Backward errors and pseudospectra for structured nonlinear eigenvalue problems †

Sk. Safique Ahmad‡ Volker Mehrmann*

December 27, 2013

Abstract

Minimal structured perturbations are constructed such that an approximate eigenpair of a nonlinear eigenvalue problem in homogeneous form is an exact eigenpair of an appropriately perturbed nonlinear matrix function. Structured and unstructured backward errors are compared. These results extend previous results for (structured) matrix polynomials to more general functions. Structured and unstructured pseudospectra for nonlinear eigenvalue problems are also discussed.

Keywords. nonlinear eigenvalue problem, backward error, symmetric/skew symmetric eigenvalue problem, Hermiitian/skew-Hermitian eigenvalue problem
AMS subject classification. 65F15, 15A18, 65F35, 15A12.

1 Introduction

In this paper we consider the problem of computing complex pairs $(c, s) \in \mathbb{C}^2 \setminus \{0\}$ with $|c|^2 + |s|^2 = 1$ and vectors $x \in \mathbb{C}^n$ such that the nonlinear matrix equation

$$\mathcal{M}(c, s)x = 0 \tag{1}$$

*Institut für Mathematik, Ma 4-5, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG; `email: mehrmann@math.tu-berlin.de`. Supported by Deutsche Forschungsgemeinschaft through Collaborative Research Center 910, *Control of self-organizing nonlinear systems: Theoretical methods and concepts of application.*

‡School of Mathematics, Indian Institute of Technology, Indore, IET, DAVV Campus, Khandwa Road, Indore-452017, India, `email: safique@iiti.ac.in`

holds, where the matrix valued function $\mathcal{M}$ has the form

$$\mathcal{M}(c,s) := \sum_{j=1}^{m} M_j f_j(c,s), \tag{2}$$

and where we assume that the functions $f_1(c,s), f_2(c,s), \dots, f_m(c,s)$ and the coefficient matrices $M_1, M_2, \dots, M_m \in \mathbb{C}^{n,n}$ are given data.

We call pairs (c_i, s_i) satisfying (1) *eigenvalues* and the associated vectors x_i *eigenvectors* of (1) and if $s_i \neq 0$ then we sometimes write $\lambda_i = \frac{c_i}{s_i}$. We also sometimes (for abbreviation and in abuse of notation) write $\mathcal{M}(c,s) = Mz$, where $M = [M_1, \dots, M_m]$ and $f = [f_1, \dots, f_m]$.

Nonlinear eigenvalue problems of the described form arise in many applications, see [38, 47] for surveys with a large number of applications and [10] for a collection of benchmark examples. Let us consider a few specific examples.

A nonrational eigenvalue problems of the form

$$\left(K + \imath\sqrt{\kappa^2 - \kappa_c^2}\, D - \kappa^2 M\right) x = 0$$

has been studied in [51]. Here κ is an unknown, κ_c is a fixed reference frequency, and K, M, D are large and sparse symmetric stiffness, mass and damping matrices, respectively. This problem can be turned into a polynomial eigenvalue problem by introducing $\lambda = \sqrt{\kappa^2 - \kappa_c^2}$.

In [13] a rational eigenvalue problem arising in the numerical solution of a fluid-structure interaction is introduced. It has the form

$$\left(\frac{\lambda^2}{a^2} M + K + \frac{\lambda^2}{\lambda\beta + \alpha} D\right) x = 0, \tag{3}$$

where a is the speed of sound in the given material, and α, β are positive constants. The matrices M, K are large sparse symmetric positive definite mass and stiffness matrices, respectively, and the symmetric positive semidefinite matrix D describes the effect of an absorbing wall. Clearing out the denominator in (3) leads to a cubic eigenvalue problem

$$(\lambda^3 \beta M + \lambda^2(\alpha M + a^2 A) + \lambda(a^2\beta K) + a^2\alpha K)x = 0. \tag{4}$$

Although the leading coefficient is positive definite and thus there are no infinite eigenvalues, in other acoustic problems, see e.g., [37], the mass matrix may be singular.

In the polynomial setting, in order to avoid some of the difficulties with infinite eigenvalues, one may use the homogeneous framework and study

$$(s^3\beta M + s^2c(\alpha M + a^2 A) + sc^2(a^2\beta K) + c^3a^2\alpha K)x = 0. \tag{5}$$

In the general nonlinear setting this may still not cure all the difficulties with infinite eigenvalues as the homogeneous version of (3) yields

$$\mathcal{M}(c,s)x = (\frac{s^2}{c^2a^2}M + K + \frac{s^2}{c(s\beta + c\alpha)}A)x = 0, \tag{6}$$

where $c = 0$ is still problematic.

Rational eigenvalue problems arising in the finite element simulation of mechanical problems, see [42, 49] for several applications, often have the form

$$\left(P(\lambda) + Q(\lambda)\sum_{i=1}^{\ell} \frac{\lambda}{\lambda - \sigma_i} E_i \right) x = 0, \tag{7}$$

where P and Q are real symmetric matrix polynomials (with usually large and sparse coefficients), and $E_i \in \mathbb{R}^{n,r_i}$ are low rank matrices for $i = 1, 2, \ldots, \ell$. Classical examples arising for P, Q are $P = \lambda A - B,\ Q = I$, $P = \lambda A - B, Q = \lambda^2 I$, or $P = \lambda^2 A + B,\ Q = I$ with $A, B \in \mathbb{R}^{n,n}$ being real symmetric and sparse and with different definiteness structure. It is again obvious that this problem can be turned into a high degree polynomial eigenvalue problem by clearing out the denominators.

Once a nonlinear eigenvalue problem of the form (1) can be converted into a polynomial eigenvalue problem, it can then subsequently can be converted into a linear eigenvalue problem by one of the usual linearization approaches, see e.g., [17, 22, 35, 36, 23]. It has demonstrated that this approach of turning a rational problem into a larger linear problem is successful in many practical applications, see e.g. [26, 27, 34]. However, the size of the problem may substantially increase and, moreover, typically extra un-physical eigenvalues are introduced. These have to be recognized and removed from the computed spectrum.

Example 1.1 Consider the symmetric rational eigenvalue problem

$$R(\lambda)x := \begin{bmatrix} \lambda - \alpha + \frac{1}{\lambda - 1} & 1 \\ 1 & 0 \end{bmatrix} x = 0,$$

which has only infinite eigenvalues, since $\det R(\lambda) = -1$ for all $\lambda \in \mathbb{C}$. Scaling the problem by $d(\lambda) = \lambda - 1$, the rational eigenvalue problem becomes a polynomial eigenvalue problem with symmetric coefficients, which has further eigenvalues at 1 associated with the roots of d. We obtain the polynomial eigenvalue problem

$$P(\lambda)x = \begin{bmatrix} (\lambda-1)(\lambda-\alpha)+1 & \lambda-1 \\ (\lambda-1) & 0 \end{bmatrix} x = 0,$$

which has a double eigenvalue at ∞ and also a double eigenvalue at 1. Thus, turning the rational problem into a polynomial one has added two eigenvalues that were not there before.

Considering a symmetric linearization [22, 35] of the polynomial problem one obtains the symmetric linear eigenvalue problem

$$L(\lambda)z = \left(\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -(\alpha+1) & 1 & \alpha & 1 \\ -1 & 0 & -1 & 0 \\ \alpha & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right) z = 0.$$

Analyzing $L(\lambda)$ for different α, one sees that it has a Jordan block of size 2 at ∞ and two Jordan blocks of size 1 for the eigenvalue $\lambda = \alpha$.

Due to the Jordan block at ∞ this problem is very sensitive to perturbations. If, e.g., we perturb the problem to

$$P_\epsilon(\lambda)x = \left(\begin{bmatrix} (\lambda-1)(\lambda-\alpha)+1 & \lambda-1 \\ (\lambda-1) & \epsilon \end{bmatrix} \right) x = 0,$$

then the problem has two finite eigenvalues as roots of $(\lambda^2 - (\alpha+1)\lambda + \alpha + 1)\epsilon + 1$ and clearly by appropriate choices of ϵ and α any value in the complex plane can be achieved.

Example 1.1 shows some of the difficulties that may arise in eigenvalue problems of the form (1) and it shows the need for a homogeneous formulation to have a uniform treatment of finite and infinite eigenvalues. It also motivates the desire for a careful perturbation analysis on the original data that avoids turning the rational problem into a polynomial problem.

There also exist practical problems where a nonlinear eigenvalue problem cannot be turned into a polynomial eigenvalue problem. Consider, e.g., the non-rational eigenvalue problem of the form

$$\left(\lambda M_0 + M_1 + M_2 e^{-\tau\lambda}\right) x = 0, \tag{8}$$

where the M_i are real matrix coefficients, and τ is a real parameter. Such problems arise in the stability analysis of single delay differential-algebraic equations [15, 19, 25, 28, 39, 43], where τ describes the delay time. In this case we also cannot transform this problem into homogeneous form easily. However, if good rational approximations to the non-rational functions are available, then these can be used.

Example 1.2 Consider for example the exponential eigenvalue problem (8) and replace the exponential term by a continued fraction expansion. Then we can make these expressions rational and use the homogenization of the rational approximations. For the exponential function, this method is applicable as long as every finite eigenvalue of $\mathcal{M}$ lies in the left half plane. Introduce the sequence $\{h_k(z)\} = \left\{\dfrac{n_k(z)}{d_k(z)}\right\}$, with

$$\begin{aligned} n_0 &= 1,\ n_1 = 1,\ d_0 = 0,\ d_1 = 1 \\ n_k &= \begin{cases} (k-1)n_{k-1} - zn_{k-2}, & \text{if } k = 2,4,6,\ldots \\ 2n_{k-1} + zn_{k-2}, & \text{if } k = 3,5,7,\ldots \end{cases} \\ d_k &= \begin{cases} (k-1)d_{k-1} - zd_{k-2}, & \text{if } k = 2,4,6,\ldots \\ 2d_{k-1} + zd_{k-2}, & \text{if } k = 3,5,7,\ldots \end{cases} \end{aligned}$$

The sequence $\{h_k(z)\}$ converges uniformly to e^z as $k \to \infty$ in any finite domain of the complex plane, see e.g., [40].

We can express h_k in homogeneous form as

$$h_k(c,s) = \frac{n_k(c,s)}{d_k(c,s)} = \frac{s(k-1)n_{k-1} + c\tau s_{k-2}}{s(k-1)d_{k-1} + c\tau d_{k-2}},$$

when k is even and

$$h_k(c,s) = \frac{n_k(c,s)}{d_k(c,s)} = \frac{2sn_{k-1} + c\tau n_{k-2}}{2sd_{k-1} + c\tau d_{k-2}},$$

for odd k. Using these rational expressions to approximate the exponential terms in $\mathcal{M}(\lambda)x = (zI - A_1 - A_2 e^{-z\tau})x = 0$, setting $z = \dfrac{c}{s}$ and clearing out the denominator, we obtain the homogeneous approximations,

$$\begin{aligned} &(c[s(k-1)n_{k-1} + c\tau n_{k-2}]I - s[s(k-1)n_{k-1} + c\tau n_{k-2}]A_1 \\ &\quad -[s^2(k-1)d_{k-1} + c\tau d_{k-2}]A_2\big)\, x = 0, \end{aligned}$$

for even k and

$$\left(c[2sn_{k-1}+c\tau n_{k-2}]I - s[2sn_{k-1}+c\tau n_{k-2}]A_1 - [2s^2d_{k-1}+cs\tau d_{k-2}]A_2\right)x = 0$$

for odd k.

On the other hand it is not all clear whether the computed eigenvalues are of the desired accuracy, since the perturbation and error analysis for these kind of problems is still mainly open. Even for polynomial problems the perturbation theory and the computation of backward errors and even more the structured perturbation theory and backward errors, is only very recent, see [1, 2, 5, 7, 8, 11, 12, 21, 22, 24, 45, 46]. Such a perturbation analysis is also needed, when the nonlinear eigenvalue problem is solved directly by a nonlinear eigenvalue method, see e.g., [38, 41, 42, 50].

Classical perturbation analysis would consider the question that we perturb the nonlinear functions f_j as $f_j+\delta f_j$ and the coefficients M_j as $M_j+\delta M_j$, $j=1,\ldots,m$ and consider instead of (2) the perturbed nonlinear function

$$\tilde{\mathcal{M}}(c,s) := (M+\Delta M)(f+\delta f) = \sum_{j=1}^{m}(M_j+\Delta M_j)(f_j+\delta f_j), \qquad (9)$$

to study how the eigenvalues change under these perturbations.

This problem is extremely difficult for general sets of functions f_j. Instead, in this paper we assume that the perturbations in the functions f_j are known (or can be bounded), i.e., that we have given functions $\tilde{f}_j = f_j+\delta f_j$ and consider only perturbations in the coefficient matrices M_j so that the perturbed problem is of the form

$$\tilde{\mathcal{M}}(c,s) := (M+\Delta M)\tilde{f} = \sum_{j=1}^{m}(M_j+\Delta M_j)\tilde{f}_j. \qquad (10)$$

This is a reasonable assumption in many applications, since the f_j are typically elementary scalar functions and thus the perturbation analysis is well understood. Thus, we assume in the following that our original eigenvalue problem has the form

$$\mathcal{M}(c,s) := \sum_{j=1}^{m} M_j\tilde{f}_j(c,s), \qquad (11)$$

with known perturbed functions $\tilde{f}_j = f_j + \delta f_j$, $j = 1, \ldots, m$, where the specific perturbation on the nonlinear functions with all $\delta f_i = 0$ is the original problem.

An important part of perturbation analysis is the construction of *backward errors*, i.e., for given perturbed pair of eigenvalue, eigenvector (which in the following we call *eigenpair*) to construct the nearest problem of the same type which has this pair as its eigenvalue and eigenvector, respectively. For a given approximate eigenpair and assuming that we know the perturbed function values $\tilde{f}_j(\lambda, \mu)$, then the backward error is the smallest perturbation (in an appropriate norm) to the coefficient matrices $\Delta M = [\Delta M_1, \ldots, \Delta M_m]$ such that $((\lambda, \mu), x))$ becomes the exact eigenpair of the perturbed problem $(\mathcal{M} + \Delta\mathcal{M})x = 0$.

There is very little literature that deals with the perturbation analysis of rational or more general nonlinear eigenvalue problems, see e.g., [13, 16, 44], but in these articles usually only problems without infinite eigenvalues are considered. But, as we will see below, it turns out that for the discussed class of backward errors the theory developed in [7, 8] for the polynomial case can be easily extended.

The main goal of this paper is therefore to derive backward errors for the problem in homogeneous form (thus including infinite eigenvalues) under the assumption that the perturbations in the functions f_j are known, and to compare structured and unstructured backward errors. For our analysis we assume that the functions $f_j, \tilde{f}_j$ are sufficiently smooth in the neighborhood of the perturbed eigenvalues, so that all necessary derivatives are locally available.

The paper is organized as follows. In Section 2, we introduce the notation and recall some of the techniques for polynomial eigenvalue problems from [7, 8]. In Sections 3 and 4 we then construct structured backward errors for complex symmetric/skew-symmetric and Hermitian/skew-Hermitian problems, respectively, and compare these to the corresponding unstructured backward errors. These results cover finite and infinite eigenvalues and are studied in a homogeneous framework. In the last section we discuss and compare unstructured and structured pseudospectra for the discussed class of nonlinear eigenvalue problems. In all the constructions we exclude poles of the $\tilde{f}_j$.

2 Notation and preliminaries

For a nonnegative vector $w = [w_1, w_2, \dots, w_n]^T \in \mathbb{R}^n$, and a vector $x \in \mathbb{C}^n$ we introduce the weighted (semi-)norm

$$\|x\|_{w,2} := \|[w_1 x_1, w_2 x_2, \dots, w_n x_n]^T\|_2,$$

where $\| \; \|_2$ denotes the classical Euclidean norm in $\mathbb{C}^n$. If w is strictly positive, then this is a norm, and if w has zero components then it is a semi-norm. For a nonnegative vector $w \in \mathbb{R}^n$, we define the componentwise inverse via $w^{-1} := [w_1^{-1}, w_2^{-1}, \dots, w_n^{-1}]^T$, where we use the convention that $w_i^{-1} = 0$ if $w_i = 0$. By $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$, we denote the largest and smallest singular values of a matrix A, respectively. The identity matrix is denoted by I, A^T stands for transpose and A^H for the conjugate transpose of a matrix $A \in \mathbb{C}^{n,n}$. For $x \in \mathbb{C}^n$ with $x^H x = 1$, we frequently use the projector $P_x := I - xx^H$ onto the orthogonal complement of the space spanned by x.

We will construct structured and unstructured backward errors both in spectral and Frobenius norm on $\mathbb{C}^{n,n}$, which are defined by

$$\|A\|_2 := \max_{\|x\|=1} \|Ax\|_2, \;\; \|A\|_F := (\mathrm{trace} A^H A)^{1/2},$$

respectively and we sometimes use $\|A\|_q$, where $q \in \{2, F\}$.

The vector space of all tuples $M = [M_1, M_2, \dots, M_m]$ with coefficients in $M_i \in \mathbb{C}^{n,n}$, is denoted by $\mathcal{M}_m(\mathbb{C}^{n,n})$. With a nonnegative weight vector $w \in \mathbb{R}^n$, it can be equipped with a weighted norm/seminorm $|\!|\!|.|\!|\!|_{w,q}$ given by

$$|\!|\!|M|\!|\!|_{w,q} := \|[M_1, \dots, M_m]\|_{w,q} = (w_1^2 \|M_1\|_q^2 + \dots + w_m^2 \|M_m\|_q^2)^{1/2},$$

for $q \in \{2, F\}$, respectively. For convenience, if $w := [1, 1, \dots, 1]^T$ then we leave off the subscript w.

In the following we consider matrix functions of the form $\mathcal{M}(c, s)$ as in (11), with eigenvalues on the *Riemann sphere* $\mathcal{R} = \{(c, s) \in \mathbb{C}^2 \setminus \{0\} \mid |c|^2 + |s|^2 = 1\}$. Such a matrix function is called *regular* if $\det(\mathcal{M}(\lambda, \mu)) \neq 0$ for some $(\lambda, \mu) \in \mathcal{R}$, otherwise it is called *singular*. The *spectrum* of such a matrix function is defined as

$$\Lambda(\mathcal{M}) := \{(c, s) \in \mathcal{R} \mid \mathrm{rank}(\mathcal{M}(c, s)) < n\}.$$

Let $(\lambda, \mu) \in \mathcal{R}$ be an approximation to an eigenvalue (10) and corresponding approximate right eigenvector $x \neq 0$, and suppose that we know the perturbations in the functions $\tilde{f}_j = f_j + \delta f_j$, $j = 1, \dots, m$, then we construct

Frobenius and spectral norm backward errors

$$\begin{aligned}&\eta_{w,q}(\lambda,\mu,x,\mathcal{M})\\&:=\inf\{|\!|\!|\Delta M|\!|\!|_{w,q},\ \Delta M\in\mathcal{M}_m(\mathbb{C}^{n,n}),\ (\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x=0\},\end{aligned}$$

where

$$\Delta\mathcal{M}(c,s)=\sum_{j=1}^{m}\Delta M_j\ \tilde{f}_j(c,s). \tag{12}$$

If the problem has coefficients that are structured in a subset $\mathbf{S}\subset\mathcal{M}_m(\mathbb{C}^{n,n})$, then we *construct structured backward errors*

$$\begin{aligned}&\eta^{\mathbf{S}}_{w,q}(\lambda,\mu,x,\mathcal{M})\\&:=\inf\{|\!|\!|\Delta M|\!|\!|_{w,q},\ \Delta M\in\mathbf{S},\ (\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x=0\}.\end{aligned}$$

Such backward errors were introduced for matrix polynomials in [20, 45], but here we follow [2, 4, 5].

In order to compute the backward errors, we will need the partial gradient $\nabla_i\|z\|_{w,2}$ of a map $\mathbb{C}^m\to\mathbb{R}, z\mapsto\|z\|_{w,2}$ which is just the gradient of the map $\mathbb{C}\to\mathbb{R},\ z_i\mapsto\|[z_1,\ldots,z_m]^T\|_{w,2}$ which fixes the variables $z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_m$ as constants. The gradient of the map $\mathbb{C}^m\to\mathbb{R}, z\mapsto\|z\|_{w,2}$, is then defined as

$$\nabla(\|z\|_{w,2})=\left[\nabla_1\|z\|_{w,2},\nabla_2\|z\|_{w,2},\ldots,\nabla_m\|z\|_{w,2}\right]^T.$$

With these definitions we have the following proposition, see [4, 8].

Proposition 2.1 *Consider the map $H_{w,2}:\mathbb{C}^m\to\mathbb{R}$ given by $H_{w,2}(z):=\|[z_1,\ldots,z_m]^T\|_{w,2}$. Then $H_{w,2}$ is differentiable on $\mathbb{C}^m$ and*

$$\nabla_j H_{w,2}(z)=\frac{w_j^2 z_j}{H_{w,2}(z)},\ j=1,2,\ldots,m.$$

Furthermore,

$$\sum_{j=1}^{m} z_j\frac{\overline{\nabla_j H_{w,2}(z)}}{H_{w,2}(z)}=1,\ \sum_{j=1}^{m} w_j^{-2}|\nabla_j H_{w,2}(z)|^2=1.$$

In order to simplify the presentation, in the following we use the abbreviations

$$\tilde{f}(\lambda,\mu) := (\tilde{f}_1(\lambda,\mu),\dots,\tilde{f}_m(\lambda,\mu)),\ z_{M_j} := \frac{\nabla_j H_{w,2}}{H_{w,2}}|_{\tilde{f}(\lambda,\mu)},\ j=1,\dots,m. \tag{13}$$

We will construct backward errors for the following structured nonlinear eigenvalue problems, which extend the polynomial classes that were introduced in non-homogeneous form in [35]. We say that an eigenvalue problem of the form (11) is complex *symmetric/skew-symmetric* if $\mathcal{M}^T(c,s) = \pm\mathcal{M}(c,s)$, and *Hermitian/skew-Hermitian* if $\mathcal{M}^H(c,s) = \pm\mathcal{M}(\overline{c},\overline{s})$. For symmetric/skew-symmetric problems of the form (10), if $x \in \mathbb{C}^n$ is a right eigenvector of $\mathcal{M}(c,s)$ corresponding to an eigenvalue $(\lambda,\mu) \in \mathcal{R}$, then $\overline{x}$ is a left eigenvector. For Hermitian/skew Hermitian eigenvalue problems, if $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ are right and left eigenvectors corresponding to an eigenvalue $(\lambda,\mu) \in \mathcal{R}$ of $\mathcal{M}$, then y and x are right and left eigenvector corresponding to the eigenvalue $(\overline{\lambda},\overline{\mu})$, respectively.

For a given eigenvalue (λ,μ) we can determine the smallest perturbation that makes this an eigenvalue, and when this is known we can determine a concrete perturbation with this norm and a given right eigenvector x. This follows from the following proposition.

Proposition 2.2 *Consider a structured eigenvalue problem of the form (10), with $\mathcal{M} \in \mathcal{M}_m^{\mathbf{S}}(\mathbb{C}^{n\times n})$ and a given set of sufficiently smooth perturbed functions $\tilde{f}_j := f_j + \delta f_j$, $j=1,\dots,m$. For a given approximate eigenvalue (λ,μ), set*

$$H_{w,2}(\tilde{f}(\lambda,\mu)) = \|[w_1\tilde{f}_1(\lambda,\mu),\dots,w_m\tilde{f}_m(\lambda,\mu)]^T\|_2. \tag{14}$$

Then the backward error, i.e., the size of the smallest perturbation that makes this eigenvalue an eigenvalue of the perturbed problem satisfies

$$\eta_{w,2}^{\mathbf{S}}(\lambda,\mu,\mathcal{M}) = \min_{\|x\|=1} \frac{\|\mathcal{M}(\lambda,\mu)x\|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))}.$$

Proof. With

$$\mathcal{M}(\lambda,\mu) = \sum_{i=0}^{m} M_i\tilde{f}_i(\lambda,\mu),$$

the backward error satisfies $(\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x = 0$ for some normalized vector x, which implies that

$$\mathcal{M}(\lambda,\mu)x = -\Delta\mathcal{M}(\lambda,\mu)x.$$

Hence we have that

$$\|\mathcal{M}(\lambda,\mu)x\| \le \|\!|\Delta M|\!\|_{w,2} H_{w^{-1},2}(\tilde{f}(\lambda,\mu))$$

which can reformulated as

$$\frac{\|\mathcal{M}(\lambda,\mu)x\|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))} \le \|\!|\Delta M|\!\|_{w,2}$$

i.e., we have

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,\mathcal{M}) \ge \frac{\|\mathcal{M}(\lambda,\mu)x\|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))}.$$

To show equality, consider any normalized vector x, a normalized vector y with $y^H x = 1$, the rank one matrix $\mathcal{M}(\lambda,\mu)xy^H$, and choose

$$\Delta M_i = \frac{w_i^{-2}\mathrm{sign}\tilde{f}_i(\lambda,\mu)|\tilde{f}_i(\lambda,\mu)|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu)^2}\mathcal{M}(\lambda,\mu)xy^H,$$

where $\mathrm{sign}(\lambda) := \dfrac{\overline{\lambda}}{|\lambda|^2}$ if $\lambda \neq 0$ and $\mathrm{sign}(\lambda) := 0$ if $\lambda = 0$. Then we have that $(\mathcal{M}(\lambda,\mu) + \Delta\mathcal{M}(\lambda,\mu))x = 0$, which implies that

$$\|\Delta M x\| = \frac{\|\mathcal{M}(\lambda,\mu)xy^H x\|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))} = \frac{\|\mathcal{M}(\lambda,\mu)x\|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))}.$$

Minimizing over all possible normalized vectors x then gives the desired inequality. □

From Proposition 2.2 it is clear that $\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,\mathcal{M}) \le \eta_{w,2}(\lambda,\mu,\mathcal{M})$, and since the constructed minimal perturbation is of rank one, we also have $\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,\mathcal{M}) \le \eta_{w,F}(\lambda,\mu,\mathcal{M})$.

We will also make use of the following completion result which is a direct corollary of Theorem 1.2, [14].

Proposition 2.3 *1. Let $A = \pm A^T$, $C = \pm B^T \in \mathbb{C}^{n,n}$ and $\chi := \sigma_{\max}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right)$. Then there exists a symmetric/skew-symmetric matrix $X \in \mathbb{C}^{n,n}$ such that $\sigma_{max}\left(\begin{bmatrix} A & \pm B^T \\ B & X \end{bmatrix}\right) = \chi$, and X has the form*

$$X := -K\overline{A}K^T + \chi(I - KK^H)^{1/2} Z (I - \overline{K}K^T)^{1/2},$$

where $K := B(\chi^2 I - \overline{A}A)^{-1/2}$ and $Z = \pm Z^T \in \mathbb{C}^{n,n}$ is an arbitrary matrix such that $\|Z\|_2 \le 1$.

2. *For $A = \pm A^H, B = \pm B^H$, set $\chi := \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2$. Then there exists an Hermitian/skew-Hermitian matrix D, respectively, such that $\left\| \begin{bmatrix} A & \pm B^H \\ B & D \end{bmatrix} \right\|_2 = \chi$ and D is of the form $D := -KAK^H + \chi(I - KK^H)^{1/2} Z (I - KK^H)^{1/2}$, where $K := B(\chi^2 I - A^2)^{-1/2}$ and $Z = \pm Z^H$ is an arbitrary matrix such that $\|Z\|_2 \leq 1$.*

For our theory, we always fix the arbitrary part to be $Z = 0$, because this minimizes the spectral and Frobenius norm.

After these preliminary results in the following section we derive backward errors for the different classes of structured nonlinear matrix functions.

3 Backward errors for symmetric/skew-symmetric nonlinear eigenvalue problems

In this section we will construct backward error formulas for homogeneous symmetric/skew-symmetric nonlinear eigenvalue problems.

Theorem 3.1 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n,n})$ be a regular symmetric/skew-symmetric nonlinear matrix equation of the form (11), and let $x \in \mathbb{C}^n$ with $x^H x = 1$.*

With $k := -\mathcal{M}(\lambda, \mu)x$, introduce the perturbation matrices

$$\Delta M_j := \begin{cases} -\overline{x}x^T M_j x x^H + \overline{z_{M_j}} \left[\overline{x} k^T + k x^H - 2(x^T k)\overline{x} x^H \right] & \text{if } M_j = M_j^T, \\ \overline{z_{M_j}} \left[\overline{x} k^T + k x^H - (x^T k)\overline{x} x^H \right] & \text{if } M_j = -M_j^T, \end{cases}$$

for $j = 1, \ldots, m$, where z_{M_j} is as in (13), and form

$$\Delta\mathcal{M}(c, s) = \sum_{j=0}^{m} \tilde{f}_j(c, s) \Delta M_j.$$

Then $\Delta\mathcal{M}$ has the desired symmetry structure and satisfies $(\mathcal{M}(\lambda, \mu) + \Delta\mathcal{M}(\lambda, \mu))x = 0$.

Proof. The proof is a slight modification of the proof for the polynomial case in [7]. In the symmetric case we have for all $j = 1, \ldots, m$ that $\Delta M_j = \Delta M_j^T$.

Hence $\Delta\mathcal{M}$ is symmetric, and we have that

$$(\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x = \sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)(M_j+\Delta M_j)x$$
$$= \sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)\left[M_jx - \overline{x}x^TM_jx + \overline{z_{M_j}}\left[\overline{x}k^Tx + k - 2(x^Tk)\overline{x}\right]\right]$$
$$= (I-\overline{x}x^T)(\sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)M_j)x + \left[\overline{x}k^Tx + k - 2(x^Tk)\overline{x}\right]\sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)\overline{z_{M_j}}.$$

By Proposition 2.1, we have $\sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)\overline{z_{M_j}} = 1$. Hence

$$\begin{aligned}(\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x &= -(I-\overline{x}x^T)k + \overline{x}k^Tx + k - 2(x^Tk)\overline{x}\\ &= -k + \overline{x}(x^Tk) + \overline{x}(k^Tx) + k - 2(x^Tk)\overline{x}\\ &= \overline{x}(x^Tk) + \overline{x}(x^Tk) - 2(x^Tk)\overline{x} = 0,\end{aligned}$$

since $k^Tx = x^Tk$.

The proof for the skew-symmetric case follows analogously. □

Using Theorem 3.1, we now obtain the following backward errors for complex symmetric nonlinear eigenvalue problems.

Theorem 3.2 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n,n})$ be as in (11) with complex symmetric coefficients, let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$ and let $k := -\mathcal{M}(\lambda,\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathcal{M}) = \frac{\sqrt{2\|k\|_2^2 - |x^Tk|^2}}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))}$$

and there exists a unique complex symmetric $\Delta\mathcal{M}(c,s) := \sum_{j=1}^{m}\tilde{f}_j\Delta M_j$ with coefficients

$$\Delta M_j = \overline{z_{M_j}}\left[\overline{x}k^T + kx^H - (x^Tk)\overline{x}x^H\right],\ j=1,\dots,m,$$

such that the structured backward error satisfies $\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathcal{M}) = \|\Delta\mathcal{M}\|_{w,F}$ and $\overline{x}$, x are left and right eigenvectors corresponding to the eigenvalue (λ,μ) of $\mathcal{M}+\Delta\mathcal{M}$, respectively.

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathcal{M}) = \frac{\|k\|_2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))}$$

and there exists a complex symmetric $\Delta\mathcal{M}(c,s) := \sum_{j=1}^{m} \tilde{f}_j \Delta M_j$ with coefficients

$$\Delta M_j := \overline{z_{M_j}} \left[\overline{x}k^T + kx^H - (k^T x)\overline{x}x^H - \frac{\overline{x^T k}(I - \overline{x}x^T)kk^T(I - xx^H)}{\|k\|_2^2 - |x^T k|^2} \right].$$

such that $\|\Delta\mathcal{M}\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathcal{M})$ and $(\mathcal{M}(\lambda,\mu) + \Delta\mathcal{M}(\lambda,\mu))x = 0$.

Proof. The proof is a slight modification of the proof for the polynomial case [7]. By Theorem 3.1 we have that $(\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x = 0$ and hence $k = \Delta\mathcal{M}(\lambda,\mu)x$. Now we construct a unitary matrix U which has x as its first column, i.e., $U = [x, U_1] \in \mathbb{C}^{n\times n}$ and let $\widetilde{\Delta M_j} := U^T \Delta M_j U = \begin{bmatrix} d_{j,j} & d_j^T \\ d_j & D_{j,j} \end{bmatrix}$, where $D_{j,j} = D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}$. Then

$$\overline{U}\widetilde{\Delta\mathcal{M}}(\lambda,\mu)U^H = \overline{U}U^T(\Delta\mathcal{M}(\lambda,\mu))U^H U = \Delta\mathcal{M}(\lambda,\mu),$$

and hence

$$\overline{U}\widetilde{\Delta\mathcal{M}}(\lambda,\mu)U^H x = \Delta\mathcal{M}(\lambda,\mu)x = k,$$

which implies that

$$\widetilde{\Delta\mathcal{M}}(\lambda,\mu)U^H x = U^T k = \begin{bmatrix} x^T k \\ U_1^T k \end{bmatrix}.$$

Therefore, we get that

$$\begin{bmatrix} \sum_{j=1}^{m} \tilde{f}_j(\lambda,\mu) d_{j,j} \\ \sum_{j=1}^{m} \tilde{f}_j(\lambda,\mu) d_j \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m} w_j d_{j,j} \dfrac{\tilde{f}_j(\lambda,\mu)}{w_j} \\ \sum_{j=1}^{m} w_j \tilde{f}_j(\lambda,\mu) \dfrac{d_j}{w_j} \end{bmatrix} = \begin{bmatrix} x^T k \\ U_1^T k \end{bmatrix}.$$

To minimize the norm of the perturbation, we use the same procedure as in the polynomial case [7] and solve this system for the parameters $d_{j,j}, d_j$ in a least squares sense which, together with Proposition 2.1, yields

$$d_{j,j} = \overline{z_{M_j}} x^T k,\ d_j = \overline{z_{M_j}} U_1^T k,\ j = 1, 1, \ldots, m,$$

and thus

$$\begin{aligned}
\Delta M_j &= \overline{U}\widetilde{\Delta M}U^H = \overline{x}d_{j,j}x^H + \overline{U_1}d_jx^H + \overline{x}d_j^TU_1^H + \overline{U_1}D_{j,j}U_1^H \\
&= \overline{z_{M_j}}[(\overline{x}x^Tkx^H) + \overline{U_1}U_1^Tkx^H + \overline{x}k^TU_1U_1^H)] + \overline{U_1}D_{j,j}U_1^H \\
&= \overline{z_{M_j}}[(\overline{x}x^Tkx^H) + (I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H))] + \overline{U_1}D_{j,j}U_1^H \\
&= \overline{z_{M_j}}\left[kx^H + \overline{x}k^T - (k^Tx)\overline{x}x^H\right] + \overline{U_1}D_{j,j}U_1^H. \qquad (15)
\end{aligned}$$

In Frobenius norm, the unique minimal perturbation is obtained by taking $D_{j,j} = 0$ and hence we get

$$\begin{aligned}
\|\Delta M_j\|_F^2 &= |d_{j,j}|^2 + 2\|d_j\|_2^2 = |z_{M_j}|^2(|x^Tk|^2 + 2\|U_1^Tk\|_2^2) \\
&= |\nabla_j H_{w^{-1},2}(\tilde{f}(\lambda,\mu))|^2 \frac{2\|k\|_2^2 - |x^Tk|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2},
\end{aligned}$$

since $\|U^Tk\|_2^2 = |x^Tk|^2 + \|U_1^Tk\|_2^2$. By Proposition 2.1, we have that

$$\sum_{j=1}^m w_j^2|\nabla_j H_{w^{-1},2}(\tilde{f}(\lambda,\mu))|^2 = 1,$$

and thus

$$|||\Delta\mathcal{M}|||_{w,F}^2 = \frac{2\|k\|_2^2 - |x^Tk|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2},$$

and hence,

$$|||\Delta\mathcal{M}|||_{w,F} = \sqrt{\frac{2\|k\|_2^2 - |x^Tk|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}}.$$

As k^Tx is a scalar constant, it follows that all ΔM_j and thus also $\Delta\mathcal{M}$ are symmetric and

$$\begin{aligned}
(\mathcal{M}(\lambda,\mu) + \Delta\mathcal{M}(\lambda,\mu))x &= \sum_{j=0}^m z_j(\lambda,\mu)(M_j + \Delta M_j)x \\
&= -k + (\sum_{j=0}^m z_j\Delta M_j)x \\
&= -k + \sum_{j=0}^m z_j\overline{z_{M_j}}[kx^H + \overline{x}k^T - \overline{x}k^Txx^H]x \\
&= -k + k + \overline{x}k^Tx - \overline{x}k^Tx = 0,
\end{aligned}$$

where we have again used Proposition 2.1.

For the spectral norm we can apply Proposition 2.3 to (15) and get

$$\begin{aligned} D_{j,j} &= -\frac{\overline{z_{M_j}}}{P^2}\left[\overline{x^T k}(U_1^T k)(U_1^T k)^T\right] \\ &+ \chi\left[I - \frac{(U_1^T k)(U_1^T k)^H}{P^2}\right]^{1/2} Z \left[I - \frac{\overline{U_1^T k}(U_1^T k)^T}{P^2}\right]^{1/2}, \end{aligned}$$

where $Z = Z^T$ and $\|Z\|_2 \leq 1$, $P^2 = \|k\|_2^2 - |x^T k|^2$, $\chi := \sqrt{\|d_{j,j}\|^2 + \|d_j\|_2^2}$. With the special choice $Z = 0$ we get $D_{j,j} = -\frac{\overline{z_{M_j}}}{P^2}\left[\overline{x^T k}(U_1^T k)(U_1^T k)^T\right]$ and

$$\overline{U_1} D_{j,j} U_1^H = -\frac{\overline{z_{M_j}}}{P^2}\overline{U_1}U_1^T k k^T U_1 U_1^H = -\frac{\overline{z_{M_j}}}{P^2}(I - \overline{x}x^T)kk^T(I - xx^H).$$

Hence,

$$\Delta M_j = \overline{z_{M_j}}\left[kx^H + \overline{x}k^T - \overline{x}(k^T x)x^H\right] - \frac{\overline{z_{M_j}}}{P^2}(I - \overline{x}x^T)kk^T(I - xx^H),$$

$\Delta\mathcal{M}(c, s)$ is symmetric, and $(\mathcal{M}(\lambda, \mu) + \Delta\mathcal{M}(\lambda, \mu))x = 0$. With

$$\begin{aligned} \chi &:= \sigma_{\max}\left(\begin{bmatrix} d_{j,j} \\ d_j \end{bmatrix}\right) = |z_{M_j}|\sqrt{|x^T k|^2 + \|U_1^T k\|^2} \\ &= \frac{|\nabla_j H_{w^{-1},2}(\tilde{f}(\lambda, \mu))|}{H_{w^{-1},2}(\tilde{f}(\lambda, \mu))}\|k\|_2, \end{aligned}$$

then by Proposition 2.3 we have $\chi = \|\Delta M_j\|_2$, and again by Proposition 2.1,

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathcal{M}) = |||\Delta\mathcal{M}|||_{w,2} = \frac{\|k\|_2}{H_{w^{-1},2}(\tilde{f}(\lambda, \mu))}.$$

□

As a corollary we have the following relations between structured and unstructured backward errors.

Corollary 3.3 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n\times n})$ as in (11) be regular with symmetric coefficients, let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, and let $x \in \mathbb{C}^n$ be such that $x^H x = 1$. Then,*

$$\begin{aligned} \eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathcal{M}) &\leq \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathcal{M}), \\ \eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathcal{M}) &= \eta_{w,2}(\lambda, \mu, x, \mathcal{M}). \end{aligned}$$

We obtain an analogous result in the case of real problems and real perturbations which we omit here for brevity, we just mention that we need that the function evaluations $\tilde{f}_j(\lambda,\mu)$ yield real values to obtain a real backward error. In this case the minimal perturbation has the form

$$\Delta\mathcal{M}(\lambda,\mu) := \sum_{j=1}^{m} \tilde{f}_j(\lambda,\mu)\Delta E_j,$$

with coefficients

$$\Delta E_j = z_{M_j}\left[xk^T + kx^T - (x^T k)xx^T\right],\ j = 1, 1, \ldots, m.$$

The same technique of proof also applies in the complex-skew symmetric case. We state the results here for completeness.

Theorem 3.4 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n,n})$ of the form (11) be complex skew-symmetric, let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ such that $x^H x = 1$ and let $k := -\mathcal{M}(\lambda,\mu)x$. Introduce the perturbation matrices*

$$\Delta M_j := -\overline{z_{M_j}}\left[\overline{x}k^T - kx^H\right],\ j = 0, 1, 2, \ldots, m.$$

Then $\Delta\mathcal{M}(c,s) = \sum_{j=1}^{m} \tilde{f}_j(c,s)\Delta M_j$ is complex skew-symmetric and $(\mathcal{M}(\lambda,\mu) + \Delta\mathcal{M}(\lambda,\mu))x = 0$.

Theorem 3.5 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n,n})$ of the form (11) be complex skew-symmetric, let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and let $k := -\mathcal{M}(\lambda,\mu)x$. The structured backward errors with respect to the Frobenius norm and spectral norm are given by*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathcal{M}) = \frac{\sqrt{2\|k\|_2^2}}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))},$$

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathcal{M}) = \frac{\|k\|_2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))},$$

respectively.

The relation between structured and unstructured backward errors is then clearly the same as in the symmetric case.

In this section we have shown that the backward error results for symmetric and skew-symmetric matrix functions carry over from the polynomial case to the more general case (11) with very little modifications.

4 Backward errors for Hermitian/skew-Hermitian nonlinear eigenvalue problems

In this section we present the results for the Hermitian/skew Hermitian case. The proofs follow in the same way as for the symmetric/skew symmetric problems, just with a slight modification of the proof in the construction of the backward errors.

Theorem 4.1 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n,n})$ of the form (11) be either Hermitian or skew-Hermitian. Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and let $k := -\mathcal{M}(\lambda, \mu)x$. Introduce the perturbation matrices*

$$\Delta M_j := \begin{cases} -xx^H M_j xx^H + \left[z_{M_j} x k^H P_x + \overline{z_{M_j}} P_x k x^H\right], & \text{if } M_j = M_j^H, \\ -xx^H M_j xx^H - \left[z_{M_j} x k^H P_x - \overline{z_{M_j}} P_x k x^H\right], & \text{if } M_j = -M_j^H, \end{cases}$$

and consider

$$\Delta\mathcal{M}(c, s) = \sum_{j=1}^{m} \tilde{f}_j(\lambda, \mu)\Delta M_j \in \mathcal{M}(\mathbb{C}^{n\times n}).$$

Then $\Delta\mathcal{M}$ is Hermitian/skew Hermitian and $(\mathcal{M} + \Delta\mathcal{M})(\lambda, \mu)x = 0$.

Proof. The proof follows the same line as the proof of Theorem 3.1. □

For the construction of the backward errors we introduce

$$T := \begin{bmatrix} \frac{\Re(\tilde{f}_1(\lambda,\mu))}{w_1} & \dots & \frac{\Re(\tilde{f}_m(\lambda,\mu))}{w_m} \\ \frac{\Im(\tilde{f}_1(\lambda,\mu))}{w_1} & \dots & \frac{\Im(\tilde{f}_m(\lambda,\mu))}{w_m} \end{bmatrix},$$

and set

$$t = [t_1, \dots, t_m]^T := T^+ \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix}, \tag{16}$$

where T^+ denotes the Moore-Penrose inverse of T, [18]. By e_j we denote the j-th unit vector.

Then we have the following structured backward errors.

Theorem 4.2 *Let $\mathcal{M} \in \mathcal{M}(\mathbb{C}^{n,n})$ of the form (11) be a Hermitian, let $x \in \mathbb{C}^n$ such that $x^H x = 1$ and set $k := -\mathcal{M}(\lambda, \mu)x$.*

i) The structured backward error in Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathcal{M}) = \frac{\sqrt{2\|k\|_2^2 - |x^H k|^2}}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))},$$

if all $\tilde{f}_j(\lambda,\mu)$, $j = 1,\ldots,m$ *are real and*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathcal{M}) = \sqrt{\sum_{j=1}^{m} \frac{\|e_j t\|_2^2}{w_j^2} + 2\frac{\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}},$$

otherwise.

ii) The structured backward error in spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathcal{M}) = \frac{\|k\|_2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))},$$

if all $\tilde{f}_j(\lambda,\mu)$, $j = 1,\ldots,m$ *are real and*

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathcal{M}) = \sqrt{\sum_{j=1}^{m} \frac{\|e_j t\|_2^2}{w_j^2} + \frac{\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}},$$

otherwise.

Proof. By Theorem 4.1, we have $(\mathcal{M}(\lambda,\mu) + \Delta\mathcal{M}(\lambda,\mu))x = 0$, and hence we have $k = \Delta\mathcal{M}(\lambda,\mu)x$. To construct a minimal perturbation, we complete x to a unitary matrix $U = [x, U_1], U_1 \in \mathbb{C}^{n\times n-1}$ and let

$$\widetilde{\Delta M_j} := U^H \Delta M_j U = \begin{bmatrix} d_{j,j} & d_j^H \\ d_j & \Delta D_{j,j} \end{bmatrix},$$

where $\Delta D_{j,j} = \Delta D_{j,j}^H \in \mathbb{C}^{n\times n-1}$. Then $U\widetilde{\Delta\mathcal{M}}(\lambda,\mu)U^H = UU^H(\Delta\mathcal{M}(\lambda,\mu))U^H U = \Delta\mathcal{M}(\lambda,\mu)$, and this implies that $U\widetilde{\Delta\mathcal{M}}(\lambda,\mu)U^H x = \Delta\mathcal{M}(\lambda,\mu)x = k$, and hence

$$\widetilde{\Delta\mathcal{M}}(\lambda,\mu)U^H x = U^H k = \begin{bmatrix} x^H k \\ U_1^H k \end{bmatrix}.$$

Since $U^H x = e_1$, we get

$$\begin{bmatrix} \sum_{j=1}^{m} w_j d_{j,j} \frac{\tilde{f}_j(\lambda,\mu)}{w_j} \\ \sum_{j=1}^{m} w_j \tilde{f}_j(\lambda,\mu) \frac{d_j}{w_j} \end{bmatrix} = \begin{bmatrix} x^H k \\ U_1^H k \end{bmatrix}.$$

To minimize the perturbation, we solve this system for the parameters $d_{j,j}, d_j$ in a least square sense, and applying Proposition 2.1, we obtain

$$\begin{bmatrix} d_{1,1} \\ \vdots \\ d_{m,m} \end{bmatrix} = \begin{bmatrix} \overline{z_{M_1}} \\ \vdots \\ \overline{z_{M_m}} \end{bmatrix} x^H k, \quad \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} \overline{z_{M_1}} \\ \vdots \\ \overline{z_{M_m}} \end{bmatrix} U_1^H k.$$

If $\tilde{f}_j(\lambda,\mu)$ is real then $d_{j,j} = z_{M_j} x^H k$ is real as well. Thus if all the $\tilde{f}_j(\lambda,\mu)$ are real then the Frobenius norm is minimized by taking $\Delta D_{j,j} = 0$ and hence,

$$\|\Delta M_j\|_F^2 = |z_{M_j}|^2 (2\|k\|_2^2 - |x^H k|^2) = |\nabla_j H_{w^{-1},2}(\tilde{f}(\lambda,\mu))|^2 \frac{2\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}$$

as $\|U^H k\|^2 = |x^H k|^2 + \|U_1^H k\|_2^2$, and making use of Proposition 2.1. Then

$$|||\Delta \mathcal{M}|||_{w,F} = \sqrt{\sum_{j=0}^{m} w_j^2 \|\Delta M_j\|_F^2} = \sqrt{\frac{2\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}}.$$

Thus,

$$\begin{aligned} \Delta M_j &= U \widetilde{\Delta M_j} U^H = \begin{bmatrix} x & U_1 \end{bmatrix} \begin{bmatrix} d_{j,j} & d_j^H \\ d_j & \Delta D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\ &= \begin{bmatrix} x d_{j,j} + U_1 d_j & x d_j^H + U_1 \Delta D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\ &= (x d_{j,j} + U_1 d_j) x^H + (x d_j^H + U_1 \Delta D_{j,j}) U_1^H \\ &= z_{M_j} [(x x^H k x^H) + U_1 U_1^H k x^H + x k^H U_1 U_1^H)] + U_1 \Delta D_{j,j} U_1^H \\ &= z_{M_j} [(x x^H k x^H) + P_x k x^H + x k^H P_x] + U_1 \Delta D_{j,j} U_1^H, \end{aligned}$$

and after simplification we get

$$\Delta M_j = z_{M_j} \left[k x^H + x k^H - (k^H x) x x^H \right] + U_1 \Delta D_{j,j} U_1^H. \quad (17)$$

For the norm minimization we take

$$\Delta M_j = z_{M_j}\left[kx^H + xk^H - xk^H xx^H\right]. \tag{18}$$

Since ΔM_j is Hermitian and $k^H x$ is a constant, also $\Delta\mathcal{M}$ is Hermitian and

$$\begin{aligned}(\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x &= \sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)(M_j+\Delta M_j)x \\ &= -k+\sum_{j=1}^{m}\tilde{f}_j(\lambda,\mu)z_{M_j}[kx^H+xk^H-xk^Hxx^H]x \\ &= -k+k+xk^Hx-xk^Hx=0,\end{aligned}$$

where we have again used Proposition 2.1.

This implies that $(\mathcal{M}(\lambda,\mu)+\Delta\mathcal{M}(\lambda,\mu))x=0$ and hence x is an eigenvector corresponding to (λ,μ). From (17) we have that

$$\Delta M_j = z_{M_j}\left[kx^H + xk^H - (k^H x)xx^H\right] + U_1\Delta D_{j,j}U_1^H,$$

and using Proposition 2.3 we have $\Delta D_{j,j} = -z_{M_j}\dfrac{x^H k U_1^H kk^H U_1}{P^2}$. Hence for the spectral norm we define

$$\Delta E_j = \Delta M_j - \frac{z_{M_j}x^H kP_x kk^H P_x}{P^2},$$

where ΔM_j is defined in (18).

If one of the $\tilde{f}_j(\lambda,\mu)$ is not real, then from

$$\sum_{j=1}^{m}\frac{\tilde{f}_j(\lambda,\mu)}{w_j}w_j d_{j,j} = x^H k$$

we obtain

$$\begin{bmatrix}\displaystyle\sum_{j=1}^{m}\frac{\Re(\tilde{f}_j(\lambda,\mu))}{w_j}w_j d_{j,j}\\ \displaystyle\sum_{j=1}^{m}\frac{\Im(\tilde{f}_j(\lambda,\mu))}{w_j}w_j d_{j,j}\end{bmatrix} = \begin{bmatrix}\Re(x^H k)\\ \Im(x^H k)\end{bmatrix},$$

which implies that

$$\begin{bmatrix} \frac{\Re(\tilde{f}_1(\lambda,\mu)))}{w_1} & \dots & \frac{\Re(\tilde{f}_m(\lambda,\mu)))}{w_m} \end{bmatrix} \begin{bmatrix} w_1 d_{1,1} \\ \vdots \\ w_m d_{m,m} \end{bmatrix} = \Re(x^H k),$$

and

$$\begin{bmatrix} \frac{\Im(\tilde{f}_1(\lambda,\mu)))}{w_1} & \dots & \frac{\Im(\tilde{f}_m(\lambda,\mu)))}{w_m} \end{bmatrix} \begin{bmatrix} w_1 d_{1,1} \\ \vdots \\ w_m d_{m,m} \end{bmatrix} = \Im(x^H k),$$

which we write as

$$\begin{bmatrix} \frac{\Re(\tilde{f}_1(\lambda,\mu)))}{w_1} & \dots & \frac{\Re(\tilde{f}_m(\lambda,\mu)))}{w_m} \\ \frac{\Im(\tilde{f}_1(\lambda,\mu)))}{w_1} & \dots & \frac{\Im(\tilde{f}_m(\lambda,\mu)))}{w_m} \end{bmatrix} \begin{bmatrix} w_1 d_{1,1} \\ \vdots \\ w_m d_{m,m} \end{bmatrix} = \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix},$$

and hence,

$$\begin{bmatrix} w_1 d_{1,1} \\ \vdots \\ w_m d_{m,m} \end{bmatrix} = T^{+} \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix} = t.$$

Then it follows that

$$\begin{aligned} \Delta M_j &= U\widetilde{\Delta M_j}U^H = \begin{bmatrix} x & U_1 \end{bmatrix} \begin{bmatrix} w_j^{-1} e_j^T t & \left(\overline{z_{M_j}} U_1^H k\right)^H \\ \overline{z_{M_j}} U_1^H k & \Delta D_{j,j} \end{bmatrix} \begin{bmatrix} x & U_1 \end{bmatrix}^H \\ &= \left[w_j^{-1} x e_j^T t + U_1 \overline{z_{M_j}} U_1^H k\right] x^H + \left[x z_{M_j} (U_1^H k)^H + U_1 \Delta D_{j,j}\right] U_1^H \\ &= w_j^{-1} x e_j^T t x^H + \overline{z_{M_j}} U_1 U_1^H k x^H + z_{M_j} x k^H U_1 U_1^H + U_1 \Delta D_{j,j} U_1^H \\ &= w_j^{-1} x e_j^T t x^H + \overline{z_{M_j}} P_x k x^H + z_{M_j} x k^H P_x + U_1 \Delta D_{j,j} U_1^H, \end{aligned}$$

setting $\Delta D_{j,j} = 0$.

For the Frobenius norm, the perturbation matrix and the backward error is then given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathcal{M}) = \sqrt{\sum_{j=1}^{m} w_j^{-2}\|e_j t\|_2^2 + 2\frac{\|k\|_2^2 - |k^H x|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}}.$$

For the spectral norm, by Proposition 2.3, we have

$$U_1 \Delta D_{j,j} U_1^H = -\frac{w_j^{-1} e_j^T t P_x k k^H P_x}{\|k\|_2^2 - |x^H k|^2}$$

and thus the perturbation matrix and the backward error are given by

$$\Delta E_j = \Delta M_j - \frac{w_j^{-1} e_j^T t P_x k k^H P_x}{\|k\|_2^2 - |x^H k|^2},$$

and thus

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathcal{M}) = \sqrt{\sum_{j=1}^{m} w_j^{-2} \|e_j t\|_2^2 + \frac{\|k\|_2^2 - |k^H x|^2}{H_{w^{-1},2}(\tilde{f}(\lambda, \mu))^2}}.$$

□

The relationship between the structured and unstructured backward errors is the same as in the symmetric and skew-symmetric case, and also the results in the skew-Hermitian case are analogous and omitted here.

In this section we have demonstrated that the backward errors for Hermitian/skew Hermitian nonlinear eigenvalue problems behave analogous as for the polynomial case, except that the distinction between the real and complex case for the function evaluation has to be considered.

Corollary 4.3 *Let $\mathcal{M} \in \mathcal{M}_m(\mathbb{C}^{n,n})$ be an Hermitian/skew-Hermitian matrix polynomial of the form (10), let $(\lambda, \mu) \in \mathcal{R}$, and let $x \in \mathbb{C}^n$ be such that $x^H x = 1$. Then we have the following relations between the structured and unstructured backward errors for an approximate eigenpair of $\mathcal{M}$.*

i) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathcal{M}) = \eta_{w,2}(\lambda, \mu, x, \mathcal{M}),$ if $\tilde{f}_j(\lambda, \mu) \in \mathbb{R}$ *for* $0 \le j \le m$,

ii) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathcal{M}) = \eta_{w,2}(\lambda, \mu, x, \mathcal{M}),$ if $\tilde{f}_j(\lambda, \mu) \in \imath\mathbb{R}$ *for* $0 \le j \le m$,

iii) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathcal{M}) \le \eta_{w,2}(\lambda, \mu, x, \mathcal{M}),$ *otherwise, if* $H_{w^{-1},2}(\tilde{f}(\lambda, \mu))\|T^+\| \le 1$.

iv) $\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathcal{M}) \le \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathcal{M}),$ if $\tilde{f}_j(\lambda, \mu) \in \mathbb{R}$ *for* $0 \le j \le m$,

v) $\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathcal{M}) \le \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathcal{M}),$ *otherwise, if* $H_{w^{-1},2}(\tilde{f}(\lambda, \mu))\|T^+\| \le \sqrt{2}$.

Proof. By Theorem 4.2, we have

$$\begin{aligned}
\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathcal{M}) &= \sqrt{\sum_{j=0}^{m}\left|\frac{t_j}{w_j}\right|^2 + 2\frac{\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}} \\
&\leq \sqrt{\|T^+\||x^H k|^2 - 2\frac{|x^H k|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2} + 2\frac{\|k\|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}} \\
&= \sqrt{|x^H k|^2\left[\|T^+\|^2 - \frac{2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}\right] + 2\frac{\|k\|^2}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}} \\
&\leq \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathcal{M})
\end{aligned}$$

if $H_{w^{-1},2}(\tilde{f}(\lambda,\mu))\|T^+\| \leq \sqrt{2}$, where $\sum_{j=0}^{m}|t_j|^2 = \|t\|^2 = |x^H k|^2\|T^+\|^2$ using (16). The other results follow from Theorem 4.2. □

We illustrate the results with some examples.

Example 4.4 Consider the delay differential equation $\dot{x}(t) = A_1 x(t) + A_2 x_1(t-\tau_1)$, where

$$A_1 = \begin{bmatrix} -5 & 1+i \\ 1-i & -6 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 4 \\ 4 & -1 \end{bmatrix},$$

the delay is $\tau_1 = 1000$, and we set $\mathcal{M}(\lambda) = \lambda I - A_1 - A_2 e^{-\lambda\tau_1}$. The coefficient matrices A_1, A_2 are Hermitian and we have the following backward errors.

If $\lambda \in \mathbb{R}$ such that $\tilde{f}_j(\lambda) \in \mathbb{R}$ then we obtain

λ	$\eta_2(\lambda,x,\mathcal{M})$	$\eta_2^{\mathbf{S}}(\lambda,x,\mathcal{M})$	$\eta_F^{\mathbf{S}}(\lambda,x,\mathbf{L})$
0.3	0.3703	0.3703	0.5104
3	0.9000	0.9000	0.9074
10	0.9780	0.9780	0.9787
10^{-5}	140.5133	140.5133	168.3081

If $\lambda \in \mathbb{C}$ such that $\tilde{f}_j(\lambda) \in \mathbb{C}$ then we obtain

λ	$\eta_2(\lambda,x,\mathcal{M})$	$\eta_2^{\mathbf{S}}(\lambda,x,\mathcal{M})$	$\eta_F^{\mathbf{S}}(\lambda,x,\mathbf{L})$
$20+3i$	0.9903	1.0158	1.0158
$3+5i$	0.9725	1.0176	1.0176
$0.3+10i$	0.9953	1.0163	1.0163
$3i$	60.279	112.5229	112.5229

Example 4.5 Consider the rational eigenvalue problem in homogeneous form

$$\left(\mu A_0 + \lambda A_1 + \frac{\lambda\mu}{\mu-\lambda} A_2\right) x = 0,$$

with Hermitian coefficients

$$A_0 = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -149 & -50i & -154i+1 \\ 50i & 7 & 4+i \\ 154i+1 & 4-i & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 2 & 3i \\ -2i & -3i & 2 \end{bmatrix}.$$

If $(\lambda,\mu) \in \mathbb{C}^2 \backslash \{0\}$ such that $\tilde{f}_j(\lambda,\mu) \in \mathbb{R}$, then we obtain the backward errors

(λ,μ)	$\eta_2(\lambda,\mu,\mathcal{M})$	$\eta_2^{\mathbf{S}}(\lambda,\mu,\mathcal{M})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,\mathbf{L})$
$(2i,3i)$	55.7863	55.7863	66.4099
$(0,2)$	0.4076	0.4076	0.5483
$(4,3)$	62.8229	62.8229	75.6904
$(4,0)$	200.0725	200.0725	239.7038

If $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{0\}$ such that $\tilde{f}_j(\lambda,\mu) \in \mathbb{C} \setminus \mathbb{R}$ then we obtain

(λ,μ)	$\eta_2(\lambda,\mu,\mathcal{M})$	$\eta_2^{\mathbf{S}}(\lambda,\mu,\mathcal{M})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,\mathbf{L})$
$(0.1+0.2i,-0.3+0.9i)$	48.0418	89.9193	89.9193
$(2-3i,-4+3i)$	110.4397	159.6463	159.6463
$(-2-5i,3+7i)$	109.2312	159.4161	159.4161

5 Pseudospectra for nonlinear eigenvalue problems

Pseudospectra are well studied for matrices, matrix pencils, and matrix polynomials, see e.g., [3, 4, 5, 6, 19, 24, 29, 30, 31, 32, 39, 46, 48], and the references therein. In this section we will discuss the determination of pseudospectra for general nonlinear eigenvalue problems such as those associated with the nonlinear function (1) in homogeneous form.

Example 5.1 Consider the use of pseudospectra in the stability analysis of delay differential-algebraic equations from [39]. There the spectral properties of $Q(\lambda) = \lambda M_1 + M_2 + M_3 e^{-\lambda\tau}$ are studied, with coefficient matrices $M_1 = I, M_2, M_3 \in \mathbb{R}^{n\times n}$ and a delay term $\tau \geq 0$. The associated pseudospectrum under perturbations in M_2, M_3 was defined as

$$\Lambda_{\epsilon,w}(\mathcal{M}) := \{\lambda \in \mathbb{C} : (\mathcal{M}(\lambda) + \Delta\mathcal{M}(\lambda))x = 0 \text{ for some } x \neq 0 \text{ and } \Delta\mathcal{M}(\lambda) = \Delta M_2 + \Delta M_3 e^{-\lambda\tau} \text{ with } \|\Delta M_i\| \leq \alpha_i, i = 2, 3\}.$$

Thus, as in our analysis of the backward errors in the previous section, the perturbations are only associated with the coefficients.

It is well known that structured pseudospectra may behave differently than unstructured ones, see [9, 29, 30, 31, 32] for the case of matrix polynomials with even or palindromic structure. Here we discuss pseudospectra for structured nonlinear eigenvalue problems of the form (10), where the coefficients are either symmetric, skew-symmetric, Hermitian, or skew-Hermitian and we consider perturbed problems of the form (9), where we again assume that the perturbations in the functions f_j are known or can be bounded, so that we are dealing with known perturbed functions $\tilde{f}_j$, $j = 1, \ldots, m$. Then for $q \in \{2, F\}$ we define the *pseudospectra* and *structured pseudospectra*

$$\Lambda_{\epsilon,w,q}(\mathcal{M}) := \{(\lambda, \mu) \in \mathcal{R} : \det((M + \Delta M)\tilde{f}(\lambda, \mu)) = 0 \text{ with } \Delta M \in \mathcal{M}_m(\mathbb{C}^{n\times n}) \text{ and } |||\Delta M|||_{w,q} \leq \epsilon\}, \tag{19}$$

$$\Lambda^{\mathbf{S}}_{\epsilon,w,q}(\mathcal{M}) := \{(\lambda, \mu) \in \mathcal{R} : \det((\mathcal{M} + \Delta\mathcal{M})\tilde{f}(\lambda, \mu)) = 0 \text{ with } \Delta M \in \mathcal{M}^{\mathbf{S}}_m(\mathbb{C}^{n\times n}) \text{ and } |||\Delta M|||_{w,q} \leq \epsilon\}, \tag{20}$$

respectively, where $\mathbf{S}$ denotes again the considered structure class for the coefficients.

Consider a structured eigenvalue problem of the form (1). Then for a given value $(\lambda, \mu) \in \mathcal{R}$ that is not a pole of any of the functions $\tilde{f}_j$, $j = 1, \ldots, m$, the *structured distance to a given eigenvalue* in spectral norm is defined as

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, \mathcal{M}) := \min\{|||\Delta M|||_{w.2} : \Delta M \in \mathcal{M}^{\mathbf{S}}_m(\mathbb{C}^{n\times n}) : \ \det((M + \Delta M)\tilde{f}(\lambda, \mu)) = 0\}.$$

This means that the structured distance is equal to the norm of the backward error, i.e., we have

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, \mathcal{M}) = \frac{\|\mathcal{M}(\lambda, \mu)x\|}{H_{w^{-1},2}(\tilde{f}(\lambda, \mu))} = \frac{\|\mathcal{M}(\lambda, \mu)^{-1}\|^{-1}}{H_{w^{-1},2}(\tilde{f}(\lambda, \mu))}. \tag{21}$$

As usual, the ϵ-pseudospectrum can be characterized by η.

Proposition 5.2 *Consider a structured eigenvalue problem of the form (10) with $\mathcal{M} \in \mathcal{M}_m^{\mathbf{S}}(\mathbb{C}^{n\times n})$. Then the structured ϵ-pseudospectrum with respect to the spectral norm is given by*

$$\Lambda_{\epsilon,w}^{\mathbf{S}}(\mathcal{M}) = \{(c,s) \in \mathcal{R} : \eta_{w,2}^{\mathbf{S}}(c,s,\mathcal{M}) \le \epsilon\}.$$

Proof. Let $\mathcal{R}_\epsilon = \{(c,s) \in \mathcal{R} : \eta_{w,2}^{\mathbf{S}}(c,s,\mathcal{M}) \le \epsilon\}$.

We first show that $\Lambda_{\epsilon,w}^{\mathbf{S}}(\mathcal{M}) \subset \mathcal{R}_\epsilon$. Suppose that $(\lambda,\mu) \in \Lambda_{\epsilon,w}^{\mathbf{S}}(\mathcal{M})$ is not an eigenvalue of $\mathcal{M}$. If ΔM is chosen such that

$$\mathcal{M}(\lambda,\mu) + \Delta\mathcal{M}(\lambda,\mu) = \mathcal{M}(\lambda,\mu)[1 + \mathcal{M}(\lambda,\mu)^{-1}\Delta\mathcal{M}(\lambda,\mu)]$$

is singular, then we have

$$\begin{aligned} 1 &\le \|\mathcal{M}(\lambda,\mu)^{-1}\Delta\mathcal{M}(\lambda,\mu)\| \\ &\le \|\mathcal{M}(\lambda,\mu)^{-1}\|\|\Delta\mathcal{M}(\lambda,\mu)\| \\ &\le \|\mathcal{M}(\lambda,\mu)^{-1}\|\epsilon H_{w^{-1},2}(\tilde{f}(\lambda,\mu)). \end{aligned}$$

This implies that

$$\frac{\|\mathcal{M}(\lambda,\mu)^{-1}\|^{-1}}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))} \le \epsilon$$

and hence $(\lambda,\mu) \in \mathcal{R}_\epsilon$.

To prove the converse inclusion, let $(\lambda,\mu) \in \mathcal{R}_\epsilon$ be such that it is not an eigenvalue of $\mathcal{M}$ and not a pole of any of the $\tilde{f}_j$. Choose y with $\|y\|_2 = 1$ such that $\|\mathcal{M}(\lambda,\mu)^{-1}y\| = \|\mathcal{M}(\lambda,\mu)^{-1}\|$ and set

$$x = \frac{\mathcal{M}(\lambda,\mu)^{-1}y}{\|\mathcal{M}(\lambda,\mu)^{-1}\|}$$

which clearly has $\|x\|_2 = 1$. Let W be a matrix with $\|W\|_2 = 1$ and $Wx = y$, and define

$$E := \frac{-W}{\|\mathcal{M}(\lambda,\mu)^{-1}\|}.$$

Then $(\mathcal{M}(\lambda,\mu) + E)x = 0$, and thus $\eta_{w,2}^{\mathbf{S}}(\lambda,\mu,\mathcal{M}) \le \epsilon$, i.e.,

$$\frac{\|\mathcal{M}(\lambda,\mu)^{-1}\|^{-1}}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))} \le \epsilon.$$

Setting then

$$\Delta M_i :== \frac{w_i^{-1}\text{sign}(\tilde{f}_i(\lambda,\mu))|\tilde{f}_i(\lambda,\mu)|}{H_{w^{-1},2}(\tilde{f}(\lambda,\mu))^2}E,$$

we have a desired perturbation, where $\text{sign}(\lambda) := \dfrac{\overline{\lambda}}{|\lambda|^2}$ if $\lambda \neq 0$, and $\text{sign}(\lambda) :=$ 0 if $\lambda = 0$. □

We can also use the relations between the structured and unstructured backward errors to obtain the relationship between the structured and unstructured pseudospectra.

Theorem 5.3 *For symmetric/skew-symmetric eigenvalue problems of the form (10) and provided that none of the functions $\tilde{f}_j$ has a pole at any λ in the pseudospectral sets, we have*

$$\Lambda^{\mathbf{S}}_{\epsilon,w,F}(\mathcal{M}) \subset \Lambda^{\mathbf{S}}_{\sqrt{2}\epsilon,w,2}(\mathcal{M}), \quad \Lambda^{\mathbf{S}}_{\epsilon,w,2}(\mathcal{M}) = \Lambda_{\epsilon,w,2}(\mathcal{M})$$

To make use of the backward errors for the Hermitian/skew-Hermitian case we have to make sure that for all those λ in the pseudospectrum for which not all of the $\tilde{f}_j(\lambda)$ are real or purely imaginary, we have that $H_{w^{-1},2}(\tilde{f}(\lambda)) \leq 1$ in the case of the spectral norm $H_{w^{-1},2}(\tilde{f}(\lambda)) \leq \sqrt{2}$ in the case of the Frobenius norm, see Theorem 3.5 in [8] in the polynomial case.

Theorem 5.4 *For symmetric/skew-symmetric eigenvalue problems of the form (10) we have the following relations between the structured and unstructured pseudospectra.*

(*a*) $\Lambda_{\epsilon,w,2}(\mathcal{M}) \subset \Lambda^{\mathbf{S}}_{\sqrt{2}\epsilon,w,2}(\mathcal{M})$,

(*b*) $\Lambda^{\mathbf{S}}_{\epsilon,w,2}(\mathcal{M}) = \Lambda_{\epsilon,2}(\mathcal{M})$.

For Hermitian/skew-Hermitian eigenvalue problems of the form (10) we have the following relations

(*c*) $\Lambda_{\epsilon,w,F}(\mathcal{M}) \subset \Lambda^{\mathbf{S}}_{\sqrt{2}\epsilon,2}(\mathcal{M})$,

(*d*) $\Lambda^{\mathbf{S}}_{\epsilon,w,F}(\mathcal{M}) \subset \Lambda_{\sqrt{2}\epsilon,w,2}(\mathcal{M})$.

Let us demonstrate these results with an example.

Example 5.5 Consider the eigenvalue problem in Example 5.1 which has the form

$$\mathcal{M}(\lambda) = \lambda M_1 + M_2 + M_3 e^{-\lambda\tau}$$

with coefficients $M_1 = I, M_2, M_3$. Suppose that only the coefficient M_3 of the delay term is perturbed and none of functions $f_1(\lambda) = \lambda$, $f_2(\lambda) = 1$ and $f_3(\lambda) = e^{-\lambda\tau}$, i.e., we choose $w_1 = w_2 = 0$, $w_3 = 1$. Then for a given eigenvalue λ we obtain a perturbation $\Delta\mathcal{M}(\lambda) = \Delta M_3 e^{-\lambda\tau}$. Let $E = \dfrac{-W}{\Delta\mathcal{M}(\lambda)^{-1}}$, with W constructed as in Proposition 5.2, then we obtain

$$\Delta M_3 = \overline{e^{-\tau\lambda}} E\phi,$$

where in this case

$$\phi = H_{w^{-1},2}(f_1(\lambda), f_2(\lambda), f_3(\lambda)) = |e^{-\lambda\tau}|,$$

and the ϵ-pseudospectrum is given by all those ΔM_3 with norm less or equal to ϵ.

6 Conclusion

We have extended the construction of structured backward errors from polynomial eigenvalue problems to nonlinear eigenvalue problems that are linear in the matrix coefficients and have derived a systematic framework for the construction of appropriately structured backward errors for the classes of complex symmetric, complex skew-symmetric, Hermitian, and Skew-Hermitian problems. The resulting minimal perturbation is unique in the case of Frobenius norm and there are infinitely many solution for the case of spectral norm. We have used these results to determine structured pseudospectra and have compared these and also the backward errors to the unstructured case. The results show no surprise, the relation between structured and unstructured backward errors and pseudospectra is as in the polynomial case.

References

[1] B. Adhikari. *Backward perturbation and sensitivity analysis of structured polynomial eigenvalue problem.* PhD thesis, IIT Guwahati, Dept. of Mathematics, 2008.

[2] B. Adhikari and R. Alam. On backward errors of structured polynomial eigenproblems solved by structure preserving linearizations. Technical report, arxiv.org/pdf/0907.2545.

[3] B. Adhikari and R. Alam. Structured backward errors and pseudospectra of structured matrix pencils. *SIAM J. Matrix Anal. Appl.*, 31:331–359, 2009.

[4] S.S. Ahmad. *Pseudospectra of matrix pencils and their applications in the perturbation analysis of eigenvalues and eigendecompositions.* PhD thesis, IIT Guwahati, Dept. of Mathematics, 2007.

[5] S.S. Ahmad and R. Alam. Pseudospectra, critical points and multiple eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 430:1171–1195, 2009.

[6] S.S. Ahmad, R. Alam, and R. Byers. On pseudospectra, critical points and multiple eigenvalues of matrix pencils. *SIAM J. Matrix Anal. Appl.*, 31:1915–1933, 2010.

[7] S.S. Ahmad and V. Mehrmann. Perturbation analysis for complex symmetric, skew symmetric, even and odd matrix polynomials. *Electr. Trans. Num. Anal.*, 38:275–302, 2011.

[8] S.S. Ahmad and V. Mehrmann. Backward errors for eigenvalues and eigenvectors of hermitian, skew-hermitian, H-even and H-odd matrix polynomials. *Lin. Multilin. Alg.*, 1:1–23, 2012.

[9] R. Alam, S. Bora, M. Karowa, V. Mehrmann, and J. Moro. Perturbation theory for Hamiltonian matrices and the distance to bounded-realness. *SIAM J. Matrix Anal. Appl.*, 32:484–514, 2011.

[10] T. Betcke, N.J. Higham, V. Mehrmann, C. Schröder, and F. Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. *J. ACM TOMS*, 39(2):7:1–7:28, February 2013.

[11] S. Bora. Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 31:900–917, 2009.

[12] S. Bora and V. Mehrmann. Perturbation theory for structured matrix pencils arising in control theory. *SIAM J. Matrix Anal. Appl.*, 28:148–169, 2006.

[13] S.-H. Chou, T.-M. Huang, W.-Q. Huang, , and W.-W. Lin. Efficient arnoldi-type algorithms for rational eigenvalue problems arising in fluid-solid systems. *J. Comput. Phys.*, 230:2189–2206, 2011.

[14] C. Davis, W. Kahan, and H. Weinberger. Norm-preserving dialations and their applications to optimal error bounds. *SIAM J. Math. Anal.*, 19:445–469, 1982.

[15] N.H. Du, V.H. Linh, V. Mehrmann, and D.D. Thuan. Stability and robust stability of linear time-invariant delay differential-algebraic equations. *SIAM J. Matrix Anal. Appl.*, 34:1631–1654, 2013. Doi: http://dx.doi.org/10.1137/130926110.

[16] C. Effenberger, D. Kressner, and C. Engström. Linearization techniques for band structure calculations in absorbing photonic crystals. *Int. J. Numer. Meth. Engineering*, 89:180–191, 2012.

[17] I. Gohberg, P. Lancaster, and L. Rodman. Spectral analysis of self adjoint matrix polynomials. *Ann. of Math.(2)*, 112:33–71, 1980.

[18] G.H. Golub and C.F. Van Loan. *Matrix Computations.* The Johns Hopkins University Press, Baltimore, third edition, 1996.

[19] K. Green and T. Wagenknecht. Pseudospectra and delay differential equations. *J. Comput. Appl. Math.*, 196:567–578, 2006.

[20] D.J. Higham and N.J. Higham. Structured backward error and condition of generalized eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 20:493–512, 1998.

[21] N.J. Higham, R.C. Li, and F. Tisseur. Backward error of polynomial eigenproblems solved by lineariation. *SIAM J. Matrix Anal. Appl.*, 29:1218–1241, 2007.

[22] N.J. Higham, D. S. Mackey, and F. Tisseur. The conditioning of linearizations of matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28:1005–1028, 2006.

[23] N.J. Higham, D.S. Mackey, N. Mackey, and F. Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29:143–159, 2006.

[24] N.J. Higham and F. Tisseur. More on pseudospectra for polynomial eigenvalue problems and applications in control theory. *Linear Algebra Appl.*, 351–352:435–453, 2002.

[25] H.Y. Hu and Z.H. Wang. *Dynamics of controlled Mechanical Systems with Delayed Feedback*. Springer-Verlag, Berlin, 2002.

[26] T.M. Hwang, W.W. Lin, J.L. Liu, and W. Wang. Jacobi-Davidson methods for cubic eigenvalue problems. *Numer. Lin. Alg. Appl.*, 12:605–624, 2005.

[27] T.M. Hwang, W.W. Lin, W.C. Wang, and W. Wang. Numerical simulation of three dimensional quantom dot. *J. Comp. Physics*, 196:208–232, 2004.

[28] E. Jarlebring. *The Spectrum of Delay-Differential Equations: Numerical Methods, Stability and Perturbation*. Dissertation thesis, TU Braunschweig, 2008.

[29] M. Karow. Structured pseudospectra and the condition of a nonderogatory eigenvalue. *SIAM J. Matrix Anal. Appl.*, 31:2860–2881, 2010.

[30] M. Karow. μ-values and spectral value sets for linear perturbation classes defined by a scalar product. *SIAM J. Matrix Anal. Appl.*, 32:845–865, 2011.

[31] M. Karow. Structured pseudospectra for small perturbations. *SIAM J. Matrix Anal. Appl.*, 32:1383–1398, 2011.

[32] M. Karow, E. Kokiopoulou, and D. Kressner. On the computation of structured singular values and pseudospectra. *Systems Control Lett.*, 59:122–129, 2010.

[33] D. Kressner, E. Mengi, I. Nakic, and N. Truhar. Generalized eigenvalue problems with specified eigenvalues. Preprint, http://arxiv.org/abs/1009.2222, 2012.

[34] l. Mazurenko and H. Voss. Low rank rational perturbations of linear symmetric eigenproblems. *Z. Angew. Math. Mech.*, 86:606–616, 2006.

[35] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28:1029–1051, 2006.

[36] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28:971–1004, 2006.

[37] V. Mehrmann and C. Schröder. Nonlinear eigenvalue and frequency response problems in industrial practice. *J. Math. in Industry*, 1:7, 2011. http://www.mathematicsinindustry.com/.

[38] V. Mehrmann and H. Voss. Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods. *GAMM Mitteilungen*, 27:121–152, 2004.

[39] W. Michiels, K. Green, T. Wagenknecht, and S. Niculescu. Pseudospectra and stability radii for analytic matrix functions with application to time-delay systems. *Linear Algebra Appl.*, 418:315–335, 2006.

[40] M. Mori. Approximation of exponential functions of a matrix by continued fraction expression. *Publ. RIMS, Kyoto Univ.*, 10:257–269, 1974.

[41] A. Ruhe. Algorithm for nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 10:674–689, 1973.

[42] S.I. Solov'ev. Preconditioned iterative methods for a class of nonlinear eigenvalue problems. *Linear Algebra Appl.*, 415:210–229, 2006.

[43] G. Stepan. *Retarded Dynamical Systems: Stability and characteristic functions.* Longman, New York, 1989.

[44] Y. Su and Z. Bai. Solving rational eigenvalue problems via linearization. Tech. Report 16, Dept. Computer Science, UC Davis, CA, USA, 2008. `url: http://www.cs.ucdavis.edu/research/tech-report/inda.html/`.

[45] F. Tisseur. Backward error and condition of polynomial eigenvalue problem. *Linear Algebra Appl.*, 309:339–361, 2000.

[46] F. Tisseur and N.J. Higham. Structured pseudospectra for polynomial eigenvalue problems, with applications. *SIAM J. Matrix Anal. Appl.*, 23:187–208, 2001.

[47] F. Tisseur and K. Meerbergen. A survey of the quadratic eigenvalue problem. *SIAM Rev.*, 43:234–286, 2001.

[48] L.N. Trefethen and M. Embree. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators.* Princeton Univ. Press, Princeton, N.J., 2005.

[49] H. Voss. A rational spectral in fluid solid vibration. *Electr. Trans. Num. Anal.*, 16:94–106, 2003.

[50] H. Voss. An Arnoldi method for nonlinear eigenvalue problems. *BIT*, 44:387–401, 2004.

[51] C. Yang. Solving large-scale eigenvalue problems in scidac applications. *J. Conference Series*, 16:425–434, 2005.