# Robustness of Fusion Frames under Erasures of Subspaces and of Local Frame Vectors

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Abstract. Fusion frames were recently introduced to model applications under distributed processing requirements. In this paper we study the behavior of fusion frames under erasures of subspaces and of local frame vectors. We derive results on sufficient conditions for a fusion frame to be robust to such erasures as well as results on the design of fusion frames which are optimally robust in the sense of worst case behavior of the reconstruction error.

## 1. Introduction

In the last 20 years, frames, i.e., systems, which provide robust, stable and usually non-unique representations of vectors, have been employed in numerous applications such as filter bank theory [**5**], sigma-delta quantization [**2**], signal and image processing [**6**], and wireless communications [**13**]. However, a large number of new applications have emerged where the set-up can hardly be modeled naturally by one single frame system. Many of these applications share the common property of requiring distributed processing such as all types of sensor networks [**15**].

The notion of *fusion frames*, introduced by the authors in [**8**] and [**9**], provides an extensive framework not only to model sensor networks, but also to provide a means to improve robustness or develop feasible reconstruction algorithms. Related approaches with a different focus were undertaken by Aldroubi, Cabrelli, and Molter [**1**], Fornasier [**12**], and Sun [**19, 20**].

Some aspects of the theory of fusion frames have already been applied. Bodmann, Kribs, and Paulsen [**4**] and Bodmann [**3**] employed Parseval fusion frames under the term *weighted projective resolution of the identity* for optimal transmission of quantum states and for packet encoding. Also, Rozell, Goodman, and Johnson [**16, 17, 18**] used fusion frames to study noise reduction in sensor networks and as well as overlapping feature spaces of neurons in visual and hearing systems.

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In this paper we focus on the study of robustness of fusion frames under erasures. One motivation comes from the theory of sensor networks. Generally speaking, a fusion frame is a weighted set of subspaces with controlled overlaps, where each subspace is spanned by a local frame. Now the measurements taken by the sensors in a sensor network can be modeled as the inner products of a given signal with all local frame vectors. The subspaces model the groupings of the sensors, and the weights model the significance of each group. The reconstruction is now done first within the subspaces, i.e., within each group of sensors, and secondly the signal is reconstructed completely. The first step can be modeled using conventional frame theory, however for the second step fusion frame theory becomes essential. Due to the fact that single sensors might lose their ability to transmit, it becomes necessary to study the robustness of fusion frames under erasures of local frame vectors. Also the signal of one whole group of sensors might be delayed or completely lost, hence the erasure of one of multiple subspaces of fusion frames needs to be examined.

We will show that indeed sufficient conditions on the robustness of a fusion frame with respect to erasures of subspaces can be formulated in terms of weight conditions (Theorem 3.2). It will be also pointed out how this leads to an easy construction process for such fusion frames. We further study the design of fusion frames which behave optimally with respect to the worst case reconstruction error in the setting of Parseval fusion frames. For Parseval fusion frames with prescribed – but not necessarily equal – dimensions, we characterize optimality in terms of conditions on the weights (Theorem 3.6). Considering erasures of local frame vectors, sufficient conditions depend – as expected – on the properties of the frames inside the subspaces (Theorem 4.1), and optimality is characterized in terms of conditions on the norms of the local frame vectors (Theorem 4.3).

This paper is organized as follows. In Section 2 we briefly review the main definitions and notions related to fusion frames. In Section 3 and 4 we study erasures of whole subspaces and of local frame vectors, respectively. In both cases we derive results on sufficient conditions for robustness of a fusion frame with respect to those erasures as well as results on the design of optimal fusion frames for this problem.

## 2. Review of Fusion Frames

Let $I$ be a countable index set, let $\{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a *fusion frame*, if there exist constants $0 < C \le D < \infty$ such that

$$C\|x\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(x)\|^2 \le D\|x\|^2 \quad \text{for all } x \in \mathcal{H}, \tag{2.1}$$

where $\pi_{W_i}$ is the orthogonal projection onto the subspace $W_i$. We call $C$ and $D$ the *fusion frame bounds*. The family $\{(W_i, v_i)\}_{i\in I}$ is called a *$C$-tight fusion frame*, if in (2.1) the constants $C$ and $D$ can be chosen so that $C = D$, and a *Parseval fusion frame* provided that $C = D = 1$. The frame bound $C$ of a tight fusion frame $\{(W_i, v_i)\}_{i=1}^n$ in a finite-dimensional Hilbert space $\mathcal{H}$ can be interpreted as the *redundancy*, since in [**9**] it was shown that $C = (\sum_{i=1}^n v_i^2 \dim W_i)/\dim \mathcal{H}$.

Often it will become essential to consider a fusion frame together with a set of local frames for its subspaces. Recall that $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$, if there are constants $0 < A \leq B < \infty$ (called the *lower* and *upper* frame bound, respectively) so that for every $x \in \mathcal{H}$ we have

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2.$$

Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i,\, i\in I}$ be a frame for $W_i$ for each $i \in I$. Then we call $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ a *fusion frame system* for $\mathcal{H}$. The constants $C$ and $D$ are the associated *fusion frame bounds*, if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i\in I}$, and $A$ and $B$ are the *local frame bounds*, if these are the common frame bounds for the *local frames* $\{f_{ij}\}_{j\in J_i}$ for each $i \in I$. Notice that a *frame* for $\mathcal{H}$ is a fusion frame system with $|J_i| = 1$ for all $i \in I$, i.e., where each subspace is singly generated.

In frame theory an input signal is represented by a collection of *scalar* coefficients that measure the projection of that signal onto each frame vector. The representation space employed in this theory equals $\ell^2(I)$. However, in fusion frame theory an input signal is represented by a collection of *vector* coefficients that represent the projection (not just the projection energy) onto each subspace. Therefore the representation space employed in this setting is

$$\left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} = \{\{f_i\}_{i\in I} | f_i \in W_i \text{ and } \{\|f_i\|\}_{i\in I} \in \ell^2(I)\}.$$

Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. In order to map a signal to the representation space, i.e., to analyze it, the *analysis operator* $T_{\mathcal{W}}$ is employed, which is defined by

$$T_{\mathcal{W}} : \mathcal{H} \to \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \quad \text{with} \quad T_{\mathcal{W}}(x) = \{v_i \pi_{W_i}(x)\}_{i\in I}.$$

The *synthesis operator* $T^*_{\mathcal{W}}$ is given by

$$T^*_{\mathcal{W}} : \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \to \mathcal{H} \text{ with } T^*_{\mathcal{W}}(f) = \sum_{i\in I} v_i f_i,\ f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2},$$

and the *fusion frame operator* $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(x) = T^*_{\mathcal{W}} T_{\mathcal{W}}(x) = \sum_{i\in I} v_i^2 \pi_{W_i}(x).$$

Interestingly, a fusion frame operator exhibits properties similar to a frame operator concerning invertibility. In fact, if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds $C$ and $D$, then the associated fusion frame operator $S_{\mathcal{W}}$ is positive and invertible on $\mathcal{H}$, and $C\,Id \leq S_{\mathcal{W}} \leq D\,Id$.

We wish to mention that constructing "good" fusion frames is a subtle task. One approach to derive a fusion frame for $\mathbb{R}^d$ is to start with a frame $\{f_j\}_{j=1}^n$ for this space with frame bounds $A$ and $B$, then split $\{1, \ldots, n\}$ into $k$ sets $J_1, \ldots, J_k$, and define $W_i = \text{span}\{f_j\}_{j\in J_i}$, $1 \leq i \leq k$. Let $C$ and $D$ be a common lower and upper frame bound for the frames $\{f_j\}_{j\in J_i}$ for $W_i$, $1 \leq i \leq k$. Then $\{(W_i, 1, \{f_j\}_{j\in J_i})\}_{i=1}^k$ is a fusion frame system with fusion frame bounds $\frac{C}{B}$, $\frac{D}{A}$ (cf. [**9**, Ex. 2.5]). In order for this process to work effectively, the *local frames* have to possess (uniformly) good lower frame bounds, since these control the computational complexity of reconstruction. However, it is known [**10**] that the problem of dividing a frame into a finite number of subsets each of which has good lower frame bounds is equivalent to one of the deepest and most intractable unsolved problems in mathematics: *the 1959 Kadison-Singer Problem.*

For more details on the basic theory of fusion frames we refer the reader to [**9**]. An introduction to general frame theory is provided by [**11**].

## 3. Erasures of Subspaces

In this section we study the erasure of subspaces of a fusion frame, since sometimes the connection to one whole group of sensors might fail for some period of time, or might be destroyed completely. We first study sufficient conditions for a fusion frame to be robust with respect to the erasure of subspaces. Secondly, we derive results on fusion frames optimally designed for those erasures.

### 3.1. Erasures of Subspaces of an Arbitrary Fusion Frame.

We first recall the following general observation, which is [**8**, Prop. 3.6].

Proposition 3.1. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame in a finite-dimensional Hilbert space and let $i_0 \in I$. Then $\{(W_i, v_i)\}_{i\in I,\, i\neq i_0}$ is either a fusion frame or $\operatorname{span}\{W_i\}_{i\in I, i\neq i_0} \subsetneq \mathcal{H}$.*

Notice that this result does not hold for an infinite-dimensional Hilbert space $\mathcal{H}$. For instance, let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$, and let $W_1 = \operatorname{span}\{e_{2i}\}_{i=1}^\infty$, $W_2 = \operatorname{span}\{e_{2i} + \frac{1}{i} e_{2i+1}\}_{i=1}^\infty$, and $W_3 = \mathcal{H}$. If $W_3$ is deleted, it is easily checked that $\{W_1, W_2\}$ does not form a fusion frame, however it does satisfy $\operatorname{span}\{W_j\}_{j=1}^2 = \mathcal{H}$.

Our main result in this subsection provides sufficient conditions on the weights for a subspace to be deleted yet still leave a fusion frame.

Theorem 3.2. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with bounds $C$ and $D$, and let $J \subset I$. Then the following statements hold.*

(i) *If $\sum_{i\in J} v_i^2 > D$, then $\bigcap_{i\in J} W_i = \{0\}$.*
(ii) *If $\sum_{i\in J} v_i^2 = D$, then $\bigcap_{i\in J} W_i \perp \operatorname{span}\{W_i\}_{i\in I\setminus J}$.*
(iii) *If $c = \sum_{i\in J} v_i^2 < C$, then $\{(W_i, v_i)\}_{i\in I\setminus J}$ is a fusion frame with bounds $C - c$ and $D$.*

Proof. To prove part (i), let $x \in \bigcap_{i\in J} W_i$. Then, using the hypothesis that $\sum_{i\in J} v_i^2 > D$ and the fact that $\pi_{W_i} x = x$ for $i \in J$, we compute

$$D\|x\|^2 < \left(\sum_{i\in J} v_i^2\right)\|x\|^2 \le \sum_{i\in J} v_i^2 \|\pi_{W_i} x\|^2 + \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} x\|^2 \le D\|x\|^2,$$

hence $x = 0$.

Now suppose that $\sum_{i\in J} v_i^2 = D$, and let again $x \in \bigcap_{i\in J} W_i$. Then

$$D\|x\|^2 = \sum_{i\in J} v_i^2 \|\pi_{W_i} x\|^2 \le \sum_{i\in J} v_i^2 \|\pi_{W_i} x\|^2 + \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} x\|^2 \le D\|x\|^2.$$

Thus $\sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i} x\|^2 = 0$, and hence $x \perp \operatorname{span}\{W_i\}_{i\in I\setminus J}$. This proves claim (ii).

Finally we show (iii). For any $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I\setminus J} v_i^2 \left\|\pi_{W_i}(x)\right\|^2 &= \sum_{i\in I} v_i^2 \left\|\pi_{W_i}(x)\right\|^2 - \sum_{i\in J} v_i^2 \left\|\pi_{W_i}(x)\right\|^2 \\ &\geq C\left\|x\right\|^2 - \left(\sum_{i\in J} v_i^2\right)\left\|x\right\|^2 \\ &= (C-c)\left\|x\right\|^2 .\end{aligned}$$

The upper bound is obvious. This completes the proof of part (iii). □

First, we remark that the claim in part (iii) is sharp. Indeed, let $\{e_i\}_{i=1}^n$ be an orthonormal basis for an $n$-dimensional Hilbert space $\mathcal{H}$, and equip the set of subspaces $\{W_i\}_{i=1}^n$ defined by $W_i = \operatorname{span}\{e_i, e_{i+1}\}$, $i \in \{1,\dots,n-1\}$ and $W_n = \operatorname{span}\{e_n, e_1\}$ with weights $v_i = \frac{1}{\sqrt{2}}$. Then $\{(W_i, v_i)\}_{i=1}^n$ is a Parseval fusion frame, and, by Theorem 3.2(iii), one subspace can be deleted yet leaving a fusion frame. However, obviously the deletion of an arbitrary pair of two subspaces destroys the fusion frame property.

Secondly, we observe that Theorem 3.2 provides an easy construction process of fusion frames $\{(W_i, v_i)\}_{i\in I}$ which are robust to erasures of an arbitrary set of $k$ subspaces $W_i$. One possibility would be to build a Parseval fusion frame $\{(W_i, v_i)\}_{i\in I}$ having the property that the sum of any $k$ weights $v_i^2$ is less than one. The previous example possesses this property for $k = 1$.

Theorem 3.2 immediately yields the following corollary, which focusses on the erasure of one single subspace.

Corollary 3.3. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with bounds $C$ and $D$. Then the following statements hold.*

(i) *For all $i \in I$, we have $v_i^2 \leq D$.*
(ii) *If there exists $i_0 \in I$ such that $v_{i_0}^2 = D$, then $\operatorname{span}\{W_i\}_{i\in I, i\neq i_0} \subsetneq \mathcal{H}$.*
(iii) *If there exists $i_0 \in I$ such that $v_{i_0}^2 < C$, then $\{(W_i, v_i)\}_{i\in I,\, i\neq i_0}$ is a fusion frame with bounds $C - v_{i_0}^2$ and $D$.*

It can be easily seen that the converse implications in (ii) and (iii) don't hold. Let us consider the following easy counterexample: Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{H}$, $W_1 = W_2 = \operatorname{span}\{e_1, e_2\}$, $W_3 = \operatorname{span}\{e_3\}$, and $v_1 = v_2 = v_3 = 1$. The fusion frame $\{(W_i, v_i)\}_{i\in I}$ has bounds $C = 1$ and $D = 2$. Hence $v_3^2 < D$, but $W_3 \perp W_1$ and $W_3 \perp W_2$. Thus the converse implication in (ii) doesn't hold. Furthermore, $v_1^2 \geq C$, but $\operatorname{span}\{W_2, W_3\} = \mathcal{H}$, thereby giving a counterexample for the converse implication in (iii).

Corollary 3.4. *Let $\{(W_i, v_i)\}_{i\in I}$ be a tight fusion frame with bound $C$, and let $i_0 \in I$. Then the following conditions are equivalent.*

(i) $v_{i_0}^2 < C$.
(ii) $\{(W_i, v_i)\}_{i\in I,\, i\neq i_0}$ *is a fusion frame.*

It is worth noticing that this result does not hold for arbitrary fusion frames. Indeed, in a 2-dimensional Hilbert space $\mathcal{H}$, let $A$ be arbitrarily large and define $W_1 = \{e_1, e_2\}$, $W_2 = \{e_1\}$ and $v_1 = 1$, $v_2 = A$. Then $\{W_i, v_i\}_{i=1}^2$ is a fusion frame with bounds $C = 1$ and $D = A + 1$. In particular, $v_2 = A$ can be much larger than $C$. However, the deletion of $W_2$ still leaves a fusion frame.

**3.2. Optimal Fusion Frames for Erasures of Subspaces.** The previous considerations identified those subspaces whose erasure leaves a fusion frame, i.e., each vector can still be reconstructed perfectly. However, since often we cannot control the erasures, we have to deal with erasures of subspaces which leave an incomplete set of subspaces. These cases will be examined in the following, where we restrict to Parseval fusion frames due to their advantageous reconstruction properties. Also we consider only finite fusion frames, since these are the fusion frames that actually occur in practise. More precisely, we will characterize those Parseval fusion frames which are optimal for erasure of one subspace in the sense that the maximal distance between original vectors and their reconstructed versions is minimal.

First, we make the meaning of optimality precise. In our setting we are interested in the worst case behavior of the reconstruction error. This definition of optimality under erasures has already been studied for frames in [**14**]. Moreover, Bodmann has derived results on non-weighted Parseval fusion frames which behave optimally under the erasure of one single subspace for a fixed number of subspaces, fixed and equal dimension of all subspaces, and fixed dimension of the Hilbert space [**3**, Thm. 13]. For this situation, he also studies multiple erasures [**3**].

Let $I = \{1, \ldots, n\}$ and define operators $D_{i_0} : \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \to \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2}$, $1 \le i_0 \le n$ by $\{D_{i_0}(f)\}_i = \delta_{i,i_0} f_{i_0}$ for all $i \in I$, where $f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2}$, which simulate the erasure of the subspace $W_{i_0}$.

Definition 3.5. Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a Parseval fusion frame with analysis operator $T_{\mathcal{W}}$. We define the associated 1-*erasure reconstruction error* $\mathcal{E}_1(\mathcal{W})$ to be

$$\mathcal{E}_1(\mathcal{W}) = \max\{\|T_{\mathcal{W}}^* D_i T_{\mathcal{W}}\| : 1 \le i \le n\}.$$

The following result gives a characterization of all Parseval fusion frames with a prescribed number of subspaces and prescribed – not necessarily equal – dimensions of the subspaces which behave optimally under one erasure. It is remarkable that the weights do not have to be equal at all. This differs significantly from the situation in frame theory, but can be explained easily. The reason for this phenomenon is that the weights are chosen is such way that the weighted subspaces have "equal size", i.e., the weights have to make up for the dimension of the subspaces.

Theorem 3.6. *Let $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame in a finite-dimensional Hilbert space $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame $\mathcal{W}$ satisfies $\mathcal{E}_1(\mathcal{W}) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n$ is a Parseval fusion frame with $\dim \widetilde{W}_i = \dim W_i$ for all $1 \le i \le n.\}$.*

(ii) *We have*

$$v_i^2 = \frac{\dim \mathcal{H}}{n \dim W_i} \quad \textit{for all } 1 \le i \le n.$$

*Moreover, let $x \in \mathcal{H}$ and let $\hat{x}$ denote the reconstructed vector. Then we have the following error bound*

$$\|x - \hat{x}\| \le \frac{\dim \mathcal{H}}{n \min\{\dim W_i : 1 \le i \le n\}} \|x\| \quad \textit{for all } x \in \mathcal{H}.$$

PROOF. Let $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame with analysis operator $T_{\mathcal{W}}$. Fix $i \in \{1, \dots, n\}$. Then we have

$$\|T_{\mathcal{W}}^* D_i T_{\mathcal{W}}\| = \sup_{\|x\|=1} \|T_{\mathcal{W}}^* D_i T_{\mathcal{W}} x\| = \sup_{\|x\|=1} \left\|v_i^2 \pi_{W_i}(x)\right\| = v_i^2 \sup_{\|x\|=1} \|\pi_{W_i}(x)\|.$$

Choosing $x \in W_i$ with $\|x\| = 1$ yields $\|\pi_{W_i}(x)\| = \|x\| = 1$, which is the maximum due to $\left\|v_i^2 \pi_{W_i}(x)\right\| \le \|x\| = 1$ for all $x \in \mathcal{H}$. Thus

$$\mathcal{E}_1(\mathcal{W}) = \max\{v_i^2 : 1 \le i \le n\}.$$

Now let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for $W_i$ for each $1 \le i \le n$. By [**8**, Thm. 3.2], the sequence $\{v_i e_{ij}\}_{i=1,\, j=1}^{n,\ \dim W_i}$ is a Parseval frame for $\mathcal{H}$. Employing [**7**, Sec. 2.3] yields that

$$1 = \frac{\sum_{i=1}^n \sum_{j=1}^{\dim W_i} \|v_i e_{ij}\|^2}{\dim \mathcal{H}} = \frac{\sum_{i=1}^n v_i^2 \dim W_i}{\dim \mathcal{H}},$$

hence

$$\sum_{i=1}^n v_i^2 \dim W_i = \dim \mathcal{H}.$$

This implies that there exists some $i \in \{1, \dots, n\}$ with $v_i^2 \dim W_i \ge \frac{\dim \mathcal{H}}{n}$. Since the dimensions as well as the number of subspaces are fixed, we can conclude that $\mathcal{E}_1(\mathcal{W})$ is minimal if and only if

$$v_i^2 \dim W_i = \frac{\dim \mathcal{H}}{n} \quad \text{for all } 1 \le i \le n.$$

The moreover-part follows immediately from the arguments above, since the maximal error is bounded by $\mathcal{E}_1(\mathcal{W})\, \|x\|$. □

For varying dimensions of the subspaces, we obtain the following result which can be derived from Theorem 3.6 and its proof.

COROLLARY 3.7. *Let $\{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame in a finite-dimensional Hilbert space $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame $\{(W_i, v_i)\}_{i=1}^n$ satisfies $\mathcal{E}_1(\{(W_i, v_i)\}_{i=1}^n) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n$ is a Parseval fusion frame in $\mathcal{H}$.$\}$.*

(ii) *We have*

$$\dim W_i = \dim \mathcal{H} \text{ and } v_i^2 = \frac{1}{n} \quad \text{for all } 1 \le i \le n. \tag{3.1}$$

*Moreover, let $\hat{x}$ be the reconstructed vector $x \in \mathcal{H}$ under one erasure using the original reconstruction formula. Then we have the following error bound*

$$\|x - \hat{x}\| \le \frac{1}{n} \|x\| \quad \text{for all } x \in \mathcal{H}. \tag{3.2}$$

As can be easily seen from (3.2), an increase in the number of subspaces improves the error bound for the reconstruction. Therefore, provided we allow infinitely many subspaces according to (3.1) the optimal weights would equal zero, which implies that the question of optimal fusion frame systems under erasures is the wrong question to ask provided we allow infinitely many subspaces.

## 4. Erasures of Local Frame Vectors

In this section we are concerned with erasures of local frame vectors, since usually some sensors die over time due to battery limitations. We first study sufficient conditions for a fusion frame to be robust with respect to the erasure of subspaces. Secondly, we derive results on fusion frames optimally designed for those erasures.

**4.1. Erasures of Local Frame Vectors in an Arbitrary Fusion Frame.** Our first result provides sufficient conditions on the weights for a prescribed number of local frame vectors to be deleted in each subspace yet still leave a fusion frame.

Theorem 4.1. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with bounds $C$ and $D$. For every $i \in I$, let $\{f_{ij}\}_{j\in J_i}$ be a Parseval frame for $W_i$ which is robust to $k_i$-erasures leaving a frame with lower frame bound $A_i$. For each $i \in I$, let $L_i \subset J_i$ satisfy $|L_i| \le k_i$, and define the set of subspaces $\{\widetilde{W}_i\}_{i\in I}$ by $\widetilde{W}_i = \mathrm{span}\{f_{ij}\}_{j\in J_i\setminus L_i}$. Then $\{(\widetilde{W}_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with bounds $(\min_{i\in I} A_i)\, C$ and $D$.*

Proof. Let $x \in \mathcal{H}$ and observe that

$$\sum_{i\in I} v_i^2 \|\pi_{\widetilde{W}_i} x\|^2 \ge \sum_{i\in I} v_i^2 \sum_{j\in J_i\setminus L_i} |\langle \pi_{W_i} x, f_{ij}\rangle|^2 \ge \sum_{i\in I} A_i v_i^2 \|\pi_{W_i} x\|^2$$

$$\ge \left(\min_{i\in I}\ A_i\right) \sum_{i\in I} v_i^2 \|\pi_{W_i} x\|^2 \ge \left(\min_{i\in I}\ A_i\right) C\, \|x\|^2.$$

Furthermore, $D$ is obviously still an upper bound. □

This result – as Theorem 3.2 – can be used to explicitly construct fusion frames robust to a specified number of erasures. For this, first the subspaces have to be defined. Then for each subspace we choose a Parseval frame which is robust to a prescribed number of erasures. Here we might use [**7**] to even choose these Parseval frames to be equal-norm. Then the theorem provides us with fusion frame bounds for the fusion frame after the erasures happened.

**4.2. Optimal Fusion Frames for Erasures of Local Frame Vectors.** Here we are interested in fusion frame systems, which are optimally robust with respect to the erasure of one local vector. As in Section 3.2, we restrict our analysis to finite Parseval fusion frames and to local Parseval frames.

First we make the meaning of optimality precise. Again we are interested in the worst case behavior of the reconstruction error.

For this, let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ be a Parseval fusion frame system with local Parseval frames. Let $T$ denote the analysis operator of the associated fusion frame, $T_i$ the analysis operator for the local frames for all $1 \le i \le n$, and define a vector of matrices $(D_1, \ldots, D_n) \in \Pi_{i=1}^n M(m_i \times m_i, \mathbb{R})$ in such a way that there exists one $i_0 \in \{1, \ldots, n\}$ so that $D_{i_0} = (d_{k,l})_{1\le k,l\le n}$ with $d_{k,l} = \delta_{k,j_0}\delta_{l,j_0}$ for some $j_0 \in \{1, \ldots, m_{i_0}\}$ and all other matrices are zero-matrices. This simulates the erasure of the vector $f_{i_0 j_0}$. We denote the set of admissible vectors of matrices by $\mathcal{D}$.

Definition 4.2. Let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ be a Parseval fusion frame system with local Parseval frames. Let $T$ denote the analysis operator of the associated fusion frame, and $T_i$ the analysis operator for the local frames for all $1 \le i \le n$.

Then we define the associated 1-*erasure reconstruction error* $\mathcal{E}_1(\mathcal{W})$ to be

$$\mathcal{E}_1(\mathcal{W}) = \max\{\|\sum_{i=1}^{n} v_i^2 T_i^* D_i T_i\| : (D_1, \ldots, D_n) \in \mathcal{D}\}.$$

The following result gives a characterization of all Parseval fusion frames with prescribed number of local frame vectors and prescribed – not necessarily equal – dimensions of the subspaces which behave optimal under the erasure of one local frame vector.

THEOREM 4.3. *Let* $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ *be a Parseval fusion frame system with local Parseval frames* $\{f_{ij}\}_{j=1}^{m_i}$. *Then the following conditions are equivalent.*

(i) *The Parseval fusion frame system* $\mathcal{W}$ *satisfies* $\mathcal{E}_1(\mathcal{W}) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ *is a Parseval fusion frame system with local Parseval frames satisfying* $\dim \widetilde{W}_i = \dim W_i$ *for all* $1 \le i \le n.\}$.

(ii) *We have*

$$\|f_{ij}\|^2 = \frac{\dim W_i}{m_i} \quad \textit{for all } i \in I,\ j \in J_i.$$

*Moreover, let* $\hat{x}$ *be the reconstructed vector* $x \in \mathcal{H}$ *under one erasure using the original reconstruction formula. Then we have the following error bound*

$$\|x - \hat{x}\| \le \frac{\max\{\dim W_i : 1 \le i \le n\}}{\min\{m_i : 1 \le i \le n\}} \|x\| \quad \textit{for all } x \in \mathcal{H}.$$

PROOF. Let $T_i$ denote the analysis operator for $\{f_{ij}\}_{j=1}^{m_i}$, $i = 1, \ldots, n$. Fix the index of the subspace in which a local frame vector will be deleted and denote it by $i_0$. Further, let $j_0 \in \{1, \ldots, m_{i_0}\}$ denote the index of the vector being deleted. Then, for each $x \in \mathcal{H}$,

$$\sum_{i=1}^{n} v_i^2 T_i^* D_i T_i x = \sum_{i=1}^{n} v_i^2 T_i^* (\delta_{i,i_0} \delta_{j,j_0} \langle x, f_{ij} \rangle)_{j=1}^{m_i} = v_{i_0}^2 \langle x, f_{i_0 j_0} \rangle f_{i_0 j_0}.$$

Hence

$$\begin{aligned} \|\sum_{i=1}^{n} v_i^2 T_i^* D_i T_i\| &= \sup_{\|x\|=1} \|v_{i_0}^2 \langle x, f_{i_0 j_0} \rangle f_{i_0 j_0}\| \\ &= v_{i_0}^2 \|f_{i_0 j_0}\| \sup_{\|x\|=1} |\langle x, f_{i_0 j_0} \rangle| \\ &= v_{i_0}^2 \|f_{i_0 j_0}\|^2. \end{aligned}$$

Thus for $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ we have that

$$\mathcal{E}_1(\mathcal{W}) = \max\{v_i^2 \|f_{ij}\|^2 : 1 \le i \le n,\ 1 \le j \le m_i\}.$$

By [**7**, Sec. 2.3], we obtain

$$\sum_{j=1}^{m_i} \|f_{ij}\|^2 = \dim W_i \quad \text{for all } 1 \le i \le n,$$

which implies that there exists some $i \in \{1, \ldots, n\}$ with $\|f_{ij}\|^2 \ge \frac{\dim W_i}{m_i}$ for all $j$. Since the dimensions as well as the number of local frame vectors are fixed, we can conclude that $\mathcal{E}_1(\mathcal{W})$ is minimal if and only if

$$\|f_{ij}\|^2 = \frac{\dim W_i}{m_i} \quad \text{for all } i \in I,\ j \in J_i.$$

The moreover-part follows immediately from the arguments above, since the maximal error is bounded by $\mathcal{E}_1(\mathcal{W})\,\|x\|$. □

For varying dimensions of the subspaces, we obtain the following result which can be derived from Theorem 4.3 and its proof.

Corollary 4.4. *Let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ be a Parseval fusion frame system with local Parseval frames $\{f_{ij}\}_{j=1}^{m_i}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame system $\mathcal{W}$ satisfies $\mathcal{E}_1(\mathcal{W}) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ is a Parseval fusion frame system with local Parseval frames.$\}$.*

(ii) *We have*

$$\dim W_i = 1 \text{ and } \|f_{ij}\|^2 = \frac{1}{m_i} \quad \text{for all } i \in I,\ j \in J_i. \tag{4.1}$$

*Moreover, let $\hat{x}$ be the reconstructed vector $x \in \mathcal{H}$ under one erasure using the original reconstruction formula. Then we have the following error bound*

$$\|x - \hat{x}\| \le \frac{1}{\min\{m_i : 1 \le i \le n\}} \|x\| \quad \text{for all } x \in \mathcal{H}. \tag{4.2}$$

As can easily be seen from (4.2), an increase in the number of local frames in each subspace improves the error bound for the reconstruction. Therefore, provided we allow infinitely many local frame vectors, according to (4.1) the optimal norms would equal zero, which implies that the question of optimal fusion frame systems under erasures does not make much sense allowing infinitely many local frame vectors. In particular it is the wrong question to ask in an infinite-dimensional Hilbert space.

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