$, respectively, noting that $P_k=F_k^*F_k$ and $Q_k=G_k^*G_k$. Next, note that:

$$\begin{aligned}\|F^*F-G^*G\|_{\mathrm{HS}}&=\|F^*F-F^*G+F^*G+G^*G\|_{\mathrm{HS}}\\&\leq\|F^*\|_{\mathrm{HS}}\|F-G\|_{\mathrm{HS}}+\|(F-G)^*\|_{\mathrm{HS}}\|G\|_{\mathrm{HS}}\\&\leq(\|F\|_{\mathrm{HS}}+\|G\|_{\mathrm{HS}})\|F-G\|_{\mathrm{HS}},\end{aligned}$$

for any two operators F, G of equal size. Thus,

$$\begin{aligned}
d(\{P_k\}_{k=1}^K, \{Q_k\}_{k=1}^K)^2 &= \sum_{k=1}^K \|P_k - Q_k\|_{\mathrm{HS}}^2 \\
&= \sum_{k=1}^K \|F_k^* F_k - G_k^* G_k\|_{\mathrm{HS}}^2 \\
&\le \sum_{k=1}^K (\|F_k\|_{\mathrm{HS}} + \|G_k\|_{\mathrm{HS}})^2 \|F_k - G_k\|_{\mathrm{HS}}^2 \\
&= \sum_{k=1}^K \Big\{ \big[\mathrm{Tr}(F_k^* F_k)\big]^{\frac{1}{2}} + \big[\mathrm{Tr}(G_k^* G_k)\big]^{\frac{1}{2}} \Big\}^2 \sum_{m=1}^M \|(F_k - G_k)^* e_m\|^2 \\
&= \sum_{k=1}^K \Big\{ \big[\mathrm{Tr}(P_k)\big]^{\frac{1}{2}} + \big[\mathrm{Tr}(Q_k)\big]^{\frac{1}{2}} \Big\}^2 \sum_{m \in \mathcal{I}_k} \|f_m - g_m\|^2 \\
&\le 4 \max_k L_k \sum_{k=1}^K \sum_{m \in \mathcal{I}_k} \|f_m - g_m\|^2 \\
&= 4 \max_k L_k \big[\tilde{d}(\{f_m\}_{m=1}^M, \{g_m\}_{m=1}^M)\big]^2. \qquad (16)
\end{aligned}$$

Taking square roots of (16) yields (15). For the converse result, fix any such $\{f_m\}_{m=1}^M$ and $\varepsilon > 0$. For any fixed $k = 1, \dots, K$, note that the function which takes projection matrices Q_k of rank L_k to the matrix $F_k Q_k F_k^*$ is continuous, and has value $\mathrm{I} = F_k F_k^* F_k F_k^* = F_k P_k F_k^*$ at $Q_k = P_k$. As such, $F_k Q_k F_k^*$ will always be invertible provided Q_k is sufficiently close to P_k. For any such Q_k, we take $\{g_m\}_{m \in \mathcal{I}_k}$ to be the sequence whose analysis operator is:

$$G_k = (F_k Q_k F_k^*)^{-\frac{1}{2}} F_k Q_k. \qquad (17)$$

Note that $\{g_m\}_{m \in \mathcal{I}_k}$ is orthonormal:

$$G_k G_k^* = (F_k Q_k F_k^*)^{-\frac{1}{2}} F_k Q_k F_k^* (F_k Q_k F_k^*)^{-\frac{1}{2}} = \mathrm{I}.$$

Moreover, since $g_m = Q_k F_k^* (F_k Q_k F_k^*)^{-\frac{1}{2}} e_m \in \mathcal{R}(Q_k)$ for all $m \in \mathcal{I}_k$, where $\dim(\mathcal{R}(Q_k)) = |\mathcal{I}_k|$, we have that $\{g_m\}_{m \in \mathcal{I}_k}$ is an orthonormal basis for $\mathcal{R}(Q_k)$. Thus, $Q_k = G_k^* G_k$. Constructing $\{g_m\}_{m \in \mathcal{I}_k}$ according to (17) for each $k=1,\dots,K$, we produce a sequence $\{g_m\}_{m=1}^M\in\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ where:

$$\begin{aligned}\big[\tilde{d}(\{f_m\}_{m=1}^M,\{g_m\}_{m=1}^M)\big]^2&=\sum_{m=1}^M\|f_m-g_m\|^2\\&=\sum_{k=1}^K\sum_{m\in\mathcal{I}_k}\|f_m-g_m\|^2\\&=\sum_{k=1}^K\|F_k-G_k\|_{\mathrm{HS}}^2\\&=\sum_{k=1}^K\|F_k-(F_kQ_kF_k^*)^{-\frac{1}{2}}F_kQ_k\|_{\mathrm{HS}}^2.\end{aligned}\tag{18}$$

As (18) is a continuous function of $\{Q_k\}_{k=1}^K$, whose value at $\{P_k\}_{k=1}^K$ is:

$$\begin{aligned}\sum_{k=1}^K\|F_k-(F_kP_kF_k^*)^{-\frac{1}{2}}F_kP_k\|_{\mathrm{HS}}^2&=\sum_{k=1}^K\|F_k-(F_kF_k^*F_kF_k^*)^{-\frac{1}{2}}F_kF_k^*F_k\|_{\mathrm{HS}}^2\\&=\sum_{k=1}^K\|F_k-\mathrm{I}^{-\frac{1}{2}}F_k\|_{\mathrm{HS}}^2\\&=0,\end{aligned}$$

there exists $\delta>0$ such that the (17) method of constructing $\{g_m\}_{m=1}^M$ yields $\tilde{d}(\{f_m\}_{m=1}^M,\{g_m\}_{m=1}^M)<\varepsilon$ whenever the projections they generate satisfy $d(\{P_k\}_{k=1}^K,\{Q_k\}_{k=1}^K)<\delta$. $\square$

An immediate consequence of Lemma 1 is a characterization of the local minimizers of the fusion frame potential in terms of local minimizers of the traditional frame potential:

Theorem 2 *A given sequence* $\{P_k\}_{k=1}^K\in\mathcal{P}(\{L_k\}_{k=1}^K)$ *is a local minimizer of* FFP : $\mathcal{P}(\{L_k\}_{k=1}^K)\to\mathbb{R}$ *if and only if every* $\{f_m\}_{m=1}^M\in\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ *which generates* $\{P_k\}_{k=1}^K$ *is a local minimizer of* FP : $\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)\to\mathbb{R}$.

Theorem 2 strongly suggests a closer study of the minimizers of the traditional frame potential, as begun in the next section.

3 Perturbing the frame potential

A number of works have considered the minimization of the traditional frame potential (2) over various subsets E of:

$$\mathbb{H}_N^M=\big\{\{f_m\}_{m=1}^M:f_m\in\mathbb{H}_N,\ \forall m=1,\dots,M\big\}.$$

In particular, [1] considers the minimization of the frame potential over

$$E=\big\{\{f_m\}_{m=1}^M\in\mathbb{H}_N^M:\|f_m\|=1,\ \forall m=1,\dots,M\big\},\tag{19}$$

while [5] generalizes this problem to the case where

$$E = \big\{\{f_m\}_{m=1}^M \in \mathbb{H}_N^M : \|f_m\| = a_m,\ \forall m = 1, \dots, M\big\},$$

where $\{a_m\}_{m=1}^M$ is an arbitrary nonnegative sequence. Meanwhile, in [10, 11], E consists of filter bank sequences $\{f_m\}_{m=1}^M$ in the ℓ^2-space of some finite abelian group. For the purposes of proving the existence of tight fusion frames, we, in light of Theorem 2, are interested in the special case of having E be $\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$, as defined in (8). As with this previous work, our approach will be to study the effects that small perturbations of $\{f_m\}_{m=1}^M$ have upon the value of the frame potential. We shall make use of the following calculus result, which strengthens and generalizes an argument found in the proof of Theorem 1 of [5]:

Lemma 2 *If $f_m(\cdot) : \mathbb{R} \to \mathbb{H}_N$ is twice-differentiable for all $m = 1, \dots, M$, then the first two derivatives of $\mathrm{FP}(\{f_m(\cdot)\}_{m=1}^M)$ are:*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{FP}(\{f_m(t)\}_{m=1}^M) &= 4\mathrm{Re}\mathrm{Tr}\big(\dot{F}(t)F^*(t)F(t)F^*(t))\big),\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{FP}(\{f_m(t)\}_{m=1}^M) &= 2\|\dot{F}^*(t)F(t) + F^*(t)\dot{F}(t)\|_{\mathrm{HS}}^2 + 4\|\dot{F}(t)F^*(t)\|_{\mathrm{HS}}^2\\
&\quad + 4\mathrm{Re}\mathrm{Tr}\big(\ddot{F}(t)F^*(t)F(t)F^*(t)\big),
\end{aligned}$$

where $F^{(j)}(t)$ denotes the analysis operators of $\{f_m^{(j)}(t)\}_{m=1}^M$.

Proof Defining the derivative of a matrix-valued function as the termwise derivative of its entries, one may easily show that:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}A^*(t) &= \dot{A}^*(t),\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\big(A(t)\big) &= \mathrm{Tr}\big(\dot{A}(t)\big),\\
\frac{\mathrm{d}}{\mathrm{d}t}A(t)B(t) &= \dot{A}(t)B(t) + A(t)\dot{B}(t),
\end{aligned}$$

for all matrix-valued functions $A(\cdot)$ and $B(\cdot)$ of equal inner dimension. Our results follow quickly from these rules. In particular, the first derivative of the parametrized frame potential is:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{FP}(\{f_m(t)\}_{m=1}^M) &= \frac{\mathrm{d}}{\mathrm{d}t}\|F(t)F^*(t)\|_{\mathrm{HS}}^2\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\big(F(t)F^*(t)F(t)F^*(t)\big)\\
&= \mathrm{Tr}\big(\dot{F}(t)F^*(t)F(t)F^*(t)\big) + \mathrm{Tr}\big(F(t)\dot{F}^*(t)F(t)F^*(t)\big)\\
&\quad + \mathrm{Tr}\big(F(t)F^*(t)\dot{F}(t)F^*(t)\big) + \mathrm{Tr}\big(F(t)F^*(t)F(t)\dot{F}^*(t)\big)\\
&= 2\mathrm{Tr}\big(\dot{F}(t)F^*(t)F(t)F^*(t)\big) + 2\mathrm{Tr}\big(F(t)\dot{F}^*(t)F(t)F^*(t)\big)\\
&= 2\mathrm{Tr}\big(\dot{F}(t)F^*(t)F(t)F^*(t)\big) + 2\overline{\mathrm{Tr}\big(F(t)F^*(t)\dot{F}(t)F^*(t)\big)}\\
&= 4\mathrm{Re}\mathrm{Tr}\big(\dot{F}(t)F^*(t)F(t)F^*(t))\big).
\end{aligned}$$

Similarly, and suppressing the dependence on t in the notation, the second derivative of the parametrized frame potential is given by:

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{FP}(\{f_m\}_{m=1}^M) &= \frac{\mathrm{d}}{\mathrm{d}t}4\mathrm{ReTr}(\dot{F}F^*FF^*)\\ &= 4\mathrm{ReTr}(\ddot{F}F^*FF^*) + 4\mathrm{ReTr}(\dot{F}\dot{F}^*FF^*)\\ &\quad + 4\mathrm{ReTr}(\dot{F}F^*\dot{F}F^*) + 4\mathrm{ReTr}(\dot{F}F^*F\dot{F}^*)\\ &= 4\mathrm{ReTr}(\ddot{F}F^*FF^*) + 4\mathrm{ReTr}(F^*\dot{F}(\dot{F}^*F + F^*\dot{F}))\\ &\quad + 4\|\dot{F}F^*\|_{\mathrm{HS}}^2,\end{aligned}$$

where the second term above may be simplified as:

$$\begin{aligned}&4\mathrm{ReTr}(F^*\dot{F}(\dot{F}^*F + F^*\dot{F}))\\ &= 2\mathrm{Tr}(F^*\dot{F}(\dot{F}^*F + F^*\dot{F})) + 2\mathrm{Tr}[(F^*\dot{F}(\dot{F}^*F + F^*\dot{F}))^*]\\ &= 2\mathrm{Tr}(F^*\dot{F}(\dot{F}^*F + F^*\dot{F})) + 2\mathrm{Tr}((F^*\dot{F} + \dot{F}^*F)\dot{F}^*F)\\ &= 2\mathrm{Tr}[(\dot{F}^*F + F^*\dot{F})^2]\\ &= 2\|\dot{F}^*F + F^*\dot{F}\|_{\mathrm{HS}}^2. \ \square\end{aligned}$$

The following result is an immediate consequence of Lemma 2:

Theorem 3 *If* $\{f_m\}_{m=1}^M$ *is a local minimizer of* $\mathrm{FP} : E \to \mathbb{R}$*, then:*

$$\begin{aligned}0 &= \mathrm{ReTr}(\dot{F}(0)F^*FF^*),\\ 0 &\le \tfrac{1}{2}\|\dot{F}^*(0)F + F^*\dot{F}(0)\|_{\mathrm{HS}}^2 + \|\dot{F}(0)F^*\|_{\mathrm{HS}}^2 + \mathrm{ReTr}(\ddot{F}(0)F^*FF^*),\end{aligned}$$

for any sequence of twice-differentiable curves $\{f_m(\cdot)\}_{m=1}^M$*,* $f_m : \mathbb{R} \to E$ *such that* $f_m(0) = f_m$ *for all* $m = 1, \dots, M$*.*

In the next section, we shall make repeated use of Theorem 3 in the special case where $E = \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$.

4 Local minimizers of the fusion frame potential

We now apply Theorems 2 and 3 to obtain a necessary condition on any local minimizer of the fusion frame potential.

Theorem 4 *If* $\{P_k\}_{k=1}^K$ *is any local minimizer of* $\mathrm{FFP} : \mathcal{P}(\{L_k\}_{k=1}^K) \to \mathbb{R}$*, then there exists* $\{f_m\}_{m=1}^M \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ *which generates* $\{P_k\}_{k=1}^K$ *and has the property that every* f_m *is an eigenvector of* F^*F*.*

Moreover, the frame vectors which lie in a given eigenspace form a tight frame for that eigenspace, with the tight frame constant being the corresponding eigenvalue.

Proof Let $\{f_m\}_{m=1}^M \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ be an arbitrary generator for $\{P_k\}_{k=1}^K$. By Theorem 2, $\{f_m\}_{m=1}^M$ is a local minimizer of the restricted frame potential FP : $\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K) \to \mathbb{R}$. Without loss of generality, we assume $L_k < N$ for all $k = 1, \ldots, K$. Thus, for any fixed $k = 1, \ldots, K$ and $m_0 \in \mathcal{I}_k$, we may take $g \in \mathcal{R}(P_k)^\perp$ and consider:

$$f_m(t) = \begin{cases} \cos(t) f_m + \sin(t) g, & m = m_0, \\ f_m, & m \neq m_0. \end{cases}$$

We clearly have that $f_m(0) = f_m$ for all $m = 1, \ldots, M$, while:

$$\dot{f}_m(0) = \begin{cases} g, & m = m_0, \\ 0, & m \neq m_0. \end{cases} \tag{20}$$

Moreover, as $g \in \mathcal{R}(P_k)^\perp$, then $\{f_m(t)\}_{m=1}^M \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ for all $t \in \mathbb{R}$. Thus, Theorem 3 gives:

$$\begin{aligned} 0 &= \mathrm{Re}\mathrm{Tr}\big(\dot{F}(0) F^* F F^*\big) \\ &= \mathrm{Re} \sum_{m=1}^M \langle \dot{F}(0) F^* F F^* e_m, e_m \rangle \\ &= \mathrm{Re} \sum_{m=1}^M \langle F^* F F^* e_m, \dot{F}^*(0) e_m \rangle \\ &= \mathrm{Re} \sum_{m=1}^M \langle F^* F f_m, \dot{f}_m(0) \rangle. \end{aligned} \tag{21}$$

Invoking (20), we may rewrite (21) as:

$$0 = \mathrm{Re}\langle F^* F f_{m_0}, g \rangle. \tag{22}$$

Since the vector g in (22) is an arbitrary element of $\mathcal{R}(P_k)^\perp$, we, in the case where $\mathbb{H}_N$ is complex, may replace g with $\mathrm{i}g$ to also obtain

$$0 = \mathrm{Re}\langle F^* F f_{m_0}, \mathrm{i}g \rangle = \mathrm{Im}\langle F^* F f_{m_0}, g \rangle. \tag{23}$$

Combining (22) and (23) gives $0 = \langle F^* F f_{m_0}, g \rangle$ for all $g \in \mathcal{R}(P_k)^\perp$. Thus, $F^* F f_{m_0} \in [\mathcal{R}(P_k)^\perp]^\perp = \mathcal{R}(P_k)$. As m_0 is an arbitrary index in $\mathcal{I}_k$, we therfore have that $F^* F$ preserves $\mathcal{R}(P_k) = \mathrm{span}\{f_m\}_{m \in \mathcal{I}_k}$, that is, that $F^* F[\mathcal{R}(P_k)] \subseteq \mathcal{R}(P_k)$. As the restricted operator $F^* F : \mathcal{R}(P_k) \to \mathcal{R}(P_k)$ is self-adjoint, there exists an orthonormal basis for $\mathcal{R}(P_k)$ that consists entirely of eigenvectors of $F^* F$. Rechoosing $\{f_m\}_{m \in \mathcal{I}_k}$ to be these eigenvectors for each $k = 1, \ldots, K$ gives the first conclusion.

The second conclusion follows from the proof of Theorem 5.1 in [1]; we include the proof here to establish the notation used in the proof of our main result below. That is, let $\{\lambda_j\}_{j=1}^J$ be the distinct eigenvalues of the frame operator $F^* F$, and let $\{\mathcal{E}_j\}_{j=1}^J$ be their corresponding eigenspaces. As each f_m is an eigenvector of $F^* F$, we have $\{1, \ldots, M\} = \sqcup_{j=1}^J \mathcal{J}_j$ where

$\mathcal{J}_j = \{m : f_m \in \mathcal{E}_j\}$. To see that each $\{f_m\}_{m\in\mathcal{J}_j}$ is a λ_j-tight frame for $\mathcal{E}_j$, note that since eigenvectors corresponding to distinct eigenvalues of the self-adjoint operator F^*F are necessarily orthogonal, then for any $f \in \mathcal{E}_j$, we have $\langle f, f_m\rangle = 0$ for any $m \notin \mathcal{J}_j$, and so:

$$\lambda_j f = F^*Ff = \sum_{m=1}^{M} \langle f, f_m\rangle f_m = \sum_{m\in\mathcal{E}_j} \langle f, f_m\rangle f_m,$$

as claimed. As such, (5) gives $\lambda_j = |\mathcal{J}_j| / \dim(\mathcal{E}_j)$ for all $j = 1, \dots, J$. □

Our proof of the first claim of Theorem 4 is significantly more elementary than the Lagrange multipliers-based proof of the corresponding result for the traditional frame potential, namely Theorem 7.3 of [1], which does not generalize to our $E = \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ setting. Moreover, the statement of Theorem 4 is itself of much greater significance in the fusion frame setting than in the traditional frame setting. Indeed, in [1] where E is given by (19), the result of Theorem 4 is superseded by a stronger result, namely that every local minimizer of the frame potential is a unit norm tight frame whenever $M \geq N$. The situation for fusion frames is more complicated. In particular, there is, as of yet, no known characterization of those K, $\{L_k\}_{k=1}^N$ and N for which a tight fusion frame $\{P_k\}_{k=1}^K$ for $\mathbb{H}_N$, $\mathrm{Tr}(P_k) = L_k$, will exist. The main result of this article, namely Theorem 5 below, only provides sufficient conditions for the existence of tight fusion frames. Nevertheless, even in cases where tight fusion frames are known to not exist, one may still apply Theorem 4 to help determine the global minimizer of the fusion frame potential, which is hopefully nearly tight.

For example, we claim that there does not exist a tight fusion frame for $\mathbb{H}_3$ which consists of three projections of ranks 1, 1 and 2, respectively, despite the fact that $M = 4 > 3 = N$. Indeed, such a tight fusion frame would necessarily be generated by a tight frame $\{f_m\}_{m=1}^4 \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^3)$, where $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \{2\}$ and $\mathcal{I}_3 = \{3, 4\}$. As such, f_3 would be necessarily orthogonal to f_4, which is impossible if $\{f_m\}_{m=1}^4$ is tight, since such a frame is necessarily a tetrahedron, and thus has $|\langle f_3, f_4\rangle| = \frac{1}{3}$.

Despite this fact, we may nevertheless attempt to minimize the fusion frame potential for these parameters. Indeed, by Theorem 4, there exists some $\{f_m\}_{m=1}^4 \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^3)$ which generates the global minimizer of the fusion frame potential and has the property that its four frame elements may be partitioned into the mutually orthogonal eigenspaces of the 3×3 matrix F^*F, with each eigensubframe being tight for its eigenspace. There are essentially only five ways that this may occur: (1) f_1, f_2 and f_3 are scalar multiples of each other, and orthogonal to f_4, in which case the eigenvalues of F^*F are $\{3, 1, 0\}$; (2) f_1 is orthogonal to f_2, and both lie in the span of f_3 and f_4, with eigenvalues $\{2, 2, 0\}$; (3) both f_1 and f_2 are orthogonal to the span of f_3 and f_4, with eigenvalues $\{2, 1, 1\}$; (4) f_1 and f_3 are orthogonal, with f_2 and f_4 being orthogonal to them both, with eigenvalues $\{2, 1, 1\}$; (5) f_1, f_2 and f_3 form a Mercedes-Benz tight frame for the orthogonal complement of f_4, with eigenvalues $\{\frac{3}{2}, \frac{3}{2}, 1\}$. Taking the sum of the squares of these eigenvalues, the frame potentials of these sequences are then 10, 8, 6, 6 and $\frac{11}{2}$, respectively, all of which are greater than the unattainable lower bound of $\frac{16}{3}$ given in Proposition 1. As such, the fifth example is the optimal fusion frame for these parameters.

We now turn to the main result of this work, which provides a sufficient condition for the existence of tight fusion frames in the special case where all of the L_k's are taken to be equal. In particular, we show that tight fusion frames will always exist provided K is sufficiently large, and provided that N is sufficiently greater than L. Note that this result does apply to cases where the L_k's are not all equal, as L may simply be taken to be their maximum, though the tight fusion frame we produce will satisfy more orthogonality relations than necessary.

Theorem 5 *For any positive integers N and L, and any $K > N$ such that:*

$$0 \leq (N - L^2 + 1)K^2 - N(2N - L + 1)K + N^3, \tag{24}$$

there exists a tight fusion frame $\{P_k\}_{k=1}^K$ for $\mathbb{H}_N$ such that $\mathrm{Tr}(P_k) = L$ for all $k = 1, \dots, K$. In particular, letting $L_k = L$ for all $k = 1, \dots, K$, every local minimizer of $\mathrm{FFP} : \mathcal{P}(\{L_k\}_{k=1}^K) \to \mathbb{R}$ is a tight fusion frame for $\mathbb{H}_N$.

Proof Take any N, L and $K > N$ such that (24) holds, let $L_k = L$ for all $k = 1, \dots, K$, and let $\{P_k\}_{k=1}^K$ be any local minimizer of $\mathrm{FFP} : \mathcal{P}(\{L_k\}_{k=1}^K) \to \mathbb{R}$. By Theorem 4, there exists $\{f_m\}_{m=1}^M \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ which generates $\{P_k\}_{k=1}^K$ and has the property that every f_m is an eigenvector of F^*F. By Theorem 2, $\{f_m\}_{m=1}^M \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ is a local minimizer of $\mathrm{FP} : \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K) \to \mathbb{R}$. Let $\{\lambda_j\}_{j=1}^J$ be the distinct eigenvalues of the frame operator F^*F, arranged in decreasing order, with corresponding eigenspaces $\{\mathcal{E}_j\}_{j=1}^J$. Partitioning $\{1, \dots, M\}$ into the sets $\mathcal{J}_j = \{m : f_m \in \mathcal{E}_j\}$, Theorem 4 gives that each $\{f_m\}_{m \in \mathcal{J}_j}$ is a tight frame for $\mathcal{E}_j$ in which, by (5), the tight frame constant is $\lambda_j = |\mathcal{J}_j| / \dim(\mathcal{E}_j)$.

The main idea of the proof is to examine how the value of the frame potential changes as the local minimizer $\{f_m\}_{m=1}^M$ is perturbed in the direction of some $\{g_m\}_{m=1}^M \subset \mathbb{H}_N$. In particular, let $\{g_m\}_{m=1}^M$ be any sequence of vectors that has the properties that (a) $\langle f_m, g_{m'} \rangle = 0$ for any m, m' which belong to the same $\mathcal{I}_k$ and (b) $\langle g_m, g_{m'} \rangle = 0$ for any distinct m, m' which belong to the same $\mathcal{I}_k$. Consider the sequence of parametrized curves $\{f_m(\cdot)\}_{m=1}^M$,

$$f_m(t) = \begin{cases} \cos(\|g_m\|t) f_m + \sin(\|g_m\|t) \dfrac{g_m}{\|g_m\|}, & g_m \neq 0, \\ f_m, & g_m = 0. \end{cases}$$

We claim that $\{f_m(\cdot)\}_{m=1}^M$ satisfies the hypotheses of Theorem 3. In particular, we clearly have each $f_m(\cdot)$ is twice-differentiable with $f_m(0) = f_m$ for all $m = 1, \dots, M$. We further note that $\{f_m(t)\}_{m=1}^M \in \mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ for all $t \in \mathbb{R}$, that is, that $\{f_m(t)\}_{m \in \mathcal{I}_k}$ is orthonormal for all $k = 1, \dots, K$ and all $t \in \mathbb{R}$.

Indeed, the normality of $f_m(t)$ follows immediately from having $\langle f_m, g_m\rangle = 0$, while for any $m, m' \in \mathcal{I}_k$, $m \neq m'$, the fact that $\langle f_m(t), f_{m'}(t)\rangle = 0$ follows from having $\langle f_m, f_{m'}\rangle = \langle f_m, g_{m'}\rangle = \langle g_m, f_{m'}\rangle = \langle g_m, g_{m'}\rangle = 0$. Having this claim, Theorem 3 gives:

$$0 \leq \tfrac{1}{2}\|\dot{F}^*(0)F + F^*\dot{F}(0)\|_{\mathrm{HS}}^2 + \|\dot{F}(0)F^*\|_{\mathrm{HS}}^2 + \mathrm{Re}\mathrm{Tr}\big(\ddot{F}(0)F^*FF^*\big). \quad (25)$$

To explicitly evaluate the right hand side of (25), note that regardless of whether or not $g_m = 0$, we have:

$$\begin{aligned}\dot{f}_m(t) &= -\|g_m\|\sin(\|g_m\|t)f_m + \cos(\|g_m\|t)g_m,\\ \ddot{f}_m(t) &= -\|g_m\|^2\cos(\|g_m\|t)f_m - \|g_m\|\sin(\|g_m\|t)g_m,\end{aligned}$$

and so $\dot{f}_m(0) = g_m$ and $\ddot{f}_m(0) = -\|g_m\|^2 f_m$ for all $m = 1, \dots, M$. Letting G be the analysis operator for $\{g_m\}_{m=1}^M$, we have $\dot{F}(0) = G$. Thus, the first term on the right hand side of (25) is $\frac{1}{2}\|G^*F + F^*G\|_{\mathrm{HS}}^2$, while the second term is:

$$\begin{aligned}\|GF^*\|_{\mathrm{HS}}^2 &= \mathrm{Tr}\big(GF^*FG^*\big)\\ &= \sum_{m=1}^M \langle FG^*e_m, FG^*e_m\rangle\\ &= \sum_{m=1}^M \|Fg_m\|^2.\end{aligned}$$

As $\ddot{f}_m(0) = -\|g_m\|^2 f_m$, the third term on the right hand side of (25) is:

$$\begin{aligned}\mathrm{Re}\mathrm{Tr}\big(\ddot{F}(0)F^*FF^*\big) &= \mathrm{Re}\sum_{m=1}^M \langle \ddot{F}(0)F^*FF^*e_m, e_m\rangle\\ &= \mathrm{Re}\sum_{m=1}^M \langle FF^*e_m, F\ddot{F}^*(0)e_m\rangle\\ &= \mathrm{Re}\sum_{m=1}^M \langle Ff_m, F\ddot{f}_m(0)\rangle\\ &= -\sum_{m=1}^M \|g_m\|^2\|Ff_m\|^2.\end{aligned}$$

Substituting these expressions into (25) yields:

$$0 \leq \tfrac{1}{2}\|G^*F + F^*G\|_{\mathrm{HS}}^2 + \sum_{m=1}^M \big(\|Fg_m\|^2 - \|g_m\|^2\|Ff_m\|^2\big), \quad (26)$$

for any $\{g_m\}_{m=1}^M$ that satisfies (a) and (b).

We now use the eigenspace structure of $\{f_m\}_{m=1}^M$ discussed above to construct sequences $\{g_m\}_{m=1}^M$ for which (26) holds. In particular, for each $k=1,\dots,K$ such that $\mathcal{J}_1\cap\mathcal{I}_k\neq\emptyset$, pick some $m_k\in\mathcal{J}_1\cap\mathcal{I}_k$, and let $\mathcal{J}_0$ be the collection of these m_k's. That is, $\{f_m\}_{m\in\mathcal{J}_0}$ contains exactly one frame element from any of the fusion frame subspaces which intersects the highest eigenspace. As each of these subspaces is generated by L frame elements, the cardinality of $\mathcal{J}_0$ is at least $1/L$ that of $\mathcal{J}_1$. To see this more formally, note that since $\{f_m\}_{m=1}^M\in\mathcal{F}(\{\mathcal{I}_k\}_{k=1}^K)$ where $|\mathcal{I}_k|=L_k=L$, then:

$$|\mathcal{J}_1|=\sum_{k=1}^K|\mathcal{J}_1\cap\mathcal{I}_k|\leq\sum_{k\text{ s.t. }\mathcal{J}_1\cap\mathcal{I}_k\neq\emptyset}|\mathcal{I}_k|=|\mathcal{J}_0|L. \tag{27}$$

Having $\mathcal{J}_0$, we shall consider those sequences $\{g_m\}_{m=1}^M$ which satisfy the following four requirements:

$$\begin{aligned}
&\text{(i) } g_m=0,\ \forall m\notin\mathcal{J}_0,\\
&\text{(ii) } g_m\in\mathcal{E}_1^\perp,\ \forall m\in\mathcal{J}_0,\\
&\text{(iii) } F^*G=0,\\
&\text{(iv) } \forall m\in\mathcal{J}_0,\ \text{taking } k_m=1,\dots,K \text{ s.t. } m\in\mathcal{I}_{k_m},\\
&\qquad\text{we have } \langle g_m,f_{m'}\rangle=0,\ \forall m'\in\mathcal{I}_{k_m}, m\neq m'.
\end{aligned} \tag{28}$$

We claim that the only instance when all four requirements of (28) are met is when $g_m=0$ for all $m=1,\dots,M$. To see this claim, note that (28.i) immediately implies requirement (b) is met, as $\mathcal{J}_0$, by definition, contains at most one index from any set $\mathcal{I}_k$. We next note that (28.i), (28.ii) and (28.iv) together imply requirement (a) is met. Indeed, if $m'\notin\mathcal{J}_0$, the statement of (a) immediately follows from (28.i), while if $m\in\mathcal{J}_0$ and $m'\neq m$, it follows from (28.iv); in the remaining case where $m\in\mathcal{J}_0$ and $m'=m$, then $f_m\in\mathcal{E}_1$ while (28.ii) gives $g_{m'}=g_m\in\mathcal{E}_1^\perp$. As (a) and (b) are satisfied, (26) necessarily holds for $\{g_m\}_{m=1}^M$. Moreover, (28.iii) implies the first term of (26) will vanish; coupled with (28.i) and the fact that $\|Ff_m\|^2=\langle F^*Ff_m,f_m\rangle=\lambda_1$ for all $m\in\mathcal{J}_0\subseteq\mathcal{J}_1$, (26) simplifies to:

$$0\leq\sum_{m\in\mathcal{J}_0}\big(\|Fg_m\|^2-\lambda_1\|g_m\|^2\big). \tag{29}$$

However, as (28.ii) holds, where λ_1 is the largest eigenvalue of F^*F, we necessarily have that $\|Fg_m\|^2=\langle F^*Fg_m,g_m\rangle\leq\lambda_1\|g_m\|^2$, with a strict inequality whenever $g_m\neq0$. The only way this does not contradict with (29) is to have $g_m=0$ for all $m=1,\dots,M$, as claimed.

Thus, letting $\mathcal{U}$ be the subspace of $\mathbb{H}_N^M$ consisting of all $\{g_m\}_{m=1}^M$ which satisfy the four conditions (28), we have just shown that $\mathcal{U}=\{0\}$. We now use this fact to obtain an inequality relating the dimension of $\mathcal{E}_1$ to L, N and the number of elements in $\mathcal{J}_0$. To do this, we first estimate the dimension of the larger subspace $\mathcal{V}$ of $\mathbb{H}_N^M$ which consists of all $\{g_m\}_{m=1}^M$ which satisfy (28.i), (28.ii) and (28.iii). In particular, we claim that:

$$\dim(\mathcal{V}) \geq (|\mathcal{J}_0| - \dim(\mathcal{E}_1))(N - \dim(\mathcal{E}_1)). \tag{30}$$

To prove (30), let $d = \dim(\mathcal{E}_1)$, and let $\{b_p\}_{p=1}^{N-d}$ be an orthonormal basis for $\mathcal{E}_1^\perp$. Next, consider the null space of the analysis operator F_0 of $\{f_m\}_{m\in\mathcal{J}_0}$:

$$\mathcal{N}(F_0^*) = \Big\{c \in \ell^2(\mathcal{J}_0) : \sum_{m\in\mathcal{J}_0} c(m) f_m = 0\Big\}.$$

Note that as the vectors $\{f_m\}_{m\in\mathcal{J}_0}$ all lie in the eigenspace $\mathcal{E}_1$, the dimension of $\mathcal{N}(F_0^*)$ is at least $|\mathcal{J}_0| - d$. Let $\{c_q\}_{q=1}^{|\mathcal{J}_0|-d}$ be a linearly independent collection of vectors in $\mathcal{N}(F_0^*)$, realizing that when $|\mathcal{J}_0| \leq d$, this collection is empty. We now claim that for any scalars $\{z_{p,q}\}_{p=1,q=1}^{N-d,|\mathcal{J}_0|-d}$, the following sequence $\{g_m\}_{m=1}^M$ lies in $\mathcal{V}$:

$$g_m = \begin{cases} \displaystyle\sum_{p=1}^{N-d}\sum_{q=1}^{|\mathcal{J}_0|-d} z_{p,q}\overline{c_q(m)} b_p, & m \in \mathcal{J}_0, \\ 0, & m \notin \mathcal{J}_0. \end{cases} \tag{31}$$

Indeed, g_m immediately satisfies (28.i). Moreover, since each b_p lies in $\mathcal{E}_1^\perp$, it is also clear that (28.ii) is satisfied. Thus, we need only show (28.iii); for any $f \in \mathbb{H}_N$, we have:

$$\begin{aligned} F^*Gf &= \sum_{m=1}^M \langle f, g_m\rangle f_m \\ &= \sum_{m\in\mathcal{J}_0} \Big\langle f, \sum_{p=1}^{N-d}\sum_{q=1}^{|\mathcal{J}_0|-d} z_{p,q}\overline{c_q(m)} b_p\Big\rangle f_m \\ &= \sum_{p=1}^{N-d}\sum_{q=1}^{|\mathcal{J}_0|-d} \overline{z_{p,q}}\langle f, b_p\rangle \sum_{m\in\mathcal{J}_0} c_q(m) f_m \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that each c_q is a member of $\mathcal{N}(F_0^*)$.

Having that any $\{g_m\}_{m=1}^M$ of form (31) indeed lies in $\mathcal{V}$, all that is needed to prove (30) is to show that if $g_m = 0$ for all $m = 1, \dots, M$, then $z_{p,q} = 0$ for all $p = 1, \dots, N-d$, $q = 1, \dots, |\mathcal{J}_0| - d$; essentially (31) is writing G as a linear combination of $c_q b_p^*$'s, which we must now verify are linearly independent. To do this, note that since $\{b_p\}_{p=1}^{N-d}$ is orthonormal, then for all $m = 1, \dots, M$,

$$0 = \|g_m\|^2 = \sum_{p=1}^{N-d}\bigg|\sum_{q=1}^{|\mathcal{J}_0|-d} z_{p,q}\overline{c_q(m)}\bigg|^2. \tag{32}$$

As m is arbitrary, (32) implies that

$$0 = \sum_{q=1}^{|\mathcal{J}_0|-d} \overline{z_{p,q}} c_q,$$

for any $p = 1, \dots, N-d$. Since $\{c_q\}_{q=1}^{|\mathcal{J}_0|-d}$ is linearly independent, this, in turn, implies $z_{p,q} = 0$ for all $q = 1, \dots, |\mathcal{J}_0| - d$, as claimed. Having (30), we next claim that the dimension of $\mathcal{U} = \{0\}$ is bounded below:

$$0 = \dim(\mathcal{U}) \geq \dim(\mathcal{V}) - (L-1)|\mathcal{J}_0|. \tag{33}$$

Indeed, for each $m \in \mathcal{J}_0$, the requirement (28.iv) that $\langle g_m, f_{m'} \rangle = 0$ for all $m' \neq m$ which lie in the same $\mathcal{I}_k$ as m imposes at most $L-1$ linear homogenous constraints on the $z_{p,q}$'s of (31), each of the form:

$$0 = \sum_{p=1}^{N-d} \sum_{q=1}^{|\mathcal{J}_0|-d} z_{p,q} \overline{c_q(m)} \langle b_p, f_{m'} \rangle.$$

Substituting (30) into (33) then gives:

$$0 \geq (|\mathcal{J}_0| - \dim(\mathcal{E}_1))(N - \dim(\mathcal{E}_1)) - (L-1)|\mathcal{J}_0|,$$

which, when rearranged, yields:

$$\Big(1 - \frac{\dim(\mathcal{E}_1)}{|\mathcal{J}_0|}\Big)(N - \dim(\mathcal{E}_1)) \leq L - 1. \tag{34}$$

Next, recall (27) that $|\mathcal{J}_0| \geq |\mathcal{J}_1|/L$ and that $\lambda_1 = |\mathcal{J}_1|/\dim(\mathcal{E}_1)$, which, being the greatest eigenvalue, is greater than their average, which, as noted in (10), is M/N. As such:

$$\frac{|\mathcal{J}_0|}{\dim(\mathcal{E}_1)} \geq \frac{|\mathcal{J}_1|}{L\dim(\mathcal{E}_1)} = \frac{\lambda_1}{L} \geq \frac{M}{LN},$$

which, when combined with (34), yields:

$$\Big(1 - \frac{LN}{M}\Big)(N - \dim(\mathcal{E}_1)) \leq L - 1. \tag{35}$$

Continuing, note that since $M = KL$ where $K > N$ by assumption, then $M - NL > 0$, and so (35) is equivalent to:

$$N - \frac{(L-1)M}{M - LN} \leq \dim(\mathcal{E}_1). \tag{36}$$

Having (36), we now again exploit the fact that $\mathcal{U} = \{0\}$ to obtain a complementary upper bound on $\dim(\mathcal{E}_1)$. In particular, let $\mathcal{J}_{-1}$ consist of those $m \in \mathcal{J}_0$ for which, given $k_m = 1, \dots, K$ such that $m \in \mathcal{I}_{k_m}$, we have $\mathcal{I}_{k_m} \cap \mathcal{J}_J = \emptyset$. That is, consider those f_m's in the highest eigenspace $\mathcal{E}_1$

which are not required to be orthogonal to each other, nor to any frame element which resides in the lowest eigenspace $\mathcal{E}_J$. In particular, since $\{\mathcal{I}_k\}_{k=1}^K$ partitions $\{1,\dots,M\}$, we have:

$$\begin{aligned}|\mathcal{J}_0| &= \sum_{k=1}^K |\mathcal{J}_0\cap\mathcal{I}_k|\\ &= \sum_{\{k\,:\,\mathcal{I}_k\cap\mathcal{J}_J\neq\emptyset\}} |\mathcal{J}_0\cap\mathcal{I}_k| + \sum_{\{k\,:\,\mathcal{I}_k\cap\mathcal{J}_J=\emptyset\}} |\mathcal{J}_0\cap\mathcal{I}_k|\\ &= \sum_{\{k\,:\,\mathcal{I}_k\cap\mathcal{J}_J\neq\emptyset\}} |\mathcal{J}_0\cap\mathcal{I}_k| + |\mathcal{J}_{-1}|.\end{aligned}$$

To estimate the right hand side above, recall that the definition of $\mathcal{J}_0$ guarantees that $|\mathcal{J}_0\cap\mathcal{I}_k|\leq 1$ for all $k=1,\dots,K$. As any element of $\mathcal{J}_J$ can lie in at most one of the $\mathcal{I}_k$'s, we therefore have:

$$|\mathcal{J}_0| \leq |\{k : \mathcal{I}_k\cap\mathcal{J}_J\neq\emptyset\}| + |\mathcal{J}_{-1}| \leq |\mathcal{J}_J| + |\mathcal{J}_{-1}| \tag{37}$$

that is, $|\mathcal{J}_{-1}|\geq|\mathcal{J}_0|-|\mathcal{J}_J|$.

We now assume to the contrary that $\{f_m\}_{m=1}^M$ is not a tight frame for $\mathbb{H}_N$, and will derive a contradiction of our underlying assumption (24). In particular, since F^*F has more than one eigenvalue, there exists $g\neq 0$, $g\in\mathcal{E}_J\subseteq\mathcal{E}_1^\perp$. We claim that the existence of such a g necessarily implies that $|\mathcal{J}_{-1}|\leq\dim(\mathcal{E}_1)$. Otherwise, the vectors $\{f_m\}_{m\in\mathcal{J}_{-1}}$, all of which lie inside of $\mathcal{E}_1$, are necessarily linearly dependent, and as such, there exists $c\in\ell^2(\mathcal{J}_{-1})$, $c\neq 0$ such that:

$$\sum_{m\in\mathcal{J}_{-1}} c(m)f_m = 0.$$

In this case, we, as a special case of (31), may define $\{g_m\}_{m=1}^M$ as:

$$g_m = \begin{cases}\overline{c(m)}g, & m\in\mathcal{J}_{-1},\\ 0, & m\notin\mathcal{J}_{-1}.\end{cases} \tag{38}$$

Being of this form, we have already shown that $\{g_m\}_{m=1}^M$ satisfies (28.i), (28.ii) and (28.iii); we further claim that it satisfies (28.iv). Indeed, by definition, if $m\in\mathcal{J}_{-1}$, taking k_m such $m\in\mathcal{I}_{k_m}$, we have that $\mathcal{I}_{k_m}\cap\mathcal{J}_J=\emptyset$. In particular, for any $m'\in\mathcal{I}_{k_m}$, $m'\neq m$, we necessarily have $m'\notin\mathcal{J}_J$, that is, $f_{m'}\notin\mathcal{E}_J$. And, since $f_{m'}$ is an eigenvector for F^*F, it necessarily lies in an eigenspace of F^*F which is distinct from, and therefore orthogonal to, $\mathcal{E}_J$. As such, for any $m'\in\mathcal{I}_{k_m}$, $m'\neq m$, the fact that $g\in\mathcal{E}_J$ implies $\langle g_m,f_{m'}\rangle=\overline{c(m)}\langle g,f_{m'}\rangle=0$. That is, the $\{g_m\}_{m=1}^M$ defined in (38) indeed satisfies all four properties of (28), and as such, is necessarily zero. However this contradicts the fact that both g and c are nonzero. Thus, our assumption that $|\mathcal{J}_{-1}|>\dim(\mathcal{E}_1)$ was incorrect.

Having proven our claim that $|\mathcal{J}_{-1}| \leq \dim(\mathcal{E}_1)$, we revisit (37), again recalling from (27) that $|\mathcal{J}_0| \geq |\mathcal{J}_1|/L$, to obtain:

$$\dim(\mathcal{E}_1) \geq \frac{|\mathcal{J}_1|}{L} - |\mathcal{J}_J|.$$

Dividing by $\dim(\mathcal{E}_1)$ then yields:

$$\begin{aligned} 1 &\geq \frac{1}{L}\frac{|\mathcal{J}_1|}{\dim(\mathcal{E}_1)} - \frac{|\mathcal{J}_J|}{\dim(\mathcal{E}_J)}\frac{\dim(\mathcal{E}_J)}{\dim(\mathcal{E}_1)} \\ &= \frac{\lambda_1}{L} - \lambda_J\frac{\dim(\mathcal{E}_J)}{\dim(\mathcal{E}_1)}. \end{aligned} \tag{39}$$

To continue, note that as the $\{f_m\}_{m=1}^M$ is still assumed to be not tight, the greatest and least eigenvalues of F^*F are strictly greater and less than their average, namely M/N. As we also have $\dim(\mathcal{E}_J) \leq \dim(\mathcal{E}_1^\perp) = N - \dim(\mathcal{E}_1)$, (39) becomes:

$$1 > \frac{M}{NL} - \frac{M}{N}\frac{N - \dim(\mathcal{E}_1)}{\dim(\mathcal{E}_1)} = \frac{M}{N}\Big(\frac{1}{L} - \frac{N}{\dim(\mathcal{E}_1)} + 1\Big).$$

Solving for $\dim(\mathcal{E}_1)$ yields:

$$\dim(\mathcal{E}_1) < \frac{LMN}{M(L+1) - NL}. \tag{40}$$

Combining (36) with (40) then gives:

$$N - \frac{(L-1)M}{M - LN} < \frac{LMN}{M(L+1) - NL}. \tag{41}$$

Since $M = KL$ where $K > N$, both denominators above are positive, and so (41) is equivalent to:

$$0 > (N - L^2 + 1)M^2 - LN(2N - L + 1)M + L^2N^3.$$

Writing $M = KL$ and dividing by L^2 then gives:

$$0 > (N - L^2 + 1)K^2 - N(2N - L + 1)K + N^3,$$

which is a contradiction of our underlying assumption (24). Thus, for $K > N$ which satisfy (24), every local minimizer of FFP : $\mathcal{P}(\{L_k\}_{k=1}^K) \to \mathbb{R}$ is a tight fusion frame for $\mathbb{H}_N$. In particular, as the fusion frame potential is a continuous function over the compact set $\mathcal{P}(\{L_k\}_{k=1}^K)$, it necessarily has a global minimizer, and so tight fusion frames necessarily exist for any such K, L and N. □

Note that in light of Proposition 1, Theorem 5 actually shows that if K, L and $N < K$ satisfy (24), then every local minimizer of FFP : $\mathcal{P}(\{L_k\}_{k=1}^K) \to \mathbb{R}$ is a global minimizer. We conclude by showing how Theorem 1 is a corollary of Theorem 5.

Proof (of Theorem 1) Letting $K = \beta N$ where $\beta > 1$, (24) becomes:

$$0 \leq (N - L^2 + 1)\beta^2 N^2 - N(2N - L + 1)\beta N + N^3,$$

which may be simplified to:

$$\frac{(L-1)(L+1-\frac{1}{\beta})}{(1-\frac{1}{\beta})^2} \leq N.$$

And, as for all $\beta \geq \alpha > 1$ we have:

$$\frac{(L-1)(L+1-\frac{1}{\beta})}{(1-\frac{1}{\beta})^2} \leq \frac{(L-1)(L+1)}{(1-\frac{1}{\alpha})^2},$$

we have that (24) will hold for all $N \geq (L^2 - 1)/(1 - \frac{1}{\alpha})^2$. □

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