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Robert Altmann   Christoph Zimmer

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RUNGE-KUTTA METHODS FOR LINEAR SEMI-EXPLICIT OPERATOR DIFFERENTIAL-ALGEBRAIC EQUATIONS

R. ALTMANN*, C. ZIMMER†

Abstract. As a first step towards time-stepping schemes for constrained PDE systems, this paper presents convergence results for the temporal discretization of operator DAEs. We consider linear, semi-explicit systems which includes e.g. the Stokes equations or applications with boundary control. To guarantee unique approximations, we restrict the analysis to algebraically stable Runge-Kutta methods for which the stability functions satisfy $R(\infty) = 0$. As expected from the theory of DAEs, the convergence properties of the single variables differ and depend strongly on the assumed smoothness of the data.

Key words. operator DAEs, PDAEs, Runge-Kutta methods, implicit Euler scheme, regularization

AMS subject classifications. 65J10, 65L80, 65M12

1. Introduction

The mixture of partial-differential equations (PDEs) and differential-algebraic equations (DAEs) provides a promising modeling approach for the simulation of (coupled) physical systems. These so-called *PDAEs* or *operator DAEs* follow the paradigm of including all available information to the system rather than implicitly eliminating variables. Typical applications are given by the Navier-Stokes equations with the incompressibility as constraint [Tem77, EM13], flexible multibody systems [Sim00, Sim13], circuit networks constrained by Kirchhoff's laws [Tis96, Tis03], or the gas transfer in pipeline networks [GJH$^+$14, JT14].

Simultaneously, these simplifications in the modeling lead to difficulties in the mathematical treatment, i.e., within the analysis and numerical simulation of such systems. It is well-known that DAEs suffer from instabilities, drift-off phenomena, and ill-posedness [GM86, KM06, LMT13] which carries over to the infinite-dimensional PDAE case. Thus, regularization techniques are needed which are called *index reduction* in the DAE case [HW96]. The corresponding procedure for operator equations was introduced in [Alt13, AH15].

In this paper, we consider linear operator DAEs of semi-explicit structure, i.e., systems of the form

$$\dot u(t) + \mathcal{K}u(t) - \mathcal{B}^*p(t) = \mathcal{F}(t), \tag{1.1a}$$

$$\mathcal{B}u(t) = \mathcal{G}(t) \tag{1.1b}$$

with linear, continuous operators $\mathcal{K}$, $\mathcal{B}$ and abstract functions $\mathcal{F}$, $\mathcal{G}$ with values in duals of some Hilbert spaces $\mathcal{V}$ and $\mathcal{Q}$, respectively. The solution consists of the *abstract functions* $u\colon [0,T] \to \mathcal{V}$ and $p\colon [0,T] \to \mathcal{Q}$. This includes the linear Stokes equations in which

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the Lagrange multiplier p equals the pressure as well as parabolic PDEs with boundary control.

The aim of this paper is to provide convergence results for a specific class of Runge-Kutta schemes applied to operator DAEs of the form (1.1). In contrast to ODEs, the application of Runge-Kutta schemes to DAEs may lead to a reduction of the convergence order or even a loss of convergence, cf. [Pet86, HLR89] or [KM06, Ch. 5.2]. For DAEs of index 2 the convergence of implicit Runge-Kutta schemes is often preserved. However, the order of convergence may be limited by two [Arn98].

The convergence of Runge-Kutta methods applied to parabolic PDEs was already discussed in [LO93, LO95]. However, these papers mainly work in the framework of semi-groups and not in the here presented setting with Gelfand triples and weak regularity assumptions on the data. The here presented analysis is mainly based on [ET10], where stiffly accurate Runge-Kutta methods were applied to nonlinear evolution equations with an hemicontinuous, monotone, and coercive operator.

First results on the convergence of the implicit Euler scheme for semi-explicit operator DAEs were presented in [Alt15, Ch. 10]. For the special case of the Navier-Stokes equations, the implicit Euler scheme and a two-step BDF method were analyzed in [Emm01]. Of special interest for the constrained operator case (1.1) is the convergence of the Lagrange multiplier p. To show the convergence towards p we need to assume more regularity than for the convergence of u.

The paper is structured as follows. In Section 2 we introduce the problem class and the used notation within the paper. Since the considered operator DAEs are unstable in terms of perturbations, we consider a regularization of the system equations in Section 3. We use the resulting system to prove the convergence of the implicit Euler scheme in Section 4 and consider general implicit Runge-Kutta schemes in Section 5. Finally, we conclude and give an outlook in Section 6.

2. Preliminaries

For a better understanding of the convergence analysis in Sections 4 and 5, we summarize some properties of Runge-Kutta methods applied to DAEs. Following, we provide the functional analytical framework for the formulation and analysis of semi-explicit operator DAEs of the form (1.1). For this, we define suitable Sobolev-Bochner spaces for the solution and discuss necessary properties of the involved operators and right-hand sides.

2.1. **Runge-Kutta Methods for DAEs.** A Runge Kutta method is defined by the *Butcher tableau*

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array} \tag{2.1}$$

with $\mathbf{b}, \mathbf{c} \in \mathbb{R}^s$ and $\mathbf{A} \in \mathbb{R}^{s\times s}$, see also [HW96, Ch. IV.3]. Therein, s denotes the number of stages. It is well-known that for the numerical treatment of DAEs it is necessary that $\mathbf{A}$ is invertible, i.e., that the method is implicit [KM06, Ch. 5.2]. This remains true when the methods are applied to infinite-dimensional operator equations with constraints.

Consider an initial value problem of a regular, linear, and time-invariant DAE

$$E\dot{y} = Ky + f(t), \qquad y(0) = y^0$$

with a sufficiently smooth right-hand side $f\colon [0,T] \to \mathbb{R}^n$ and a unique solution $y \in C^1([0,T];\mathbb{R}^n)$. Note that, considering the DAE case, we do not assume the matrix E to be invertible.

With the Kronecker product [HJ91, Def. 4.2.1] given by $\otimes$, one step of an implicit Runge-Kutta method with constant step size τ leads to the iteration scheme

$$y_j = (1 - \mathbf{b}^T \mathbf{A}^{-1} \mathbb{1}_s) y_{j-1} + (\mathbf{b}^T \mathbf{A}^{-1} \otimes I_n) \boldsymbol{y}_j \tag{2.2a}$$

$$\frac{1}{\tau_j} (\mathbf{A}^{-1} \otimes E)(\boldsymbol{y}_j - \mathbb{1}_s \otimes y_{j-1}) = (I_s \otimes K) \boldsymbol{y}_j + \boldsymbol{F}_j. \tag{2.2b}$$

Therein, $y_j \in \mathbb{R}^n$ is an approximation of $y(t_j)$ and $\boldsymbol{y}_j \in \mathbb{R}^{s \cdot n}$ are the so-called *internal stages*. Furthermore, $\boldsymbol{F}_j \in \mathbb{R}^{s \cdot n}$ is defined by

$$\boldsymbol{F}_j = \left[f(t_{j-1} + \tau \mathbf{c}_1)^T, \ldots, f(t_{j-1} + \tau \mathbf{c}_s)^T \right]^T \in \mathbb{R}^{s \cdot n} \quad \text{and} \quad \mathbb{1}_s = \left[1, \ldots, 1\right]^T \in \mathbb{R}^s.$$

In the following, we consider Runge-Kutta schemes that satisfy

$$R(\infty) := 1 - \mathbf{b}^T \mathbf{A}^{-1} \mathbb{1}_s = 0.$$

In this way, the stability function R vanishes in the limit which controls the damping of the stiff components in the system [HW96, Ch. IV.3]. This assumption guarantees that $(1 - \mathbf{b}^T \mathbf{A}^{-1} \mathbb{1}_s) y_{j-1}$ vanishes in equation (2.2a) such that the values of the previous step are not needed for variables which are in the kernel of E. Thus, algebraic variables are not treated as differential variables.

An important class of Runge-Kutta schemes in the numerical treatment of DAEs are so-called stiffly accurate methods [KM06, Ch. 5.2].

Definition 2.1 (Stiffly accurate)**.** A Runge-Kutta scheme with s stages and Butcher tableau $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ is called *stiffly accurate* if $\mathbf{b}$ satisfies $\mathbf{b}^T = e_s^T \mathbf{A}$ with $e_s = [0, \ldots, 0, 1]^T \in \mathbb{R}^s$.

Note that stiffly accurate schemes automatically satisfy the above mentioned assumption, since

$$R(\infty) = 1 - \mathbf{b}^T \mathbf{A}^{-1} \mathbb{1}_s = 1 - e_s^T \mathbb{1}_s = 0.$$

In particular, the approximation y_j is given by the last n components of $\boldsymbol{y}_j$.

Example 2.2. A stiffly accurate Runge-Kutta method of second order with two stages is defined by the Butcher tableau

$$\mathbf{A} = \begin{bmatrix} -3.25 & 6.25 \\ -0.25 & 1.25 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -0.25 \\ 1.25 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Definition 2.3 (Algebraically stable [BB79])**.** A Runge-Kutta scheme with Butcher tableau $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ is called *algebraically stable* if $\mathbf{b}$ has only non-negative entries and $\mathbf{BA} + \mathbf{A}^T \mathbf{B} - \mathbf{bb}^T$ is positive semidefinite with the diagonal matrix $\mathbf{B} \in \mathbb{R}^{s \times s}$ given by $\mathbf{B}_{jj} = \mathbf{b}_j$.

Within this paper, we will assume that the operator $\mathcal{K}$ is positive definite. The property of being algebraically stable then ensures that this remains true for the time discretized problem. This characteristic has proven to be crucial for the time discretization of parabolic equations [LO95].

2.2. **Spaces and Embeddings.** The spaces $\mathcal{V}$, $\mathcal{H}$, and $\mathcal{Q}$ appearing in the analysis of the operator DAE (1.1) are Hilbert spaces. The Hilbert space $\mathcal{V}$ is continuously and densely embedded in the pivot space $\mathcal{H}$ such that we have a Gelfand triple [Zei90, Ch. 23.4] given by

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}^* \hookrightarrow \mathcal{V}^*,$$

where $\mathcal{V}^*$ and $\mathcal{H}^*$ denote the dual spaces of $\mathcal{V}$ and $\mathcal{H}$, respectively. Note that $\hookrightarrow$ denotes a continuous embedding throughout the paper. The space $\mathcal{V}$ serves as ansatz space for the variable u, whereas $\mathcal{Q}$ is the ansatz space for p. The dual space of $\mathcal{Q}$ is denoted by $\mathcal{Q}^*$.

As a solution of the operator DAE (1.1) we search for abstract functions $u\colon [0,T] \to \mathcal{V}$, $p\colon [0,T] \to \mathcal{Q}$. For this, we introduce $L^2(0,T;\mathcal{X})$ as the space of quadratic Bochner integrable functions with values taken in the Hilbert space $\mathcal{X}$, see e.g. [Zei90, Ch. 23.2] or [Rou05, Ch. 1.7] for an introduction. Furthermore, we consider the space of Bochner integrable functions which have a time derivative in the distributional sense, i.e.,

$$W^{1,2}(0,T;\mathcal{X},\mathcal{Y}) := \big\{x \in L^2(0,T;\mathcal{X}) \;\big|\; \dot{x} \text{ exists in } L^2(0,T;\mathcal{Y})\big\}$$

for Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ with $\mathcal{X} \hookrightarrow \mathcal{Y}$. In the case $\mathcal{X} = \mathcal{Y}$, we also write $H^1(0,T;\mathcal{X}) := W^{1,2}(0,T;\mathcal{X},\mathcal{X})$. Of great importance for the convergence analysis are the subspaces of $\mathcal{V}$ and $\mathcal{H}$ including the kernel of the linear constraint operator $\mathcal{B}\colon \mathcal{V} \to \mathcal{Q}^*$. For this, we define

$$\mathcal{V}_\mathcal{B} := \{v \in \mathcal{V} \mid \mathcal{B}v = 0\} = \ker \mathcal{B}$$

and introduce its *polar set* by

$$\mathcal{V}_\mathcal{B}^0 := \big\{f \in \mathcal{V}^* \mid \langle f, v_\mathcal{B}\rangle = 0 \text{ for all } v \in \mathcal{V}_\mathcal{B}\big\} \subseteq \mathcal{V}^*.$$

Since we assume the operator $\mathcal{B}$ to be linear and continuous, $\mathcal{V}_\mathcal{B}$ is a closed subspace of $\mathcal{V}$. The closed subspace of $\mathcal{H}$ given by the closure of $\mathcal{V}_\mathcal{B}$ in $\mathcal{H}$ is denoted by $\mathcal{H}_\mathcal{B}$. We emphasize that the spaces $\mathcal{V}_\mathcal{B}$, $\mathcal{H}_\mathcal{B}$, $\mathcal{V}_\mathcal{B}^*$ form again a Gelfand triple. Under the assumptions on the operator $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$, which we discuss in Section 2.3, we note that the space

$$\mathcal{V}_\mathrm{c} := \{v \in \mathcal{V} \mid \mathcal{K}v \in \mathcal{V}_\mathcal{B}^0\}$$

is a closed subspace of $\mathcal{V}$ and forms a complement to $\mathcal{V}_\mathcal{B}$.

For the application of Runge-Kutta methods to operator equations, we need the above spaces in s components. This is necessary in order to define generalized state vectors. For this, we introduce

$$\mathcal{V}_s := [\mathcal{V}]^s, \quad \mathcal{H}_s := [\mathcal{H}]^s, \quad \mathcal{Q}_s := [\mathcal{Q}]^s, \quad \mathcal{V}_{\mathcal{B},s} := [\mathcal{V}_\mathcal{B}]^s, \quad \mathcal{V}_{\mathrm{c},s} := [\mathcal{V}_\mathrm{c}]^s.$$

Accordingly, we define the dual spaces $\mathcal{V}_s^*$, $\mathcal{V}_{\mathcal{B},s}^*$, and $\mathcal{Q}_s^*$.

2.3. **Norms and Operators.** For the norms and inner products of the Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$ we use the abbreviations

$$\|\cdot\| := \|\cdot\|_\mathcal{V}, \qquad |\cdot| := \|\cdot\|_\mathcal{H}, \qquad (\cdot,\cdot) := (\cdot,\cdot)_\mathcal{H}.$$

All dual spaces are equipped with the standard operator norm. The operators $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$ and $\mathcal{B}\colon \mathcal{V} \to \mathcal{Q}^*$ in the operator DAE (1.1) are assumed to be linear, continuous, and time-independent. Furthermore, $\mathcal{K}$ is elliptic on $\mathcal{V}_\mathcal{B}$, i.e., there exits a constant $\alpha > 0$ with

$$\alpha\,\|v\|^2 \le \langle \mathcal{K}v, v\rangle$$

for all $v, w \in \mathcal{V}_\mathcal{B}$. With these assumptions on the operators, we obtain a decomposition of the space $\mathcal{V}$.

Lemma 2.4. *Let $\mathcal{V}_\mathcal{B}$ and $\mathcal{V}_c$ be defined as in Section 2.2. Furthermore, let $\mathcal{B}$ be linear and continuous and $\mathcal{K}$ linear, continuous, and elliptic on $\mathcal{V}_\mathcal{B}$. Then, $\mathcal{V}_\mathcal{B}$ and $\mathcal{V}_c$ are closed subspaces of $\mathcal{V}$ and we have the splitting $\mathcal{V} = \mathcal{V}_\mathcal{B} \oplus \mathcal{V}_c$.*

Proof. By the linearity and continuity of $\mathcal{B}$ and $\mathcal{K}$ it follows that $\mathcal{V}_\mathcal{B}$ and $\mathcal{V}_\mathrm{c}$ are closed subspaces of $\mathcal{V}$. The ellipticity of $\mathcal{K}$ shows that $v \in \mathcal{V}_\mathcal{B} \cap \mathcal{V}_\mathrm{c}$ implies $v = 0$. It remains to show that $\mathcal{V} \subseteq \mathcal{V}_\mathcal{B} \oplus \mathcal{V}_\mathrm{c}$. Let $u \in \mathcal{V}$ be given. By the Lax-Milgram theorem there exists a unique $u_\mathcal{B} \in \mathcal{V}_\mathcal{B}$ with $\langle \mathcal{K}u_\mathcal{B}, v_\mathcal{B}\rangle = \langle \mathcal{K}u, v_\mathcal{B}\rangle$ for all $v_\mathcal{B} \in \mathcal{V}_\mathcal{B}$ [Bre10, Cor. 5.8]. We define $u_\mathrm{c} := u - u_\mathcal{B}$ and observe

$$\langle \mathcal{K}u_\mathrm{c}, v_\mathcal{B}\rangle = \langle \mathcal{K}u, v_\mathcal{B}\rangle - \langle \mathcal{K}u_\mathcal{B}, v_\mathcal{B}\rangle = 0$$

for all $v_\mathcal{B} \in \mathcal{V}_\mathcal{B}$ such that $\mathcal{K}u_\mathrm{c} \in \mathcal{V}_\mathcal{B}^0$ and thus, $u_\mathrm{c} \in \mathcal{V}_\mathrm{c}$ and $\mathcal{V} \subseteq \mathcal{V}_\mathcal{B} \oplus \mathcal{V}_\mathrm{c}$. □

In addition to the linearity and continuity, the operator $\mathcal{B}\colon \mathcal{V} \to \mathcal{Q}^*$ is required to fulfill an inf-sup condition, i.e., there exists a positive constant $\beta \in \mathbb{R}$ such that

$$\inf_{q\in\mathcal{Q}\setminus\{0\}} \sup_{v\in\mathcal{V}\setminus\{0\}} \frac{\langle \mathcal{B}v, q\rangle}{\|v\|\|q\|_{\mathcal{Q}}} \geq \beta > 0. \tag{2.3}$$

This condition implies that $\mathcal{B}$ is surjective and thus, its adjoint operator $\mathcal{B}^*$ is injective. Therefore, we get isomorphisms if we restrict the domain of definition of $\mathcal{B}$ and the range of $\mathcal{B}^*$.

Lemma 2.5. *Let $\mathcal{V}_c$ and $\mathcal{V}^0_{\mathcal{B}}$ be defined as in Section 2.2 and let $\mathcal{B}$ satisfy the inf-sup condition* (2.3)*. Then, the restriction $\mathcal{B}\colon \mathcal{V}_c \to \mathcal{Q}^*$ is an isomorphism as well as $\mathcal{B}^*\colon \mathcal{Q} \to \mathcal{V}^0_{\mathcal{B}}$.*

Proof. The second statement can be found in [Bra07, Lem. III.4.2]. The quoted lemma also proves the existence of a right inverse $\mathcal{B}^-_\perp\colon \mathcal{Q}^* \to \mathcal{V}^\perp_{\mathcal{B}} \subseteq \mathcal{V}$ from $\mathcal{B}$ where $\mathcal{V}^\perp_{\mathcal{B}}$ denotes the orthogonal complement of $\mathcal{V}_{\mathcal{B}}$ in $\mathcal{V}$. Since $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}_{\mathrm{c}}$ are closed subspaces of $\mathcal{V} = \mathcal{V}_{\mathcal{B}} \oplus \mathcal{V}_{\mathrm{c}}$, there exists a projector $\mathcal{P}_{\mathcal{B}}\colon \mathcal{V} \to \mathcal{V}_{\mathcal{B}} \subseteq \mathcal{V}$ with kernel $\mathcal{V}_{\mathrm{c}}$ [BK14, Th. 4.42]. The definition of $\mathcal{B}^-\colon \mathcal{Q}^* \to \mathcal{V}_{\mathrm{c}} \subseteq \mathcal{V}$ by $\mathcal{B}^- := (\mathrm{id}_{\mathcal{V}} - \mathcal{P}_{\mathcal{B}})\mathcal{B}^-_\perp$ then implies

$$\mathcal{B}\mathcal{B}^- = \mathcal{B}\,\mathrm{id}_{\mathcal{V}}\,\mathcal{B}^-_\perp - \mathcal{B}\mathcal{P}_{\mathcal{B}}\mathcal{B}^-_\perp = \mathcal{B}\mathcal{B}^-_\perp = \mathrm{id}_{\mathcal{Q}^*}.$$

Thus, $\mathcal{B}^-$ is a right inverse of $\mathcal{B}$. As a result, there exists a bounded inverse of the restriction $\mathcal{B}\colon \mathcal{V}_{\mathrm{c}} \to \mathcal{Q}^*$. □

Remark 2.6. Since $\mathcal{V}_{\mathcal{B}}$ is densely embedded in $\mathcal{H}_{\mathcal{B}}$, we may define $\mathcal{B}h := 0$ for $h \in \mathcal{H}_{\mathcal{B}}$. For this, consider a sequence $v_n \in \mathcal{V}_{\mathcal{B}}$ such that $v_n \to h$ in $\mathcal{H}_{\mathcal{B}}$. This sequence then satisfies $0 = \mathcal{B}v_n$.

Remark 2.7. Consider $u \in W^{1,2}(0,T;\mathcal{V},\mathcal{V}^*)$ with the unique decomposition $u = u_{\mathcal{B}} + u_{\mathrm{c}}$ and $\dot u_{\mathcal{B}} \in \mathcal{H}_{\mathcal{B}}$ and $\dot u_{\mathrm{c}} \in \mathcal{V}_{\mathrm{c}}$. Then, we write $u \in W^{1,2}(0,T;\mathcal{V},\mathcal{H}_{\mathcal{B}} + \mathcal{V}_{\mathrm{c}})$ and define $\mathcal{B}\dot u$ by

$$\mathcal{B}\dot u = \mathcal{B}\dot u_{\mathcal{B}} + \mathcal{B}\dot u_{\mathrm{c}} = \mathcal{B}\dot u_{\mathrm{c}}$$

which is then well-defined by the previous Remark 2.6.

3. Regularization

The spatial discretization of the linear semi-explicit operator DAE (1.1) leads to a DAE of *differentiation index* 2. Recall that the index measures, loosely speaking, the distance of a DAE from an ODE and thus, provides a measure of difficulty [Meh13].

Motivated by the GGL formulation for multibody systems [GGL85], we include the hidden constraint by the introduction of an additional Lagrange multiplier. Such a regularization is necessary, since operator DAEs are highly sensitive to perturbations [Alt15]. The proposed regularization makes the system more robust and achieves that a spatial discretization leads to a DAE of index 1 rather than index 2.

3.1. Formulation as Operator DAE.

With the introduction of suitable ansatz spaces for the solution in Section 2.2, we formulate once more problem (1.1): For given right-hand sides $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$ and $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$ find $u \in W^{1,2}(0,T;\mathcal{V},\mathcal{V}^*)$ and $p \in L^2(0,T;\mathcal{Q})$ such that for a.e. $t \in [0,T]$ it holds that

$$\dot u(t) + \mathcal{K}u(t) - \mathcal{B}^*p(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}^*, \tag{3.1a}$$

$$\mathcal{B}u(t) = \mathcal{G}(t) \quad \text{in } \mathcal{Q}^*. \tag{3.1b}$$

In addition, u should satisfy an initial condition of the form $u(0) = a \in \mathcal{H}$. Note that the embedding $W^{1,2}(0,T;\mathcal{V},\mathcal{V}^*) \hookrightarrow C([0,T];\mathcal{H})$ implies that $u(0)$ is well-defined in $\mathcal{H}$.

We assume the initial data to be consistent, i.e, a should be compatible with the constraint (3.1b), see the discussion in [Alt15, Rem 6.9]. This means that a has the form $a = a_{\mathcal{B}} + \mathcal{B}^- \mathcal{G}(0)$ with $a_{\mathcal{B}} \in \mathcal{H}_{\mathcal{B}}$. As introduced in Section 2.3, the operator $\mathcal{B}$ in linear, bounded, and fulfills an inf-sup condition such that there exists a right inverse $\mathcal{B}^-\colon \mathcal{Q}^* \to \mathcal{V}_{\mathrm{c}}$ by Lemma 2.5. The operator $\mathcal{K}$ is linear, bounded, and elliptic on $\mathcal{V}_{\mathcal{B}}$. For a discussion on the existence of a unique solution (u, p), we refer to [EM13].

Example 3.1 (Unsteady Stokes equations)**.** The weak formulation of the linear unsteady Stokes equations, which characterize the evolution of a Newtonian fluid [Tem77], can be written as an operator DAE of the form (3.1). The variable u then describes the velocity of the fluid whereas p denotes the pressure which is assumed to have zero mean. For this, we consider the weak formulation, i.e., the operator $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$ corresponds to the Laplace operator and is defined by

$$\langle \mathcal{K}u, v\rangle := \nu \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x.$$

The operators $\mathcal{B}\colon \mathcal{V} \to \mathcal{Q}^*$ and its dual $\mathcal{B}^*\colon \mathcal{Q} \to \mathcal{V}^*$ correspond to the divergence and (minus) the gradient operator, respectively. For the application of the Stokes equation, we consider the Hilbert spaces

$$\mathcal{V} := [H^1_0(\Omega)]^d, \qquad \mathcal{H} := [L^2(\Omega)]^d, \qquad \mathcal{Q} := L^2(\Omega)/\mathbb{R}.$$

Therein, $\Omega \subseteq \mathbb{R}^d$ denotes the bounded computational domain with Lipschitz boundary. The space $\mathcal{V}_{\mathcal{B}}$ are the divergence-free functions of $[H^1_0(\Omega)]^d$. Its closure $\mathcal{H}_{\mathcal{B}}$ is the subset of functions in $[L^2(\Omega)]^d$ with a vanishing divergence in the distributional sense and a well-defined trace in normal direction [Tem77, Ch. 1.4].

Example 3.2 (Heat equation with boundary control)**.** The constraint may also be used for boundary control [HPUU09]. For this, $\mathcal{B}$ equals the trace operator, i.e., $\mathcal{B}\colon \mathcal{V} := H^1(\Omega) \to \mathcal{Q}^* := H^{1/2}(\Omega)$, cf. [Tar07, Ch. 13]. The operator $\mathcal{K}$ would again correspond to the Laplace operator as in Example 3.1. Since the closure of $\mathcal{V}_{\mathcal{B}} = H^1_0(\Omega)$ in $\mathcal{H} := L^2(\Omega)$ equals $\mathcal{H}$ itself, the initial data only has to satisfy $a \in \mathcal{H}$, i.e., there is no consistency condition.

For the regularization of system (3.1) we extend the system by a Lagrange multiplier $\lambda\colon [0, T] \to \mathcal{Q}$. With this, we enforce the system to satisfy additionally the hidden constraint, i.e., the derivative of constraint (3.1b).

3.2. **Finite-dimensional Case.** Consider the DAE which results from a spatial discretization of system (3.1) by finite elements. With the positive definite mass matrix $M \in \mathbb{R}^{n_q,n_q}$ as discretized version of $(\cdot,\cdot)$, the matrix $K \in \mathbb{R}^{n_q,n_q}$ as discrete version of $\mathcal{K}$, and the constraint matrix $B \in \mathbb{R}^{n_r,n_q}$, which we assume to be of full rank, the DAE has the form

$$M\dot q(t) + Kq(t) - B^T r(t) = f(t), \tag{3.2a}$$

$$Bq(t) \qquad\qquad = g(t). \tag{3.2b}$$

Therein, $q = [q_i] \in \mathbb{R}^{n_q}$ denotes the coefficient vector to a given basis of the finite element space which approximates the solution $u \in \mathcal{V}$. The vector $r = [r_i] \in \mathbb{R}^{n_r}$ corresponds to the variable $p \in \mathcal{Q}$ in the continuous setting. The initial condition is given by $q(0) = q_0$ and is consistent if $Bq_0 = g(0)$. It is well-known that the DAE (3.2) is of index 2 [HW96, Ch. VII.1].

As mentioned above, we reduce the index, and thus regularize the system equations, by adding the hidden constraint and an additional Lagrange multiplier $\mu \in \mathbb{R}^{n_\mu}$. With some regular matrix $C \in \mathbb{R}^{n_\mu, n_\mu}$, the extended DAE reads

$$M\dot{q}(t) + Kq(t) - B^T r(t) - B^T \mu(t) = f(t), \tag{3.3a}$$

$$Bq(t) \qquad - \quad C\mu(t) = g(t), \tag{3.3b}$$

$$B\dot{q}(t) \qquad\qquad = \dot{g}(t). \tag{3.3c}$$

In the following lemma we show that this system is equivalent to the DAE (3.2) but lowers the index.

Lemma 3.3. *The DAE* (3.3) *has index 1. For consistent initial data $q_0 \in \mathbb{R}^n$, i.e., $Bq_0 = g(0)$, the DAEs* (3.2) *and* (3.3) *are equivalent in the following sense. A solution pair (q, r) of system* (3.2) *implies the solution $(q, r, 0)$ of* (3.3)*. On the other hand, a solution (q, r, μ) of system* (3.3) *satisfies $\mu = 0$ and (q, r) solves the original DAE* (3.2)*.*

Proof. To show the index-1 property, we write system (3.3) in block structure,

$$\left[\begin{array}{cc|c} M & B^T & B^T \\ B & 0 & 0 \\ \hline 0 & 0 & C \end{array}\right] \begin{bmatrix} \dot{q} \\ -r \\ -\mu \end{bmatrix} = \begin{bmatrix} f - Kq \\ \dot{g} \\ g - Bq \end{bmatrix}.$$

Since M is positive definite and B is of full rank, the left upper 2-by-2 block in the matrix on the left-hand side is regular. The invertibility of C then implies that the whole matrix on the left-hand side is invertible. Thus, we obtain an ODE for q and algebraic equations for r and μ without any differentiations.

For the stated equivalence let (q, r) be a solution of the DAE (3.2). Obviously, $(q, r, 0)$ solves the DAE (3.3) since (q, r) also has to satisfy the hidden constraint (3.3c). For the reverse direction let (q, r, μ) be a solution of system (3.3). Since C is invertible, we obtain the explicit formula

$$\mu = C^{-1}\big(Bq - g\big).$$

Equation (3.3c) then implies $\dot{\mu} = 0$ such that μ has to be constant. Because of the consistency of the initial data, we have

$$\mu \equiv \mu(0) = C^{-1}\big(Bq(0) - g(0)\big) = 0. \qquad \square$$

Remark 3.4. An alternative strategy to reduce the index of system (3.2) is given by the introduction of a small term εr in the constraint equation (3.2b). Note that this is known as *penalty method* in the field of fluid dynamics [HV95, She95]. However, this kind of methods strongly depend on a wise choice of the parameter ε, particularly if iterative solvers are used [BH15].

3.3. **Infinite-dimensional Case.** The index reduction procedure from the previous subsection motivates to apply the same ideas also to the operator DAE (3.2). This then leads to an extended system of the form: find $u \in W^{1,2}(0, T; \mathcal{V}, \mathcal{H}_\mathcal{B} + \mathcal{V}_c)$, $p \in L^2(0, T; \mathcal{Q})$, and $\lambda \in L^2(0, T; \mathcal{Q})$ such that for a.e. $t \in [0, T]$ it holds that

$$\dot{u}(t) + \mathcal{K}u(t) - \mathcal{B}^* p(t) - \mathcal{B}^*\lambda(t) = \mathcal{F}(t) \qquad \text{in } \mathcal{V}^*, \tag{3.4a}$$

$$\mathcal{B}u(t) \qquad - \quad \mathcal{C}\lambda(t) = \mathcal{G}(t) \qquad \text{in } \mathcal{Q}^*, \tag{3.4b}$$

$$\mathcal{B}\dot{u}(t) \qquad\qquad = \dot{\mathcal{G}}(t) \qquad \text{in } \mathcal{Q}^* \tag{3.4c}$$

with consistent initial value $u(0) = a$. The right-hand sides are still assumed to satisfy $\mathcal{F} \in L^2(0, T; \mathcal{V}^*)$ and $\mathcal{G} \in H^1(0, T; \mathcal{Q}^*)$ whereas the linear operator $\mathcal{C}\colon \mathcal{Q} \to \mathcal{Q}^*$ is assumed to be elliptic and bounded. Recall that equation (3.4c) is well-defined by Remark 2.7.

Remark 3.5. Compared to the extended system proposed in [AH15] we need here additional regularity of the velocity u, namely $\dot{u} \in L^2(0,T;\mathcal{H}_\mathcal{B} + \mathcal{V}_\mathrm{c})$. The formulation in [AH15] gets away with $\dot{u} \in L^2(0,T;\mathcal{V}^*)$ where only the derivative of the component in $\mathcal{V}_\mathrm{c}$ has to take values in $\mathcal{V}$. Note that this means no restriction, since this follows directly from the regularity of $\mathcal{G}$. However, the resulting DAE in the present approach is better structured in the sense that the linear systems one has to solve in every time step are better suited for iterative solvers. This is caused by the preservation of the saddle point structure which allows to apply effective solution algorithms, cf. [BWY90, BGL05] and the references therein.

From the construction of the operator DAE (3.4) and the results of the previous subsection, we already know that a spatial discretization leads to a DAE of the form (3.3) and thus, is of index 1. It remains to show the equivalence of the original and extended operator DAE. This goes hand in hand with Lemma 3.3 for the finite-dimensional case.

Lemma 3.6. *Consider right-hand sides $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$ and consistent initial data $a \in \mathcal{H}$, i.e., $a = a_\mathcal{B} + a_c$ with $a_\mathcal{B} \in \mathcal{H}_\mathcal{B}$ and $a_c = \mathcal{B}^-\mathcal{G}(0) \in \mathcal{V}_c$. Then, the operator DAEs* (3.1) *and* (3.4) *are equivalent in the following sense. Every solution (u,p) of* (3.1) *with $\dot{u} \in L^2(0,T;\mathcal{H}_\mathcal{B} + \mathcal{V}_c)$ implies a solution $(u,p,0)$ of the operator DAE* (3.4)*. On the other hand, if (u,p,λ) solves the extended system, then $\lambda \equiv 0$ and (u,p) is a solution of system* (3.1)*.*

Proof. Let (u,p) be a solution of the operator DAE (3.1). Since (3.4c) is just the time derivative of $\mathcal{B}u = \mathcal{G}$, the triple $(u,p,0)$ solves system (3.4) if $\mathcal{B}\dot{u}$ is well-defined. This is the case if we assume the additional regularity $\dot{u} \in L^2(0,T;\mathcal{H}_\mathcal{B} + \mathcal{V}_\mathrm{c})$. For the reverse direction let (u,p,λ) denote a solution of the extended system (3.4). As in the proof of Lemma 3.3, the consistency condition and equations (3.4b)-(3.4c) imply that $\mathcal{C}\lambda = 0$. Since the operator $\mathcal{C}$ is injective by its ellipticity, we obtain $\lambda \equiv 0$ and thus, (u,p) solves the operator DAE (3.1). □

Remark 3.7. Within this paper we always assume to have a consistent initial condition, i.e., $u(0) = a = a_\mathcal{B} + \mathcal{B}^-\mathcal{G}(0)$ with $a_\mathcal{B} \in \mathcal{H}_\mathcal{B}$. Lemma 3.6 then implies $\lambda_0 := \lambda(0) = 0$.

4. Convergence of the Implicit Euler Scheme

As first step towards the convergence for Runge-Kutta schemes, we prove in this section the convergence of the implicit Euler method. For this, we show first that the semi-discrete system has a unique solution for every time step. With these approximations, we construct global approximations on $[0,T]$ of the solution of system (3.4) and investigate the convergence behavior.

4.1. **Temporal Discretization.** We formally apply the implicit Euler scheme to the operator DAE (3.4). For this, consider a uniform partition of the interval $[0,T]$ with step size $\tau = T/n$. The time-discrete system which has to be solved for each time step $t_j = \tau j$, $j = 1,\dots,n$, is given by the stationary system

$$Du_j + \mathcal{K}u_j - \mathcal{B}^*p_j - \mathcal{B}^*\lambda_j = \mathcal{F}_j \quad \text{in } \mathcal{V}^*, \tag{4.1a}$$

$$\mathcal{B}u_j - \mathcal{C}\lambda_j = \mathcal{G}_j \quad \text{in } \mathcal{Q}^*, \tag{4.1b}$$

$$\mathcal{B}Du_j = \dot{\mathcal{G}}_j \quad \text{in } \mathcal{Q}^*. \tag{4.1c}$$

Therein, D denotes the discrete derivative, defined by $Du_j := (u_j - u_{j-1})/\tau$. For $j = 1$, equation (4.1c) includes the term $\mathcal{B}u_0$. Assuming $u_0 = a$ to be consistent, we understand $\mathcal{B}u_0$ as $\mathcal{G}(0)$ by Remark 2.6. Note that system (4.1) gives an implicit formula for u_j,

p_j, and λ_j in terms of a given approximation u_{j-1}. There is no dependence on previous approximations of p and λ.

Since the right-hand sides are assumed to be Sobolev-Bochner functions of the form $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$ and $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*) \hookrightarrow C([0,T];\mathcal{Q}^*)$, function evaluations are typically not defined. Thus, only for $\mathcal{G}$ we may define $\mathcal{G}_j := \mathcal{G}(t_j)$. For $\mathcal{F}_j$ and $\dot{\mathcal{G}}_j$, however, we define

$$\mathcal{F}_j := \frac{1}{\tau}\int_{t_{j-1}}^{t_j} \mathcal{F}(s)\,\mathrm{d}s \in \mathcal{V}^* \qquad \text{and} \qquad \dot{\mathcal{G}}_j := \frac{1}{\tau}\int_{t_{j-1}}^{t_j} \dot{\mathcal{G}}(s)\,\mathrm{d}s \in \mathcal{Q}^*. \tag{4.2}$$

We emphasize that $\dot{\mathcal{G}}_j$ is not the derivative of $\mathcal{G}_j$, but it holds that $D\mathcal{G}_j = \dot{\mathcal{G}}_j$. The introduced approximations $\mathcal{F}_j$, $\mathcal{G}_j$, and $\dot{\mathcal{G}}_j$ are of first order but may be replaced by any other approximation, especially for more regular data $\mathcal{F}$ and $\mathcal{G}$. Nevertheless, we require certain convergence properties which we summarize in the following assumption.

Assumption 4.1. Let $\mathcal{F}_\tau\colon [0,T] \to \mathcal{V}^*$ denote the piecewise constant function with $\mathcal{F}_\tau(t) := \mathcal{F}_j$ for $t \in (t_{j-1}, t_j]$ and $\mathcal{F}(0) := \mathcal{F}_1$. Analogously, we define the piecewise constant functions $\mathcal{G}_\tau$ and $\dot{\mathcal{G}}_\tau$ via $\mathcal{G}_j$ and $\dot{\mathcal{G}}_j$, respectively. We assume that $\mathcal{F}_\tau$, $\mathcal{G}_\tau$, and $\dot{\mathcal{G}}_\tau$ converge for $\tau \to 0$ in the strong sense, i.e.,

$$\mathcal{F}_\tau \to \mathcal{F} \text{ in } L^2(0,T,\mathcal{V}^*), \qquad \mathcal{G}_\tau \to \mathcal{G} \text{ in } L^\infty(0,T,\mathcal{Q}^*), \qquad \dot{\mathcal{G}}_\tau \to \dot{\mathcal{G}} \text{ in } L^2(0,T,\mathcal{Q}^*).$$

Note that Assumption 4.1 is fulfilled for the discretization given in (4.2) as shown in [Emm01, Th. 4.2.5]. With the proposed discretization of the right-hand sides, system (4.1) is well-defined. It remains to check the solvability of this system.

Lemma 4.2 (Solvability of the time-discrete system)**.** *Let u_{j-1} be an element of $\mathcal{H}_\mathcal{B} + \mathcal{V}_c$ such that the operator $\mathcal{B}$ is applicable, $j \in \{1,\dots,n\}$. The right-hand sides satisfy $\mathcal{F}_j \in \mathcal{V}^*$, $\mathcal{G}_j \in \mathcal{Q}^*$, and $\dot{\mathcal{G}}_j \in \mathcal{Q}^*$. Then, system* (4.1) *has a unique solution $(u_j, p_j, \lambda_j) \in \mathcal{V} \times \mathcal{Q} \times \mathcal{Q}$.*

Proof. Consider the sum of equation (4.1b), tested by $q \in \mathcal{Q}$, and equation (4.1c), tested by $-\tau q \in \mathcal{Q}$, i.e.,

$$-\langle \mathcal{C}\lambda_j, q\rangle = \langle \mathcal{G}_j - \tau\dot{\mathcal{G}}_j, q\rangle - \langle \mathcal{B}u_{j-1}, q\rangle, \tag{4.3}$$

By the Lax-Milgram theorem [Bre10, Cor. 5.8] there exits a unique solution λ_j of (4.3). Thus, it remains to show that the system given by the equations (4.1a) and (4.1c) has a unique solution. Since $\mathrm{id} + \tau\mathcal{K}$ is bounded and elliptic on $\mathcal{V}_\mathcal{B}$ and $\mathcal{B}$ is surjective, the reduced problem and thus also system (4.1) have a unique solution [BF91, Ch. II.1]. □

4.2. **Convergence Results.** Due to Lemma 4.2, for a given consistent initial value $u_0 := a$ system (4.1) provides discrete approximations at time points t_j, namely u_j, p_j, and λ_j. With these, we define global approximations of the weak solution u on the interval $[0,T]$. More precisely, we define U_τ, $\widehat{U}_\tau\colon [0,T] \to \mathcal{H}_\mathcal{B} + \mathcal{V}_\mathrm{c}$ by

$$U_\tau(t) := \begin{cases} a, & \text{if } t = 0 \\ u_j, & \text{if } t \in (t_{j-1}, t_j] \end{cases}, \qquad \widehat{U}_\tau(t) := \begin{cases} a, & \text{if } t = 0 \\ u_j + Du_j(t - t_j), & \text{if } t \in (t_{j-1}, t_j] \end{cases}. \tag{4.4}$$

Analogously, we define piecewise constant approximations of the Lagrange multipliers λ and p which we denote by Λ_τ and P_τ, respectively. As starting value we set $\Lambda_\tau(0) := \lambda_0$ and $P_\tau(0)$ arbitrarily. By $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$ we denote the generalized time derivative of $\widehat{U}_\tau$ which is piecewise constant with values Du_j. With this, the stationary system (4.1) may be reformulated as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau + \mathcal{K}U_\tau - \mathcal{B}^*P_\tau - \mathcal{B}^*\Lambda_\tau = \mathcal{F}_\tau \qquad \text{in } \mathcal{V}^*, \tag{4.5a}$$

$$\mathcal{B}U_\tau - \mathcal{C}\Lambda_\tau = \mathcal{G}_\tau \qquad \text{in } \mathcal{Q}^*, \tag{4.5b}$$

$$\mathcal{B}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau\big) = \dot{\mathcal{G}}_\tau \qquad \text{in } \mathcal{Q}^*. \tag{4.5c}$$

Here, we assume that the discrete right-hand sides $\mathcal{F}_\tau$, $\mathcal{G}_\tau$, and $\dot{\mathcal{G}}_\tau$ satisfy Assumption 4.1. The following result presents the convergence of the introduced approximations.

Theorem 4.3 (Convergence of the implicit Euler scheme)**.** *Suppose right-hand sides $\mathcal{F} \in L^2(0,T,\mathcal{V}^*)$, $\mathcal{G} \in H^1(0,T,\mathcal{Q}^*)$ and initial data $a \in \mathcal{H}_\mathcal{B} + \mathcal{B}^-\mathcal{G}(0)$ are given. Let $(u,p,0)$ be the solution of the operator DAE (3.4). If the approximations of the right-hand sides $\mathcal{F}_\tau$, $\mathcal{G}_\tau$, and $\dot{\mathcal{G}}_\tau$ fulfill Assumption 4.1, then*

$$\begin{aligned} U_\tau &\to u \quad \text{in } L^2(0,T,\mathcal{V}), & \widehat{U}_\tau &\to u \quad \text{in } L^2(0,T,\mathcal{H}),\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau &\to \dot{u} \quad \text{in } L^2(0,T,\mathcal{V}_\mathcal{B}^*), & \Lambda_\tau &\to 0 \quad \text{in } L^\infty(0,T,\mathcal{Q}) \end{aligned}$$

as $\tau \to 0$. Furthermore, the primitive of P_τ, namely $\widetilde{P}_\tau := \int_0^t P_\tau(s)\,ds$, converges to a function $\widetilde{p}$ in $L^2(0,T;\mathcal{Q})$ whose distributional derivative is p.

Proof. In the first step we show the convergence of the Lagrange multiplier Λ_τ. With this, we are able to show the weak and afterwards even the strong convergence of U_τ and the derivative of $\widehat{U}_\tau$. Finally, we prove the assertions for $\widehat{U}_\tau$ and P_τ.

Step 1: (Convergence of Λ_τ) With the initial value a, equation (4.1b), and a successive application of equation (4.1c), we obtain

$$\begin{aligned} \mathcal{C}\lambda_j = -\mathcal{G}_j + \sum_{k=1}^{j} \tau\dot{\mathcal{G}}_k + \mathcal{G}(0) &= \sum_{k=1}^{j} \tau\dot{\mathcal{G}}_k - \int_0^{t_j} \dot{\mathcal{G}}(s)\,\mathrm{d}s + \mathcal{G}(t_j) - \mathcal{G}_j \\ &= \int_0^{t_j} \Big[\dot{\mathcal{G}}_\tau(s) - \dot{\mathcal{G}}(s)\Big]\,\mathrm{d}s + \mathcal{G}(t_j) - \mathcal{G}_j. \end{aligned} \tag{4.6}$$

Since $\mathcal{C}$ is elliptic and bounded, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|\Lambda_\tau\|_{L^\infty(0,T;\mathcal{Q})} &= \max_{j=1,\dots,n} \|\lambda_j\|_\mathcal{Q} \overset{(4.6)}{\lesssim} \max_{j=1,\dots,n} \Big\| \int_0^{t_j} \Big[\dot{\mathcal{G}}_\tau(s) - \dot{\mathcal{G}}(s)\Big]\,\mathrm{d}s + \mathcal{G}(t_j) - \mathcal{G}_j \Big\|_{\mathcal{Q}^*} \\ &\le \sqrt{T}\|\dot{\mathcal{G}}_\tau - \dot{\mathcal{G}}\|_{L^2(0,T;\mathcal{Q}^*)} + \|\mathcal{G} - \mathcal{G}_\tau\|_{L^\infty(0,T;\mathcal{Q}^*)}. \end{aligned} \tag{4.7}$$

Thus, by Assumption 4.1 it follows that $\|\Lambda_\tau\|_{L^\infty(0,T;\mathcal{Q})} \to 0$.

Step 2: (Weak convergence of U_τ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$): We use the splitting $\mathcal{V} = \mathcal{V}_\mathcal{B} \oplus \mathcal{V}_\mathrm{c}$ as discussed in Section 2.2 and decompose u_j and Du_j for $j = 1,\dots,n$ as well as the initial value $a = a_\mathcal{B} + a_\mathrm{c}$ with $a_\mathcal{B} \in \mathcal{H}_\mathcal{B}$ and $a_\mathrm{c} = \mathcal{B}^-\mathcal{G}(0) \in \mathcal{V}_\mathrm{c}$. We also split the global approximations of u into

$$U_\tau = U_{\tau,\mathcal{B}} + U_{\tau,\mathrm{c}}, \qquad \widehat{U}_\tau = \widehat{U}_{\tau,\mathcal{B}} + \widehat{U}_{\tau,\mathrm{c}}, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau = \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} + \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}}. \tag{4.8}$$

The exact solution u is decomposed into $u_\mathcal{B} \in \mathcal{V}_\mathcal{B}$ and $u_\mathrm{c} \in \mathcal{V}_\mathrm{c}$. Equation (4.5), Assumption 4.1, and the convergence of Λ_τ imply

$$U_{\tau,\mathrm{c}} = \mathcal{B}^-\mathcal{B}U_\tau = \mathcal{B}^-(\mathcal{G}_\tau + \mathcal{C}\Lambda_\tau) \;\to\; \mathcal{B}^-\mathcal{G} = u_\mathrm{c} \qquad \text{in } L^2(0,T;\mathcal{V}_\mathrm{c}), \tag{4.9a}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}} = \mathcal{B}^-\mathcal{B}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau\big) = \mathcal{B}^-\dot{\mathcal{G}}_\tau \;\to\; \mathcal{B}^-\dot{\mathcal{G}} = \dot{u}_\mathrm{c} \qquad \text{in } L^2(0,T;\mathcal{V}_\mathrm{c}). \tag{4.9b}$$

Furthermore, the linearity of the discrete derivative yields $(Du_j)_\mathcal{B} = Du_{j,\mathcal{B}}$. Thus, we can rewrite equation (4.1a) as

$$Du_{j,\mathcal{B}} + \mathcal{K}u_{j,\mathcal{B}} - \mathcal{B}^* p_j - \mathcal{B}^*\lambda_j = \mathcal{F}_j - Du_{j,\mathrm{c}} - \mathcal{K}u_{j,\mathrm{c}} \quad \text{in } \mathcal{V}^*. \tag{4.10}$$

Since $Du_{j,\mathcal{B}} \in \mathcal{V}_\mathcal{B} = \ker \mathcal{B}$ for $j > 1$ and $Du_{1,\mathcal{B}} \in \mathcal{H}_\mathcal{B}$, we conclude with $u_{j,\mathrm{c}} \in \mathcal{V}_\mathrm{c}$ that $u_{j,\mathcal{B}}$ is already fully determined by

$$Du_{j,\mathcal{B}} + \mathcal{K}u_{j,\mathcal{B}} = \mathcal{F}_j - Du_{j,\mathrm{c}} \quad \text{in } \mathcal{V}_\mathcal{B}^*, \tag{4.11}$$

where $\mathcal{K}u_{j,\mathrm{c}}$ vanishes by the definition of $\mathcal{V}_\mathrm{c}$. By the convergence result in (4.9) $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}}$ can be seen as approximation of $\dot{u}_\mathrm{c}$. Therefore, equation (4.11) is the implicit Euler discretization of the unconstrained equation

$$\dot{u}_\mathcal{B} + \mathcal{K}u_\mathcal{B} = \mathcal{F} - \dot{u}_\mathrm{c} \quad \text{in } \mathcal{V}_\mathcal{B}^*. \tag{4.12}$$

As initial value we use $a_\mathcal{B} \in \mathcal{H}_\mathcal{B}$. It is well-known [ET10, Th. 5.1 and Rem. 5.3] that for the exact solution and its approximation it holds that

$$U_{\tau,\mathcal{B}} \rightharpoonup u_\mathcal{B} \quad \text{in } L^2(0,T,\mathcal{V}_\mathcal{B}) \hookrightarrow L^2(0,T;\mathcal{V}), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} \overset{*}{\rightharpoonup} \dot{u}_\mathcal{B} \quad \text{in } L^2(0,T;\mathcal{V}_\mathcal{B}^*). \tag{4.13}$$

The combination of (4.9) and (4.13) shows the weak (respectively weak-$*$) convergence of U_τ in $L^2(0,T;\mathcal{V})$ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$ in $L^2(0,T;\mathcal{V}_\mathcal{B}^*)$.

Step 3: (Strong convergence of U_τ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$): It remains to prove that the sequences $U_{\tau,\mathcal{B}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$ converge strongly. For this, we note that equation (4.11) may be written in the continuous form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} + \mathcal{K}U_{\tau,\mathcal{B}} = \mathcal{F}_\tau - \mathcal{B}^-\dot{\mathcal{G}}_\tau \quad \text{in } \mathcal{V}_\mathcal{B}^*.$$

This equation leads to the estimate

$$\begin{aligned}
&\big\|U_{\tau,\mathcal{B}} - u_\mathcal{B}\big\|^2_{L^2(0,T;\mathcal{V})} \\
&\quad\lesssim \int_0^T \big\langle \mathcal{K}U_{\tau,\mathcal{B}}(s) - \mathcal{K}u_\mathcal{B}(s), U_{\tau,\mathcal{B}}(s) - u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s \\
&\quad= -\int_0^T \big\langle \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}(s),\ U_{\tau,\mathcal{B}}(s) - u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s + \int_0^T \big\langle \dot{u}_\mathcal{B}(s),\ U_{\tau,\mathcal{B}}(s) - u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s \\
&\qquad+ \int_0^T \big\langle \mathcal{F}_\tau(s) - \mathcal{F}(s) + \mathcal{B}^-\big(\dot{\mathcal{G}}_\tau(s) - \dot{\mathcal{G}}(s)\big),\ U_{\tau,\mathcal{B}}(s) - u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s.
\end{aligned} \tag{4.14}$$

The second integral convergences to zero because of the weak convergence $U_{\tau,\mathcal{B}} \rightharpoonup u_\mathcal{B}$ and the third integral because of the assumption on the right-hand sides and the boundedness of $U_{\tau,\mathcal{B}} - u_\mathcal{B}$. For the first integral we use

$$\liminf_{n\to\infty} \int_0^T \big\langle \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}(s), U_{\tau,\mathcal{B}}(s)\big\rangle \,\mathrm{d}s \geq \int_0^T \big\langle \dot{u}_\mathcal{B}(s), u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s \tag{4.15}$$

which follows from $u_{\mathcal{B},n} \rightharpoonup u_\mathcal{B}(T)$ in $\mathcal{H}_\mathcal{B}$, shown in [ET10, Th. 5.1], and the identity

$$2\langle Du_j, u_j\rangle = D\langle u_j, u_j\rangle + \tau\langle Du_j, Du_j\rangle. \tag{4.16}$$

For more details on the proof of (4.15), we refer to [Alt15, Lem. 11.5]. With this and the weak-$*$ convergence of $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$, estimate (4.14) implies

$$\begin{aligned}
0 &\leq \limsup_{\tau\to 0} \|U_{\tau,\mathcal{B}} - u_\mathcal{B}\|^2_{L^2(0,T;\mathcal{V})} \\
&\overset{(4.14)}{\lesssim} \limsup_{\tau\to 0} \int_0^T \big\langle \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}(s),\ u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s - \liminf_{\tau\to 0} \int_0^T \big\langle \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}(s),\ U_{\tau,\mathcal{B}}(s)\big\rangle \,\mathrm{d}s \\
&\leq \int_0^T \big\langle \dot{u}_\mathcal{B}(s),\ u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s - \int_0^T \big\langle \dot{u}_\mathcal{B}(s),\ u_\mathcal{B}(s)\big\rangle \,\mathrm{d}s = 0.
\end{aligned}$$

This shows the strong convergence $U_{\tau,\mathcal{B}} \to u_\mathcal{B}$ as well as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} = \mathcal{F}_\tau - \mathcal{B}^-\dot{\mathcal{G}}_\tau - \mathcal{K}U_{\tau,\mathcal{B}} \ \to\ \mathcal{F} - \mathcal{B}^-\dot{\mathcal{G}} - \mathcal{K}u_\mathcal{B} = \dot{u}_\mathcal{B} \quad \text{in } L^2(0,T;\mathcal{V}_\mathcal{B}^*).$$

By the triangle inequality we obtain the claimed convergence of $U_\tau = U_{\tau,\mathcal{B}} + U_{\tau,\mathrm{c}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau = \frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} + \frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}}$.

Step 4: (Convergence of $\widehat{U}_\tau$): We observe that

$$\widehat{U}_{\tau,\mathrm{c}}(t) = a_\mathrm{c} + \int_0^t \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}}(s)\,\mathrm{d}s$$

for all $t \in [0,T]$. With this, $\widehat{U}_{\tau,\mathrm{c}}(0) = u_\mathrm{c}(0)$, and a Poincaré-Friedrichs inequality [Rou05, Ch. 1.4] we get

$$\big\|\widehat{U}_{\tau,\mathrm{c}} - u_\mathrm{c}\big\|^2_{L^2(0,T;\mathcal{V})} \leq T\,\big\|\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}} - \dot{u}_\mathrm{c}\big\|^2_{L^2(0,T;\mathcal{V})}.$$

With the strong convergence of $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}}$, we get $\widehat{U}_{\tau,\mathrm{c}} \to u_\mathrm{c}$ in $L^2(0,T;\mathcal{V}) \hookrightarrow L^2(0,T;\mathcal{H})$. For the convergence of $\widehat{U}_{\tau,\mathcal{B}}$ we obtain by Young's inequality

$$\begin{aligned}
\frac{1}{2\tau}\Big(|u_{j,\mathcal{B}}|^2 - |u_{j-1,\mathcal{B}}|^2 + |u_{j,\mathcal{B}} - u_{j-1,\mathcal{B}}|^2\Big) &\overset{(4.16)}{=} \langle Du_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle \\
&\overset{(4.11)}{=} \langle \mathcal{F}_j, u_{j,\mathcal{B}}\rangle - \langle Du_{j,\mathrm{c}}, u_{j,\mathcal{B}}\rangle - \langle \mathcal{K}u_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle \\
&\lesssim \|\mathcal{F}_j\|^2_{\mathcal{V}^*} + |Du_{j,\mathrm{c}}|^2 + \|u_{j,\mathcal{B}}\|^2.
\end{aligned}$$

With the telescope sum $\sum_{j=1}^n \big(|u_{j,\mathcal{B}}|^2 - |u_{j-1,\mathcal{B}}|^2\big) = |u_{n,\mathcal{B}}|^2 - |u_{0,\mathcal{B}}|^2$, this estimate yields

$$\begin{aligned}
\|\widehat{U}_{\tau,\mathcal{B}} - U_{\tau,\mathcal{B}}\|^2_{L^2(0,T;\mathcal{H})} &= \frac{\tau}{3}\sum_{j=1}^n \big|u_{j,\mathcal{B}} - u_{j-1,\mathcal{B}}\big|^2 \\
&\lesssim \tau\,\bigg(|a_\mathcal{B}|^2 + \tau\sum_{j=1}^n \|\mathcal{F}_j\|^2_{\mathcal{V}^*} + \tau\sum_{j=1}^n |Du_{j,\mathrm{c}}|^2 + \tau\sum_{j=1}^n \|u_{j,\mathcal{B}}\|^2\bigg).
\end{aligned}$$

Note that the terms in brackets are bounded independently of τ, since the right-hand sides are bounded by Assumption 4.1 and $U_{\tau,\mathcal{B}}$ is a convergent sequence. Thus, $\widehat{U}_{\tau,\mathcal{B}}$ and $U_{\tau,\mathcal{B}}$ have the same limit $u_\mathcal{B}$ in $L^2(0,T;\mathcal{H})$ which implies the strong convergence $\widehat{U}_\tau \to u$ in $L^2(0,T;\mathcal{H})$.

Step 5: (Convergence of P_τ): Let $\widetilde{P}_\tau$, $\widetilde{U}_\tau$, $\widetilde{\Lambda}_\tau$, and $\widetilde{\mathcal{F}}_\tau$ denote the primitives of P_τ, U_τ, Λ_τ, and $\mathcal{F}_\tau$, respectively, with zero initial condition at $t = 0$. An integration of equation (4.5a) then leads to

$$\mathcal{B}^*\widetilde{P}_\tau = \widehat{U}_\tau + \mathcal{K}\widetilde{U}_\tau - \mathcal{B}^*\widetilde{\Lambda}_\tau - \widetilde{\mathcal{F}}_\tau - a \qquad \text{in } AC([0,T],\mathcal{V}^*), \tag{4.17}$$

where $AC([0,T],\mathcal{V}^*) \hookrightarrow L^2(0,T;\mathcal{V}^*)$ denotes the space of absolutely continuous functions with values in $\mathcal{V}^*$. The inf-sup condition of $\mathcal{B}$ implies

$$\beta\|\widetilde{P}_\tau(t)\|_{\mathcal{Q}} \leq \sup_{v\in\mathcal{V}} \frac{\langle \mathcal{B}v, \widetilde{P}_\tau(t)\rangle}{\|v\|} \lesssim |a| + |\widehat{U}_\tau(t)| + \|\widetilde{U}_\tau(t)\| + \|\widetilde{\Lambda}_\tau(t)\|_{\mathcal{Q}} + \ \|\widetilde{\mathcal{F}}_\tau(t)\|_{\mathcal{V}^*}$$

and thus,

$$\beta^2\|\widetilde{P}_\tau\|^2_{L^2(0,T;Q)} \lesssim \|\widehat{U}_\tau\|^2_{L^2(0,T;\mathcal{H})} + T\,\Big(|a|^2 + \|U_\tau\|^2_{L^2(0,T;\mathcal{V})} + \|\Lambda_\tau\|^2_{L^2(0,T;\mathcal{Q})} + \|\mathcal{F}_\tau\|^2_{L^2(0,T;\mathcal{V}^*)}\Big).$$

Inserting $\widetilde{P}_{\tau_1} - \widetilde{P}_{\tau_2}$ instead of $\widetilde{P}_\tau$ for two different time step sizes τ_1, τ_2, we obtain that $\widetilde{P}_\tau$ is a Cauchy sequence in $L^2(0,T;\mathcal{Q})$. Thus, there exists a unique limit $\widetilde{p} \in L^2(0,T;\mathcal{Q})$. Finally, a straightforward calculation with equations (4.5a) and (4.17) shows that the exact solution p is the distributional derivative of $\widetilde{p}$. □

4.3. **Convergence Results for more Regular Data.** In Theorem 4.3 we could only prove the convergence of p in the distributional sense. In this subsection, we consider additional assumptions on the right-hand sides and the initial data which yield an improved convergence result. We distinguish the two cases of the right-hand sides having more regularity in space or in time.

Theorem 4.4 (Convergence for more regular data)**.** *In addition to the assumptions of Theorem 4.3 suppose that $a \in \mathcal{V}$ with $\mathcal{B}a = \mathcal{G}(0)$ and one of the following conditions holdes:*

- *(i) The right-hand side $\mathcal{F}$ is element of $L^2(0,T;\mathcal{H}^*)$ and its approximation $\mathcal{F}_\tau$ satisfies Assumption 4.1 in $L^2(0,T;\mathcal{H}^*)$. Furthermore, $\mathcal{K}$ is symmetric.*
- *(ii) The right-hand sides satisfy $\mathcal{F} \in H^1(0,T;\mathcal{V}^*)$ and $\mathcal{G} \in H^2(0,T;\mathcal{Q}^*)$ and the compatibility condition $\mathcal{F}(0) - \mathcal{B}^-\dot{\mathcal{G}}(0) - \mathcal{K}a_{\mathcal{B}} \in \mathcal{H}^*_{\mathcal{B}}$ is fulfilled.*

Then, the piecewise constant approximations $\frac{d}{dt}\widehat{U}_\tau$ and P_τ satisfy

$$\tfrac{d}{dt}\widehat{U}_\tau \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;\mathcal{H}), \qquad\qquad P_\tau \rightharpoonup p \quad \text{in } L^2(0,T;\mathcal{Q}).$$

Proof. For the proof of the convergence of $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$ we split u_j and Du_j into their components in $\mathcal{V}_{\mathcal{B}}$ and $\mathcal{V}_{\mathrm{c}}$. We show that $\dot{u}_{\mathcal{B}} \in L^2(0,T;\mathcal{H}_{\mathcal{B}})$ and that $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$ converges weakly to $\dot{u}_{\mathcal{B}}$ in $L^2(0,T;\mathcal{H})$. The weak convergence of P_τ is then a direct implication.

Proof for condition (i): Since $Du_{j,\mathcal{B}}$ is an element of $\mathcal{V}_{\mathcal{B}}$, it follows by (4.11) that

$$\langle Du_{j,\mathcal{B}}, Du_{j,\mathcal{B}}\rangle + \langle \mathcal{K}u_{j,\mathcal{B}}, Du_{j,\mathcal{B}}\rangle = \langle \mathcal{F}_j, Du_{j,\mathcal{B}}\rangle - \langle Du_{j,\mathrm{c}}, Du_{j,\mathcal{B}}\rangle. \tag{4.18}$$

The symmetry of $\mathcal{K}$ implies similarly to (4.16) that

$$2\langle \mathcal{K}u_{j,\mathcal{B}}, Du_{j,\mathcal{B}}\rangle = D\langle \mathcal{K}u_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle + \tau\langle \mathcal{K}Du_{j,\mathcal{B}}, Du_{j,\mathcal{B}}\rangle.$$

With this, a multiplication of equation (4.18) by τ and the summation over all discrete time points leads to

$$\begin{aligned}\sum_{j=1}^n \tau|Du_{j,\mathcal{B}}|^2 + \frac{1}{2}\langle \mathcal{K}u_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle + \frac{\tau}{2}\sum_{j=1}^n \tau\langle \mathcal{K}Du_{j,\mathcal{B}}, Du_{j,\mathcal{B}}\rangle \\ = \frac{1}{2}\langle \mathcal{K}a_{\mathcal{B}}, a_{\mathcal{B}}\rangle + \tau\sum_{j=1}^n \Big(\langle \mathcal{F}_j, Du_{j,\mathcal{B}}\rangle - \langle Du_{j,\mathrm{c}}, Du_{j,\mathcal{B}}\rangle\Big).\end{aligned}$$

With the assumed properties on $\mathcal{K}$ we get $\langle \mathcal{K}a_{\mathcal{B}}, a_{\mathcal{B}}\rangle \lesssim \|a_{\mathcal{B}}\|^2 \lesssim \|a\|^2$. As in the proof of Theorem 4.3, the sum $\sum_{j=1}^n \tau|Du_{j,\mathrm{c}}|^2$ is bounded in terms of $\|\dot{G}_\tau\|^2_{L^2(0,T;\mathcal{Q}^*)}$. In conclusion, we obtain with the triangle inequality and Young's inequality that there exists a constant $c > 0$ such that

$$\begin{aligned}2\int_0^T \big|\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}(s)\big|^2\,\mathrm{d}s &= 2\tau\sum_{j=1}^n |Du_{j,\mathcal{B}}|^2\\ &\le c\Big(\|a\|^2 + \|\mathcal{F}_\tau\|^2_{L^2(0,T;\mathcal{H}^*)} + \|\dot{\mathcal{G}}_\tau\|^2_{L^2(0,T;\mathcal{Q}^*)}\Big) + \tau\sum_{j=1}^n |Du_{j,\mathcal{B}}|^2.\end{aligned}$$

This shows that $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$ is bounded in $L^2(0,T;\mathcal{H}_\mathcal{B}) \subseteq L^2(0,T;\mathcal{H})$ independently of τ such that there exists a weakly converging subsequence with weak limit $V_\mathcal{B}$. Together with the convergence of $\widehat{U}_{\tau,\mathcal{B}}$, for arbitrary $h \in \mathcal{H}$ and $\phi \in \mathcal{C}_0^\infty(0,T)$ it holds that

$$0 = \int_0^T \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\mathcal{B},\tau}(t), h\big)\phi(t) + \big(\widehat{U}_{\mathcal{B},\tau}(t), h\big)\dot{\phi}(t)\,\mathrm{d}t \to \int_0^T \big(V_\mathcal{B}(t), h\big)\phi(t) + \big(u_\mathcal{B}(t), h\big)\dot{\phi}(t)\,\mathrm{d}t$$

if τ (or rather a subsequence) tends to zero. This means that $V_\mathcal{B}$ is the generalized time derivative of $u_\mathcal{B}$ in $L^2(0,T;\mathcal{H}_\mathcal{B})$. Note that since the derivative is unique and every subsequence has a converging subsubsequence, the entire sequence $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$ converges weakly to $\dot{u}_\mathcal{B} = V_\mathcal{B}$ in $L^2(0,T;\mathcal{H}_\mathcal{B})$. For the proof of the convergence of P_τ, we obtain

$$\mathcal{B}^*P_\tau = -\mathcal{F}_\tau + \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau + \mathcal{K}U_\tau - \mathcal{B}^*\Lambda_\tau \overset{*}{\rightharpoonup} -\mathcal{F} + \dot{u} + \mathcal{K}u \quad \text{in } L^2(0,T;\mathcal{V}^*). \tag{4.19}$$

Since $\mathcal{B}^*P_\tau$ vanishes if tested by elements of $\mathcal{V}_\mathcal{B}$, the right-hand side satisfies $-\mathcal{F}+\dot{u}+\mathcal{K}u \in L^2(0,T;\mathcal{V}_\mathcal{B}^0)$. By Lemma 2.5, there exists a unique $p \in L^2(0,T;\mathcal{Q})$ with $\mathcal{B}^*p = -\mathcal{F}+\dot{u}+\mathcal{K}u$, namely the solution component of system (3.4). By the continuity of the left inverse of $\mathcal{B}^*$, which we denote by $\mathcal{B}^{-*}$, it follows that

$$P_\tau = \mathcal{B}^{-*}\mathcal{B}^*P_\tau \rightharpoonup \mathcal{B}^{-*}\mathcal{B}^*p = p \quad \text{in } L^2(0,T;\mathcal{Q}).$$

Proof for condition (ii): With the given assumptions, the result in [Emm04, Th. 8.5.1] implies the existence of a generalized time derivative of $u_\mathcal{B}$ in $L^2(0,T;\mathcal{H}_\mathcal{B})$. Further, the weak convergence $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} \rightharpoonup \dot{u}_\mathcal{B}$ can be concluded from the convergence of $\widehat{U}_{\tau,\mathcal{B}}$, since for every $h \in \mathcal{H}$ and $\phi \in \mathcal{C}_0^\infty(0,T)$ it holds that

$$\int_0^T \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}(t) - \dot{u}_\mathcal{B}(t),\ h\big)\phi(t)\,\mathrm{d}t = -\int_0^T \big(\widehat{U}_{\tau,\mathcal{B}}(t) - u_\mathcal{B}(t),\ h\big)\dot{\phi}(t)\,\mathrm{d}t \to 0.$$

The convergence of P_τ follows by the same arguments as in the first part of the proof. □

Remark 4.5. Given the assumptions of Theorem 4.4 (i), we can even show the strong convergence of $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$ and P_τ. For this, one shows that for every $v_\mathcal{B} \in L^2(0,T;\mathcal{V}_\mathcal{B})$ with $\mathcal{K}v_\mathcal{B} \in L^2(0,T;\mathcal{H}_\mathcal{B}^*)$ and generalized derivative $\dot{v}_\mathcal{B} \in L^2(0,T;\mathcal{H}_\mathcal{B})$ it holds that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{K}v_\mathcal{B}, v_\mathcal{B}\rangle = 2\langle \mathcal{K}v_\mathcal{B}, \dot{v}_\mathcal{B}\rangle.$$

For the strong convergence of $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$ one argues similarly as for the convergence of U_τ in Theorem 4.3. Equation (4.5a) then implies the strong convergence of P_τ.

5. Convergence of Implicit Runge-Kutta Schemes

In this section, we analyse the convergence of a special class of Runge-Kutta schemes applied to operator DAEs. Note that in general, an implicit Runge-Kutta scheme may not even provide a unique approximation which then leads to unbounded solutions and thus, to divergence. Thus, we first give sufficient conditions on the approximation scheme which guarantee a unique solution in every time step.

We consider an s-stage Runge-Kutta scheme as presented in Section 2.1, given by the Butcher tableau $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$. As mentioned before, we assume $\mathbf{A}$ to be regular and $R(\infty) = 1 - \mathbf{b}^T\mathbf{A}^{-1}\mathbb{1}_s = 0$. In this case, the approximations of p and λ are independent of the approximations from the previous time step.

5.1. **Temporal Discretization.** Similar to the finite-dimensional case u_j, p_j, and λ_j are approximations of u, p, and λ at time $t_j = j\tau$, respectively. We introduce the internal stages

$$\boldsymbol{u}_j = \begin{bmatrix} \boldsymbol{u}_{j,1} \\ \vdots \\ \boldsymbol{u}_{j,s} \end{bmatrix} \in \mathcal{V}_s, \qquad \boldsymbol{p}_j = \begin{bmatrix} \boldsymbol{p}_{j,1} \\ \vdots \\ \boldsymbol{p}_{j,s} \end{bmatrix} \in \mathcal{Q}_s, \qquad \boldsymbol{\lambda}_j = \begin{bmatrix} \boldsymbol{\lambda}_{j,1} \\ \vdots \\ \boldsymbol{\lambda}_{j,s} \end{bmatrix} \in \mathcal{Q}_s.$$

These stage vectors call for corresponding operators such as $\mathcal{K}_s \colon \mathcal{V}_s \to \mathcal{V}_s^*$ which is induced by $\mathcal{K} \colon \mathcal{V} \to \mathcal{V}^*$ by a componentwise application. In the sequel, we do not distinguish between these two operators such that for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}_s$ we write

$$\langle \mathcal{K}\boldsymbol{u}, \boldsymbol{v}\rangle := \langle \mathcal{K}_s\boldsymbol{u}, \boldsymbol{v}\rangle := \sum_{j=1}^{s} \langle \mathcal{K}\boldsymbol{u}_j, \boldsymbol{v}_j\rangle.$$

In a corresponding manner, the operators $\mathcal{B}$ and $\mathcal{C}$ can be applied componentwise to elements with s components.

Finally, we denote for an arbitrary matrix $M \in \mathbb{R}^{r\times s}$ and an element $\boldsymbol{u} \in \mathcal{V}_s$ by $M\boldsymbol{u} \in \mathcal{V}^r$ the formal matrix-vector multiplication $(M\boldsymbol{u})_k := \sum_{j=1}^{s} M_{kj}\boldsymbol{u}_j \in \mathcal{V}$ for $k = 1, \dots, r$.

Lemma 5.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. Consider a matrix $M \in \mathbb{R}^{s\times s}$ and a linear operator $\mathcal{A} \colon \mathcal{X} \to \mathcal{Y}^*$ which induces a linear operator $\mathcal{A} \colon \mathcal{X}^s \to (\mathcal{Y}^s)^*$ by a componentwise application. Then, for all $\boldsymbol{x} \in \mathcal{X}^s$ and $\boldsymbol{y} \in \mathcal{Y}^s$ it holds that*

$$\big\langle \mathcal{A}M\boldsymbol{x}, \boldsymbol{y}\big\rangle = \big\langle M\mathcal{A}\boldsymbol{x}, \boldsymbol{y}\big\rangle = \big\langle \mathcal{A}\boldsymbol{x}, M^T\boldsymbol{y}\big\rangle. \tag{5.1}$$

Proof. The result follows by a simple calculation,

$$\begin{aligned} \langle \mathcal{A}M\boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{k=1}^{s} \big\langle \mathcal{A}\sum_{j=1}^{s} M_{kj}\boldsymbol{x}_j, \boldsymbol{y}_k\big\rangle &= \sum_{k,j=1}^{s} M_{kj}\langle \mathcal{A}\boldsymbol{x}_j, \boldsymbol{y}_k\rangle \\ &= \sum_{j=1}^{s} \big\langle \mathcal{A}\boldsymbol{x}_j, \sum_{k=1}^{s} M_{kj}\boldsymbol{y}_k\big\rangle = \langle \mathcal{A}\boldsymbol{x}, M^T\boldsymbol{y}\rangle. \end{aligned}$$ □

Also the approximation of the right-hand sides $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$ and $\mathcal{G} \in H^1(0,T;\mathcal{Q}^*)$ need to be extended for elements with s components. For this, we introduce $\boldsymbol{\mathcal{F}}_j \in \mathcal{V}_s^*$ and $\boldsymbol{\mathcal{G}}_j, \dot{\boldsymbol{\mathcal{G}}}_j \in \mathcal{Q}_s^*$, $j = 1, \dots, n$. As in Section 4.1, the specific definition of $\boldsymbol{\mathcal{F}}_j$, $\boldsymbol{\mathcal{G}}_j$, and $\dot{\boldsymbol{\mathcal{G}}}_j$ is not of importance as long as it satisfies the following assumption.

Assumption 5.2. Let $\boldsymbol{\mathcal{F}}_\tau$, $\boldsymbol{\mathcal{G}}_\tau$, and $\dot{\boldsymbol{\mathcal{G}}}_\tau$ denote the piecewise constant functions defined on $[0,T]$ with

$$\boldsymbol{\mathcal{F}}_\tau(t)|_{(t_{j-1},t_j]} \equiv \boldsymbol{\mathcal{F}}_j, \qquad \boldsymbol{\mathcal{G}}_\tau(t)|_{(t_{j-1},t_j]} \equiv \boldsymbol{\mathcal{G}}_j, \qquad \dot{\boldsymbol{\mathcal{G}}}_\tau(t)|_{(t_{j-1},t_j]} \equiv \dot{\boldsymbol{\mathcal{G}}}_j,$$

for $j = 1, \dots, n$ and a continuous extension at time point $t = 0$. We assume that for $\tau \to 0$ it holds that

$$\boldsymbol{\mathcal{F}}_\tau \to \mathcal{F}\mathbb{1}_s \text{ in } L^2(0,T,\mathcal{V}_s^*), \quad \boldsymbol{\mathcal{G}}_\tau \to \mathcal{G}\mathbb{1}_s \text{ in } L^\infty(0,T,\mathcal{Q}_s^*), \quad \dot{\boldsymbol{\mathcal{G}}}_\tau \to \dot{\mathcal{G}}\mathbb{1}_s \text{ in } L^2(0,T,\mathcal{Q}_s^*).$$

An example which satisfies Assumption 5.2 is given by $\boldsymbol{\mathcal{F}}_j := \mathcal{F}_j \mathbb{1}_s$, $\boldsymbol{\mathcal{G}}_j := \mathcal{G}_j \mathbb{1}_s$, and $\dot{\boldsymbol{\mathcal{G}}}_j := \dot{\mathcal{G}}_j \mathbb{1}_s$, $j = 1, \dots, n$, if $\mathcal{F}_j$, $\mathcal{G}_j$, and $\dot{\mathcal{G}}_j$ fulfill Assumption 4.1. Recall that for continuous $\mathcal{G}$ we could define $\mathcal{G}_j$ by the function evaluation $\mathcal{G}_j := \mathcal{G}(t_j)$. However, given the Butcher tableau, we may also define $\boldsymbol{\mathcal{G}}_j$ componentwise by $\boldsymbol{\mathcal{G}}_{j,\ell} := \mathcal{G}(t_{j-1}+\mathbf{c}_\ell \tau)$. Also this approach satisfies Assumption 5.2, since $\mathcal{G}$ is absolutely continuous on $[0,T]$. In any case, we are able to prove the convergence to the solution of the operator DAE (3.4). Recall that we do not aim for convergence orders.

With the introduced notation, the temporal discretization of system (3.4) yields the time-discrete problem

$$u_j = \mathbf{b}^T \mathbf{A}^{-1} \boldsymbol{u}_j, \qquad p_j = \mathbf{b}^T \mathbf{A}^{-1} \boldsymbol{p}_j, \qquad \lambda_j = \mathbf{b}^T \mathbf{A}^{-1} \boldsymbol{\lambda}_j \tag{5.2}$$

where $\boldsymbol{u}_j$, $\boldsymbol{p}_j$, and $\boldsymbol{\lambda}_j$ satisfy the operator equation

$$\mathbf{A}^{-1} D\boldsymbol{u}_j + \mathcal{K}\boldsymbol{u}_j - \mathcal{B}^*\boldsymbol{p}_j - \mathcal{B}^*\boldsymbol{\lambda}_j = \boldsymbol{\mathcal{F}}_j \qquad \text{in } \mathcal{V}_s^*, \tag{5.3a}$$

$$\mathcal{B}\boldsymbol{u}_j - \mathcal{C}\boldsymbol{\lambda}_j = \boldsymbol{\mathcal{G}}_j \qquad \text{in } \mathcal{Q}_s^*, \tag{5.3b}$$

$$\mathcal{B}\mathbf{A}^{-1} D\boldsymbol{u}_j = \dot{\boldsymbol{\mathcal{G}}}_j \qquad \text{in } \mathcal{Q}_s^*. \tag{5.3c}$$

Therein, the discrete derivative $D\boldsymbol{u}_j$ is given by $(\boldsymbol{u}_j - u_{j-1}\mathbb{1}_s)/\tau$.

Unfortunately, $\boldsymbol{u}_j$, $\boldsymbol{p}_j$, and $\boldsymbol{\lambda}_j$ are not bounded in terms of the right-hand sides for all Runge-Kutta schemes, even for an arbitrarily small step size τ as we show by means of the following example.

Example 5.3. Consider the discretization (5.3) with vanishing right-hand sides and $a = 0$. Furthermore, we assume that $\mathcal{V}$ is compactly embedded in $\mathcal{H}$ and that the operator $\mathcal{K}$ is symmetric. We show that the discrete solution given by the 2-stage stiffly accurate Runge-Kutta scheme from Example 2.2 may be non-zero no matter how small τ is chosen and thus, not stable. For this, we note that $\mathbf{A}^{-1}$ has a negative eigenvalue $\alpha \in \mathbb{R}$ with eigenvector $w \in \mathbb{R}^2$ which satisfies $\mathbf{b}^T w \neq 0$.

Since $\langle \mathcal{K}\cdot, \cdot \rangle$ defines an elliptic, bounded, and symmetric bilinear form on $\mathcal{V}_\mathcal{B}$, there exist countable many eigenpairs $(\beta_k, v_k) \in \mathbb{R} \times \mathcal{V}_\mathcal{B}$ of the infinite-dimensional eigenvalue problem $\beta v = \mathcal{K}v$ in $\mathcal{V}_\mathcal{B}^*$. More precisely, all β_k are positive and tend to infinity as $k \to \infty$ and $v_k \neq 0$ for all $k \in \mathbb{N}$ [Mic62, Ch. 4.34]. Let $\varepsilon > 0$ be arbitrarily small and choose k large enough such that $\tau := -\alpha/\beta_k < \varepsilon$ and set $\boldsymbol{u} := v_k w \in \mathcal{V}_{\mathcal{B},s}$. The given eigenvalue problems imply $(\mathbf{A}^{-1} + \tau\mathcal{K})\boldsymbol{u} \in \mathcal{V}_{\mathcal{B},s}^0$ such that there exists a unique $\boldsymbol{p}$ with

$$\mathcal{B}^*\boldsymbol{p} = \big(\tau^{-1}\mathbf{A}^{-1} + \mathcal{K}\big)\boldsymbol{u} \qquad \text{in } \mathcal{V}_s^*.$$

Thus, the tuple $(\boldsymbol{u}, \boldsymbol{p}, 0)$ satisfies system (5.3) and we obtain as approximation in the first time step

$$u_1 = \mathbf{b}^T \mathbf{A}^{-1} \boldsymbol{u} = \alpha \mathbf{b}^T w v_k \neq 0.$$

In summary, one step of the given Runge-Kutta scheme with step size τ yields an approximation which is unbounded.

Example 5.3 shows that it is not sufficient to require the discretization scheme to satisfy $R(\infty) = 0$. We introduce a class of Runge-Kutta methods which provide a unique and bounded solution for every discrete time point. For this, we state further assumptions on the Runge-Kutta scheme.

Assumption 5.4. The Runge-Kutta method (2.1) is algebraically stable, i.e., the matrix $\mathbf{B}\mathbf{A} + \mathbf{A}^T\mathbf{B} - \mathbf{b}\mathbf{b}^T$ is positive semidefinite with the diagonal matrix $\mathbf{B}_{ii} = \mathbf{b}_i$ and $R(\infty) = 0$. Furthermore, all weights $\mathbf{b}_i$ are assumed to be positive and its order is at least one, i.e., $\sum_{i=1}^s \mathbf{b}_i = \mathbb{1}_s^T \mathbf{b} = 1$.

Example 5.5. Radau IA, Radau IIA, and Lobatto IIIC methods satisfy Assumption 5.4, c.f. [HW96, Th. IV.12.9 and Pro. IV.3.8].

With the given assumptions on the discretization scheme, we are able to show the unique solvability for every time step.

Lemma 5.6 (Solvability of the time-discrete system)**.** *Consider $u_{j-1} \in \mathcal{H}_{\mathcal{B}} + \mathcal{V}_c$, $j \in \{1, \dots, n\}$, and right-hand sides $\boldsymbol{\mathcal{F}}_j \in \mathcal{V}_s^*$ and $\boldsymbol{\mathcal{G}}_j, \dot{\boldsymbol{\mathcal{G}}}_j \in \mathcal{Q}_s^*$. If the Runge-Kutta method satisfies Assumption 5.4, then system* (5.3) *has a unique solution of internal stages $(\boldsymbol{u}_j, \boldsymbol{p}_j, \boldsymbol{\lambda}_j) \in \mathcal{V}_s \times \mathcal{Q}_s \times \mathcal{Q}_s$ and thus, there exists a unique approximation $(u_j, p_j, \lambda_j) \in \mathcal{V} \times \mathcal{Q} \times \mathcal{Q}$.*

Proof. Since $\mathbf{M} := \mathbf{BA} + \mathbf{A}^T\mathbf{B} - \mathbf{bb}^T$ is positive semidefinite by Assumption 5.4, it follows for arbitrary $\boldsymbol{x} \in \mathbb{R}^s$ that

$$\boldsymbol{x}^T\mathbf{BA}^{-1}\boldsymbol{x} = \frac{1}{2}(\mathbf{A}^{-1}\boldsymbol{x})^T[\mathbf{BA} + \mathbf{A}^T\mathbf{B}](\mathbf{A}^{-1}\boldsymbol{x}) \geq \frac{1}{2}(\mathbf{A}^{-1}\boldsymbol{x})^T\mathbf{M}(\mathbf{A}^{-1}\boldsymbol{x}) \geq 0$$

and consequently $\mathbf{BA}^{-1}$ is also positive semidefinite. If we multiply equations (5.3a) and (5.3b) by $\mathbf{B}$ from the left and equation (5.3c) by $\mathbf{BA}$, then it results in the system

$$\begin{aligned}
&(5.4a) && \mathbf{BA}^{-1}D\boldsymbol{u}_j + \mathcal{K}\mathbf{B}\boldsymbol{u}_j - \mathcal{B}^*\mathbf{B}\boldsymbol{p}_j - \mathcal{B}^*\mathbf{B}\boldsymbol{\lambda}_j = \mathbf{B}\boldsymbol{\mathcal{F}}_j && \text{in } \mathcal{V}_s^*,\\
&(5.4b) && \mathcal{B}\mathbf{B}\boldsymbol{u}_j - \mathcal{C}\mathbf{B}\boldsymbol{\lambda}_j = \mathbf{B}\boldsymbol{\mathcal{G}}_j && \text{in } \mathcal{Q}_s^*,\\
&(5.4c) && \mathcal{B}\mathbf{B}D\boldsymbol{u}_j = \mathbf{BA}\dot{\boldsymbol{\mathcal{G}}}_j && \text{in } \mathcal{Q}_s^*.
\end{aligned}$$

Note that we have used $\mathbf{B}\mathcal{K} = \mathcal{K}\mathbf{B}$ as well as similar results for the other operators. Let $\mathbf{B}^{1/2}$ be the diagonal matrix with $\mathbf{B}_{ii}^{1/2} = \sqrt{\mathbf{b}_i}$. Since

$$\langle \mathbf{BA}^{-1}\boldsymbol{u} + \tau\mathcal{K}\mathbf{B}\boldsymbol{u}, \boldsymbol{u}\rangle \geq \langle \tau\mathcal{K}\mathbf{B}^{1/2}\boldsymbol{u}, \mathbf{B}^{1/2}\boldsymbol{u}\rangle \geq \tau\alpha\|\mathbf{B}^{1/2}\boldsymbol{u}\|_{\mathcal{V}_s}^2 \geq \tau\alpha \min_{i=1,\dots,s} \mathbf{b}_i\|\boldsymbol{u}\|_{\mathcal{V}_s}^2$$

for all $\boldsymbol{u} \in \mathcal{V}_s$, the operator $\mathbf{BA}^{-1} + \tau\mathcal{K}\mathbf{B}$ is elliptic. The solvability then follows by the invertibility of $\mathbf{B}$ and a similar argumentation as in the implicit Euler case in Lemma 4.2. □

Before we investigate the convergence of implicit Runge-Kutta schemes applied to operator DAEs, we summarize results on the convergence for unconstrained operator equations.

5.2. **Convergence Results for Linear Operator Equations.** We consider a linear parabolic PDE in the weak form which corresponds to an (unconstrained) operator equation. More precisely, we consider an operator equation of the form

$$\dot{v}(t) + \mathcal{A}v(t) = \mathcal{F}(t) \quad \text{in } \mathcal{V}^* \tag{5.5}$$

with initial condition $v(0) = v_0 \in \mathcal{H}$ and right-hand side $\mathcal{F} \in L^2(0, T; \mathcal{V}^*)$. The linear operator $\mathcal{A}\colon \mathcal{V} \to \mathcal{V}^*$ is assumed to be elliptic and bounded and the Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$ form a Gelfand triple. These assumptions then guarantee a unique solution $v \in L^2(0, T; \mathcal{V}) \cap C([0, T], \mathcal{H})$ with derivative $\dot{v} \in L^2(0, T; \mathcal{V}^*)$ [Wlo87, Ch. IV]. The following convergence analysis is based on the paper [ET10] which investigates the behavior of stiffly accurate and algebraically stable Runge-Kutta schemes of first order applied to the evolution problem (5.5). Note that such methods fulfill Assumption 5.4.

Lemma 5.7 (Generalization of [ET10, Lem. 3.4])**.** *Let the Runge-Kutta method with Butcher tableau* $\mathbf{A}$*,* $\mathbf{b}$*,* $\mathbf{c}$ *satisfy Assumption 5.4. Then, it holds that*

$$2\boldsymbol{x}^T\mathbf{BA}^{-1}(\boldsymbol{x} - x_0\mathbb{1}_s) \geq (\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{x})^2 - x_0^2 \tag{5.6}$$

for all $x_0 \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^s$.

Proof. Consider the matrices $\mathbf{M}$, $\mathbf{M}' \in \mathbb{R}^{s\times s}$ given by $\mathbf{M} := \mathbf{B}\mathbf{A} + \mathbf{A}^T\mathbf{B}^T - \mathbf{b}\mathbf{b}^T$ and $\mathbf{M}' := \mathbf{A}^{-T}\mathbf{M}\mathbf{A}^{-1} - \mathbf{B}\mathbf{A}^{-1}\mathbb{1}_s\mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}$. Then, it holds that

$$2\boldsymbol{x}^T\mathbf{B}\mathbf{A}^{-1}(\boldsymbol{x} - x_0\mathbb{1}_s) + x_0^2 - (\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{x})^2 = \boldsymbol{x}^T\mathbf{M}'\boldsymbol{x} + (x_0 - \mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}\boldsymbol{x})^2 \geq \boldsymbol{x}^T\mathbf{M}'\boldsymbol{x}.$$

Thus, it remains to show that $\mathbf{M}'$ is positive semidefinite. For this, we use the splitting $\mathbb{R}^s = \ker(\mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}) \oplus \operatorname{span}\{\mathbb{1}_s\}$. This is well-defined, since the kernel of $\mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}$ is an $(s-1)$-dimensional subspace of $\mathbb{R}^s$ and $\mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}\mathbb{1}_s = \mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{b} = 1 \neq 0$ by Assumption 5.4. A simple calculation shows

$$\begin{aligned}\mathbb{1}_s^T\mathbf{M}'\mathbb{1}_s &= \mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}\mathbb{1}_s + \mathbb{1}_s^T\mathbf{B}\mathbf{A}^{-1}\mathbb{1}_s - \mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{b}\mathbf{b}^T\mathbf{A}^{-1}\mathbb{1}_s - \mathbb{1}_s^T\mathbf{B}\mathbf{A}^{-1}\mathbb{1}_s\mathbb{1}_s^T\mathbf{B}\mathbf{A}^{-1}\mathbb{1}_s \\ &= 2\left[\mathbf{b}^T\mathbf{A}^{-1}\mathbb{1}_s - (\mathbf{b}^T\mathbf{A}^{-1}\mathbb{1}_s)^2\right] = 2\left[1-1\right] = 0.\end{aligned}$$

For an arbitrary element $\boldsymbol{x} \in \ker(\mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B})$, we obtain

$$\mathbb{1}_s^T\mathbf{M}'\boldsymbol{x} = \mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B}\boldsymbol{x} + \mathbb{1}_s^T\mathbf{B}\mathbf{A}^{-1}\boldsymbol{x} - \mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{b}\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{x} = \mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{x} - \mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{x} = 0 \tag{5.7}$$

as well as $\boldsymbol{x}^T\mathbf{M}'\boldsymbol{x} \geq 0$ by the positive semi-definiteness of $\mathbf{M}$. This shows that $\mathbf{M}'$ is semidefinte on $\mathbb{R}^s$ by the symmetry of $\mathbf{M}'$, equation (5.7), and the semi-definiteness of $\mathbf{M}'$ for every element of the complements $\ker(\mathbb{1}_s^T\mathbf{A}^{-T}\mathbf{B})$ and $\operatorname{span}\{\mathbb{1}_s\}$. □

Remark 5.8. The lines of the proof of Lemma 5.7 can be carried over to the Hilbert space $\mathcal{V}$ and a positive semidefinite, symmetric operator $\mathcal{A}$. Thus, for every $v_0 \in \mathcal{V}$ and $\boldsymbol{v} \in \mathcal{V}_s$ it holds that

$$2\langle\mathcal{A}\boldsymbol{v}, \mathbf{B}\mathbf{A}^{-1}(\boldsymbol{v} - v_0\mathbb{1}_s)\rangle \geq \langle\mathcal{A}\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{v}, \mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{v}\rangle - \langle\mathcal{A}v_0, v_0\rangle.$$

With this remark, we get the following result.

Theorem 5.9. *Consider equation* (5.5) *with* $\mathcal{F} \in L^2(0,T;\mathcal{V}^*)$*, initial data* $v_0 \in \mathcal{H}$*, and a linear, bounded, and elliptic operator* $\mathcal{A}\colon \mathcal{V} \to \mathcal{V}^*$*. The corresponding exact solution is denoted by* v*. The temporal discretization of* (5.5) *on* $[0,T]$ *with constant step size* τ *and a Runge-Kutta method which satisfies Assumption 5.4 is given by*

$$v_j = \mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{v}_j, \tag{5.8a}$$

$$\mathbf{A}^{-1}D\boldsymbol{v}_j + \mathcal{A}\boldsymbol{v}_j = \boldsymbol{\mathcal{F}}_j. \tag{5.8b}$$

Suppose that the piecewise constant function $\boldsymbol{\mathcal{F}}_\tau \in L^2(0,T;\mathcal{V}_s^*)$ *defined by* $\boldsymbol{\mathcal{F}}_\tau(t) = \boldsymbol{\mathcal{F}}_j$ *for* $t \in (t_{j-1}, t_j]$ *satisfies* $\boldsymbol{\mathcal{F}}_\tau \to \mathcal{F}\mathbb{1}_s$ *in* $L^2(0,T;\mathcal{V}_s^*)$*. Then, there exists a unique solution* $v_j \in \mathcal{V}$ *and* $\boldsymbol{v}_j \in \mathcal{V}_s$ *of system* (5.8) *for every time step* $j = 1,\dots,n$*. Furthermore, the functions* V_τ *and* $\frac{d}{dt}\widehat{V}_\tau$ *as defined in Section 4.2 and* $\boldsymbol{V}_\tau$ *defined by* $\boldsymbol{V}_\tau(0) = v_0\mathbb{1}_s$ *and*

$$\boldsymbol{V}_\tau(t) = \boldsymbol{v}_j, \qquad \text{for } t \in (t_{j-1}, t_j]$$

are weakly convergent,

$$\boldsymbol{V}_\tau \rightharpoonup v\mathbb{1}_s \text{ in } L^2(0,T;\mathcal{V}_s), \quad V_\tau(T) \rightharpoonup v(T) \text{ in } \mathcal{H}, \quad \tfrac{d}{dt}\widehat{V} \rightharpoonup \dot{v} \text{ in } L^2(0,T;\mathcal{V}^*). \tag{5.9}$$

Proof. By the same arguments as in the proof of Lemma 5.6 one shows that $\mathbf{B}\mathbf{A}^{-1} + \tau\mathcal{A}\mathbf{B}$ is elliptic and bounded. The existence of a unique solution of (5.8) then follows by the Lax-Milgram theorem. With $\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{v}_j = v_j$ and estimate (5.6) one proves the stated convergence behavior by a reconstruction of the proof of [ET10, Th. 5.1]. □

5.3. **Convergence Results for Linear Operator DAEs.** In this section, we investigate the convergence behavior of the semi-discretized system (5.3). For this, we recall the piecewise constant and piecewise linear approximations U_τ, $\widehat{U}_\tau$, $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$, P_τ, and Λ_τ from Section 4.2. For the internal stages we introduce accordingly

$$\begin{aligned} &\boldsymbol{U}_\tau(t) := \begin{cases} a\mathbb{1}_s, & \text{if } t=0 \\ \boldsymbol{u}_j, & \text{if } t\in(t_{j-1},t_j] \end{cases}, \qquad && \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_\tau(t) := \begin{cases} 0, & \text{if } t=0 \\ D\boldsymbol{u}_j, & \text{if } t\in(t_{j-1},t_j] \end{cases}, \\ &\boldsymbol{P}_\tau(t) := \boldsymbol{p}_j, \ \text{if } t\in(t_{j-1},t_j], && \boldsymbol{\Lambda}_\tau(t) := \boldsymbol{\lambda}_j, \ \text{if } t\in(t_{j-1},t_j]. \end{aligned} \tag{5.10}$$

The values for $\boldsymbol{P}_\tau$ and $\boldsymbol{\Lambda}_\tau$ at time $t=0$ can be chosen arbitrarily. We state the first main result of this paper.

Theorem 5.10 (Convergence of Runge-Kutta schemes)**.** *Consider right-hand sides $\mathcal{F}\in L^2(0,T;\mathcal{V}^*)$, $\mathcal{G}\in H^1(0,T;\mathcal{Q}^*)$ with approximations $\boldsymbol{\mathcal{F}}_\tau$, $\boldsymbol{\mathcal{G}}_\tau$, and $\dot{\boldsymbol{\mathcal{G}}}_\tau$ satisfying Assumption 5.2 and an initial value $a\in\mathcal{H}_\mathcal{B}+\mathcal{B}^-\mathcal{G}(0)$. The corresponding solution of the operator DAE (3.4) is denoted by $(u,p,0)$. Then, every Runge-Kutta scheme which satisfies Assumption 5.4 yields for $\tau\to 0$ the convergence results*

$$\begin{aligned} U_\tau &\to u \quad \text{in } L^2(0,T;\mathcal{V}), & \widehat{U}_\tau &\to u \quad \text{in } L^2(0,T;\mathcal{H}), \\ \tfrac{d}{dt}\widehat{U}_\tau &\to \dot{u} \quad \text{in } L^2(0,T;\mathcal{V}_\mathcal{B}^*), & \Lambda_\tau &\to 0 \quad \text{in } L^\infty(0,T;\mathcal{Q}). \end{aligned}$$

Furthermore, $\int_0^t \mathbf{b}^T\boldsymbol{P}_\tau(s)\,ds$ converges to a function $\widetilde{p}$ in $L^2(0,T;\mathcal{Q})$, where p is the distributional derivative of $\widetilde{p}$.

Proof. We follow the steps of the proof of Theorem 4.3 where we have shown the convergence for the implicit Euler scheme.

Step 1: (Convergence of Λ_τ) With equation (5.3b), a successive application of equation (5.3c), and $\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{u}_j = u_j$ we obtain

$$\mathcal{C}\boldsymbol{\lambda}_j = \tau\Big(\mathbf{A}\dot{\boldsymbol{\mathcal{G}}}_j - \mathbf{b}^T\dot{\boldsymbol{\mathcal{G}}}_j\mathbb{1}_s\Big) + \Big(\int_0^{t_j}\mathbf{b}^T\dot{\boldsymbol{\mathcal{G}}}_\tau(s) - \dot{\mathcal{G}}(s)\,\mathrm{d}s\Big)\mathbb{1}_s + \mathcal{G}(t_j)\mathbb{1}_s - \boldsymbol{\mathcal{G}}_j. \tag{5.11}$$

Furthermore, with $\mathbf{b}^T\mathbf{A}^{-1}\mathbb{1}_s = 1$ it holds that

$$\mathcal{C}\lambda_j = \mathcal{C}\mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{\lambda}_j = \int_0^{t_j}\mathbf{b}^T\dot{\boldsymbol{\mathcal{G}}}_\tau(s) - \dot{\mathcal{G}}(s)\,\mathrm{d}s + \mathbf{b}^T\mathbf{A}^{-1}\big(\mathcal{G}(t_j)\mathbb{1}_s - \boldsymbol{\mathcal{G}}_j\big).$$

Similarly as in the proof of Theorem 4.3, Assumption 5.2 and $\mathbf{b}^T\mathbb{1}_s = 1$ imply

$$\|\Lambda_\tau\|_{L^\infty(0,T;\mathcal{Q})} \lesssim \sqrt{T}\|\dot{\boldsymbol{\mathcal{G}}}_\tau - \dot{\mathcal{G}}\mathbb{1}_s\|_{L^2(0,T;\mathcal{Q}_s^*)} + \|\boldsymbol{\mathcal{G}}_\tau - \mathcal{G}\mathbb{1}_s\|_{L^\infty(0,T;\mathcal{Q}_s^*)} \to 0.$$

Given equation (5.11), Assumption 5.2 also implies $\boldsymbol{\Lambda}_\tau \to 0$ in $L^2(0,T;\mathcal{Q}_s)$ by the estimate

$$\begin{aligned} \|\boldsymbol{\Lambda}_\tau\|^2_{L^2(0,T;\mathcal{Q}_s)} \lesssim{}& \tau\,\|\mathbf{A}\dot{\boldsymbol{\mathcal{G}}}_\tau\|^2_{L^2(0,T;\mathcal{Q}_s^*)} + \tau\,\|\mathbf{b}^T\dot{\boldsymbol{\mathcal{G}}}_\tau\mathbb{1}_s\|^2_{L^2(0,T;\mathcal{Q}_s^*)} \\ &+ T^2\,\|\mathbf{b}^T\dot{\boldsymbol{\mathcal{G}}}_\tau - \dot{\mathcal{G}}\mathbb{1}_s\|^2_{L^2(0,T;\mathcal{Q}_s^*)} + T\,\|\mathcal{G}\mathbb{1}_s - \boldsymbol{\mathcal{G}}_\tau\|^2_{L^\infty(0,T;\mathcal{Q}_s^*)}. \end{aligned}$$

Step 2: (Weak convergence of U_τ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$): Note that the splitting $\mathcal{V} = \mathcal{V}_\mathcal{B}\oplus\mathcal{V}_\mathrm{c}$ from Section 2.2 implies the splitting $\mathcal{V}_s = \mathcal{V}_{\mathcal{B},s}\oplus\mathcal{V}_{\mathrm{c},s}$. With this, we obtain

$$\boldsymbol{u}_j = \boldsymbol{u}_{j,\mathcal{B}} + \boldsymbol{u}_{j,\mathrm{c}}, \qquad D\boldsymbol{u}_j = D\boldsymbol{u}_{j,\mathcal{B}} + D\boldsymbol{u}_{j,\mathrm{c}}, \qquad u_j = u_{j,\mathcal{B}} + u_{j,\mathrm{c}}.$$

Analogously, we split the global approximations into

$$\boldsymbol{U}_\tau = \boldsymbol{U}_{\tau,\mathcal{B}} + \boldsymbol{U}_{\tau,\mathrm{c}}, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_\tau = \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}} + \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathrm{c}}.$$

Thus, formula (5.3b) yields

$$\boldsymbol{U}_{\tau,\mathrm{c}} = \mathcal{B}^{-}\boldsymbol{\mathcal{G}}_{\tau} + \mathcal{B}^{-}\mathcal{C}\boldsymbol{\Lambda}_{\tau} \;\to\; \mathcal{B}^{-}\mathcal{G}\mathbb{1}_s \quad \text{in } L^2(0,T;\mathcal{V}_s)$$

which implies $U_{\tau,\mathrm{c}} \to \mathcal{B}^{-}\mathcal{G}$ and respectively by equation (5.3c) and $\mathbf{b}^T\mathbb{1}_s = 1$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}} = \mathbf{b}^T\mathbf{A}^{-1}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathrm{c}}\big) = \mathcal{B}^{-}\mathbf{b}^T\dot{\boldsymbol{\mathcal{G}}}_{\tau} \;\to\; \mathcal{B}^{-}\mathbf{b}^T\dot{\mathcal{G}}\mathbb{1}_s = \mathcal{B}^{-}\dot{\mathcal{G}} \quad \text{in } L^2(0,T;\mathcal{V}).$$

By a combination of equations (5.3a), (5.3c) and a restriction of the test functions to $\mathcal{V}_{\mathcal{B},s}$, we obtain

$$\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}} + \mathcal{K}\boldsymbol{u}_{j,\mathcal{B}} = \boldsymbol{\mathcal{F}}_j - \mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathrm{c}} = \boldsymbol{\mathcal{F}}_j - \mathcal{B}^{-}\dot{\boldsymbol{\mathcal{G}}}_j \quad \text{in } \mathcal{V}_{\mathcal{B},s}^*. \tag{5.12}$$

Note that (5.12) equals the Runge-Kutta approximation of an unconstrained problem such as (5.5). With the initial value $a_{\mathcal{B}} \in \mathcal{H}_{\mathcal{B}}$, the conditions of Theorem 5.9 are satisfied. Thus, $\boldsymbol{U}_{\tau,\mathcal{B}}$ converges weakly towards $u_{\mathcal{B}}\mathbb{1}_s$ in $L^2(0,T;\mathcal{V}_{\mathcal{B},s}) \subseteq L^2(0,T;\mathcal{V}_s)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$ converges weakly towards $\dot{u}_{\mathcal{B}}$ in $L^2(0,T;\mathcal{V}_{\mathcal{B}}^*)$ as $\tau \to 0$.

Step 3: (Strong convergence of U_τ and $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$): For the strong convergence we note that by equation (5.12) it holds that

(5.13)
$$\begin{aligned}
&\|\boldsymbol{U}_{\tau,\mathcal{B}} - u_{\mathcal{B}}\mathbb{1}_s\|^2_{L^2(0,T;\mathcal{V})}\\
&\lesssim \min_{i=1,\dots,s} \mathbf{b}_i \int_0^T \big\langle \mathcal{K}\big(\boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big),\ \boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big\rangle\,\mathrm{d}s\\
&\le \int_0^T \big\langle \mathcal{K}\mathbf{B}\big(\boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big),\ \boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big\rangle\,\mathrm{d}s\\
&= -\int_0^T \big\langle \mathbf{B}\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}(s), \boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big\rangle\,\mathrm{d}s\\
&\qquad + \int_0^T \big\langle \mathbf{B}\dot{u}_{\mathcal{B}}(s)\mathbb{1}_s,\ \boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big\rangle\,\mathrm{d}s\\
&\qquad + \int_0^T \big\langle \mathbf{B}\big(\boldsymbol{\mathcal{F}}_\tau(s) - \mathcal{F}(s)\mathbb{1}_s\big) - \mathcal{B}^{-}\mathbf{B}\big(\dot{\boldsymbol{\mathcal{G}}}_\tau(s) - \dot{\mathcal{G}}(s)\mathbb{1}_s\big),\ \boldsymbol{U}_{\tau,\mathcal{B}}(s) - u_{\mathcal{B}}(s)\mathbb{1}_s\big\rangle\,\mathrm{d}s,
\end{aligned}$$

since $\mathcal{K}$ is elliptic and all $\mathbf{b}_i$ are positive. As for the implicit Euler method, we only need to analyze the first integral, since the remaining terms vanish as $\tau \to 0$ by the weak convergence of $\boldsymbol{U}_{\tau,\mathcal{B}}$ and Assumption 5.2. By Remark 5.8 we obtain

$$2\tau\,\big\langle \mathbf{B}\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}},\ \boldsymbol{u}_{j,\mathcal{B}}\big\rangle \ge |u_{j,\mathcal{B}}|^2 - |u_{j-1,\mathcal{B}}|^2$$

and thus,

$$\int_0^T \big\langle \mathbf{B}\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}(s), \boldsymbol{U}_{\tau,\mathcal{B}}(s)\big\rangle\,\mathrm{d}s \ge \frac{1}{2}\sum_{j=1}^{n}|u_{j,\mathcal{B}}|^2 - |u_{j-1,\mathcal{B}}|^2 = \frac{1}{2}|u_{\mathcal{B},n}|^2 - \frac{1}{2}|a_{\mathcal{B}}|^2.$$

From Theorem 5.9 we know that $u_{n,\mathcal{B}} \rightharpoonup u_{\mathcal{B}}(T)$ which implies that

$$\liminf_{n\to\infty}\frac{1}{2}|u_{\mathcal{B},n}|^2 - \frac{1}{2}|a_{\mathcal{B}}|^2 \ge \frac{1}{2}|u_{\mathcal{B}}(T)|^2 - \frac{1}{2}|u_{\mathcal{B}}(0)|^2 = \int_0^T \big\langle \dot{u}_{\mathcal{B}}(s), u_{\mathcal{B}}(s)\big\rangle\,\mathrm{d}s.$$

On the other hand, with the convergence results for $\boldsymbol{U}_\tau$, $\boldsymbol{\mathcal{F}}_\tau$, and $\dot{\boldsymbol{\mathcal{G}}}_\tau$ as well as $\mathbb{1}_s^T\mathbf{B}\mathbb{1}_s = 1$ we get

$$\begin{aligned}\int_0^T \big\langle \mathbf{B}\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}(s), u_\mathcal{B}\mathbb{1}_s(s)\big\rangle \,\mathrm{d}s &= \int_0^T \big\langle \mathbf{B}(\boldsymbol{\mathcal{F}}_\tau - \mathcal{B}^-\dot{\boldsymbol{\mathcal{G}}}_\tau - \mathcal{K}\boldsymbol{U}_{\tau,\mathcal{B}})(s), u_\mathcal{B}\mathbb{1}_s(s)\big\rangle \,\mathrm{d}s \\ &\to \int_0^T \big\langle \mathbf{B}(\mathcal{F}\mathbb{1}_s - \mathcal{B}^-\dot{\mathcal{G}}\mathbb{1}_s - \mathcal{K}u_\mathcal{B}\mathbb{1}_s)(s), u_\mathcal{B}\mathbb{1}_s(s)\big\rangle \,\mathrm{d}s \\ &= \int_0^T \big\langle \mathbf{B}\dot{u}_\mathcal{B}\mathbb{1}_s(s), u_\mathcal{B}\mathbb{1}_s(s)\big\rangle \,\mathrm{d}s = \int_0^T \langle \dot{u}_\mathcal{B}(s), u_\mathcal{B}(s)\rangle \,\mathrm{d}s.\end{aligned}$$

As in the proof of Theorem 4.3 we conclude with (5.13) that $\boldsymbol{U}_{\tau,\mathcal{B}} \to u_\mathcal{B}\mathbb{1}_s$ in $L^2(0,T;\mathcal{V}_{\mathcal{B},s})$. A direct implication is given by

$$U_{\tau,\mathcal{B}} = \mathbf{b}^T\mathbf{A}^{-1}\boldsymbol{U}_{\tau,\mathcal{B}} \;\to\; \mathbf{b}^T\mathbf{A}^{-1}u_\mathcal{B}\mathbb{1}_s = u_\mathcal{B} \qquad \text{in } L^2(0,T;\mathcal{V}_\mathcal{B}) \subseteq L^2(0,T;\mathcal{V}).$$

Furthermore, we obtain the convergence of $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$ in $L^2(0,T;\mathcal{V}_\mathcal{B}^*)$ by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} = \mathbf{b}^T\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}} = \mathbf{b}^T(\boldsymbol{\mathcal{F}}_\tau - \mathcal{B}^-\dot{\boldsymbol{\mathcal{G}}}_\tau - \mathcal{K}\boldsymbol{U}_{\tau,\mathcal{B}}) \;\to\; \mathbf{b}^T(\mathcal{F} - \mathcal{B}^-\dot{\mathcal{G}} - \mathcal{K}u_\mathcal{B})\mathbb{1}_s = \dot{u}_\mathcal{B}.$$

Step 4: (Convergence of $\widehat{U}_\tau$): For the proof of the convergence of $\widehat{U}_{\tau,\mathcal{B}} \to u$ we argue as in the proof of Theorem 4.3, using the estimate

$$\begin{aligned}\langle Du_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle &= \langle \mathbf{b}^T\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle \\ &= \langle \mathbf{b}^T\boldsymbol{\mathcal{F}}_j, u_{j,\mathcal{B}}\rangle - \langle \mathbf{b}^T\mathcal{B}^-\boldsymbol{\mathcal{G}}_j, u_{j,\mathcal{B}}\rangle - \langle \mathbf{b}^T\mathcal{K}\boldsymbol{u}_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle \\ &\lesssim \|\boldsymbol{\mathcal{F}}_j\|_{\mathcal{V}^*}^2 + |\mathcal{B}^-\boldsymbol{\mathcal{G}}_j|^2 + \|\boldsymbol{u}_{j,\mathcal{B}}\|^2 + \|u_{j,\mathcal{B}}\|^2.\end{aligned}$$

With this, the same arguments as in the proof of Theorem 4.3 show the claim.

Step 5: (Convergence of $\mathbf{b}^T\boldsymbol{P}_\tau$): For the proof of the distributional convergence of $\mathbf{b}^T\boldsymbol{P}_\tau$ we introduce primitives for the expansions of the stages $\boldsymbol{U}_\tau$, $\boldsymbol{P}_\tau$, $\boldsymbol{\Lambda}_\tau$, and for the right-hand side $\boldsymbol{\mathcal{F}}_\tau$. We mark the absolutely continuous primitives with zero initial conditions at $t = 0$ by a tilde, e.g.,

$$\widetilde{\boldsymbol{U}}_\tau(t) = \int_0^t \boldsymbol{U}(s)\,\mathrm{d}s.$$

Recall that $\boldsymbol{\Lambda}_\tau$ converges to zero in $L^2(0,T;\mathcal{Q}_s)$ and therefore $\boldsymbol{U}_{\tau,\mathrm{c}} \to u_\mathrm{c}\mathbb{1}_s$ in $L^2(0,T;\mathcal{V}_s)$ by equation (5.3b). Now, consider the equality

$$\mathcal{B}^*\mathbf{b}^T\widetilde{\boldsymbol{P}}_\tau = \mathcal{B}^*\int_0^\cdot \mathbf{b}^T\boldsymbol{P}_\tau(s)\,\mathrm{d}s = \widehat{U}_\tau + \mathcal{K}\mathbf{b}^T\widetilde{\boldsymbol{U}}_\tau - \mathcal{B}^*\mathbf{b}^T\widetilde{\boldsymbol{\Lambda}}_\tau - \mathbf{b}^T\widetilde{\boldsymbol{\mathcal{F}}}_\tau - a \tag{5.14}$$

in $AC([0,T],\mathcal{V}^*)$, which follows from equation (5.3a). Then, the inf-sup condition of $\mathcal{B}$, and an argumentation as in the proof of Theorem 4.3 yields that $\mathbf{b}^T\widetilde{\boldsymbol{P}}_\tau$ converges to $\widetilde{p}$ with $\widetilde{p}(t) = \int_0^t p\,\mathrm{d}t$ in $L^2(0,T;\mathcal{Q})$. □

Remark 5.11. In Theorem 5.10 we have shown the convergence of $\mathbf{b}^T\widetilde{\boldsymbol{P}}_\tau$. For a proof of $\widetilde{P}_\tau \to \widetilde{p}$ in $L^2(0,T;\mathcal{Q})$ we would need a result of the form

$$\mathbf{A}\mathbb{1}_s a_\mathcal{B} + \int_0^t \tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}\,\mathrm{d}s \;\to\; \mathbf{A}\mathbb{1}_s u_\mathcal{B} \quad \text{in } L^2(0,T;\mathcal{V}_s^*).$$

With this, we could consider $\mathcal{B}^*\mathbf{b}^T\mathbf{A}^{-1}\widetilde{\boldsymbol{P}}_\tau$ similarly as in equation (5.14).

As for the implicit Euler scheme, we can prove the convergence of the variable p if we assume additional regularity of the right-hand side $\mathcal{F}$ and the initial data. This gives the second main result.

Theorem 5.12 (Convergence with more regular data)**.** *In addition to the assumptions of Theorem 5.10, consider an initial value $a \in \mathcal{V}$ with $\mathcal{B}a = \mathcal{G}(0)$ and $\mathcal{F} \in L^2(0,T;\mathcal{H}^*)$. Furthermore, let the approximation $\mathcal{F}_j$ satisfy Assumption 5.2 in $L^2(0,T;\mathcal{H}_s^*)$ and let the operator $\mathcal{K}$ be symmetric. Then, the approximations satisfy*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;\mathcal{H}), \qquad\qquad P_\tau \rightharpoonup p \quad \text{in } L^2(0,T;\mathcal{Q}).$$

Proof. We follow the ideas of the proofs of Theorems 4.4 and 5.10. With the splitting $\mathcal{V} = \mathcal{V}_\mathcal{B} \oplus \mathcal{V}_\mathrm{c}$ and the strong convergence

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathrm{c}} = \mathbf{b}^T\mathbf{A}^{-1}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathrm{c}}\big) \;\to\; \mathbf{b}^T\mathbf{A}^{-1}\dot{u}_\mathrm{c}\mathbb{1}_s = \dot{u}_\mathrm{c}$$

in $L^2(0,T;\mathcal{V}_\mathrm{c}) \hookrightarrow L^2(0,T;\mathcal{H})$, cf. the proof of Theorem 5.10, we consider the remaining part $\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}}$. For this, we test equation (5.12) by $\mathbf{B}\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}} \in \mathcal{V}_{\mathcal{B},s}$. Remark 5.8 with $\mathcal{A} = \mathcal{K}$ and the symmetry of $\mathcal{K}$ yield

$$\begin{aligned} c\,|D\boldsymbol{u}_{j,\mathcal{B}}|^2 + \frac{1}{2\tau}\big(\langle \mathcal{K}u_{j,\mathcal{B}}, u_{j,\mathcal{B}}\rangle - \langle \mathcal{K}u_{j-1,\mathcal{B}}, u_{j-1,\mathcal{B}}\rangle\big)& \\ \le \langle \mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}}, \mathbf{B}\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}}\rangle + \langle \mathcal{K}\boldsymbol{u}_{j,\mathcal{B}}, \mathbf{B}\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}}\rangle& \\ = \langle \boldsymbol{\mathcal{F}}_j, \mathbf{B}\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}}\rangle + \langle \mathcal{B}^-\dot{\boldsymbol{\mathcal{G}}}_j, \mathbf{B}\mathbf{A}^{-1}D\boldsymbol{u}_{j,\mathcal{B}}\rangle.& \end{aligned}$$

Therein, $c > 0$ denotes the smallest eigenvalue of $\mathbf{A}^{-T}\mathbf{B}\mathbf{A}^{-1}$. As in the proof of Theorem 4.4, a multiplication by τ and a summation over all time steps leads to the estimate

$$\int_0^T |\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}(s)|^2\,\mathrm{d}s \;\lesssim\; \|a_\mathcal{B}\|^2 + \sum_{j=1}^n \tau\left(|\mathbf{A}^{-T}\mathbf{B}\boldsymbol{\mathcal{F}}_j|^2 + |\mathcal{B}^-\mathbf{A}^{-T}\mathbf{B}\dot{\boldsymbol{\mathcal{G}}}_j|^2\right). \tag{5.15}$$

Since the right-hand side is bounded, $\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}$ is bounded in $L^2(0,T;\mathcal{H}_s)$ and thus, $\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_{\tau,\mathcal{B}} = \mathbf{b}^T\mathbf{A}^{-1}\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}\big)$ is bounded in $L^2(0,T;\mathcal{H})$. We conclude the weak convergence of $\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{U}_\tau$ in $L^2(0,T;\mathcal{H})$. Further, estimate (5.15) guaranties the existence of a weak converging subsequence of $\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}$ in $L^2(0,T;\mathcal{H}_{\mathcal{B},\mathrm{s}})$ with a limit denoted by $\boldsymbol{H}$. Note that by equations (4.12), (5.12), Assumption 5.2, and the convergence of $\boldsymbol{U}_\tau$ it holds that $(\boldsymbol{H} - \dot{u}_\mathcal{B}\mathbb{1}_s, \boldsymbol{V}_\mathcal{B}) = 0$ for all $\boldsymbol{V}_\mathcal{B} \in \mathcal{V}_{\mathcal{B},s}$. Since $\mathcal{V}_{\mathcal{B},s}$ is densely embedded in $\mathcal{H}_{\mathcal{B},\mathrm{s}}$, it follows that $\boldsymbol{H} = \dot{u}_\mathcal{B}\mathbb{1}_s$. The convergence can be shown for the entire sequence such that $\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}}$ converges weakly to $\dot{u}_\mathcal{B}\mathbb{1}_s$ in $L^2(0,T;\mathcal{H}_{\mathcal{B},\mathrm{s}}) \subseteq L^2(0,T;\mathcal{H}_s)$. With the continuity of the operators it holds that

$$\begin{aligned} \mathcal{B}^*P_\tau &= \mathbf{b}^T\mathbf{A}^{-1}\left(\mathbf{A}^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\boldsymbol{U}}_{\tau,\mathcal{B}} + \mathcal{B}^-\dot{\boldsymbol{\mathcal{G}}}_\tau - \boldsymbol{\mathcal{F}}_\tau\right) + \mathcal{K}U_\tau - \mathcal{B}^-\Lambda_\tau \\ &\rightharpoonup \mathbf{b}^T\mathbf{A}^{-1}\left(\dot{u}_\mathcal{B} + \mathcal{B}^-\dot{\mathcal{G}} - \mathcal{F}\right)\mathbb{1}_s + \mathcal{K}u = \dot{u} - \mathcal{F} + \mathcal{K}u \qquad \text{in } L^2(0,T;\mathcal{V}^*). \end{aligned}$$

As in the proof of Theorem 4.4, this results in the claimed convergence of P_τ. □

Remark 5.13. The condition in Assumption 5.4 that the scheme has to be algebraically stable may be weakened. It is sufficient if a positive definite matrix $M \in \mathbb{R}^{s\times s}$ exists such that $\mathbf{M} := M\mathbf{A} + \mathbf{A}^TM^T - \mathbf{b}\mathbf{b}^T$ is positive semidefinite and $M^T\mathbb{1}_s = \mathbf{b}$.

6. Conclusion

Within this paper, we have analyzed the convergence of the implicit Euler scheme and, more general, of algebraically stable Runge-Kutta schemes with $R(\infty) = 0$ applied to linear operator DAEs of semi-explicit structure. For this, we have considered a regularized version of the system equations where a spatial discretization leads directly to a DAE of index one. This implies that the system is more stable than the original formulation although the solution set remains unchanged.

Within the convergence analysis, we have distinguished several cases for the smoothness of the data which includes the right-hand sides as well as the initial data. In the weakest case, we can only prove the convergence of the Lagrange multiplier in a distributional sense, i.e., only its integral converges. Note that we cannot expect more since for the given assumptions also the solution only exists in a distributional sense. With more regularity, the Lagrange multiplier converges weakly and the derivative of the approximation of the differential variable converges in a stronger norm.

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Institut für Mathematik MA4-5, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: {raltmann,zimmer}@math.tu-berlin.de