ShearLab 3D

Manual

v1.0

August the 29th, 2013

Contents

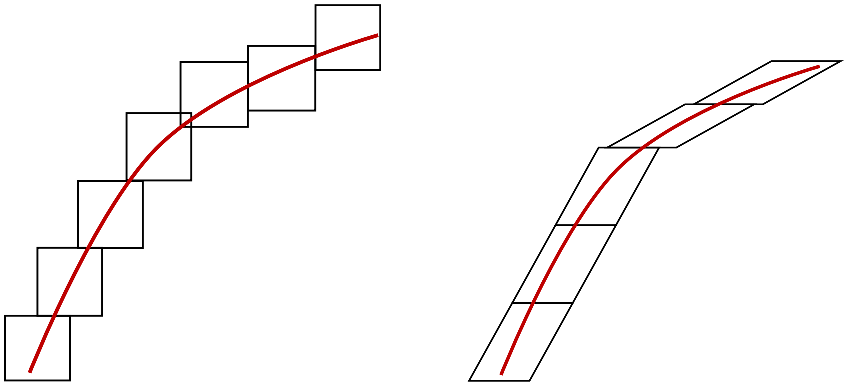


Figure 1: Anisotropically scaled and sheared atoms efficiently cover curve-like singularities.

1 Introduction

ShearLab 3D is a MATLAB Library, developed for processing two- and three-dimensional signals using a certain class of basis functions named shearlets. These functions were first described in 2005 by Labate, Lim, Kutnyiok and Weiss [10] with the goal of constructing systems of basis-functions nicely suited for representing anisotropic features (e.g. curvilinear singularities) that often are present in multivariate data.

This manual aims to give a short overview of the usage of ShearLab 3D and the concepts behind it. Section 2 provides a brief introduction to the general theory of shearlets, while chapter 3 is devoted to the discrete nonseparable shearlet transform (DNST), the main algorithm used in ShearLab 3D. Finally, chapters 4 and 5 explain the finer implementational details and contain a few sample scripts.

2 Shearlets - A Brief Overview

It is a well established fact that wavelets provide optimally sparse representations for 1-D functions that are smooth away from point singularities. In higher dimensions, however, this optimality can not be retained, as wavelets, due to their isotropic nature, are not ideally suited for covering anisotropic features such as curve-like singularities (see figure 1).

Shearlets have been constructed with the aim of improving on this shortcoming by applying anisotropic scaling to the generating function (i.e. different dimensions can have different scaling factors). Naturally, introducing directional selectivity to a system of representing functions demands the capability of varying the direction. In the theory of shearlets, this is achieved by applying a so-called shearing operator along with the (anisotropic) scaling operator. In short, shearlets are very similar to wavelets in the sense that both are constructed from generating functions that can be modified with a

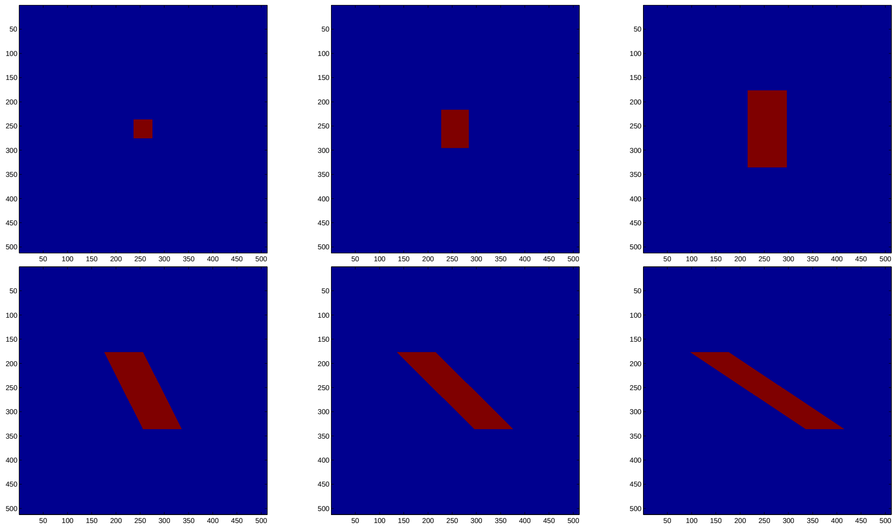


Figure 2: A square is (parabolically) scaled twice and sheared three times.

certain operator. But while for wavelets, only isotropic scaling is possible, shearlets are subject to both anisotropic scaling and shearing (see figure 2).

2.1 Continuous Shearlet System

Let

$$A_a = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}, \qquad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \tag{1}$$

be a (parabolic) scaling matrix and a shearing matrix with $a > 0$, then, assuming the existence of a generating function $\psi \in L_2(\mathbb{R}^2)$, a two-dimensional continuous shearlet system

$$\mathrm{SH}_{\mathrm{cont}}(\psi) = \{\psi_{a,s,t} = a^{3/4}\psi(A_a^{-1}S_s^{-1}(\cdot - t)) \mid a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2\}, \tag{2}$$

and the associated shearlet transform

$$f \mapsto \mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{a,s,t} \rangle \tag{3}$$

with $f \in L^2(\mathbb{R}^2)$ and $(a,s,t) \in \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}^2$ can be defined. So $\mathcal{SH}_\psi$ maps a function $f \in L_2(\mathbb{R}^2)$ to a set of coefficients where each coefficient is indexed by a scaling parameter a, a shearing parameter s and a translation parameter t.

2.2 Cone-Adapted Shearlet System

One look at figure 2 reveals a significant problem of the shearlet system just introduced. In order to capture horizontally aligned anisotropic structures, one would have to apply the shearlet matrix a great number of times (leading to shearlets that are almost

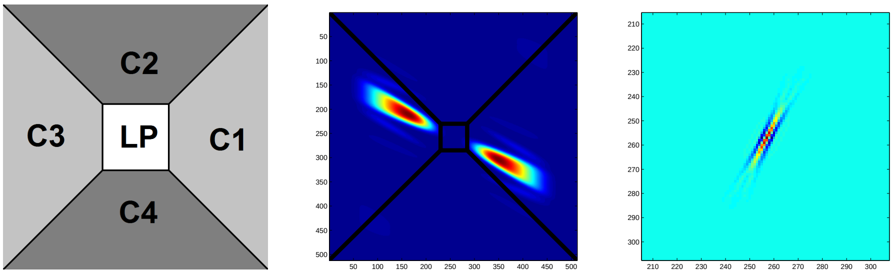


Figure 3: The leftmost image shows the tiling of the frequency-domain used in the cone-adapted shearlet systems. The picture in the center shows the magnitude response of a shearlet within the horizontal cones of the frequency domain while the image to the right shows the same shearlet in the time-domain.

horizontally aligned but also very alongated). To avoid this issue, which would be very difficult to handle in any digital implementation, the so-called cone-adapted continuous shearlet system was introduced. Here, the Fourier-domain is partitioned into four cones (two horizontal, two vertical) and a square-shaped low-pass region (see figure 3). The horizontal and vertical cones are now associated with their own generating functions $\psi, \tilde{\psi}$ and additionally, we introduce a scaling function ϕ, covering the low-pass region.

Let

$$\tilde{A}_a = \begin{pmatrix} a^{1/2} & 0 \\ 0 & a \end{pmatrix} \tag{4}$$

be another scaling matrix, shearlet generators $\psi, \tilde{\psi} \in L_2(\mathbb{R}^2)$ and a scaling function $\phi \in L_2(\mathbb{R}^2)$ be given. The cone-adapted continuous shearlet system $\mathcal{SH}_{cont}(\phi, \psi, \tilde{\psi})$ is then given by the union of the following sets:

$$\begin{aligned}
\Phi &= \left\{\phi_t = \phi(\cdot - t) : t \in \mathbb{R}^2\right\} \\
\Psi &= \left\{\psi_{a,s,t} = a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(\cdot - t)) : a \in (0,1], |s| \le 1 + a^{1/2}, t \in \mathbb{R}^2\right\} \\
\tilde{\Psi} &= \left\{\tilde{\psi}_{a,s,t} = a^{-\frac{3}{4}}\tilde{\psi}(\tilde{A}_a^{-1}S_s^{-T}(\cdot - t)) : a \in (0,1], |s| \le 1 + a^{1/2}, t \in \mathbb{R}^2\right\}.
\end{aligned}$$

The respective transform can be defined analogous to (3).

2.3 Discrete Shearlet Transform

A discrete cone-adapted shearlet system can now be defined as a countable subset of the continuous cone-adapted shearlet system.

Let $\phi \in L_2(\mathbb{R}^2)$ be a scaling function, $\psi, \tilde{\psi} \in L_2(\mathbb{R}^2)$ be shearlet generators and $c = (c_1, c_2) \in \mathbb{R}^2_{>0}$ be sampling constants then the regular cone-adapted discrete shearlet system $\mathcal{SH}(\phi,\psi,\tilde{\psi},c)$ is defined by the union of the following sets:

$$\Phi = \left\{\phi_m = \phi(\cdot - c_1 m) : m \in \mathbb{Z}^2\right\},$$
$$\Psi = \left\{\psi_{j,k,m} = 2^{\frac{3}{4}j}\psi(S_k A_{2^j}\cdot - M_c m) : j \geq 0\, |k| \leq \lceil 2^{j/2}\rceil, m \in \mathbb{Z}^2\right\},$$
$$\tilde{\Psi} = \left\{\tilde{\psi}_{j,k,m} = 2^{\frac{3}{4}j}\tilde{\psi}(S_k^T \tilde{A}_{2^j}\cdot - \tilde{M}_c m) : j \geq 0\, |k| \leq \lceil 2^{j/2}\rceil, m \in \mathbb{Z}^2\right\}$$

with $A_{2^j} = \mathrm{diag}(2^j,(2^j)^{1/2})$,$\tilde{A}_{2^j} = \mathrm{diag}((2^j)^{1/2},2^j)$, $M_c = \mathrm{diag}(c_1,c_2)$ and $\tilde{M}_c = \mathrm{diag}(c_2,c_1)$.

Note that the set Φ is associated with the low-frequency region, the Ψ with the horizontal cones and $\tilde{\Psi}$ with the vertical cones. Also note that this definition can be extended to α-shearlets (these shearlets are not restricted to parabolic scaling and use factors $2^{\alpha j}$ instead of 2^j), introduced by Grohs, Keiper, Kutyniok and Schaefer in 2013 [3], which can also be used in ShearLab 3D.

A shearlet transform can again be defined analogues to (3).

2.4 Frame Property

A set of functions $\Psi \subset L_2(\mathbb{R}^2)$ fulfilling

$$A\|f\|^2 \leq \sum_{\psi\in\Psi} |\langle f,\psi\rangle|^2 \leq B\|f\|^2 \tag{5}$$

with certain constants $A, B > 0$ (independent of f) for all $f \in L_2(\mathbb{R}^2)$ is said to be a frame of $L_2(\mathbb{R}^2)$. If $A = B$, Ψ is called a tight frame whereas a tight frame with $A = B = 1$ is called a Parseval frame. Note that every orthonormal basis of $L_2(\mathbb{R}^2)$ is necessarily a Parseval frame and that the notion of a frame can be considered a generalization of the concept of orthonormal basis for redundant systems.

It has been shown in [4] that one can find band-limited shearlet generators $\psi,\tilde{\psi}$ and a shearlet scaling function ϕ such that the discrete cone-adapted shearlet system defines a Parseval frame of $L_2(\mathbb{R}^2)$. Also, it has been shown in [5] that there exist compactly supported shearlet generators $\psi,\tilde{\psi}$ such that the discrete cone-adapted shearlet system forms a frame in $L_2(\mathbb{R}^2)$. Whether there also exists a tight frame constructed from compactly supportet shearlets is still an open question.

2.5 Sparse Approximation sing Shearlets

In the spirit of orthonormal bases, each frame Ψ in $L_2(\mathbb{R}^2)$ is associated with the frame operator

$$S : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2) : f \mapsto \sum_{\psi\in\Psi} \langle f,\psi\rangle\psi. \tag{6}$$

which is positive, self-adjoint and invertible.

Clearly, any $f \in L_2(\mathbb{R}^2)$ can now be written as

$$f = \sum_{\psi \in \Psi} \langle f, \psi \rangle S^{-1} \psi \tag{7}$$

where S^{-1} is the dual frame operator and the sequence $(S^{-1}\psi)_{\psi \in \Psi}$ is called the canonical dual frame.

Now, one can define the best n-term approximation of $f \in L_2(\mathbb{R}^2)$ to be

$$f_n = \sum_{i \in I} \langle f, \psi_i \rangle S^{-1} \psi_i \tag{8}$$

where I is the index-set associated with the n largest shearlet coefficients $\langle f, \psi_i \rangle$ and (of course, $|I| = n$).

It was shown in [9] that there exist compactly supported shearlet generators $\psi, \tilde{\psi}$ and a shearlet scaling function ϕ such that the best n-term approximation of functions f in the class of cartoon like images (a subset of $L_2(\mathbb{R}^2)$ defined by functions of the form $f = f_0 + f_1 \chi_B$ where $f_0, f_1 \in C^2(\mathbb{R}^2)$ and δB is a closed C^2 curve) obeys

$$\|f - f_n\|^2 = O(n^{-2} (\log n)^3) \tag{9}$$

which is (besides the log factor) the optimal decay rate achievable. This also gives a mathematical justification of the superiority of shearlet systems over wavelet or Fourier bases which cannot guarantee optimal decay rates (see [9]).

2.6 References

For a extensive introduction to the theory of shearlets, see the book published by Gitta Kutyniok and Demetrio Labate in 2012 [6] and the articles referenced at the and of this manual.

3 Digital Shearlet Transform

ShearLab 3D implements a cone (in the 3D case: pyramid) adapted discrete shearlet system based on compactly supported shearlets (see [8]). The specific algorithm was published in 2013 by Wang-Q Lim [11] and will be quickly summarized here.

3.1 2D Data

The goal of the algorithm is to compute shearlet coefficients

$$\langle \psi^{2D}_{j,k,m}, f \rangle \tag{10}$$

where $\psi^{2D}_{j,k,m}$ is a two-dimensional digital shearlet from the cone-adapted discrete shearlet system indexed by a scale paramter j, a shearing paramter k and a translation parameter

m and $f \in \ell_2(\mathbb{Z}^2)$.

We use a nonseparable shearlet generator

$$\hat{\psi}(\xi) = P(\xi_1/2, \xi_2)\widehat{\psi_1 \otimes \phi_1}(\xi) \tag{11}$$

where P is a 2D directional filter, ϕ_1 is a 1D scaling function and ψ_1 is the wavelet function of the corresponding multiresolution analysis.

Because of

$$\psi_{j,k,m}(\cdot) = \psi_{j,0,m}(S_{k/2^{j/2}}\cdot), \tag{12}$$

we need two ingredients for the digitization of shearlets $\psi_{j,k,m}$: Digitial shearlet filters $\psi^{2D}_{j,0}$ and a digital shear operator $S^d_{k/2^{j/2}}$.

Let $J \in \mathbb{N}$ be highest scale to be considered (i.e. $j < J$ for all shearlets $\psi_{j,k,m}$), then the digital shearlet filter $\psi^{2D}_{j,0}$ is given by:

$$\psi^{2D}_{j,0} = p_j * \left(g_{J-j} \otimes h_{J-j/2}\right) \tag{13}$$

where p_j are the Fourier coefficients of $P(2^{J-j-1}\xi_1, 2^{J-j/2}\xi_2)$, h_{J-j} is a low-pass filter associated with the scaling function ϕ_1 and g_{J-j} is the corresponding high-pass filter, associated with the wavelet function ψ_1.

Furthermore, we define the digital shear operator $S^d_{k/2^{j/2}}$ to be

$$S^d_{k/2^{j/2}}(x) = \left(\left(x_{\uparrow 2^{j/2}} *_1 h_{j/2}\right)(S_k\cdot) *_1 \overline{h_{j/2}}\right)_{\downarrow 2^{j/2}} \tag{14}$$

with $x \in \ell^2(\mathbb{Z}^2)$ and the digital shearlet filter

$$\psi^{2D}_{j,k} = S^d_{k/2^{j/2}}\psi^{2D}_{j,0}. \tag{15}$$

The 2D digital shearlet transform for a signal $f \in \ell^2(\mathbb{Z}^2)$ is now given by:

$$DST^{2D}_{j,k,m}(f) = (\overline{\psi^{2D}_{j,k}} * f)(m) \tag{16}$$

for $j \in \{0, J-1\}$ and $|k| < \lceil 2^{j(\alpha-1)/2} \rceil$.

3.2 3D Data

Each 3D digital shearlet filter can be constructed by combining two 2D digital shearlet filters in the frequency domain:

$$\hat{\psi}^{3D}_{j,k}(\xi) = \hat{\psi}^{2D}_{j,k_1}(\xi_1, \xi_2)\hat{\psi}^{2D}_{j,k_2}(\xi_1, \xi_3) \tag{17}$$

The 3D digital shearlet transform for a signal $f \in \ell^2(\mathbb{Z}^3)$ is thus given by:

$$DST^{3D}_{j,k,m}(f) = (\overline{\psi^{3D}_{j,k}} * f)(m) \tag{18}$$

where $j \in \{0, J-1\}$ and $|k| < \lceil 2^{j(\alpha-1)/2} \rceil$.

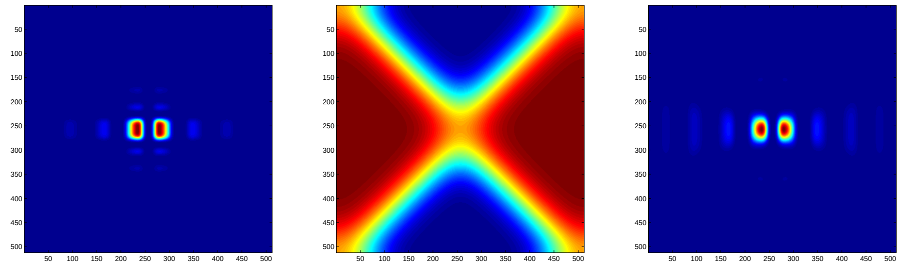


Figure 4: Magnitude responses in the frequency domain of the separable shearlet generator $g_J \otimes h_J$ (left), the 2D fan filter P (middle) and the nonseparable shearlet generator $\psi^{2D}_{0,0}$ with $J = 4$.

3.3 Inverse Digital Shearlet Transform

The dual digital shearlet filters $\tilde{\psi}^{2D}_{j,k}$ and $\tilde{\psi}^{3D}_{j,k}$ are given by:

$$\hat{\tilde{\psi}}^{2D}_{\tilde{j},\tilde{k}}(\xi) = \frac{\hat{\psi}^{2D}_{\tilde{j},\tilde{k}}(\xi)}{\sum_{j,k} \left|\hat{\psi}^{2D}_{j,k}\right|^2}, \qquad \hat{\tilde{\psi}}^{3D}_{\tilde{j},\tilde{k}}(\xi) = \frac{\hat{\psi}^{3D}_{\tilde{j},\tilde{k}}(\xi)}{\sum_{j,k} \left|\hat{\psi}^{3D}_{j,k}\right|^2}. \tag{19}$$

4 ShearLab 3D

4.1 Implementation of the Digital Shearlet Transform

ShearLab3D provides an implementation of the digital transform summarized in chapter 3. Hence, it can be used to compute the digital shearlet transform of arbitrarily sized two- and three dimensional signals according to formulas (16), (18) as well as the inverse shearlet transform (19).

Applying the convolution theorem, formulas (16), (18) and (19) can be computed by multiplying conjugated digital shearlet filters $\overline{\psi^{2D}_{j,k}}$, their duals $\tilde{\psi}^{2D}_{j,k}$ and the given signal f_J in the frequency domain. This means that both the decomposition and the reconstruction algorithm reduce to multiple computations of the fast fourier transform. Thus, their complexity is given by $\mathcal{O}(R \cdot N \log(N))$, where $R \in \mathbb{N}$ is the redundancy of the specific digital shearlet system (i.e. the number of filters $\psi^{2D}_{j,k}$).

Following equation (13), the construction of a 2D digital shearlet filter $\psi^{2D}_{j,k}$, requieres a 1D lowpass filter h_1 and a 2D directional filter P. The 1D filter h_1 defines a wavelet multiresolution analysis (and thereby the highpass filter g_1) whereas the trigonometric polynomial P will be used to ensure the wedge shape of the essential frequency support of $\psi^{2D}_{j,k}$ depicted in figure 3.1. Please note that the proper choice of these filters is essential for generating an optimal digital shearlet system as it will influence important properties like frame bounds and directional selectivity.

Our choice for h_1, from now on denoted by $h_{ShearLab}$[1] (see figure 5), is a maximally flat (i.e. a maximum number of derivatives of the magnitude frequency response at 0 and π vanish) and symmetric 9-tap lowpass filter that is normalized such that $\sum_n h_{ShearLab}(n) = 1$. It has two vanishing moments (i.e. $\int \phi(x)x^k = 0$ for $k \in \{0, 1\}$) and while there is no symmetric, compactly supported and orthogonal wavelet besides the Haar wavelet, the renormalized filter $\sqrt{2}h_{ShearLab}$ at least approximately fulfils the orthonormality condition, that is

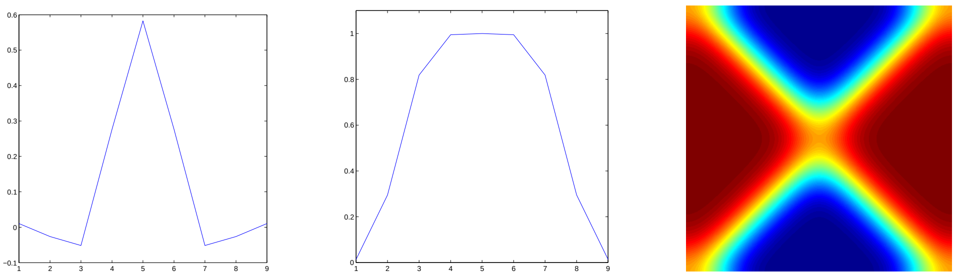
$$\left| 2 \sum_n h_{ShearLab}(n) h_{ShearLab}(n + 2l) - \delta_{l0} \right| \leq 0.0018$$

for all $\in \mathbb{Z}$ with δ denoting Kronecker's delta. By choosing $h_{ShearLab}$ to be maximally flat, the amount of ripple in the digital filter $\psi^{2D}_{j,k}$ is significantly reduced. This leads to a better localiziation of the shearlets in the frequency domain. The highpass filter g_1, hereafter denoted by $g_{ShearLab}$, is of course chosen to be the associated mirror filter, that is

$$g_{ShearLab}(n) = (-1)^n \cdot h_{ShearLab}(n).$$

Please note that the filter coefficients $h_{ShearLab}$ are quite similar to those of the Cohen-Daubechies-Feauveau 9/7 wavelet [1], famously used in the JPEG 2000 standard. While the CDF 9/7 wavelet has four vanishing moments and higher degrees of regularity both in the Hölder and Sobolev sense, trading these advantageous properties for maximal flatnass seems to be the optimal choice for most applications.

For the trigonometric polynomial $P_{ShearLab}$[2] (see figure 5), we use the maximally flat 2D fan filter described in [2].



(a) (b) (c)

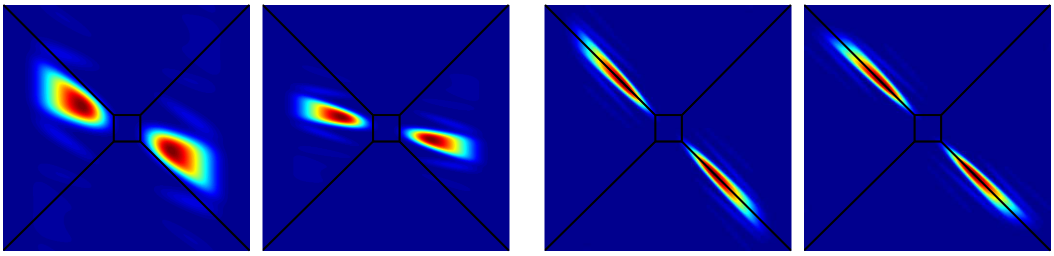
Figure 5: (a) The coefficients of the 1D lowpass filter $h_{ShearLab}$. (b) Magnitude frequency response of $h_{ShearLab}$. (c) Magnitude response of the 2D fan filter $P_{ShearLab}$.

In the original definition of the cone-adapted discrete shearlet system [7, p. 26], the parabolic scaling matrix was defined as $A_{2^j} = \mathrm{diag}(2^j, 2^{j/2})$ and the shearing parameter

[1]The 9-tap filter $h_{ShearLab}$ can be generated in MATLAB with `design(fdesign.lowpass('N,F3dB',8,0.5),'maxflat');`. The approximate values are $h_{ShearLab} = (0.01049, -0.02635, -0.05178, 0.27635, 0.58257, ...)$.

[2]The 2D fan filter $P_{ShearLab}$ can be obtained in MATLAB using the Nonsubsampled Contourlet Toolbox by the statement `fftshift(fft2(modulate2(dfilters('dmaxflat4','d')./sqrt(2),'c')));`

$k \in \mathbb{Z}$ ranged from $-\lceil 2^{j/2} \rceil$ to $\lceil 2^{j/2} \rceil$ in each cone for each scale $j \in \mathbb{N}$. However, this definition was generalized in [3] using a factor $\alpha \in (1, \infty)$ such that $A_{2^j} = \mathrm{diag}(2^j, 2^{\alpha j/2})$ and $|k| \leq \lceil 2^{(\alpha-1)j/2} \rceil$ in each cone and for each scale $j \in \mathbb{N}$. ShearLab 3D supports this generalization by allowing the user to specify the exact number of shearings (in powers of 2) occuring on each scale of a shearlet system[3]. Note that an increased number of differently oriented shearlets on a certain scale effectively leads to a finer partitioning of the frequency domain (see figure 6).



(a) (b) (c) (d)

Figure 6: Plots (a) and (b) show the magnitude frequency response of a shearlet in the horizontal frequency cones on the third scale of a four-scale system. Each shearlet was sheared once but the system in (a) contains only five differently oriented shearlets in the horizontal cones, while the system in (b) has nine differently sheared horizontal shearlets. The magnitude frequency response of the maximally sheared shearlet in the vertical cones (d) is almost equal to the response of the corresponding shearlet in the horizontal cones (c). In most cases, the digital shearlet filter (d) can be omitted to decrease the redundancy of a shearlet system.

4.2 How to Use ShearLab 3D

In order to use ShearLab 3D, just download the packge from www.shearlab.org and add anything in it to your MATLAB path.

Please note that ShearLab 3D requires the Signal Processing Toolbox and the Image Processing Toolbox. If additionally the Parallel Computing Toolbox is available, CUDA-compatibale NVidia graphics cards can be used to gain a significant speed up.

4.2.1 Compute a Shearlet Decomposition

In order to compute the shearlet decomposition of arbitrariliy sized two- or three-dimensional data, you just have to construct a shearlet system and then call the *SLsheardec* method:

[3] e.g.: For a three-scale 2D shearlet system, the vector `shearLevels = [0 1 1]` means that $2^0 = 1$ shearings occur in both directions in both cones on the first scale and $2^1 = 2$ shearings occur in both directions in both conse on the second and third scale. Including the unsheared shearlets, this configuration defines a system of $2 \cdot ((2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 2 + 1)) = 26$ different shearlets. The default value of `shearLevels` in ShearLab 3D is `ceil((1:nScales)/2);`

```
% 2D
sizeX = 500;
sizeY = 300;
useGPU = 0;

data = randn(sizeX,sizeY);

system = SLgetShearletSystem2D(useGPU,sizeX,sizeY,4);

shearletCoefficients = SLsheardec2D(data,system);

% 3D

sizeX = 70;
sizeY = 71;
sizeZ = 68;

useGPU = 0;

data = randn(sizeX,sizeY,sizeZ);

system = SLgetShearletSystem3D(useGPU,sizeX,sizeY,sizeZ,1);

shearletCoefficients = SLsheardec3D(data,system);
```

The array *shearletCoefficients* is three dimensional in the 2D and four dimensional in the 3D case. In both cases, the last dimension enumerates all shearlets within the specified system with different shearing parameters k and scaling parameters j while the first two or three dimensions are associated with the translates of one single shearlet (see figure 8).

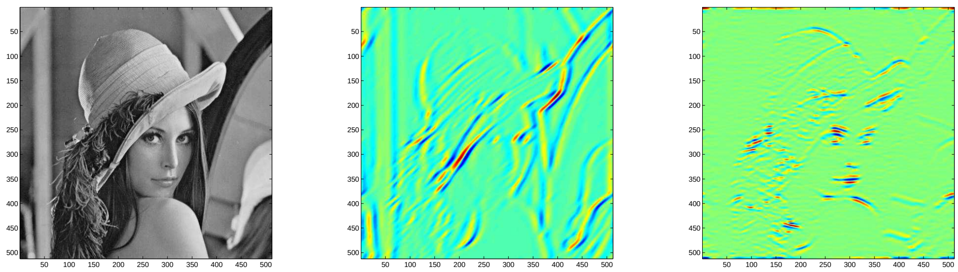


Figure 7: The two images to the right show all shearlet coefficients of the translates of two different shearlets. The used system has four scales, a redundancy of 49 and was specified with $shearLevels = [1, 1, 2, 2]$. The shearlet corresponding to the coefficients in the centered picture has a scale parameter $j = 1$ a shearing parameter $k = 2$ and lives on the horizontal frequency cones. The shearlet corresponding to the coefficients plotted in the rightmost image has a scale parameter $j = 2$, a shearing parameter $k = 0$ and lives on the vertical frequency cones.

4.2.2 Serial Decomposition for Large Data

Sometimes, especially in the three-dimensional case, the data to be analysed is so large that it's impossible to keep all shearlet coefficients in memory at the same time. In this case, it is also possible to compute the shearlet decomposition in a serialized manner. Then, only the coefficients associated to the translates of one single shearlet are available at one point in time.

```
% 2D
sizeX = 1024;
sizeY = 1024;
data = randn(sizeX,sizeY);

% prepare serial processing
[Xfreq, Xrec, preparedFilters, dualFrameWeightsCurr, shearletIdxs] = ...
SLprepareSerial2D(useGPU,data,4);


for j = 1:size(shearletIdxs,1)
    shearletIdx = shearletIdxs(j,:);

    %shearlet decomposition
    [coefficients,shearlet, dualFrameWeightsCurr,RMS] = ...
    SLsheardecSerial2D(Xfreq,shearletIdx,preparedFilters,dualFrameWeightsCurr);

    % add processing here
end

% 3D
sizeX = 192;
sizeY = 192;
sizeZ = 192;

data = randn(sizeX,sizeY,sizeZ);

% prepare serial processing
[Xfreq, Xrec, preparedFilters, dualFrameWeightsCurr, shearletIdxs] = ...
SLprepareSerial3D(useGPU,data,2);


for j = 1:size(shearletIdxs,1)
    shearletIdx = shearletIdxs(j,:);

    %shearlet decomposition
    [coefficients,shearlet, dualFrameWeightsCurr,RMS] = ...
    SLsheardecSerial3D(Xfreq,shearletIdx,preparedFilters,dualFrameWeightsCurr);

    % add processing here
end
```

4.2.3 Compute the Reconstruction

If shearlet coefficients and the corresponding system are present, it is possible to compute a reconstruction. This can be done by simply calling the *SLshearrec* method.

```
% 2D

reconstruction = SLshearrec2D(coefficients,system);

% 3D

reconstruction = SLshearrec3D(coefficients,system);
```

4.2.4 Serial Reconstruction for Large Data

Of course, data can also be constructed from the shearlet coefficients during serial processing.

```
% 2D

for j = 1:size(shearletIdxs,1)
    shearletIdx = shearletIdxs(j,:);

    %shearlet decomposition
    [coefficients,shearlet, dualFrameWeightsCurr,RMS] = ...
    SLsheardecSerial2D(Xfreq,shearletIdx,preparedFilters,dualFrameWeightsCurr);

    % add processing here

        Xrec = SLshearrecSerial2D(coefficients,shearlet,Xrec);
end

reconstruction = SLfinishSerial2D(Xrec,dualFrameWeightsCurr);

% 3D

for j = 1:size(shearletIdxs,1)
    shearletIdx = shearletIdxs(j,:);

    %shearlet decomposition
    [coefficients,shearlet, dualFrameWeightsCurr,RMS] = ...
    SLsheardecSerial3D(Xfreq,shearletIdx,preparedFilters,dualFrameWeightsCurr);

    % add processing here

        Xrec = SLshearrecSerial3D(coefficients,shearlet,Xrec);
end

reconstruction = SLfinishSerial3D(Xrec,dualFrameWeightsCurr);
```

5 Examples

5.1 Image Denoising

The following code shows how an image distorted with Gaussian white noise can be denoised using the digital shearlet transform and hard thresholding.

```
%%settings
sigma = 30;
scales = 4;
thresholdingFactors = [0 3 3 4 4];

%%load data
X = double(imread('barbara.jpg'));

%%add noise
Xnoisy = X + sigma*randn(size(X));

%%create shearlets
shearletSystem = SLgetShearletSystem2D(0,size(X,1),size(X,2),scales);

%%decomposition
coeffs = SLsheardec2D(Xnoisy,shearletSystem);
```

```
%%thresholding
for j = 1:shearletSystem.nShearlets
    shearletIdx = shearletSystem.shearletIdxs(j,:);
    coeffs(:,:,j) = coeffs(:,:,j).*(abs(coeffs(:,:,j)) > ...
    thresholdingFactors(shearletIdx(2)+1)*shearletSystem.RMS(j)*sigma);
end

%%reconstrsuction
Xrec = SLshearrec2D(coeffs,shearletSystem);

%%compute psnr
PSNR = SLcomputePSNR(X,Xrec);
```

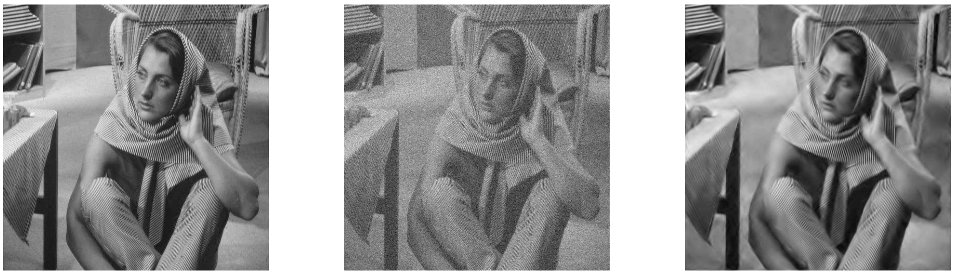


Figure 8: Image denoising using ShearLab 3D. The grayscale image (values ranging from 0 to 255) was distorted with Gaussian white noise with $\sigma = 30$ and denoised using hard tresholding on the shearlet coefficients.

5.2 Video Denoising

The following code shows how a video distorted with Gaussian white noise can be denoised using the digital shearlet transform and hard thresholding.

```
%%settings
sigma = 30;
scales = 3;
shearLevels = [0 0 1];
thresholdingFactors = [3 3 3 4];
fullSystem = 0;
useGPU = 0;

%%load data
load 'missamericaseq';
X = double(X);

%%add noise
Xnoisy = X + sigma*randn(size(X));

%%prepare serial processing
[Xfreq, Xrec, preparedFilters, dualFrameWeightsCurr, shearletIdxs] = ...
SLprepareSerial3D(useGPU,Xnoisy,scales,shearLevels,fullSystem);

for j = 1:size(shearletIdxs,1)
    shearletIdx = shearletIdxs(j,:);
```

```
    %%shearlet decomposition
    [coeffs,shearlet, dualFrameWeightsCurr,RMS] = ...
    SLsheardecSerial3D(Xfreq,shearletIdx,preparedFilters,dualFrameWeightsCurr);

    %%put processing of shearlet coefficients here, for example:
    %%hard thresholding
    coeffs = coeffs.*(abs(coeffs) > thresholdingFactors(shearletIdx(2)+1)*RMS*sigma);

    %%shearlet reconstruction
    Xrec = SLshearrecSerial3D(coeffs,shearlet,Xrec);
end

Xrec = SLfinishSerial3D(Xrec,dualFrameWeightsCurr);
%%compute psnr
PSNR = SLcomputePSNR(X,Xrec);
```

References

[1] Albert Cohen, Ingrid Daubechies, and Jean-Christophe Feauveau. Biorthogonal bases of compactly supported wavelets. Communications on Pure and Applied Mathematics, 45:1992, 1992.

[2] Arthur L. da Cunha, Jianping Zhou, and Minh N. Do. The nonsubsampled contourlet transform: Theory, design and applications. IEEE Trans. Image Proc., 15:3089–3101, 2006.

[3] Phlipp Grohs, Sandra Keiper, Gitta Kutyniok, and Martin Schäfer. Alpha molecules. preprint, 2013.

[4] Kanghui Guo, Gitta Kutyniok, and Demetrio Labate. Sparse multidimensional representations using anistropic dilation and shear oper. Wavelets and Splines (Athens, GA, 2005), pages 189–201, 2006.

[5] Pisamai Kittipoom, Gitta Kutyniok, and Wang-Q Lim. Construction of compactly supported shearlet frames. Constr. Approx., 35:21–72, 2012.

[6] Gitta Kutyniok and Demetrio Labate, editors. Shearlets Multiscale Analysis for Multivariate Data. Birkhäuser Boston, 2012.

[7] Gitta Kutyniok and Demetrio Labate. Shearlets: Multiscale Analysis for Multivariate Data, chapter Introduction to Shearlets, pages 1–38. Birkhäuser Boston, 2012.

[8] Gitta Kutyniok, Jakob Lemvig, and Wang-Q Lim. Optimally sparse approximations of 3d functions by compactly supported shearlet frames. SIAM J. Math. Anal., 44:2962–3017, 2012.

[9] Gitta Kutyniok and Wang-Q Lim. Compactly supported shearlets are optimally sparse. J. Approx. Theory, 163:1564–1589, 2011.

[10] Demetrio Labate, Wang-Q Lim, Gitta Kutyniok, and Guido Weiss. Sparse multidimensional representation using shearlets. In SPIE Proc., volume 5914, pages 254–262, 2005.

[11] Wang-Q Lim. Nonseparable shearlet transform. IEEE Trans. Image Proc., 2013. to appear.