

Fusion Frames and Robust Dimension Reduction

Ali Pezeshki
Princeton University
Princeton, NJ 08544
pezeshki@math.princeton.edu

Gitta Kutyniok
Stanford University
Stanford, CA 94305
kutyniok@stanford.edu

Robert Calderbank
Princeton University
Princeton, NJ 08544
calderbk@math.princeton.edu

Abstract—We consider the linear minimum mean-squared error (LMMSE) estimation of a random vector of interest from its *fusion frame* measurements in presence noise and subspace erasures. Each fusion frame measurement is a low-dimensional vector whose elements are inner products of an orthogonal basis for a fusion frame subspace and the random vector of interest. We derive bounds on the mean-squared error (MSE) and show that the MSE will achieve its lower bound if the fusion frame is *tight*. We prove that tight fusion frames consisting of equi-dimensional subspaces have maximum robustness with respect to erasures of one subspace, and that the optimal dimension depends on SNR. We also show that tight fusion frames consisting of equi-dimensional subspaces with equal pairwise chordal distances are most robust with respect to two and more subspace erasures, and refer to such fusion frames as *equi-distance tight fusion frames*. Finally, we show that the squared chordal distance between the subspaces in such fusion frames meets the so-called *simplex bound*, and thereby establish a connection between equi-distance tight fusion frames and *optimal Grassmannian packings*.

I. INTRODUCTION

The notion of a *fusion frame* was introduced in [1] and further developed in [2]. A fusion frame for \mathbb{R}^M is a finite collection of subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in \mathbb{R}^M such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|\mathbf{x}\|^2 \leq \sum_{i=1}^N \|\mathbf{P}_i \mathbf{x}\|^2 \leq B\|\mathbf{x}\|^2, \quad \text{for any } \mathbf{x} \in \mathbb{R}^M,$$

where \mathbf{P}_i is the orthogonal projection onto \mathcal{W}_i . Alternatively, $\{\mathcal{W}_i\}_{i=1}^N$ is a fusion frame if and only if

$$A\mathbf{I} \leq \sum_{i=1}^N \mathbf{P}_i \leq B\mathbf{I}. \quad (1)$$

The constants A and B are called fusion frame bounds. An important class of fusion frames is the class of *tight*

fusion frames, for which A and B can be chosen to be equal and hence $\sum_{i=1}^N \mathbf{P}_i = A\mathbf{I}$. We note that the definition given in [1] and [2] for fusion frames applies to closed and weighted subspaces in any Hilbert space. However, since the scope of this paper is limited to non-weighted subspaces in \mathbb{R}^M , the definition of a fusion frame is only presented for this case.

A fusion frame is a frame-like collection of low-dimensional subspaces, where a signal is represented by a collection of vectors, whose elements are the inner products of the signal and the orthogonal bases for the fusion frame subspaces. Similar to frames, fusion frames can be used to provide a redundant representation of a signal. In fact, in applications where data has to be processed in a distributed manner by combining several locally processed data vectors, fusion frames may provide a more natural mathematical framework than frames. The reader is referred to [2]–[6] for examples of applications of fusion frames in distributed sensing, parallel computing, and packet encoding.

The optimal reconstruction (in ℓ_2 norm sense) of a deterministic signal $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements is considered in [2]. Constructing frames that allow for robust reconstruction of a deterministic signal in the presence of frame element erasures has been considered by a number of authors [7]–[10]. The construction of fusion frames for robust reconstruction of deterministic signals in the presence of subspace erasures has been considered in [5] and [11].

In this paper, we consider the LMMSE estimation of a zero-mean random vector $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements in presence of additive white noise and subspace erasures. Each fusion frame measurement is a low-dimensional (smaller than M) vector whose elements are inner products of \mathbf{x} and an orthogonal basis for a fusion frame subspace. We limit our analysis to the case where the signal covariance matrix $\mathbf{R}_{xx} = E[\mathbf{x}\mathbf{x}^T]$ is $\mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$. We derive bounds on the MSE in the absence of erasures and show that the lower bound will

A. Pezeshki and R. Calderbank were supported by AFOSR Grant 00852833. G. Kutyniok was supported by Deutsche Forschungsgemeinschaft (DFG) Heisenberg-Fellowship KU 1446/8-1.

be achieved if the fusion frame is tight. We then analyze the effect of subspace erasures on the performance of LMMSE estimators. We limit our analysis to the class of tight fusion frames and aim to minimize the *maximal* MSE due to the erasure of k subspaces for $k = 1$, $k = 2$, and $k > 2$. We show that maximum robustness against one subspace erasure is achieved when all subspaces in a tight-fusion frame have equal dimensions, and that the optimal dimension depends on SNR. We also show that a tight fusion frame consisting of equi-dimensional subspaces with equal pairwise *chordal distances* is maximally robust with respect to two and more subspace erasures. We refer to such fusion frames as *equi-distance tight fusion frames*. We show that the squares of the pairwise chordal distances between the subspaces in equi-distance tight fusion frames meet the so-called *simplex bound*, and thereby establish an important connection between the construction of such fusion frames and *optimal Grassmannian packings* (cf. Conway et al. [12]). This connection indicates that optimal Grassmannian packings are fundamental for robust dimension reduction.

We note that this paper is a summary of results. We have omitted many of the proofs, and derivations have been shortened or left out entirely. A more comprehensive treatment of the subject is presented in [6].

II. LMMSE ESTIMATION WITH NO ERASURES

Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for \mathbb{R}^M with bounds $A \leq B$ and m_i be the dimension of \mathcal{W}_i , $i = 1, \dots, N$. Let $\mathbf{x} \in \mathbb{R}^M$ be a zero-mean random vector with covariance matrix $E[\mathbf{x}\mathbf{x}^T] = \mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$. We wish to estimate \mathbf{x} from N low-dimensional (smaller than M) measurement vectors $\mathbf{z}_i \in \mathbb{R}^{m_i}$ given by

$$\mathbf{z}_i = \mathbf{U}_i^T \mathbf{x} + \mathbf{n}_i, \quad i = 1, \dots, N,$$

where $\mathbf{U}_i \in \mathbb{R}^{M \times m_i}$ is a known but otherwise arbitrary *left-orthogonal* basis for \mathcal{W}_i , $i = 1, \dots, N$. That is $\mathbf{U}_i^T \mathbf{U}_i = \mathbf{I}_{m_i}$, where \mathbf{I}_{m_i} is the $m_i \times m_i$ identity matrix, and $\mathbf{U}_i \mathbf{U}_i^T = \mathbf{P}_i$, where \mathbf{P}_i is the orthogonal projection matrix onto \mathcal{W}_i . The vector $\mathbf{n}_i \in \mathbb{R}^{m_i}$ is an additive white noise vector with zero mean and covariance matrix $E[\mathbf{n}_i \mathbf{n}_i^T] = \sigma_n^2 \mathbf{I}$, $i = 1, \dots, N$. We assume that the noise vectors for different subspaces are mutually uncorrelated, and that the signal vector \mathbf{x} and the noise vectors \mathbf{n}_i , $i = 1, \dots, N$ are uncorrelated.

We define the composite measurement vector $\mathbf{z} \in \mathbb{R}^L$ and the composite basis matrix $\mathbf{U} \in \mathbb{R}^{M \times L}$ as

$$\mathbf{z} = (\mathbf{z}_1^T \quad \mathbf{z}_2^T \quad \dots \quad \mathbf{z}_N^T)^T$$

and

$$\mathbf{U} = (\mathbf{U}_1 \quad \mathbf{U}_2 \quad \dots \quad \mathbf{U}_N),$$

where $L = \sum_{i=1}^N m_i$. Then, the composite covariance matrix between \mathbf{x} and \mathbf{z} can be written as

$$E \left[\begin{pmatrix} \mathbf{x} \\ \mathbf{z} \end{pmatrix} \begin{pmatrix} \mathbf{x}^T & \mathbf{z}^T \end{pmatrix} \right] = \begin{pmatrix} \mathbf{R}_{xx} & \mathbf{R}_{xz} \\ \mathbf{R}_{zx} & \mathbf{R}_{zz} \end{pmatrix},$$

where

$$\mathbf{R}_{xz} = E[\mathbf{x}\mathbf{z}^T] = \mathbf{R}_{xx} \mathbf{U}$$

is the $M \times L$ cross-covariance matrix between \mathbf{x} and \mathbf{z} , $\mathbf{R}_{zx} = \mathbf{R}_{xz}^T$, and

$$\mathbf{R}_{zz} = E[\mathbf{z}\mathbf{z}^T] = \mathbf{U}^T \mathbf{R}_{xx} \mathbf{U} + \sigma_n^2 \mathbf{I}_L \quad (2)$$

is the $L \times L$ composite measurement covariance matrix.

The linear MSE minimizer for estimating \mathbf{x} from \mathbf{z} is the Wiener filter or the LMMSE filter $\mathbf{F} = \mathbf{R}_{xz} \mathbf{R}_{zz}^{-1}$, which estimates \mathbf{x} by $\hat{\mathbf{x}} = \mathbf{F}\mathbf{z}$ (e.g., see [13]). The error covariance matrix \mathbf{R}_{ee} in this estimation is given by

$$\begin{aligned} \mathbf{R}_{ee} &= E[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T] \\ &= \left(\mathbf{R}_{xx}^{-1} + \frac{1}{\sigma_n^2} \sum_{i=1}^N \mathbf{P}_i \right)^{-1}. \end{aligned}$$

The MSE is obtained by taking the trace of \mathbf{R}_{ee} . Since $\mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$, it follows from (1) that

$$\frac{M}{\sigma_x^2 + \frac{B}{\sigma_n^2}} \leq (MSE = \text{tr}\{\mathbf{R}_{ee}\}) \leq \frac{M}{\sigma_x^2 + \frac{A}{\sigma_n^2}}.$$

The lower bound will be achieved, if the fusion frame is tight. That is, when $A = B$ and

$$\sum_{i=1}^N \mathbf{P}_i = A \mathbf{I}. \quad (3)$$

Taking the trace from both sides of (3) yields the bound A as

$$A = \frac{\sum_{i=1}^N m_i}{M} = \frac{L}{M}. \quad (4)$$

Thus, the MSE is given by

$$MSE = \frac{M \sigma_n^2 \sigma_x^2}{\sigma_n^2 + \frac{\sigma_x^2 L}{M}}. \quad (5)$$

III. EFFECT OF SUBSPACE ERASURES

We now consider the case where subspace erasures occur, that is, when measurement vectors from one or more subspaces are lost or discarded. We wish to determine the MSE when the LMMSE filter \mathbf{F} , which is calculated based on the full composite covariance matrix in (2), is applied to a composite measurement vector with erasures. In order to maintain optimality with respect to no erasures, in our analysis we assume that the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ is tight with bound A given by (4). We consider the case where all k -subspace erasures are equally important. In other words, we aim to minimize the maximal MSE due to k erasures.

Let $\mathbb{S} \subset \{1, 2, \dots, N\}$ be the set of indices corresponding to the erased subspaces. Then, the composite measurement vector with erasures $\tilde{\mathbf{z}} \in \mathbb{R}^L$ may be expressed as

$$\tilde{\mathbf{z}} = (\mathbf{I} - \mathbf{E})\mathbf{z},$$

where \mathbf{E} is an $L \times L$ block-diagonal erasure matrix whose i th diagonal block is an $m_i \times m_i$ zero matrix, if $i \notin \mathbb{S}$, or an $m_i \times m_i$ identity matrix, if $i \in \mathbb{S}$.

The estimate of \mathbf{x} is given by $\tilde{\mathbf{x}} = \mathbf{F}\tilde{\mathbf{z}}$, where $\mathbf{F} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}$ is the (no-erasure) LMMSE filter with error covariance matrix

$$\begin{aligned} \tilde{\mathbf{R}}_{ee} &= E[(\mathbf{x} - \tilde{\mathbf{x}})(\mathbf{x} - \tilde{\mathbf{x}})^T] \\ &= E[(\mathbf{x} - \mathbf{F}(\mathbf{I} - \mathbf{E})\mathbf{z})(\mathbf{x} - \mathbf{F}(\mathbf{I} - \mathbf{E})\mathbf{z})^T]. \end{aligned}$$

The MSE for this estimate can be written as

$$MSE = \text{tr}[\tilde{\mathbf{R}}_{ee}] = MSE_0 + \overline{MSE},$$

where $MSE_0 = \text{tr}[\mathbf{R}_{ee}]$ is the no-erasure MSE in (5) and \overline{MSE} is the extra MSE due to erasures given by

$$\overline{MSE} = \alpha^2 \text{tr} \left[\sigma_x^2 \left(\sum_{i \in \mathbb{S}} \mathbf{P}_i \right)^2 + \sigma_n^2 \left(\sum_{i \in \mathbb{S}} \mathbf{P}_i \right) \right],$$

where $\alpha = \sigma_x^2 / (A\sigma_x^2 + \sigma_n^2)$.

We now show how the subspaces in the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ must be constructed so that the total MSE is minimized for a given number of erasures. We consider three scenarios: one subspace erasure, two subspace erasures, and more than two subspace erasures.

A. One Subspace Erasure

If only one of the subspaces, say the i th subspace, is erased, then MSE is given by

$$\begin{aligned} MSE &= MSE_0 + \text{tr}[\alpha^2(\sigma_x^2 + \sigma_n^2)\mathbf{P}_i] \\ &= \frac{M\sigma_x^2\sigma_n^2}{\sigma_n^2 + \frac{\sigma_x^2}{M}L} + \frac{\sigma_x^4(\sigma_x^2 + \sigma_n^2)}{(\sigma_n^2 + \frac{\sigma_x^2}{M}L)^2} m_i, \end{aligned} \quad (6)$$

where $m_i = \text{tr}[\mathbf{P}_i]$ is the dimension of \mathcal{W}_i and $L = \sum_{i=1}^N m_i$.

Since we have assumed that all one-erasures are equally important we have to choose $m_i = m$ for all $i = 1, \dots, N$, so that any one-erasure results in the same amount of performance degradation. This strategy is equivalent to minimizing the maximal MSE due to one subspace erasure and reduces the MSE expression in (6) to

$$MSE = \frac{M\sigma_x^2\sigma_n^2}{(Nm\sigma_x^2/M + \sigma_n^2)} + \frac{\sigma_x^4(\sigma_x^2 + \sigma_n^2)m}{(Nm\sigma_x^2/M + \sigma_n^2)^2}.$$

As a function of m , $MSE = MSE(m)$ has a maximum at $m = \tilde{m}$, where

$$\tilde{m} = \frac{M}{N} \frac{(N-1)\sigma_n^4 - \sigma_x^2\sigma_n^2}{((N+1)\sigma_n^2 + \sigma_x^2)(1 - 2\sigma_x^2)}.$$

The MSE is monotonically increasing for $m < \tilde{m}$ and monotonically decreasing for $m > \tilde{m}$. The smallest value m can take under the constraint that the set of m -dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^N$ remains a tight fusion frame is $m_{min} = \lceil M/N \rceil$, where $\lceil \cdot \rceil$ denotes integer ceiling. We assume that the largest value m can take is $m_{max} \leq M$. The maximum allowable dimension m_{max} is determined by practical considerations. Therefore, we have the following theorem.

Theorem 1: Let $\{\mathcal{W}_i\}_{i=1}^N$ be a tight fusion frame consisting of subspaces with dimensions $m_{min} \leq m_i \leq m_{max}$, $i = 1, \dots, N$. Then the maximal MSE due to the erasure of one subspace is minimized when all subspaces in $\{\mathcal{W}_i\}_{i=1}^N$ have equal dimension $m = m^*$, where

$$m^* = \begin{cases} m_{min}, & \text{if } m_{max} \leq \tilde{m} \text{ or} \\ & \text{if } m_{min} \leq \tilde{m} \leq m_{max} \text{ and} \\ & MSE(m_{min}) \leq MSE(m_{max}), \\ m_{max}, & \text{otherwise.} \end{cases}$$

B. Two Subspace Erasures

We now consider the case where two subspaces, say the i th subspace and the j th subspace, are erased or discarded. We take the fusion frame to be tight and assume all subspaces have equal dimension m^* to maintain MSE optimality with respect to no erasures and one-erasures. This reduces the minimization of the total MSE to minimizing the extra MSE, which is given by

$$\begin{aligned} \overline{MSE} &= \alpha^2 \text{tr}[\sigma_x^2(\mathbf{P}_i + \mathbf{P}_j)^2 + \sigma_n^2(\mathbf{P}_i + \mathbf{P}_j)] \\ &= 2\alpha^2(\sigma_x^2 + \sigma_n^2)m^* + 2\alpha^2\sigma_x^2 \text{tr}[\mathbf{P}_i\mathbf{P}_j]. \end{aligned}$$

To minimize \overline{MSE} we have to choose \mathcal{W}_i and \mathcal{W}_j , so that $\text{tr}[\mathbf{P}_i\mathbf{P}_j]$ is minimized. Since \mathbf{P}_i and \mathbf{P}_j are

orthogonal projection matrices onto \mathcal{W}_i and \mathcal{W}_j , the eigenvalues of $\mathbf{P}_i\mathbf{P}_j$ are squares of the cosines of the *principal angles* $\theta_\ell(i, j)$, $\ell = 1, \dots, M$ between \mathcal{W}_i and \mathcal{W}_j . Therefore,

$$\text{tr}[\mathbf{P}_i\mathbf{P}_j] = \sum_{\ell=1}^M \cos^2 \theta_\ell(i, j) = M - d_c^2(i, j),$$

where

$$d_c(i, j) = \left(\sum_{\ell=1}^M \sin^2 \theta_\ell(i, j) \right)^{1/2}$$

is known as the *chordal distance* [12] between \mathcal{W}_i and \mathcal{W}_j .

Thus, we need to maximize the chordal distance $d_c(i, j)$. Since we have assumed that all two subspace erasures are equally important, we have to construct the subspaces $\{\mathcal{W}_i\}_{i=1}^N$ so that any such pair has maximum chordal distance. This strategy is equivalent to minimizing the maximal MSE due to two subspace erasures.

We will show in Section IV that the subspaces in a fusion frame consisting of equi-dimensional and equi-distance subspaces have maximal chordal distance if and only if the fusion frame is tight. We refer to such a fusion frame as an *equi-distance tight fusion frame* and to the corresponding subspaces as *maximal equi-distance subspaces*. We defer the construction of maximal equi-distance subspaces to Section IV, where we explain the connection between this construction and the problem of optimal packing of N equi-dimensional subspaces in a Grassmannian space [12], [14]. We have the following result concerning maximal equi-distance subspaces.

Theorem 2: Let $\{\mathcal{W}_i\}_{i=1}^N$ be a tight fusion frame with equal dimensional subspaces, where $\dim[\mathcal{W}_i] = m^*$, $i = 1, \dots, N$. Then the maximal MSE due to two subspace erasures is minimized when the subspaces \mathcal{W}_i are maximal equi-distance subspaces.

C. More Than Two Subspace Erasures

We now consider the case where more than two subspaces are erased or discarded. Here we take the fusion frame to be an equi-distance tight fusion frame to maintain MSE optimality with respect k erasures, for $0 \leq k \leq 2$.

Let $\{\mathcal{W}_i\}_{i=1}^N$ be an equi-distance tight fusion frame, where $\dim[\mathcal{W}_i] = m^*$, $i = 1, \dots, N$ and $d_c(i, j) = d_c$ for any (i, j) pair with $i \neq j$. Then, \overline{MSE} can be written

as

$$\begin{aligned} \overline{MSE} &= \alpha^2(\sigma_x^2 + \sigma_n^2) \sum_{i \in \mathbb{S}} \text{tr}[\mathbf{P}_i] \\ &\quad + \alpha^2 \sigma_x^2 \sum_{i \in \mathbb{S}} \sum_{j \in \mathbb{S}, j \neq i} \text{tr}[\mathbf{P}_i\mathbf{P}_j] \\ &= \alpha^2(\sigma_x^2 + \sigma_n^2) |\mathbb{S}| m^* \\ &\quad + \alpha^2 \sigma_x^2 |\mathbb{S}| (|\mathbb{S}| - 1) (M - d_c^2), \end{aligned}$$

where $|\mathbb{S}|$ is the cardinality of \mathbb{S} . Noting that m^* , d_c^2 and $|\mathbb{S}|$ are fixed, we have the following theorem.

Theorem 3: Let $\{\mathcal{W}_i\}_{i=1}^N$ be an equi-distance tight fusion frame with $\dim[\mathcal{W}_i] = m^*$, $i = 1, \dots, N$. Then the MSE due to k subspace erasures, $3 \leq k < N$ is constant.

IV. CONNECTIONS WITH OPTIMAL PACKINGS

In this section, we show that tight fusion frames that consist of equi-dimensional and equi-distance subspaces are closely related to optimal packings of subspaces. We start by reviewing the classical packing problem for subspaces [12], [14], [15].

Classical Packing Problem: For given m, M, N , find a set of m -dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in \mathbb{R}^M such that $\min_{i \neq j} d_c(i, j)$ is as large as possible. In this case we call $\{\mathcal{W}_i\}_{i=1}^N$ an *optimal packing*.

The following theorem shows that if the squares of the pairwise chordal distances between a set of m -dimensional subspaces of \mathbb{R}^M meet the *simplex bound* $m(M - m)N/[M(N - 1)]$, then those subspaces form an optimal packing, as the minimum of chordal distances cannot grow any further.

Theorem 4: [12] Each packing of m -dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in \mathbb{R}^M satisfies

$$d_c^2(i, j) \leq \frac{m(M - m)}{M} \frac{N}{N - 1}, \quad i, j = 1, \dots, N.$$

We now establish a connection between tight fusion frames and optimal packings by stating the following theorem.

Theorem 5: Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame of equi-dimensional subspaces with equal pairwise chordal distances d_c . Then, the fusion frame is tight if and only if d_c^2 equals the simplex bound.

An intriguing consequence of Theorem 5 is as follows.

Corollary 1: Equi-distance tight fusion frames are optimal Grassmannian packings.

V. CONCLUSIONS

We considered the LMMSE estimation of a zero mean random vector, with covariance $\sigma_x^2 \mathbf{I}$, from its low-dimensional fusion frame measurements in the presence of noise and subspace erasures. We showed that, in the absence of erasures, the MSE will achieve its lower bound if the fusion frame is tight. We restricted our erasure analysis to the class of tight fusion frames and considered minimizing the maximal MSE due to k subspace erasures. We showed that maximum robustness against one subspace erasures is achieved when all subspaces of the tight fusion frame have equal dimensions, where the optimal dimension depends on the SNR. We also showed that equi-distance tight fusion frames are maximally robust against two and more than two subspace erasures. In addition, we established that equi-distance tight fusion frames are in fact optimal Grassmannian packings, and thereby highlighted the importance of optimal Grassmannian packings for robust dimension reduction.

ACKNOWLEDGMENT

G. Kutyniok would like to thank Peter Casazza, Shidong Li, and Christopher Rozell for discussions about fusion frames. She is indebted to Minh Do and Richard Baraniuk for a discussion about applications of fusion frames during the 2007 Von Neuman Symposium in Snow Bird, Utah. This work was conducted when G. Kutyniok was a visitor at PACM, Princeton University, and she would like to thank PACM for their hospitality.

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