

# FUSION FRAMES AND $G$ -FRAMES IN BANACH SPACES

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ABSTRACT. Fusion frames and  $g$ -frames in Hilbert spaces are generalizations of frames, and frames were extended to Banach spaces. In this article we introduce fusion frames,  $g$ -frames, Banach  $g$ -frames in Banach spaces and we show that they share many useful properties with their corresponding notions in Hilbert spaces. We also show that  $g$ -frames, fusion frames and Banach  $g$ -frames are stable under small perturbations and invertible operators.

## 1. INTRODUCTION AND PRELIMINARIES

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [11], later Daubechies, Grossmann and Meyer [10] found a fundamental new application. Nice properties of frames make them very useful in filter banks, sigma-delta quantization, signal and image processing. This notion was generalized to  $g$ -frames by Sun [16], see also [14, 15]. Fusion frames were introduced by Casazza et al in [6] and studied by many authors see [1, 4, 13]. Fusion frames have some applications in (wireless) sensor networks, see [5, 7]. Meanwhile frames were extended to Banach spaces by Grochenig [12] and studied in [2, 3].

In this article we introduce fusion frames,  $g$ -frames and Banach  $g$ -frames in Banach spaces. First we recall the following definitions from [12].

**Definition 1.1.** Let  $X_d$  be a Banach space of scalar-valued sequences. It is called a  $BK$ -space if the coordinate functionals are continuous on  $X_d$  and it is *solid* if whenever  $\{b_i\}$  and  $\{c_i\}$  are sequences with  $\{c_i\} \in X_d$  and  $|b_i| \leq |c_i|$ , then it follows that  $\{b_i\} \in X_d$  and  $\|\{b_i\}\| \leq \|\{c_i\}\|$ .

We note that  $l^2(I)$  is a solid  $BK$ -space and if  $X_d$  is a solid  $BK$ -space such that for each  $i \in I$ , there exists some  $x = \{x_j\}$  in  $X_d$  such that  $x_i \neq 0$ , then every  $e_i = \{\delta_{ij}\}_{j \in I}$  is in  $X_d$ .

**Definition 1.2.** Let  $X$  be a Banach space with dual space  $X^*$  and  $X_d$  be a  $BK$ -space. A countable family  $\{g_i\}$  in  $X^*$  is called an  $X_d$ -frame for  $X$  if there exist constants  $A, B > 0$  such that for every  $x \in X$ ,  $\{g_i(x)\} \in X_d$  and  $A\|x\| \leq \|\{g_i(x)\}\| \leq B\|x\|$ .

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$A$  and  $B$  are called  $X_d$ -frame bounds. If moreover there exists a bounded linear operator  $S : X_d \rightarrow X$  such that for every  $x \in X$ ,  $S(\{g_i(x)\}_{i \in I}) = x$ , then  $(\{g_i\}_i, S)$  is a *Banach frame* for  $X$  with respect to  $X_d$ .

If there exists a sequence  $\{x_i\}$  in  $X$  such that  $x = \sum g_i(x)x_i$ , for each  $x \in X$ , then  $(\{g_i\}, \{x_i\})$  is an atomic decomposition of  $X$  with respect to  $X_d$ .

**Definition 1.3.** Let  $X$  be a Banach space with dual space  $X^*$ . A sequence  $\{x_i\}$  in  $X$  is a *Schauder frame* for  $X$  if there exist a  $BK$ -space  $X_d$  and a sequence  $\{g_i\}$  in  $X^*$  such that  $(\{g_i\}, \{x_i\})$  is an atomic decomposition of  $X$  with respect to  $X_d$ . We also call the bounds of  $\{g_i\}$ , the bounds of  $\{x_i\}$ .

In Section 2 we study  $g$ -frames in Banach spaces and in Section 3 we study fusion frames in Banach spaces. Throughout this article  $X$  is a Banach space with dual space  $X^*$  and  $\mathbb{N}$ ,  $\mathbb{C}$  will denote the set of natural numbers and the field of complex numbers, respectively.  $I$ ,  $J$  and  $I_i$ 's will also denote a finite set or a subset of integers.

## 2. $G$ -FRAMES

In this section we introduce  $g$ -frames in Banach spaces and we generalize some of their known results in Hilbert spaces to Banach spaces. Throughout this section  $X$ ,  $Y$  are Banach spaces and  $\{Y_i : i \in I\}$  is a sequence of closed subspaces of  $Y$ , as usual  $B(X, Y_i)$  denotes the space of all bounded operators from  $X$  to  $Y_i$  and  $B(X, X)$  is abbreviated by  $B(X)$ .

**Definition 2.1.** We call a sequence  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  a  $g$ -frame for  $X$  with respect to  $\{Y_i\}$  if there exists a solid  $BK$ -space  $X_d$  and positive constants  $0 < A \leq B < \infty$  such that

- (i) for every  $f \in X$ ,  $\{\|\Lambda_i f\|\} \in X_d$ ,
- (ii) for every  $f \in X$ ,  $A\|f\|_X \leq \|\{\|\Lambda_i f\|\}\|_{X_d} \leq B\|f\|_X$ .

$A$  and  $B$  are called *bounds of  $g$ -frame* and we say that  $\{\Lambda_i\}$  is an  $(A, B)$ - $g$ -frame.

We note that every  $X_d$ -frame  $\{g_i\}$  is a  $g$ -frame for  $X$ , where  $Y_i = \mathbb{C}$  for each  $i \in I$ . The following result is essential for our next investigations.

**Theorem 2.2.** Let  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  be a  $g$ -frame for  $X$  with respect to  $\{Y_i : i \in I\}$ . Then  $W = (\oplus Y_i)_d = \{\{y_i\} : y_i \in Y_i, \{\|y_i\|\} \in X_d\}$ , with  $\|\cdot\|_W$  defined by  $\|\{y_i\}\|_W = \|\{\|y_i\|\}\|_{X_d}$  is a Banach space with coordinatewise operations and  $\theta : X \rightarrow W$ , the analysis operator, defined by  $\theta(x) = \{\Lambda_i(x)\}$  is a bounded, one-to-one, linear operator with closed range and  $\theta^{-1} : \theta(X) \rightarrow X$  is bounded.

*Proof.* For every  $w = \{w_i\}$ ,  $z = \{z_i\}$  in  $W$  and every  $i \in I$  we have  $\|z_i + w_i\| \leq \|z_i\| + \|w_i\|$ . Since  $\{\|w_i\|\}, \{\|z_i\|\} \in X_d$  and  $X_d$  is a solid space, then  $\{\|z_i + w_i\|\} \in X_d$  and so  $z + w \in W$ , moreover

$$\|z + w\|_W = \|\{\|z_i + w_i\|\}\|_{X_d} \leq \|\{\|z_i\|\}\|_{X_d} + \|\{\|w_i\|\}\|_{X_d} = \|z\|_W + \|w\|_W.$$

Plainly for each  $\lambda \in \mathbb{C}$ ,  $\|\lambda z\|_W = |\lambda| \cdot \|z\|_W$ . So  $W$  is a normed space. For the completeness of  $W$ , we consider the map  $\eta : W \rightarrow X_d$  defined by

$$\eta(\{y_i\}) = \{\|y_i\|\}, \quad \text{for all } \{y_i\}_i \in W.$$

Firstly for each  $w = \{w_i\}_i$  in  $W$ , we have  $\|\eta(w)\|_{X_d} = \|w\|_W$ . Secondly for each  $i \in I$ ,  $\|z_i\| - \|w_i\| \leq \|z_i - w_i\|$  and  $X_d$  is a solid space, then

$$(1) \quad \|\eta(z) - \eta(w)\|_{X_d} \leq \|z - w\|_W \quad \text{for all } z, w \in W.$$

Similarly by the continuity of coordinate functionals  $\pi_i : X_d \rightarrow \mathbb{C}$ , for each  $i \in I$ , the projection  $P_i : W \rightarrow Y_i$  defined by

$$P_i(w) = w_i, \quad \text{for all } w = \{w_i\}_i \text{ in } W$$

is a continuous linear map, because

$$(2) \quad \|P_i(w)\| = \|w_i\| = \pi_i(\eta(w)) \leq \|\pi_i\| \cdot \|\eta(w)\| = \|\pi_i\| \cdot \|w\|.$$

Now let  $\{w^{(n)}\}$  be a Cauchy sequence in  $W$ , where  $w^{(n)} = \{w_i^{(n)}\}_{i \in I}$  for each  $n \in \mathbb{N}$ . By (2) for each  $i \in I$ ,  $\{w_i^{(n)}\}_n$  is a Cauchy sequence in  $Y_i$  and  $Y_i$  is complete. So there exists  $w_i \in Y_i$  such that  $w_i^{(n)} \rightarrow w_i$  as  $n \rightarrow \infty$ . Therefore for each  $i \in I$ ,  $\|w_i^{(n)}\| \rightarrow \|w_i\|$  as  $n \rightarrow \infty$ .

On the other hand by (1),  $\{\eta(w^{(n)})\}_n$  is a Cauchy sequence in  $X_d$  and  $X_d$  is complete, so there exists some  $\Lambda = \{\lambda_i\}_{i \in I}$  in  $X_d$  such that  $\{\eta(w^{(n)})\}_n$  converges to  $\Lambda$ . Hence by the continuity of  $\pi_i$ 's, for each  $i \in I$ ,  $\|w_i^{(n)}\| \rightarrow \lambda_i$  as  $n \rightarrow \infty$ . Therefore for each  $i \in I$ ,  $\lambda_i = \|w_i\|$  and  $\Lambda = \{\|w_i\|\}_i$  is in  $X_d$ . So  $w = \{w_i\}$  is in  $W$ .

Now for each  $n \in \mathbb{N}$ ,  $w^{(n)} - w \in W$  and  $\{w^{(n)} - w\}_n$  is a Cauchy sequence in  $W$  and for each  $i \in I$ ,  $w_i^{(n)} - w_i \rightarrow 0$  as  $n \rightarrow \infty$ . Hence by the above proof  $\{\eta(w^{(n)} - w)\}_n$  converges to 0 in  $X_d$ . Since  $\|w^{(n)} - w\|_W = \|\eta(w^{(n)} - w)\|_{X_d}$ , then  $\{w^{(n)}\}$  converges to  $w$  in  $W$ . Hence  $W$  is complete.

For the second assertion for each  $x \in X$ ,

$$A\|x\|_X \leq \|\theta(x)\|_W \leq B\|x\|_X,$$

then  $\theta$  is bounded, one-to-one and  $\theta^{-1}$  is bounded. Moreover since  $W$  is complete,  $\theta$  has a closed range.  $\square$

**Corollary 2.3.** *With the hypothesis of Theorem 2.2 every  $Y_i$  is isomorphic to its copy  $E_i$  in  $W$ , where  $E_i = \{\{y_i \delta_{ij}\}_{j \in I} : y_i \in Y_i\}$ .*

*Proof.* First we show that  $E_i$  is a closed subspace of  $W$ . Since  $X_d$  is a solid space, then for each  $y_i \in Y_i$ , we have  $\{\|y_i\| \delta_{ij}\}_{j \in I} \in X_d$  and therefore  $\{y_i \delta_{ij}\}_{j \in I} \in W$ , which implies that  $E_i \subseteq W$ . Plainly  $E_i$  is a subspace of  $W$ . If  $\{u^{(n)}\}_{n \in \mathbb{N}}$  is a sequence in  $E_i$  converges to  $z = \{z_j\}_{j \in I}$  in  $W$ , where  $u^{(n)} = \{y_i^{(n)} \delta_{ij}\}_{j \in I}$ , then for each  $j \in I$ ,

$$\|y_i^{(n)} \delta_{ij} - z_j\| = \pi_j(\eta(u^{(n)} - z)) \leq \|\pi_j\| \cdot \|\eta(u^{(n)} - z)\|_{X_d} \leq \|u^{(n)} - z\|_W.$$

Therefore  $y_i^{(n)}\delta_{ij} \rightarrow z_j$ . Specially for  $j = i$ , we get  $y_i^{(n)} \rightarrow z_i$  and since  $Y_i$  is closed,  $z_i \in Y_i$ . If  $j \neq i$ , then  $y_i^{(n)}\delta_{ij} = 0$  and consequently  $z_j = 0$ . Hence  $z = \{z_i\delta_{ij}\}$  is in  $E_i$  which shows that  $E_i$  is closed. Now the coordinate projection  $P_i : E_i \rightarrow Y_i$  is a continuous bijection, since  $\|P_i(\{y_i\delta_{ij}\}_j)\| = \|\pi_i\eta(y)\| \leq \|\pi_i\| \cdot \|y\|$ . Therefore by the open mapping theorem it is an isomorphism.  $\square$

For our next investigation we need the following result, and for convenience we abbreviate  $\|\cdot\|_{Z_{i,d}}$  by  $\|\cdot\|_i$ .

**Corollary 2.4.** *Let for each  $i \in I$ ,  $(Z_{i,d}, \|\cdot\|_i)$  be a BK-space and  $X_d$  be a solid BK-space. Then*

$S_d = (\oplus_{i \in I} Z_{i,d}) = \{\{\Lambda_i\}_{i \in I} : \Lambda_i = \{\lambda_{i,j} : j \in I_i\} \in Z_{i,d}, \{\|\Lambda_i\|_i\}_{i \in I} \in X_d\}$   
with  $\|\cdot\|_{S_d}$  defined by  $\|\{\Lambda_i\}_i\|_{S_d} = \|\{\|\Lambda_i\|_i\}_{i \in I}\|_{X_d}$  is a BK-space and if moreover each  $Z_{i,d}$  is solid, then  $S_d$  is solid.

*Proof.* By the above theorem  $S_d$  is a Banach space and for each  $i \in I, j \in I_i$ , the coordinate functional  $\{\lambda_{ij} : j \in I_i\}_{i \in I} \rightarrow \lambda_{ij}$  is  $\pi_{ij} \circ P_i$  which is continuous, because  $X_d$  and each  $Z_{i,d}$  is a BK-space. Moreover suppose that each  $Z_{i,d}$  is a solid space and let for each  $i \in I, j \in I_i, |b_{ij}| \leq |\lambda_{ij}|$  and  $\{\lambda_{ij}\} \in S_d$ . Let  $i \in I$ . Since  $\Lambda_i = \{\lambda_{ij} : j \in I_i\}$  is in  $Z_{i,d}$  and  $Z_{i,d}$  is solid, then  $\{b_{ij} : j \in I_i\}$  is in  $Z_{i,d}$  and

$$\|\{b_{ij} : j \in I_i\}\|_i \leq \|\{\lambda_{ij} : j \in I_i\}\|_i.$$

Now  $\{\|\{\lambda_{ij} : j \in I_i\}\|_i\}_i \in X_d$  and  $X_d$  is solid, so by the above relation  $\{\|\{b_{ij} : j \in I_i\}\|_i\}_i \in X_d$ . Therefore  $\{b_{ij} : j \in I_i\}_i \in S_d$  and moreover

$$\begin{aligned} \|\{\{b_{ij} : j \in I_i\}\}_i\| &= \|\{\|\{b_{ij} : j \in I_i\}\|_i\}_i\|_{X_d} \\ &\leq \|\{\|\{\lambda_{ij} : j \in I_i\}\|_i\}_i\|_{X_d} = \|\{\{\lambda_{ij} : j \in I_i\}_i\}\|. \end{aligned}$$

$\square$

Our next result is a generalization of Theorem 3.4 of [7] and Theorem 2.2 of [15].

**Theorem 2.5.** *Let  $\{\Lambda_i \in B(X, Y) : i \in I\}$  and let for each  $i \in I, \{\Gamma_{ij} \in B(Y_i, W_{ij}) : j \in I_i\}$  be an  $(A_i, B_i)$ -g-frame for  $Y_i$  with respect to  $\{W_{ij} : j \in I_i\}$  and associated BK-space  $Z_{i,d}$  and suppose that  $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$ . Then the following conditions are equivalent:*

(i)  $\{\Lambda_i \in B(X, Y) : i \in I\}$  is a g-frame for  $X$  with associated BK-space  $X_d$ .

(ii)  $\{\Gamma_{ij} \circ \Lambda_i \in B(X, W_{ij}) : i \in I, j \in I_i\}$  is a g-frame for  $X$  with associated BK-space  $S_d$ .

*Proof.* First note that for every  $x \in X$  and every  $i \in I$ , we have  $\Lambda_i x \in Y_i$ , so  $\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : j \in I_i\} \in Z_{i,d}$  and

$$(3) \quad A\|\Lambda_i x\| \leq A_i\|\Lambda_i x\| \leq \|\{\|\Gamma_{ij} \circ \Lambda_i(x)\| : j \in I_i\}\|_i \leq B_i\|\Lambda_i(x)\| \leq B\|\Lambda_i x\|.$$

To prove (i)  $\Rightarrow$  (ii) let  $\{\Lambda_i \in B(X, Y_i); i \in I\}$  be a  $g$ -frame with bounds  $C$  and  $D$ . Then for every  $x \in X$ ,  $\{\|\Lambda_i x\|\} \in X_d$  and

$$(4) \quad C\|x\|_X \leq \|\{\|\Lambda_i x\|\}_i\|_{X_d} \leq D\|x\|.$$

Since  $\{\|\Lambda_i x\|\} \in X_d$  and  $X_d$  is solid, then by (3) we have

$$\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\| \in X_d$$

and  $\|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\|_{X_d} \leq B\|\{\|\Lambda_i(x)\|\}_i\|_{X_d}$ . Hence by (4) we conclude that  $\|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\|_{X_d} \leq BD\|x\|_X$ . Similarly we get  $AC\|x\|_X \leq \|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\|_{X_d}$ .

To prove (ii)  $\Rightarrow$  (i) let  $\{\Gamma_{ij} \circ \Lambda_i \in B(X, W_{ij}) : i \in I, j \in I_i\}$  be a  $(C', D')$ - $g$ -frame with associated  $BK$ -space  $S_d$ . Then for each  $x \in X$ ,  $\{\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i \in S_d$ ,

$$\|\{\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\|_{S_d} = \|\{\|\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\|_{X_d}$$

and  $C'\|x\|_X \leq \|\{\{\|\Gamma_{ij} \circ \Lambda_i(x)\|\} : j \in I_i\}_i\|_{S_d} \leq D'\|x\|_X$ . Again by (3) and the solidity of  $X_d$  we conclude that  $\{\|\Lambda_i x\|\} \in X_d$  and  $A\|\{\|\Lambda_i x\|\}_i\|_{X_d} \leq D'\|x\|_X$ . Similarly we get  $C'\|x\|_X \leq B\|\{\|\Lambda_i x\|\}_i\|_{X_d}$ , which completes the proof.  $\square$

If  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  is a  $g$ -frame for  $X$  with respect to  $\{Y_i : i \in I\}$  and there exists a bounded linear map  $\eta : W \rightarrow X$  such that for each  $x \in X$ ,  $\eta(\{\Lambda_i(x)\}_i) = x$ , we call  $(\{\Lambda_i\}, \eta)$  a *Banach  $g$ -frame*. If  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  is a  $g$ -frame and there exists a sequence  $\{q_i \in B(Y_i, X) : i \in I\}$  such that for each  $x \in X$ ,  $x = \sum_i q_i \circ \Lambda_i(x)$ , then  $(\{\Lambda_i\}, \{q_i\})$  is called a (linear) *decomposition* of  $X$  with respect to  $\{Y_i : i \in I\}$ .

We note that if  $(\{\Lambda_i\}, \eta)$  is a Banach  $g$ -frame, then  $\eta \circ \theta = \text{id}_X$  and  $p = \theta \circ \eta$  is a projection which implies that  $\theta(X) = p(W)$  is a complemented subspace of  $W$ .

From the above theorem we have the following consequence.

**Corollary 2.6.** *Let  $X$  be a Banach space and  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  be a  $g$ -frame. Let  $\theta : X \rightarrow W$  be the analysis operator of  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$ . Then  $(\{\Lambda_i\}, \theta^{-1})$  is a Banach  $g$ -frame.*

Our next result is a generalization of Lemma 2.13 in [13] to Banach spaces.

**Lemma 2.7.** *Let  $\{\Lambda_i \in B(X, Y_i)\}_i$  be a  $g$ -frame with bounds  $A, B$  and let  $T_i \in B(Y_i, Z_i)$  be a bounded invertible operator. Suppose that*

$$0 < m = \inf_i \frac{1}{\|T_i^{-1}\|} \leq \sup_i \|T_i\| = M < \infty.$$

*If  $T \in B(X)$  is invertible and  $\Gamma_i = T_i \Lambda_i T$ , then  $\{\Gamma_i \in B(X, Z_i)\}$  is a  $g$ -frame. If moreover  $(\{\Lambda_i\}, \{q_i\})$  is a linear decomposition of  $X$  with respect to  $\{Y_i : i \in I\}$ , then there exists a sequence  $\{P_i \in B(Z_i, X) : i \in I\}$  such that  $(\{\Gamma_i\}, \{P_i\})$  is a linear decomposition of  $X$  with respect to  $\{Z_i : i \in I\}$ .*

*Proof.* Let  $f \in X$ . For each  $j \in I$  we have

$$m\|\Lambda_j(Tf)\| \leq \|\Gamma_j f\| \leq M\|\Lambda_i(Tf)\|.$$

Since  $X_d$  is a solid space and  $\{\|\Lambda_j Tf\|\} \in X_d$ , then  $\{\|\Gamma_j f\|\} \in X_d$  and  $\|\{\|\Gamma_j f\|\}\|_{X_d} \leq M\|\{\|\Lambda_j Tf\|\}\|_{X_d}$ , similarly  $m\|\{\|\Lambda_j Tf\|\}\|_{X_d} \leq \|\{\|\Gamma_j f\|\}\|_{X_d}$ . Hence for every  $f \in X$ ,

$$\begin{aligned} mA \frac{1}{\|T^{-1}\|} \cdot \|f\|_X &\leq mA\|Tf\|_X \leq m\|\{\|\Lambda_j Tf\|\}_j\|_{X_d} \\ &\leq \|\{\|\Gamma_j f\|\}_j\|_{X_d} \leq MB\|Tf\| \leq MB\|T\| \cdot \|f\|. \end{aligned}$$

If  $(\{\Lambda_i\}, \{q_i\})$  is a linear decomposition of  $X$ , then by taking  $p_i = T^{-1}q_iT_i^{-1}$  we see that for every  $f \in X$ ,  $\sum T^{-1}q_iT_i^{-1}(T^{-1}\Lambda_i T)(f) = T^{-1}\sum q_i\Lambda_i T f = f$  and we have the result.  $\square$

Christensen and Heil in [9] showed that Banach frames are stable under small perturbations. We show that  $g$ -frames and Banach  $g$ -frames are also stable under small perturbations.

**Definition 2.8.** Let  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  be a  $g$ -frame and let  $0 \leq \lambda, \mu < 1$ . Let  $\{c_i\} \in X_d$ . We say that  $\{\Gamma_i \in B(X, Y_i) : i \in I\}$  is a  $(\lambda, \mu, \{c_i\})$ -perturbation of  $\{\Lambda_i\}$  if

$$\|\Lambda_i f - \Gamma_i f\| \leq \lambda\|\Lambda_i f\| + \mu\|\Gamma_i f\| + c_i\|f\|, \quad \text{for each } f \in X.$$

Now we have the following result, see [15, Proposition 4.3].

**Theorem 2.9.** Let  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  be a  $g$ -frame for  $X$  with respect to  $\{Y_i : i \in I\}$  with bounds  $A, B$  and let  $\{\Gamma_i \in B(X, Y_i) : i \in I\}$  be a  $(\lambda, \mu, \{c_i\})$ -perturbation of it with  $K = \|\{c_i\}\|_{X_d}$ .

(a) If  $A' = (1 - \lambda)A(1 + \mu)^{-1} - K > 0$ , then  $\{\Gamma_i\}$  is a  $g$ -frame with bounds  $A'$  and  $B' = (1 + \mu)(1 + \lambda)B(1 - \mu)^{-1} + K(1 + \mu)(1 - \mu)^{-1}$ .

(b) If  $(\{\Lambda_i\}, \eta)$  is a Banach  $g$ -frame,  $A' > 0$ , and  $\|\eta\|(B\lambda + \mu B' + K) < 1$ , then there exists  $T \in B(X)$  such that  $(\{\Gamma_i\}, T)$  is a Banach  $g$ -frame.

*Proof.* (a) By assumption for each  $f \in X$  we have  $\|\Lambda_i f - \Gamma_i f\| \leq \lambda\|\Lambda_i f\| + \mu\|\Gamma_i f\| + c_i\|f\|$ . Hence

$$(1 - \mu)\|\Gamma_i f\| \leq (1 + \lambda)\|\Lambda_i f\| + c_i\|f\|.$$

Now  $\{\|\Lambda_i f\|\}_i \in X_d$ ,  $\{c_i\} \in X_d$  and  $X_d$  is a solid space, therefore  $\{\|\Gamma_i f\|\} \in X_d$  and moreover

$$(1 - \mu)\|\{\|\Gamma_i f\|\}\|_{X_d} \leq (1 + \lambda)\|\{\|\Lambda_i f\|\}\|_{X_d} + \|f\|K.$$

Similarly we have

$$(1 - \lambda)\|\{\|\Lambda_i f\|\}\|_{X_d} \leq (1 + \mu)\|\{\|\Gamma_i f\|\}\|_{X_d} + \|f\|K.$$

Now a simple calculation shows that

$$A'\|f\|_X \leq \|\{\|\Gamma_i f\|\}\|_{X_d} \leq B'\|f\|_X, \quad \text{for each } f \in X$$

which completes the proof of (a).

(b) In (a) we proved that for each  $f \in X$ ,  $\{\|\Gamma_i f\|\} \in X_d$  and therefore  $\{\Gamma_i f\} \in W$ . So we can define  $S : X \rightarrow W$  by  $S(f) = \{\Gamma_i f\}$ . Plainly  $S$  is a linear operator and by the above relation  $\|S\| \leq B'$ . Moreover for every  $f \in X$ ,

$$\begin{aligned} \|\eta \circ S(f) - f\| &= \|\eta(S - \theta)(f)\| \leq \|\eta\| \cdot \|S(f) - \theta(f)\|_{X_d} \\ &\leq \|\eta\|(\lambda B + \mu B' + K)\|f\|. \end{aligned}$$

Hence  $\|\eta \circ S - I\| \leq \|\eta\|(\lambda B + \mu B' + K) < 1$ , therefore  $\eta \circ S$  is invertible in  $B(X)$  and for  $T = (\eta \circ S)^{-1} \circ \eta$  we have  $TS = I$ , which completes the proof of (b).  $\square$

### 3. FUSION FRAMES

Fusion frames in Hilbert spaces were introduced by Casazza et al. in [2], [3], it has been intensely studied and some of its applications are discovered [3] and [6]. In this section we introduce fusion frames in Banach spaces and we discuss some of their properties.

**Definition 3.1.** Let  $X$  be a Banach space,  $\{p_i\}_{i \in I}$  be a sequence of continuous linear projections on  $X$ ,  $W_i = p_i(X)$  for each  $i \in I$  and let  $\{v_i : i \in I\}$  be a sequence of weights, i.e.,  $v_i > 0$ . We say that  $\{(v_i, W_i) : i \in I\}$  is a *fusion frame* for  $X$  if there exists a sequence  $\{q_i \in B(W_i, X) : i \in I\}$  and an invertible operator  $S \in B(X)$  such that  $(\{v_i S^{-1} \circ q_i\}, \{v_i p_i\})$  is an atomic decomposition of  $X$  with respect to  $\{W_i : i \in I\}$ .

We note that if  $\{(v_i, W_i) : i \in I\}$  is a fusion frame for  $X$  with associated sequence  $\{q_i \in B(W_i, X) : i \in I\}$ , then for each  $x \in X$ , the series  $\sum_{i \in I} v_i^2 q_i \circ p_i(x)$  converges to  $S(x)$  and there exist a solid  $BK$ -space  $X_d$  and constants  $0 < A \leq B$  such that for every  $x \in X$ ,

$$A\|x\|_X \leq \|\{p_i(x)\}\|_{X_d} \leq B\|x\|.$$

The constants  $A$  and  $B$  are called the fusion frame bounds of  $\{(v_i, W_i) : i \in I\}$ , it is called tight if  $A = B$  and Parseval if  $A = B = 1$ . Also note that if  $\{(v_i, W_i) : i \in I\}$  is a fusion frame, where  $W_i = p_i(X)$ , then  $\{v_i p_i \in B(X, W_i) : i \in I\}$  is a  $g$ -frame.

**Lemma 3.2.** Let  $X$  be a Banach space and  $\{f_i : i \in I\}$  be a Schauder frame for  $X$  with bounds  $A$  and  $B$ . Then  $\{\|f_i\|^{-1}, \text{Span}\{f_i\}_{i \in I}\}$  is a fusion frame with bounds  $A$  and  $B$ .

*Proof.* Since  $\{f_i : i \in I\}$  is a Schauder frame with bounds  $A$  and  $B$ , there exists a sequence  $\{g_i : i \in I\}$  in  $X^*$  and a solid  $BK$ -space  $X_d$  such that for each  $x \in X$ ,  $x = \sum g_i(x) f_i$  and  $A\|x\|_X \leq \|\{g_i(x)\}\|_{X_d} \leq B\|x\|_X$ .

In a Banach space every finite dimensional subspace is closed and complemented, so  $W_i = \text{span}\{f_i\}$  is closed, complemented and its associated continuous projection  $p_i : X \rightarrow W_i$  is defined by

$$p_i(f) = g_i(f) f_i \quad \text{for all } f \in X,$$

and

$$A\|f\|_X \leq \|\{v_i p_i(f)\}\|_{X_d} \leq B\|f\|_X,$$

where  $v_i = 1/\|f_i\|$ . We can take  $q_i$  the identity map on  $X$  for each  $i \in I$ .  $\square$

The following result shows that our definition consistent with the definition in [7].

**Example 3.3.** Every fusion frame  $\{(v_i, W_i) : i \in I\}$  in a Hilbert space  $H$  is a fusion frame in  $H$  as a Banach space. To see this, we know that there exist constants  $A$  and  $B$  such that for each  $x \in H$ ,

$$A\|x\|_H^2 \leq \sum_i v_i^2 \|\pi_{W_i}(x)\|^2 \leq B\|x\|^2$$

and  $S_W$ , the fusion frame operator of  $\{(v_i, W_i) : i \in I\}$  is defined by

$$S_W(f) = \sum_i v_i^2 \pi_{W_i}(f) \quad \text{for all } f \in H,$$

is invertible. So  $\{(v_i, W_i) : i \in I\}$  is a fusion frame for the Banach space  $X = H$ , where  $p_i = \pi_{W_i} = q_i$  and  $X_d = \ell^2(I)$ .

For constructing  $g$ -frames, from Theorem 2.5 we have the following results for fusion frames and  $g$ -frames.

**Lemma 3.4.** *Let  $\{\Lambda_i \in B(X, Y_i) : i \in I\}$  be a sequence of bounded operators and let for each  $i \in I$ ,  $\{(\alpha_{ij}, W_{ij}) : j \in I_i\}$  be a fusion frame for  $Y_i$  with bounds  $A_i, B_i$  and suppose that  $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$ . Then the following conditions are equivalent:*

- (i)  $\{\Lambda_i\}_{i \in I}$  is a  $g$ -frame for  $X$  with respect to  $\{Y_i : i \in I\}$ ,
- (ii)  $\{\alpha_{ij} \pi_{ij} \circ \Lambda_i : i \in I, j \in I_i\}$  is a  $g$ -frame for  $X$  with respect to  $\{W_{ij} : i \in I, j \in I_i\}$ , where  $\pi_{ij} : Y_i \rightarrow W_{ij}$  is the projection of  $Y_i$  onto  $W_{ij}$ .

*Proof.* It is enough to note that  $\{\alpha_{ij} \pi_{ij} : j \in I_i\}$  is a  $g$ -frame for  $W_{ij}$  with bounds  $A_i, B_i$  and use Theorem 2.5.  $\square$

Another version of these combinations is as follows:

**Corollary 3.5.** *Let  $\{(\alpha_i, Y_i) : i \in I\}$  be a fusion frame with bounds  $C$  and  $D$ . Let for each  $i \in I$ ,  $\{\Lambda_{ij} \in B(Y_i, W_{ij}) : j \in I_i\}$  be an  $(A_i, B_i)$ - $g$ -frame for  $Y_i$ , such that  $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$ . Then  $\{\alpha_i \Lambda_{ij} \circ \pi_{Y_i} : i \in I, j \in I_i\}$  is an  $(AC, BD)$ - $g$ -frame.*

*Proof.* It is enough to note that  $\{\alpha_i \pi_{Y_i} : i \in I\}$  is a  $(C, D)$ - $g$ -frame for  $X$  and use Theorem 2.5.  $\square$

**Theorem 3.6.** *Let  $\{(v_i, W_i) : i \in I\}$  be a fusion frame for  $X$  with bounds  $C, D$ ; associated sequence  $\{q_i \in B(W_i, X) : i \in I\}$  and let  $\{f_{ij}\}_{j \in I_i}$  be a Schauder frame for  $W_i$  with bounds  $A_i, B_i$  for each  $i \in I$ . If  $0 < A = \inf_i A_i \leq B = \sup_i B_i < \infty$ , then  $\{\{S^{-1} q_i(f_{ij})\}_{j \in I_i}\}_{i \in I}$  is a Schauder frame for  $X$  with bounds  $AC$  and  $BD$ . In particular if  $\{(v_i, W_i) : i \in I\}$  and  $\{f_{ij} : j \in I_i\}$  are Parseval, then  $\{S^{-1} q_i(f_{ij})\}_i$  is Parseval.*



*Proof.* Since  $\{(v_i, W_i) : i \in I\}$  is a fusion frame with associated sequence  $\{q_i \in B(W_i, X) : i \in I\}$ , then there exists an invertible linear operator  $S$  in  $B(X)$  such that for each  $x \in X$ ,  $x = \sum_i v_i^2 S^{-1} \circ q_i \circ p_i(x)$ , where  $p_i$  is the projection of  $X$  onto  $W_i$ . We also know that for each  $i \in I$ ,  $\{f_{ij}\}_{j \in I_i}$  is a Schauder frame for  $W_i$ , so there exists a sequence  $\{g_{ij}\}_{j \in I_i} \subseteq W_i^*$  and a solid  $BK$ -space  $Z_{i,d}$  such that for each  $x \in X$ ,  $\{g_{ij} \circ p_i(x)\}_{j \in I_i} \in Z_{i,d}$  and  $p_i(x) = \sum_{j \in I_i} g_{ij}(p_i(x)) f_{ij}$ . Therefore for each  $x \in X$  we have

$$(5) \quad x = \sum_{i \in I} \sum_{j \in I_i} g_{ij}(p_i(x)) S^{-1} q_i(f_{ij}).$$

Now a small modification in the proof of Theorem 2.5 shows that for each  $x \in X$ ,  $\{\{v_i g_{ij} \circ p_i(x)\}_{j \in I_i}\}_{i \in I} \in S_d$  and

$$AC \|x\|_X \leq \|\{\{v_i g_{ij} \circ p_i(x)\}_{j \in I_i}\}_i\|_{X_d} \leq BD \|x\|_X.$$

Since  $\{\{v_i g_{ij} \circ p_i\}_{j \in I_i}\}_{i \in I} \subseteq X^*$ ,  $S_d$  is a solid  $BK$ -space and (5) holds for each  $x \in X$ , then  $\{\{S^{-1} q_i(f_{ij})\}_{j \in I_i}\}_{i \in I}$  is a Schauder frame for  $X$ .  $\square$

**Proposition 3.7.** *Let  $\{(v_i, W_i) : i \in I\}$  be a fusion frame for  $X$  and  $T \in B(X)$  be invertible. Then  $\{(v_i, TW_i) : i \in I\}$  is a fusion frame for  $X$ .*

*Proof.* Since  $\{(v_i, W_i) : i \in I\}$  is a fusion frame for  $X$ , then each  $W_i$  is complemented in  $X$ , there exist a solid  $BK$ -space  $X_d$ , constants  $0 < A \leq B < \infty$ , a sequence  $\{q_i \in B(W_i, X) : i \in I\}$  and an invertible operator  $S \in B(X)$  such that for each  $x \in X$ ,  $\{v_i \|p_i(x)\| : i \in I\} \in X_d$ ,  $x = \sum_i v_i^2 S^{-1} \circ q_i \circ p_i(x)$  and  $A \|x\|_X \leq \|\{v_i \|p_i(x)\|\}_i\|_{X_d} \leq B \|x\|_X$ , where  $p_i = \pi_{W_i}$  is the projection of  $X$  onto  $W_i$ .

Now since  $T \in B(X)$  is invertible and  $W_i$  is complemented, then  $TW_i$  is complemented in  $X$  and  $\pi_{TW_i} = Tp_i T^{-1}$ , for each  $i \in I$ . Moreover for each  $x \in X$  and  $i \in I$ ,  $\|Tp_i T^{-1}(x)\| \leq \|T\| \cdot \|p_i T^{-1}(x)\|$ . Since  $\{v_i \|p_i T^{-1}(x)\|\} \in X_d$  and  $X_d$  is a solid space, then  $\{v_i \|Tp_i T^{-1}(x)\|\}_i \in X_d$  and  $\|\{v_i \|Tp_i T^{-1}(x)\|\}_i\|_{X_d} \leq \|T\| \cdot \|\{v_i \|p_i T^{-1}(x)\|\}_i\|_{X_d}$ . Hence for each  $x \in X$ ,

$$\frac{1}{\|T\|} \cdot \frac{1}{\|T^{-1}\|} A \|x\|_X \leq \|\{v_i \|Tp_i T^{-1}(x)\|\}_i\|_{X_d} \leq \|T\| \cdot \|B\| \cdot \|T^{-1}\| \cdot \|x\|_X.$$

Finally  $\{q_i T^{-1} \in B(TW_i, X)\}$ ,  $ST^{-1} \in B(X)$  is invertible and for each  $x \in X$ , we have  $x = \sum_i v_i^2 T(S^{-1} \circ q_i \circ T^{-1})(T \circ p_i \circ T^{-1})(x)$ , which completes the proof.  $\square$

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