FRAMES OF SUBSPACES AND APPROXIMATION OF THE INVERSE FRAME OPERATOR

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Abstract. A frame of subspaces in a Hilbert space H allows that identity operator on H to be written as a sum of some bounded operators on H. This family of bounded operators on H is called an atomic resolution of the identity on H. We show the atomic resolution of the identity associated to a frame of subspaces have a certain minimum property relative to $\ell^2(H, I)$-norm. We further show that under extra condition every atomic resolution of the identity provides a frame of subspaces for H. We consider direct sum of frames of subspaces with respect to the same family of weights which is a frame of subspaces for their direct sum space. Frame theory of subspaces describes how one can choose the corresponding atomic resolution of the identity, which is interesting from mathematical point of view, but for applications it is a problem that requires to know the inverse frame operator $S_{W,v}^{-1}$ on H. If the underlying Hilbert space is infinite dimensional it is hard to invert the frame operator $S_{W,v}$. We show how the inverse of $S_{W,v}$ can be approximated by using the methods of linear algebra.

1. Introduction

Frames have played a very important role in wavelet analysis, signal processing, image processing and data compression in the last 20 years [6], but the concept already has been introduced by Duffin and Schaeffer [7] in 1952. The general frame theory of subspaces is a natural generalization of the frame theory in Hilbert spaces, which is introduced by P. G. Casazza and G. Kutyniok in [3] and M. Fornasier in [8]. We extend some of the known results of frames to frames of subspaces.

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In the first part of this paper we present some useful new results about frames of subspaces and the atomic resolution of the identity, most important are some certain minimum property of the atomic resolution of the identity associated to a frame of subspaces, and a result about the direct sum of the frames of subspaces.

The second part is devoted to the approximation method for inverse frame operator of subspaces which is a new tool for approximation of the atomic resolution of the identity associated to a frame of subspaces. Some of these results are generalizations of Casazza and Christensen's works [2, 4].

Throughout the paper H denotes a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and I, J and every J_i denotes countable (or finite) index sets, and I also denotes the identity operator.

A sequence $\{f_i\}_{i \in I}$ in a Hilbert space H is a frame if there exist real numbers $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \tag{1}$$

The numbers A, B are called the frame bounds.

2. Frames of subspaces and atomic resolution of the identity

First we will briefly recall the definitions and basic properties of frames and bases of subspaces. For more details we refer to Casazza and Kutyniok [3], Asgari and Khosravi [1]. Throught the paper if W is a closed subspace of a Hilbert space H, then π_W denotes the orthogonal projection of H onto W.

Let H be a Hilbert space and $\{v_i\}_{i \in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. A sequence of closed subspaces $\{W_i\}_{i \in I}$ of H is a frame of subspaces with respect to $\{v_i\}_{i \in I}$ if there exist positive real numbers C, D such that for all $f \in H$

$$C\|f\|^2 \leq \sum_{i \in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2. \tag{2}$$

The numbers C, D are called the frame bounds for the frame of subspaces. The frame $\{W_i\}_{i\in I}$ is called a tight frame of subspaces if $C = D$ and is called a Parseval frame of subspaces if $C = D = 1$. Moreover, we call a frame of subspaces with respect to $\{v_i\}_{i\in I}$ v-uniform, if $v = v_i = v_j$ for each $i, j \in I$. A sequence $\{W_i\}_{i\in I}$ of closed subspaces of H is called an orthonormal basis of subspaces if $H = \bigoplus_{i\in I} W_i$. If we only know that $\{W_i\}_{i\in I}$ satisfies the upper inequality in (2), then $\{W_i\}_{i\in I}$ is called a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in I}$ with Bessel bound D. If the sequence $\{W_i\}_{i\in I}$ satisfies the lower inequality in (2), we say that $\{W_i\}_{i\in I}$ satisfies the lower frame condition of subspaces. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ with frame bounds C and D. Then the frame operator $S_{W,v}(f) = \sum_{i\in I} v_i^2 \pi_{W_i}(f)$ for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is a positive self-adjoint, and invertible operator on H with $CI \le S_{W,v} \le DI$, where I is the identity operator on H. Further we have the reconstruction formula.

$$f = S_{W,v}^{-1} S_{W,v}(f) = \sum_{i\in I} v_i^2 S_{W,v}^{-1} \pi_{W_i}(f) = \sum_{i\in I} v_i^2 \pi_{W_i} S_{W,v}^{-1}(f). \tag{3}$$

In particular, for all $f \in H$ we have

$$\langle S_{W,v}(f), f\rangle = \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \tag{4}$$

and

$$\langle f, S_{W,v}^{-1}(f)\rangle = \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 \tag{5}$$

Notation. Let H be a Hilbert space, and let I be an index set. Then the space $\ell^2(H, I)$ defined by

$$\ell^2(H, I) = \{\{a_k\}_{k\in I} | a_k \in H \text{ and } \sum_{i\in I} \|a_i\|^2 < \infty\}. \tag{6}$$

with inner product given by

$$\langle \{a_k\}_{k\in I}, \{b_k\}_{k\in I}\rangle = \sum_{i\in I} \langle a_i, b_i\rangle \tag{7}$$

and with respect to the pointwise operations is a Hilbert space.

In the sequel we consider definition (3.1) in [1] with some modification. This definition is a generalization of bounded quasi-projectors in [8] and ℓ^2-resolution of the identity in [3].

Definition 2.1. Let I be a countable index set and let H be a separable Hilbert space. Suppose $\{v_i\}_{i\in I}$ is a family of weights. Then a family of bounded operators $\{T_i\}_{i\in I}$ on H is called an *atomic (unconditional) resolution of the identity* with respect to $\{v_i\}_{i\in I}$ for H if there exist positive real numbers C and D such that for all $f \in H$

A1) $f = \sum_{i\in I} v_i^2 T_i(f)$ (and the series converges unconditionally).

A2) $C\|f\|^2 \leq \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \leq D\|f\|^2$.

The family $\{T_i\}_{i\in I}$ is said to be *self-adjoint* if

A3) $T_i = T_i^*$ for all $i \in I$.

The numbers C and D are called the atomic resolution of the identity bounds. If we only know that $\{T_i\}_{i\in I}$ satisfies in (A1) and the upper inequality in (A2), then $\{T_i\}_{i\in I}$ is called a (unconditional) ℓ^2-*resolution of the identity* with respect to $\{v_i\}_{i\in I}$. If $\{T_i\}_{i\in I}$ satisfy only in (A1), then $\{T_i\}_{i\in I}$ is called a (unconditional) *resolution of the identity* with respect to $\{v_i\}_{i\in I}$, and if $v_i = 1$ for all $i \in I$ in (A1), then $\{T_i\}_{i\in I}$ is called a (unconditional) *resolution of the identity* on H.

The next proposition shows that every frame of subspaces for H, provides many atomic resolution of the identity on H. (See [1, Proposition 3.6]).

Proposition 2.2. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H with frame bounds C and D. Then,*

(i) For each $i \in I$ let $T_i : H \longrightarrow W_i$ be given by $T_i = \pi_{W_i} S_{W,v}^{-1}$, then $\{T_i\}_{i\in I}$ is an atomic unconditional resolution of the identity with respect to $\{v_i\}_{i\in I}$ on H with bounds $1/D,\ 1/C$.

(ii) Suppose that $T_i : H \longrightarrow H$ is given by $T_i = S_{W,v}^{-1}\pi_{W_i}$, $(i \in I)$, then $\{T_i\}_{i\in I}$ is an atomic unconditional resolution of the identity with respect to $\{v_i\}_{i\in I}$ on H with bounds $C/D^2,\ D/C^2$.

For the converse we have the following result.

Theorem 2.3. *Let* $\{T_i\}_{i\in I}$ *be a* ℓ^2*-resolution of the identity with respect to* $\{v_i\}_{i\in I}$ *for* H*, and let* $W_i = \overline{T_i(H)}$*. If there exists some* $R > 0$ *such that*

$$\sum_{i\in I} v_i^2 \|\pi_{W_i}(f) - T_i(f)\|^2 \le R\|f\|^2$$

for all $f \in H$*, then* $\{W_i\}_{i\in I}$ *is a frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* H*.*

Proof. Let D be the ℓ^2-resolution of the identity bound for $\{T_i\}_{i\in I}$. Then we have

$$\begin{aligned}
\|f\|^4 &= |\langle \sum_{i\in I} v_i^2 T_i(f), f\rangle|^2 \le (\sum_{i\in I} v_i^2 |\langle T_i(f), f\rangle|)^2 \\
&= (\sum_{i\in I} v_i^2 |\langle T_i(f), \pi_{W_i}(f)\rangle|)^2 \le (\sum_{i\in I} v_i^2 \|T_i(f)\|\|\pi_{W_i}(f)\|)^2 \\
&\le (\sum_{i\in I} v_i^2 \|T_i(f)\|^2)(\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2) \\
&\le D\|f\|^2(\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2).
\end{aligned}$$

for all $f \in H$. This shows that $\frac{1}{D}\|f\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2$. By the triangle inequality in $\ell^2(H, I)$ we have

$$\begin{aligned}
\|\{v_i \pi_{W_i}(f)\}_{i\in I}\|_{\ell^2} &\le \|\{v_i(\pi_{W_i}(f) - T_i(f))\}_{i\in I}\|_{\ell^2} + \|\{v_i T_i(f)\}_{i\in I}\|_{\ell^2} \\
&\le (\sqrt{R} + \sqrt{D})\|f\|.
\end{aligned}$$

Therefore $\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \le D(1 + \sqrt{\frac{R}{D}})^2 \|f\|^2$. □

In the next theorem we show that for each $f \in H$ the family $\{v_i \pi_{W_i} S_{W,v}^{-1}(f)\}_{i\in I}$ in $\ell^2(H, I)$ have a certain minimum property.

Theorem 2.4. *Let* $\{W_i\}_{i\in I}$ *be a frame of subspaces with respect to* $\{v_i\}_{i\in I}$ *for* H*, and let* $T_i : H \longrightarrow W_i$ *be such that* $\{T_i\}_{i\in I}$ *is a* ℓ^2*-resolution of the identity with respect to* $\{v_i\}_{i\in I}$ *on*

H. Then for each $f \in H$

$$\sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 \leq \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \tag{8}$$

and

$$\sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f) - \pi_{W_i}(f)\|^2 \leq \sum_{i\in I} v_i^2 \|T_i(f) - \pi_{W_i}(f)\|^2. \tag{9}$$

Proof. By (5) we have

$$\sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 = \langle f, S_{W,v}^{-1}(f)\rangle = \langle \sum_{i\in I} v_i^2 T_i(f), S_{W,v}^{-1}(f)\rangle$$

$$= \sum_{i\in I} v_i^2 \langle T_i(f), S_{W,v}^{-1}(f)\rangle = \sum_{i\in I} v_i^2 \langle T_i(f), \pi_{W_i} S_{W,v}^{-1}(f)\rangle$$

for all $f \in H$. We further compute

$$\begin{aligned} \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f) - T_i(f)\|^2 &= \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 + \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \\ &- \sum_{i\in I} v_i^2 \langle \pi_{W_i} S_{W,v}^{-1}(f), T_i(f)\rangle - \sum_{i\in I} v_i^2 \langle T_i(f), \pi_{W_i} S_{W,v}^{-1}(f)\rangle \\ &= \sum_{i\in I} v_i^2 \|T_i(f)\|^2 - \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2. \end{aligned} \tag{10}$$

which implies (8). To prove the second claim for all $f \in H$, we have

$$\sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f) - \pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 + \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 - 2\|f\|^2$$

and

$$\sum_{i\in I} v_i^2 \|T_i(f) - \pi_{W_i}(f)\|^2 = \sum_{i\in I} v_i^2 \|T_i(f)\|^2 + \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 - 2\|f\|^2$$

Now by using (10) we obtain

$$\sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f) - T_i(f)\|^2 + \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f) - \pi_{W_i}(f)\|^2$$

$$= \sum_{i\in I} v_i^2 \|T_i(f) - \pi_{W_i}(f)\|^2.$$

which implies (9). ☐

Corollary 2.5. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H, and let $k \in I$ be such that $W_k = H$. Then $\|S_{W,v}^{-1}\| \leq v_k^{-2}$.*

Proof. Since $W_k = H$ then $\pi_{W_k} = I$, where I is the identity operator on H. For each $i \in I$ let $T_i : H \longrightarrow W_i$ be given by $T_i = v_i^{-2}\delta_{ik}\pi_{W_i}$ where δ_{ik} is the Kronecker delta. Then $\{T_i\}_{i\in I}$ is a ℓ^2-resolution of the identity with respect to $\{v_i\}_{i\in I}$. Now by (8) and (5) we have

$$\begin{aligned} < f, S_{W,v}^{-1}(f) > &= \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2 \leq \sum_{i\in I} v_i^2 \|T_i(f)\|^2 \\ &= \sum_{i\in I} v_i^2 \|v_i^{-2}\delta_{ik}\pi_{W_i}(f)\|^2 = v_k^{-2}\|\pi_{W_k}(f)\|^2 \leq v_k^{-2}\|f\|^2 \end{aligned}$$

for all $f \in H$. Since $S_{W,v}^{-1}$ is a self-adjoint operator on H, then

$$\|S_{W,v}^{-1}\| = \sup_{\|f\|=1} |\langle f, S_{W,v}^{-1}(f)\rangle| \leq v_k^{-2}.$$

□

Definition 2.6. A family of closed subspaces $\{W_i\}_{i\in I}$ of H is called a frame *sequence of subspaces* with respect to $\{v_i\}_{i\in I}$ if it is a frame of subspaces only for its closed linear span.

Proposition 2.7. *Let $\{W_i\}_{i\in I}$ and $\{Z_j\}_{j\in J}$ be frame sequences of subspaces with respect to $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ respectively and let P_1, P_2 denote the orthogonal projections of H onto $\overline{\text{span}}\{W_i\}_{i\in I}$, $\overline{\text{span}}\{Z_j\}_{j\in J}$, respectively. Then*

(i) *the family $\{W_i, Z_j\}_{i\in I, j\in J}$ is a Bessel sequence of subspaces with respect to $\{u_i, v_j\}_{i\in I, j\in J}$.*

(ii) *$\{W_i, Z_j\}$ satisfies the lower frame condition of subspaces i.e. it satisfies the lower inequality in (2) if and only if there exist constant $K > 0$ such that for all $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$*

$$\|P_1(f)\|^2 + \|P_2(f)\|^2 \geq K\|f\|^2$$

Proof. Let C_1 and D_1 be the frame bounds for $\{W_i\}_{i\in I}$ and let C_2, D_2 be the frame bounds for $\{Z_j\}_{j\in J}$.

(i) For each $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$ we have

$$\begin{aligned}
&\sum_{i\in I} u_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{j\in J} v_j^2 \|\pi_{Z_j}(f)\|^2 \\
&\qquad = \sum_{i\in I} u_i^2 \|\pi_{W_i}(P_1(f))\|^2 + \sum_{j\in J} v_j^2 \|\pi_{Z_j}(P_2(f))\|^2 \\
&\qquad \le D_1\|P_1(f)\|^2 + D_2\|P_2(f)\|^2 \\
&\qquad \le \max\{D_1, D_2\}(\|P_1(f)\|^2 + \|P_2(f)\|^2) \\
&\qquad \le 2\ max\{D_1, D_2\}\|f\|^2.
\end{aligned}$$

(ii) Let $\|p_1(f)\|^2 + \|p_2(f)\|^2 \ge k\|f\|^2$. For any $f \in \overline{\text{span}}\{W_i, Z_j\}_{i\in I, j\in J}$ we have

$$\begin{aligned}
\sum_{i\in I} u_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{j\in J} v_j^2 \|\pi_{Z_j}(f)\|^2 &= \sum_{i\in I} u_i^2 \|\pi_{W_i}(P_1(f))\|^2 + \sum_{j\in J} v_j^2 \|\pi_{Z_j}(P_2(f))\|^2 \\
&\ge C_1\|P_1(f)\|^2 + C_2\|P_2(f)\|^2 \\
&\ge \min\{C_1, C_2\}(\|P_1(f)\|^2 + \|P_2(f)\|^2) \\
&\ge K \min\{C_1, C_2\}\|f\|^2
\end{aligned}$$

Conversely, let there exists some $C > 0$ such that for any $f \in \overline{span}\{W_i, Z_j\}_{i\in I, j\in J}$,

$$\begin{aligned}
C\|f\|^2 &\le \sum_i u_i^2 \|\pi_{W_i}(f)\|^2 + \sum_j v_j^2 \|\pi_{Z_j}(f)\|^2 \\
&\le D_1\|p_1(f)\|^2 + D_2\|p_2(f)\|^2 \\
&\le max\{D_1, D_2\}(\|p_1(f)\|^2 + \|p_2(f)\|^2).
\end{aligned}$$

□

In the next theorem we consider direct sum of frames of subspaces for their direct sum space.

Theorem 2.8. *Let $\{\{W_{ij}\}_{i\in I} : j = 1, 2, \ldots, k\}$ be a k-tuple of frames of subspaces with respect to $\{v_i\}_{i\in I}$ on Hilbert spaces H_j $(1 \leq j \leq k)$, respectively. Then $\{W_{i1} \oplus W_{i2} \oplus \ldots \oplus W_{ik}\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for the Hilbert space $H_1 \oplus H_2 \oplus \ldots \oplus H_k$.*

Proof. It is enough to prove the theorem for $k = 2$. Let C_j and D_j be the frame bounds for $\{W_{ij}\}_{i\in I}$ $(j = 1, 2,)$. Since $\pi_{W_{i1}\oplus W_{i2}} = \pi_{W_{i1}} \oplus \pi_{W_{i2}}$ $(i \in I)$, then for all $f \in H_1$ and $g \in H_2$ we have

$$\begin{aligned}
\min\{C_1, C_2\}\|f \oplus g\|^2 &= \min\{C_1, C_2\}(\|f\|^2 + \|g\|^2) \\
&\leq C_1\|f\|^2 + C_2\|g\|^2 \\
&\leq \sum_{i\in I} v_i^2\|\pi_{W_{i1}}(f)\|^2 + \sum_{i\in I} v_i^2\|\pi_{W_{i2}}(g)\|^2 \\
&\leq D_1\|f\|^2 + D_2\|g\|^2 \\
&\leq \max\{D_1, D_2\}(\|f\|^2 + \|g\|^2) \\
&= \max\{D_1, D_2\}\|f \oplus g\|^2.
\end{aligned}$$

Now we observe that

$$\sum_{i\in I} v_i^2\|\pi_{W_{i1}}(f)\|^2 + \sum_{i\in I} v_i^2\|\pi_{W_{i2}}(g)\|^2 = \sum_{i\in I} v_i^2\|\pi_{W_{i1}\oplus W_{i2}}(f \oplus g)\|^2.$$

This shows that $\{W_{i1} \oplus W_{i2}\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $H_1 \oplus H_2$ with frame bounds $\min\{C_1, C_2\}$ and $\max\{D_1, D_2\}$. □

Corollary 2.9. *Let $\{\{W_{ij}\}_{i\in I} : j = 1, 2, \ldots, k\}$ be a k-tuple of Parseval frames of subspaces with respect to $\{v_i\}_{i\in I}$ on Hilbert spaces $H_j(1 \leq j \leq k)$, respectively. Then $\{W_{i1} \oplus W_{i2} \oplus \ldots \oplus W_{ik}\}_{i\in I}$ is a Parseval frame of subspaces with respect to $\{v_i\}_{i\in I}$ for the Hilbert space $H_1 \oplus H_2 \oplus \ldots \oplus H_j$.*

3. Approximation of $S_{W,v}^{-1}$

In this section we discuss a method for approximation of the inverse frame operator $S_{W,v}^{-1}$ associated to a frame $\{W_i\}_{i\in I}$ with respect to $\{v_i\}_{i\in I}$ such that each W_i $(i \in I)$ is a finite-dimensional subspace of H. The idea is to approximate $S_{W,v}^{-1}$ using finite subsets $\{W_i\}_{i\in I_n}$ of the frame $\{W_i\}_{i\in I}$.

Definition 3.1. A frame of subspaces $\{W_i\}_{i\in I}$ is called a Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$ if there exist constants $C, D > 0$ such that for any $J \subseteq I$ the family $\{W_i\}_{i\in J}$ is a frame of subspaces with respect to $\{v_i\}_{i\in J}$ for $\overline{\text{span}}\{W_i\}_{i\in J}$ with frame bounds C and D.

Example 5.7 in [3] shows that subfamilies of a frame of subspaces are not automatically frames of subspaces for their closed linear spans.

Remark 3.2. (a) Let I be a countable index set and let $\{I_n\}_{n=1}^{\infty}$ be a family of finite subsets of I such that.

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_n \nearrow I.$$

Suppose $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Suppose that P_n $(n \in \mathbb{N})$ denote the orthogonal projection of H onto $H_n = \overline{\text{span}}\{W_i\}_{i\in I_n}$, then

$$\lim_{n\to\infty} P_n(f) = f \quad \text{for all} \quad f \in H.$$

(b) Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ and let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Then there exists a family of finite-dimensional subspaces $\{W_{ij}\}_{i\in I, j\in J_i}$ for H, such that for all $f \in H$

$$\sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i\in I}\sum_{j\in J_i} v_i^2 \|\pi_{W_{ij}}(f)\|^2.$$

Let I be a countable index set and $\{I_n\}_{n=1}^{\infty}$ be a family of finite subsets of I such that

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \nearrow I.$$

Suppose $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H, such that W_i is a finite-dimensional subspace of H for all $i \in I$. Then for $n \in \mathbb{N}$ we consider the family $\{W_i\}_{i\in I_n}$ which is a frame of subspaces with respect to $\{v_i\}_{i\in I_n}$ for $H_n = \operatorname{span}\{W_i\}_{i\in I_n}$ with frame operator

$$S_{W_n,v_n} : H_n \longrightarrow H_n, \quad S_{W_n,v_n}(f) = \sum_{i\in I_n} v_i^2 \pi_{W_i}(f) \tag{11}$$

Since H_n is a finite-dimensional subspace of H, then S_{W_n,v_n} can be inverted using linear algebra. Let P_n denote the orthogonal projection of H onto H_n, then by (3) for all $f \in H$ we have

$$P_n(f) = \sum_{i\in I_n} v_i^2 \pi_{W_i} S^{-1}_{W_n,v_n} P_n(f) \tag{12}$$

Since for each $f \in H$ we obtain

$$\lim_{n\to\infty} \sum_{i\in I_n} v_i^2 \pi_{W_i} S^{-1}_{W_n,v_n} P_n(f) = \lim_{n\to\infty} P_n(f) = f = \sum_{i\in I} v_i^2 \pi_{W_i} S^{-1}_{W,v}(f)$$

then it is natural to ask whether

$$\lim_{n\to\infty} S^{-1}_{W_n,v_n} P_n(f) = S^{-1}_{W,v}(f). \tag{13}$$

Definition 3.3. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. If for all $f \in H$

$$\lim_{n\to\infty} S^{-1}_{W_n,v_n} P_n(f) = S^{-1}_{W,v}(f),$$

then we say that the approximation method of $S^{-1}_{W,v}$ works.

Definition 3.4. Let $\{T_n\}_{n=1}^{\infty}$ be a family of bounded operators on H, and let $T \in B(H)$. Then we say that $T_n \xrightarrow{s.t} T$ in the strong operator topology if $T_n(f) \longrightarrow T(f)$ for all $f \in H$ and we say that $T_n \xrightarrow{w.t} T$ in the weak operator topology if $\langle g, T_n(f)\rangle \longrightarrow \langle g, T(f)\rangle$ for all $f, g \in H$.

The next result is a generalization of Theorem 1 in [2] to frames of subspaces.

Theorem 3.5. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Then $S^{-1}_{W_n,v_n}P_n \overset{w.t}{\longrightarrow} S^{-1}_{W,v}$ if and only if*

$$\sup_n \|S^{-1}_{W_n,v_n}P_n\| = \sup_n \|S^{-1}_{W_n,v_n}\|_{H_n} < \infty.$$

Proof. Suppose that $S^{-1}_{W_n,v_n}P_n \overset{w.t}{\longrightarrow} S^{-1}_{W_n,v_n}$. Then for all $n \in \mathbb{N}$ and $f \in H$ we have

$$\langle f, S^{-1}_{W_n,v_n}P_n(f)\rangle = \langle P_n(f), S^{-1}_{W_n,v_n}P_n(f)\rangle = \sum_{i\in I_n} v_i^2 \|\pi_{W_i}S^{-1}_{W_n,v_n}P_n(f)\|^2$$

Since $S^{-1}_{W_n,v_n}P_n$ is a positive operator, then there are positive and self-adjoint operators $T_n : H \longrightarrow H$ such that $S^{-1}_{W_n,v_n}P_n = T_n^2$. Now for any $f \in H$ we obtain

$$\|T_n(f)\|^2 = \langle T_n(f), T_n(f)\rangle = \langle f, S^{-1}_{W_n,v_n}P_n(f)\rangle \longrightarrow \langle f, S^{-1}_{W,v}(f)\rangle \quad \text{for } n \longrightarrow \infty.$$

Therefore the family $\{T_n\}_{n=1}^{\infty}$ is pointwise bounded on H. By the uniform boundedness principle the family $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded, hence

$$\sup_n \|S^{-1}_{W_n,v_n}P_n\| \le \sup_n \|T_n\|^2 < \infty.$$

For the converse suppose that $\sup_n \|S^{-1}_{W_n,v_n}P_n\| = M < \infty$, and let C and D be the frame bounds for $\{W_i\}_{i\in I}$. If for any $n \in \mathbb{N}$ and $f \in H$ we define $\phi_n = S^{-1}_{W_n,v_n}P_n(f) - S^{-1}_{W,v}(f)$, then we have

$$S_{W,v}\phi_n = P_n(f) + \sum_{i\in I-I_n} v_i^2 \pi_{W_i} S^{-1}_{W_n,v_n}P_n(f) - f.$$

Hence $\phi_n = \sum_{i\in I-I_n} v_i^2 S^{-1}_{W,v}\pi_{W_i}S^{-1}_{W_n,v_n}P_n(f) + S^{-1}_{W,v}(P_nf - f)$. Therefore for $g \in H$ we obtain

$$|\langle \phi_n, g\rangle| \le |\langle \sum_{i\in I-I_n} v_i^2 S^{-1}_{W,v}\pi_{W_i}S^{-1}_{W_n,v_n}P_n(f), g\rangle| + |\langle S^{-1}_{W,v}(P_nf - f), g\rangle|$$

The second term $|\langle S^{-1}_{W,v}(P_nf - f), g\rangle| \longrightarrow 0$ for all $f, g \in H$. So we only need to show that the first term tends to zero, therefore we have

$$|\langle \sum_{i\in I-I_n} v_i^2 S^{-1}_{W,v}\pi_{W_i}S^{-1}_{W_n,v_n}P_n(f), g\rangle|^2 = |\sum_{i\in I-I_n} v_i^2 \langle S^{-1}_{W,v}\pi_{W_i}S^{-1}_{W_n,v_n}P_n(f), g\rangle|^2$$

$$\leq \left(\sum_{i\in I-I_n} v_i^2 | < \pi_{W_i} S_{W_n,v_n}^{-1} P_n(f), \pi_{W_i} S_{W,v}^{-1}(g) > | \right)^2$$

$$\leq \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W_n,v_n}^{-1} P_n(f)\| \|\pi_{W_i} S_{W,v}^{-1}(g)\| \right)^2$$

$$\leq \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W_n,v_n}^{-1} P_n(f)\|^2 \right) \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(g)\|^2 \right)$$

$$\leq D \|S_{W_n,v_n}^{-1} P_n(f)\|^2 \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{w_i} S_{W_n,v_n}^{-1}(g)\|^2 \right)$$

$$\leq DM^2 \|f\|^2 \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W_n,v_n}^{-1}(g)\|^2 \right) \longrightarrow 0 \quad \text{for } n \longrightarrow \infty$$

which implies that $S_{W_n,v_n}^{-1} P_n \xrightarrow{w.t} S_{W,v}^{-1}$. □

Theorem 3.6. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Then the following conditions are equivalent.*

(i) *The approximation method of $S_{W,v}^{-1}$ works.*

(ii) *The sequence $\{v_i \pi_{W_i} S_{W_n,v_n}^{-1} P_n(f)\}_{i\in I_n}$ converges to $\{v_i \pi_{W_i} S_{W,v}^{-1}(f)\}_{i\in I}$ in the $\ell^2(H, I)$-norm for all $f \in H$.*

(iii) $\sup_n \|S_{W_n,v_n}^{-1} P_n\| = \sup_n \|S_{W_n,v_n}^{-1}\|_{H_n} < \infty$.

(iv) $\lim_{n\to\infty} \sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W_n,v_n}^{-1} P_n(f)\|^2 = 0$ *for all $f \in H$.*

(v) $\lim_{n\to\infty} \|(S_{W_n,v_n} - S_{W,v}) S_{W_n,v_n}^{-1} P_n(f)\| = 0$ *for all $f \in H$.*

Proof. (i)$\Longrightarrow$(ii) Let C and D be the frame bounds for $\{W_i\}_{i\in I}$. Then for any $n \in \mathbb{N}$ and $f \in H$ we have

$$\|\{v_i \pi_{W_i} S_{W_n,v_n}^{-1} P_n(f)\}_{i\in I_n} - \{v_i \pi_{W_i} S_{W,v}^{-1}(f)\}_{i\in I}\|_{\ell^2}^2$$

$$= \sum_{i\in I_n} v_i^2 \|\pi_{W_i}(S_{W_n,v_n}^{-1} P_n(f) - S_{W,v}^{-1}(f))\|^2 + \sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2$$

$$\leq D \|S_{W_n,v_n}^{-1} P_n(f) - S_{W,v}^{-1}(f)\|^2 + \sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S_{W,v}^{-1}(f)\|^2$$

The second term $\sum_{i\in I-I_n} v_i^2\|\pi_{W_i}S_{W,v}^{-1}(f)\|^2 \longrightarrow 0$ for every $f\in H$. From this (ii) follows.

(ii)$\Longrightarrow$(iii) By Theorem (4.6) there are positive and self-adjoint operators $T_n : H \longrightarrow H$ such that $S_{W_n,v_n}^{-1}P_n = T_n^2$. Suppose that $\sup_n \|S_{W_n,v_n}^{-1}P_n\| = \infty$. Since $\|S_{W_n,v_n}^{-1}P_n\| \leq \|T_n\|^2$ it follows that $\sup_n \|T_n\| = \infty$. By the principle of uniform boundedness, there is some $f\in H$ such that $\|f_0\| = 1$ and $\sup_n \|T_n(f_0)\| = \infty$. It follows that

$$\sup_n \langle S_{W_n,v_n}^{-1}P_n(f_0), f_0\rangle = \sup_n \|T_n(f_0)\|^2 = \infty.$$

Thus

$$\begin{aligned}
&\|\{v_i\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f_0)\}_{i\in I_n} - \{v_i\pi_{W_i}S_{W,v}^{-1}(f_0)\}_{i\in I}\|_{\ell^2}\\
&\geq \|\{v_i\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f_0)\}_{i\in I_n}\|_{\ell^2} - \|\{v_i\pi_{W_i}S_{W,v}^{-1}(f_0)\}_{i\in I}\|_{\ell^2}\\
&\geq \sqrt{\langle S_{W_n,v_n}^{-1}P_n(f_0), f_0\rangle} - \frac{1}{\sqrt{C}}
\end{aligned}$$

and so $\{v_i\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f_0)\}_{i\in I_n}$ does not converge to $\{v_i\pi_{W_i}S_{W,v}^{-1}(f_0)\}_{i\in I}$ in the $\ell^2(H,I)$-norm.

(iii)$\Longrightarrow$(i) Let $\sup_n \|S_{W_n,v_n}^{-1}P_n\| = M$ then for all $n\in\mathbb{N}$ and $f\in H$ we have

$$\begin{aligned}
S_{W_n,v_n}P_n(f) - S_{W,v}(f) &= \sum_{i\in I_n} v_i^2\pi_{W_i}P_n(f) - \sum_{i\in I} v_i^2\pi_{W_i}(f)\\
&= \sum_{i\in I_n} v_i^2\pi_{W_i}P_n(f) - \sum_{i\in I_n} v_i^2\pi_{W_i}(f) - \sum_{i\in I-I_n} v_i^2\pi_{W_i}(f)\\
&= \sum_{i\in I_n} v_i^2\pi_{W_i}P_n(f) - \sum_{i\in I_n} v_i^2\pi_{W_i}P_n(f) - \sum_{i\in I-I_n} v_i^2\pi_{W_i}(f)\\
&= -\sum_{i\in I-I_n} v_i^2\pi_{W_i}(f) \longrightarrow 0 \quad \text{for} \quad n\longrightarrow\infty
\end{aligned}$$

Hence $S_{W_n,v_n}P_n(f) \longrightarrow S_{W,v}(f)$. Furthermore for all $n\in\mathbb{N}$ and $f\in H$ we obtain

$$\|S_{W_n,v_n}^{-1}P_n(f) - S_{W,v}^{-1}(f)\| = \|S_{W_n,v_n}^{-1}P_n(f) - P_nS_{W,v}^{-1}(f) + P_nS_{W,v}^{-1}(f) - S_{W,v}^{-1}(f)\|$$

$$\leq \|S_{W_n,v_n}^{-1}P_n(f) - S_{W_n,v_n}^{-1}P_nS_{W_n,v_n}P_nS_{W,v}^{-1}(f)\| + \|P_nS_{W,v}^{-1}(f) - S_{W,v}^{-1}(f)\|$$

$$\leq \|S_{W_n,v_n}^{-1}P_n\|\|f - S_{W_n,v_n}P_nS_{W,v}^{-1}(f)\| + \|P_nS_{W,v}^{-1}(f) - S_{W,v}^{-1}(f)\|$$

$$\leq M\|f - S_{W_n,v_n}P_nS_{W,v}^{-1}(f)\| + \|P_nS_{W,v}^{-1}(f) - S_{W,v}^{-1}(f)\|.$$

Since $S_{W_n,v_n}P_n(f) \longrightarrow S_{W,v}(f)$ and $P_n(f) \longrightarrow f$ for all $f \in H$, then the approximation method of $S_{W,v}^{-1}$ works.

(i) $\Longrightarrow$(iv) For all $n \in \mathbb{N}$ and $f \in H$ we have

$$\left(\sum_{i\in I-I_n} v_i^2\|\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f)\|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i\in I-I_n} v_i^2\|\pi_{W_i}(S_{W_n,v_n}^{-1}P_n(f) - S_{W,v}^{-1}(f))\|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in I-I_n} v_i^2\|\pi_{W_i}S_{W,v}^{-1}(f)\|^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{D}\|S_{W_n,v_n}^{-1}P_n(f) - S_{W,v}^{-1}(f)\| + \left(\sum_{i\in I-I_n} v_i^2\|\pi_{W_i}S_{W,v}^{-1}(f)\|^2\right)^{\frac{1}{2}}$$

The second term $\sum_{i\in I-I_n} v_i^2\|\pi_{W_i}S_{W,v}^{-1}(f)\|^2 \longrightarrow 0$ for every $f \in H$. From this (iv) follows.

(iv)$\Longrightarrow$(v) For any $n \in \mathbb{N}$ and $f \in H$ we have

$$(S_{W,v} - S_{W_n,v_n})S_{W_n,v_n}^{-1}P_n(f) = \sum_{i\in I} v_i^2\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f) - \sum_{i\in I_n} v_i^2\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f)$$

$$= \sum_{i\in I-I_n} v_i^2\pi_{W_i}S_{W_n,v_n}^{-1}P_n(f).$$

Hence

$$\|(S_{W,v} - S_{W_n,v_n})S_{W_n,v_n}^{-1}P_n(f)\|^2 = \sup_{\|g\|=1} |\langle(S_{W,v} - S_{W_n,v_n})S_{W_n,v_n}^{-1}P_n(f), g\rangle|^2$$

$$= \sup_{\|g\|=1} |\sum_{i\in I-I_n} v_i^2 \langle \pi_{W_i} S^{-1}_{W_n,v_n} P_n(f), g\rangle|^2$$

$$\leq \sup_{\|g\|=1} \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S^{-1}_{W_n,v_n} P_n(f)\| \cdot \|\pi_{W_i}(g)\| \right)^2$$

$$\leq \sup_{\|g\|=1} \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S^{-1}_{W_n,v_n} P_n(f)\|^2 \right) \left(\sum_{i\in I-I_n} v_i^2 \|\pi_{W_i}(g)\|^2 \right)$$

$$\leq D \sum_{i\in I-I_n} v_i^2 \|\pi_{W_i} S^{-1}_{W_n,v_n} P_n(f)\|^2$$

From this (v) follows.

(v)$\Longrightarrow$(i) For each $n \in \mathbb{N}$ and $f \in H$ we have

$$S^{-1}_{W,v}(f) - S^{-1}_{W_n,v_n} P_n(f) = S^{-1}_{W,v}(f - P_n(f)) + S^{-1}_{W,v}(S_{W_n,v_n} - S_{W,v}) S^{-1}_{W_n,n} P_n(f)$$

Thus

$$\|S^{-1}_{W,v}(f) - S^{-1}_{W_n,v_n} P_n(f)\| \leq$$

$$\|S^{-1}_{W,v}\| \cdot \|f - P_n(f)\| + \|S^{-1}_{W,v}\| \cdot \|(S_{W_n,v_n} - S_{W,v}) S^{-1}_{W_n,v_n} P_n(f)\|$$

Since $P_n(f) \longrightarrow f$ for any $f \in H$. Therefore if (v) holds then the approximation method of $S^{-1}_{W,v}$ works. □

Corollary 3.7. *Let $\{W_i\}_{i\in I}$ be a Riesz frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Then the approximation method of $S^{-1}_{W,v}$ works.*

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