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ON THE SMOOTHING PROPERTY OF LINEAR DELAY PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. We consider linear partial differential equations with an additional delay term, which – under spatial discretization – lead to ordinary differential equations with fixed delay of retarded type. This means that the semi-discrete solution gains smoothness over time. For the concept of classical, mild, and weak solutions we analyse whether this effect also takes place in the original system. We show that some systems behave in a neutral way only. As a result, the smoothness of the exact solution remains unchanged instead of gaining smoothness over time.

Key words. linear PDEs, delay differential equations, retarded, neutral, smoothing property

AMS subject classifications. **35B65**, **35R10**, **65Q20**

1. Introduction

Partial differential equations (PDEs) with delay appear in many applications such as control theory [Wan75], population dynamics [Mur76], genetic repression [MP84], chemical reactions [Pao97], climate models [Het97], or fluid dynamics [CR01]. One specific example is given by the heat equation with delayed feedback control, i.e.,

$$\dot{u}(t,x) - \Delta u(t,x) = f(t,x) + u(t-\tau,x).$$

Therein, τ denotes the fixed delay modeling the needed reaction time of the system. In this paper, we analyse the smoothing effect of such a delay term with a single delay time τ for linear time-dependent PDEs. For the corresponding ordinary differential equation (ODE), which appears after a discretization in space, there exists a classification into retarded, neutral, and advanced type. This is based on the structure of the equation but is directly related to smoothing properties of the solution. Thus, a correct classification is also of importance for the correct numerical treatment of such systems, since the numerical scheme should be adapted to the expected regularity of the solution.

A corresponding classification for delay PDEs is not as common. However, the notion of retarded and neutral is used for example in [Sin84] and [Hal94, AE98, WG08], respectively, based on the structure of the system as in the ODE case. More precisely, this means that the system

$$\dot{u} + \alpha\dot{u}(\cdot-\tau) + \mathcal{K}u = f + \beta u(\cdot-\tau) \tag{1.1}$$

with a linear operator $\mathcal{K}$ is called *retarded* if $\beta \neq 0$, $\alpha = 0$ and *neutral* in the case of $\alpha \neq 0$. The analysis presented in this paper shows that this may not reflect the actual behaviour of the PDE if the operator $\mathcal{K}$ does not generate an analytic semigroup or satisfies a Gårding inequality.

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This paper is restricted to linear PDEs as in (1.1) with $\alpha = 0$, i.e., systems which lead to retarded delay ODEs after spatial discretization. This includes parabolic PDEs such as the heat equation mentioned above as well as the transport equation. The assumed structure implies that the semi-discrete solution attains more regularity over time. For the PDE, however, this may not be the case. We analyse this kind of smoothing property and discuss regularity assumptions on the data such that the PDE with delay behaves retardedly as well. By means of the transport equation, which is not retarded in this sense, we illustrate the obtained results also numerically.

The paper is structured as follows. In Section 2 we recap the notion of retarded, neutral, and advanced delay ODEs and illustrate the connection to the smoothing property of the solution. Afterwards, we discuss the different solution concepts for PDEs with delay in Section 3, i.e., we give the definitions for classical, mild, and weak solutions in the given setting. The main analysis of the smoothing property is then part of Section 4. Therein, we use Bellman's method of steps to analyse the change of regularity in each interval of length τ. In the numerical example of Section 5, we consider the transport equation with periodic boundary conditions. Finally, we conclude in Section 6.

Throughout this paper we consider the delay PDE on a bounded time domain $[0, T]$ and assume that the end time T is a multiple of the delay τ, i.e., $T = N\tau$. To shorten notation we define the open, left-open, and closed intervals

$$I_k^\circ := \big((k-1)\tau,\ k\tau\big), \qquad I_k := \big((k-1)\tau,\ k\tau\big], \qquad \bar{I}_k = \big[(k-1)\tau,\ k\tau\big].$$

Furthermore, we introduce the notion $u_\tau(s) := u(s-\tau)$.

2. Classification for delay ODEs

This section gives a short recap of known results for delay ODEs. In particular, we recall the definitions of retarded, neutral, and advanced systems and the connection to the smoothing property of the solution. The here considered finite-dimensional setting consists of delay differential equations of the form

$$\dot{q} + Kq = f + q_\tau \tag{2.1a}$$

with a square matrix $K \in \mathbb{R}^{n,n}$. The unique solvability of the system requires an initial value q_0 as well as a history function $\Phi \in C([-\tau, 0), \mathbb{R}^n)$, i.e., we demand

$$q(t) = \Phi(t) \text{ on } [-\tau, 0), \qquad q(0) = q_0. \tag{2.1b}$$

We say that $q \in C([0, T], \mathbb{R}^n)$ is a (classical) solution of (2.1) if is piecewise continuously differentiable, i.e., $q|_{I_k^\circ} \in C^1(I_k^\circ, \mathbb{R}^n)$, and the function

$$\tilde{q}(t) := \begin{cases} \Phi(t), & t \in [-\tau, 0) \\ q(t), & t \in [0, T] \end{cases}$$

satisfies the differential equation (2.1a) almost everywhere as well as (2.1b). Here, almost everywhere means that the differential equation is satisfied pointwise for $t \neq k\tau$.

For the classification of delay ODEs we consider a scalar equation of the form

$$\alpha_0 \dot{q} = \alpha_1 \dot{q}_\tau + \beta_0 q + \beta_1 q_\tau + f, \tag{2.2}$$

for which we distinguish three types, cf. [BZ03, Ch. 1] or [Wal14]. This classification is characterized by the smoothing properties of the solution. Note that such a classification is reasonable, since numerical methods rely on this kind of properties [AP95]. Thus, it has a direct influence on the behaviour of numerical methods.

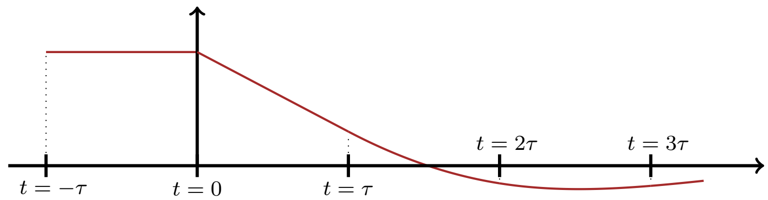

FIGURE 2.1. Solution of the delay ODE in Example 2.1.

We say that (2.2) with fixed delay τ is *retarded* if $\alpha_0, \beta_1 \neq 0$ and $\alpha_1 = 0$, which implies that the solution gets smoother with time. This can be seen as follows. Assuming $f \in C^\infty([0,T])$ and a history function $\Phi \in C([-\tau, 0])$ with $\Phi(0) = q_0$, we obtain on $\bar{I}_1 = [0, \tau]$ a differential equation with a continuous right-hand side, namely

$$\alpha_0 \dot{q}(t) = \beta_0 q(t) + \beta_1 q(t-\tau) + f(t) = \beta_0 q(t) + \beta_1 \Phi(t-\tau) + f(t).$$

This gives a solution $q_1 \in C^1(I_1)$ and we proceed to the next time interval. On $\bar{I}_2 = [\tau, 2\tau]$ we insert q_1 for the delay term and get a differential equation with a continuously differentiable right-hand side such that the solution satisfies $q_2 \in C^2(I_2)$. This can be repeated and yields $q|_{I_k} = q_k \in C^k(I_k)$.

The gain of regularity can also be seen in a different way. For $f \in C^\infty([0,T])$ one may consider the evolution of the (jump) discontinuity of the derivatives of q in $t = 0$. Even for smooth data, in general, we have $\dot{\Phi}(0^-) \neq \dot{q}(0^+)$. In $t = \tau$ the right and left derivatives coincide but the second derivative has a discontinuity. This can be seen by Bellman's method of steps, cf. [Bel61] or [BZ03, Ch. 3.4].

Example 2.1. Consider the delay equation

$$\dot{q} = -q_\tau, \qquad q(t) = 1 \text{ on } [-\tau, 0].$$

On the interval I_1 we have the ODE $\dot{q} = -1$ with initial condition $q(0) = 1$. This yields $q(t) = 1 - t$ for $t \leq \tau$. On I_2 we need to consider $\dot{q} = (t - \tau) - 1$ with initial condition $q(\tau) = 1 - \tau$. Thus, we get $q(t) = 1 - t + \frac{1}{2}(t-\tau)^2$ for $t \in I_2$. This procedure may be continued sequentially. An illustration of the solution for $\tau = 0.7$ is given in Figure 2.1. The derivatives of q at time points $t = k\tau$ satisfy the following. At $t = 0$ we have $\dot{\Phi}(0^-) = 0 \neq -1 = \dot{q}(0^+)$. At $t = \tau$ we have $\dot{q}(\tau^-) = -1 = \dot{q}(\tau^+)$ but $\ddot{q}(\tau^-) = 0 \neq 1 = \ddot{q}(\tau^+)$. In general, we get for $t = k\tau$,

$$q^{(k)}(t^-) = (-1)^k = q^{(k)}(t^+), \qquad q^{(k+1)}(t^-) = 0 \neq (-1)^{k+1} = q^{(k+1)}(t^+).$$

For equations of *neutral* type we have $\alpha_0 \neq 0$ and $\alpha_1 \neq 0$. In this case the solution retains its smoothness (neither gain nor loss of smoothness), since the equation also contains the derivative of q at prior times. Finally, the equation is called *advanced* whenever $\alpha_0 = 0$ and $\alpha_1 \neq 0$. This means, that the ODE depends on future events and that the solution looses regularity in every time step of size τ, cf. [BC63, Ch. 5.1].

Apparently, such a classification is also available for systems of delay ODEs. Here, however, it may happen that the system includes variables of different type. In this paper, we only consider delay ODEs of the form (2.1), which are retarded, i.e., all variables gain smoothness over time. This means, in particular, that the derivative of the solution of (2.1) may only be discontinuous in $t = 0$ or $t = \tau$. The aim of this paper is to analyse the smoothness properties of their infinite-dimensional analogon, i.e., time-dependent PDEs, which lead to (2.1) after spatial discretization.

3. Solution concepts

In this section, we recapitulate different solution concepts and existence results for linear PDEs without delay, i.e., for equations of the type

$$\dot{u} + \mathcal{K}u = f, \qquad u(0) = u_0. \tag{3.1}$$

For a classical solution and the weaker notion of a mild solution we assume that the operator $-\mathcal{K}\colon D(-\mathcal{K}) = D(\mathcal{K}) \subseteq \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup. Note that in this framework the operator $\mathcal{K}$ is, in general, unbounded. Furthermore, we discuss weak solutions, which correspond to the weak formulation of (3.1). In this case, the operator $\mathcal{K}$ is assumed to be bounded as mapping $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$, based on a Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$.

The various solution concepts for PDEs are then extended to the delay case. For this, we add an additional (discrete) delay term and consider

$$\dot{u} + \mathcal{K}u = f + u_\tau, \qquad u(t) = \Phi(t) \text{ on } t \in [-\tau, 0), \qquad u(0) = u_0. \tag{3.2}$$

Recall that u_τ is defined by $u_\tau(s) = u(s - \tau)$ and that Φ is the needed history function in order to make the right-hand side of the differential equation meaningful. The corresponding ODE case, i.e., equation (3.2) after a spatial disretization, equals equation (2.1) and was discussed in Section 2.

3.1. Mild solutions.

If the C_0-semigroup generated by $-\mathcal{K}$ is denoted by $S(t)$ and the right-hand side satisfies $f \in L^1(0, T; \mathcal{H})$, then the mild solution of (3.1) is defined by

$$u(t) = S(t)u_0 + \int_0^t S(t - s)f(s) \,\mathrm{d}s. \tag{3.3}$$

Note that $S(t - s)f(s)$ is integrable for $f \in L^1(0, T; \mathcal{H})$ and that this implies $u \in C([0, T], \mathcal{H})$. Furthermore, we only need $u_0 \in \mathcal{H}$, i.e., the operator $-\mathcal{K}$ may not be applicable for the initial value. More details can be found in [Paz83, Ch. 4.2].

Similarly, we can define the mild solution for the delay equation (3.2) by an explicit solution formula. Again we assume that $-\mathcal{K}$ generates the C_0-semigroup $S(t)$, an initial value $u_0 \in \mathcal{H}$, and a right-hand side $f \in L^1(0, T; \mathcal{H})$. In addition, we need an integrable history function Φ. This then leads to the following definition.

Definition 3.1 (mild solution)**.** Consider an initial value $u_0 \in \mathcal{H}$ and a history function $\Phi \in L^1(-\tau, 0; \mathcal{H})$. A function $u \in C([0, T], \mathcal{H})$ is called *mild solution* of (3.2) if $u(0) = u_0$ and $u|_{I_k} = u_k$ for $k = 1, \dots, N$ with $u_k \in C(\bar{I}_k, \mathcal{H})$ defined by

$$u_1(t) = S(t)u_0 + \int_0^t S(t - s)\big[f(s) + \Phi(s - \tau)\big] \,\mathrm{d}s$$

and

$$u_k(t) = S(t - (k-1)\tau)u_{k,0} + \int_{(k-1)\tau}^t S(t - s)\big[f(s) + u_{k-1}(s - \tau)\big] \,\mathrm{d}s$$

for $k \geq 2$, respectively. Therein, the initial value $u_{k,0}$ is given by $u_{k,0} = u_{k-1}((k-1)\tau)$.

Note that, in general, this mild solution is not differentiable. If we search for a continuously differentiable solution, then we need to switch to the notion of classical solutions

3.2. **Classical solutions.** Classical solutions of (3.1) are functions $u \in C([0,T];\mathcal{H})$, which are continuously differentiable in $(0,T]$ and satisfy $u(t) \in D(\mathcal{K})$ as well as the differential equation in $(0,T]$ and $u(0) = u_0$. In this setting, we again assume that $-\mathcal{K}$ generates a C_0-semigroup and an initial value $u_0 \in D(\mathcal{K})$. We emphasize that the additional regularity of u_0 is not sufficient for the existence of a classical solution, cf. the discussion in [Paz83, Ch. 4.2]. One sufficient condition is that – in addition to $u_0 \in D(\mathcal{K})$ – the right-hand side satisfies $f \in C^1([0,T];\mathcal{H})$. However, we analyse a slightly different situation here.

Theorem 3.2 (classical solution without delay)**.** *Assume that $-\mathcal{K}$ generates a C_0-semigroup, $u_0 \in D(\mathcal{K})$, and a right-hand side $f \in C([0,T],\mathcal{H})$ with $f(s) \in D(\mathcal{K})$ for all $s \in [0,T]$ and $\mathcal{K}f \in L^1(0,T;\mathcal{H})$. Then, equation* (3.1) *has a unique classical solution u, which satisfies in addition $\mathcal{K}u \in C([0,T],\mathcal{H}) \hookrightarrow L^1(0,T,\mathcal{H})$.*

Proof. By the given assumptions, the result in [Paz83, Ch. 4, Cor. 2.6] implies that the mild solution $u \in C([0,T],\mathcal{H})$ is continuously differentiable on $(0,T)$ and $u(t) \in D(\mathcal{K})$ for $t < T$. Since we can extend the right-hand side to a function $\tilde{f} \in C([0,T+\varepsilon],\mathcal{H})$ with $\tilde{f}(s) \in D(\mathcal{K})$ and $\mathcal{K}\tilde{f} \in L^1(0,T+\varepsilon;\mathcal{H})$, u is also differentiable in $t = T$. Applying the operator $\mathcal{K}$ to the mild solution (3.3), we get

$$\mathcal{K}u(t) = \mathcal{K}S(t)u_0 + \mathcal{K}\int_0^t S(t-s)f(s)\,\mathrm{d}s = S(t)\mathcal{K}u_0 + \int_0^t S(s)\mathcal{K}f(t-s)\,\mathrm{d}s.$$

Here we have used $u_0 \in D(\mathcal{K})$ together with basic properties of the semigroup S, cf. [Paz83, Ch. 1, Th. 2.4]. Since $\mathcal{K}f$ is integrable and S is a bounded operator, we obtain $u(t) \in D(\mathcal{K})$ for all $t \in [0,T]$. Note that this equation also implies $\mathcal{K}u \in C([0,T],\mathcal{H})$ due to the continuity of the semigroup. □

In the following, we extend the concept of a classical solution to the delay case, i.e., the solution being a continuous function u, which is piecewise continuously differentiable in $(0,T]$ and satisfies $u(t) \in D(\mathcal{K})$.

Definition 3.3 (classical solution)**.** Consider an initial value $u_0 \in D(\mathcal{K})$ and a history function $\Phi \in C([-\tau,0],\mathcal{H})$ with $\Phi(s) \in D(\mathcal{K})$ for all $s \in [-\tau,0]$, $\mathcal{K}\Phi \in L^1(-\tau,0;\mathcal{H})$, and $\Phi(0) = u_0$. We call a function $u \in C([0,T],\mathcal{H})$ with $u|_{I_k^\circ} \in C^1(I_k^\circ,\mathcal{H})$ a *classical solution* of (3.2) if $u(s) \in D(\mathcal{K})$ for all $s \in [0,T]$ and $\tilde{u} \in C([-\tau,T],\mathcal{H})$, given by

$$\tilde{u}(t) := \begin{cases} \Phi(t), & t \in [-\tau,0) \\ u(t), & t \in [0,T] \end{cases},$$

satisfies the differential equation in (3.2) almost everywhere.

We emphasize that the existence of a classical solution requires an initial value $u_0 \in D(\mathcal{K})$ as well as the consistency condition $\Phi(0) = u_0$.

3.3. **Weak solutions.** Finally we consider the weak solution concept based on a Gelfand triple $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}^*$, cf. [Zei90, Ch. 23.4]. To shorten notation we introduce the space

$$L^+(0,T) := L^2(0,T;\mathcal{V}^*) + L^1(0,T;\mathcal{H}).$$

Within the weak formulation of equation (3.1) we consider a linear, continuous operator $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$, which satisfies a Gårding inequality, i.e.,

$$\mu\|v\|_{\mathcal{V}}^2 - \kappa\|v\|_{\mathcal{H}}^2 \le \langle \mathcal{K}v, v\rangle \tag{3.4}$$

for all $v \in \mathcal{V}$ and fixed $\mu > 0$, $\kappa \ge 0$.

Remark 3.4. If the operator $\mathcal{K}$ satisfies a Gårding inequality (3.4), then $-\mathcal{K}$ generates an analytic semigroup on $\mathcal{H}$, see [Paz83, Ch. 7, Th. 2.7].

In the weak setting, the operator equation (3.1) should be understood in the variational sense, i.e.,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(u,v)_{\mathcal{H}} + \langle \mathcal{K}u, v\rangle = \langle f, v\rangle \tag{3.5}$$

for all test functions $v \in \mathcal{V}$. A function $u \in C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$ with $\dot{u} \in L^+(0,T)$, which satisfies (3.5) and $u(0) = u_0$, is then called a weak solution. An existence result for weak solutions for a right-hand side $f \in L^+(0,T)$ is given by the theorem of Lions-Tartar, see e.g. [Tar06, Lem. 19.1]. This result, together with a stability estimate, is subject of the following theorem.

Theorem 3.5 (weak solution without delay)**.** *Let the linear and bounded operator $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$ satisfy a Gårding inequality* (3.4)*. Assume an initial value $u_0 \in \mathcal{H}$ and a right-hand side $f = f_1 + f_2$ with $f_1 \in L^2(0,T;\mathcal{V}^*)$ and $f_2 \in L^1(0,T;\mathcal{H})$. Then, system* (3.1) *has a unique weak solution $u \in C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$. Furthermore, it holds for all $t \in [0,T]$ that*

$$\begin{aligned}\|u\|^2_{C([0,t],\mathcal{H})} &+ \mu\|u\|^2_{L^2(0,t;\mathcal{V})} \\ &\le \left[\left(\|u_0\|^2_{\mathcal{H}} + \int_0^t \tfrac{1}{\mu}\|f_1(s)\|^2_{\mathcal{V}^*}\, ds\right)^{1/2} + \int_0^t e^{-\kappa s}\,\|f_2(s)\|_{\mathcal{H}}\, ds\right]^2 e^{2\kappa t}.\end{aligned}$$

Proof. The existence of a unique solution is proven in [Tar06, Lem. 19.1]. For the proof of the estimates we follow the idea of [Tar06, p. 112]. For this, we test equation (3.1) with the solution u and integrate over $[0,t]$. Together with (3.4) this yields

$$\|u(t)\|^2_{\mathcal{H}} + \mu\int_0^t \|u\|^2_{\mathcal{V}}\,\mathrm{d}s \le \|u_0\|^2_{\mathcal{H}} + \int_0^t \left(\tfrac{1}{\mu}\|f_1\|^2_{\mathcal{V}^*} + 2\kappa\|u\|^2_{\mathcal{H}} + 2\|f_2\|_{\mathcal{H}}\|u\|_{\mathcal{H}}\right)\mathrm{d}s.$$

We set $\varphi(t) := \|u(t)\|^2_{\mathcal{H}}$. The previous estimate implies for every $s \in [0,t]$ that

$$\varphi(s) \le \psi(s) := A + \int_0^s \left(2\kappa\varphi(\eta) + 2\|f_2(\eta)\|_{\mathcal{H}}\sqrt{\varphi(\eta)}\right)\mathrm{d}\eta$$

with the s-independent constant $A := \|u_0\|^2_{\mathcal{H}} + \int_0^t \frac{1}{\mu}\|f_1(\eta)\|^2_{\mathcal{V}^*}\,\mathrm{d}\eta$. The derivative of ψ satisfies

$$\dot{\psi}(s) = 2\kappa\varphi(s) + 2\|f_2(s)\|_{\mathcal{H}}\sqrt{\varphi(s)} \le 2\kappa\psi(s) + 2\|f_2(s)\|_{\mathcal{H}}\sqrt{\psi(s)}.$$

Furthermore, the function

$$z(s) := \left(\sqrt{A} + \int_0^s e^{-\kappa\eta}\,\|f_2(\eta)\|_{\mathcal{H}}\,\mathrm{d}\eta\right)^2 e^{2\kappa s} \tag{3.6}$$

satisfies the differential equation $\dot{z} = 2\kappa z + 2\|f_2\|_{\mathcal{H}}\sqrt{z}$ with initial value $z(0) = A = \psi(0)$. The theory of differential inequalities thus implies $\psi(s) \le z(s)$ for all $s \in [0,t]$ and especially for $s = t$. The claimed estimates finally follow from

$$\|u(t)\|^2_{\mathcal{H}} + \mu\int_0^t \|u(s)\|^2_{\mathcal{V}}\,\mathrm{d}s \le \psi(t) \le z(t). \qquad \square$$

Including delay, we still assume a right-hand side $f \in L^+(0,T)$ and an initial value $u_0 \in \mathcal{H}$. Furthermore, we need a history function $\Phi \in L^+(-\tau,0)$.

Definition 3.6 (weak solution)**.** A function $u \in L^2(0,T;\mathcal{V})$ with $\dot{u} \in L^+(0,T)$ is called a *weak solution* of (3.2) if $u(0^+) = u_0 \in \mathcal{H}$ and the function

$$\tilde{u}(t) := \begin{cases} \Phi(t), & t \in [-\tau, 0) \\ u(t), & t \in [0,T] \end{cases}$$

satisfies the differential equation (3.2) in the variational sense.

4. Classification for delay PDEs

This section is devoted to the classification of delay PDEs of the form (3.2). The aim is to transfer the given classification of delay ODEs from Section 2 into the context of delay PDEs. Recall that in the finite-dimensional case the different types of delay were characterized only by the structure of the equation. As mentioned in the introduction, one may use a similar classification for PDEs, based on the existence of the terms $\dot{u}(t)$ and $\dot{u}(t-\tau)$. In this paper, however, we are interested in the change of smoothness of the solution, since this is the crucial property for delay differential equations. More precisely, we consider history functions, which are only continuous or even integrable in time and analyse the smoothness of the solution in the intervals $(k\tau, T]$.

Throughout this section, we assume that $-\mathcal{K}$ either generates a C_0-semigroup or satisfies a Gårding inequality, which implies the existence of an analytic semigroup according to Remark 3.4. Note that the operator $\mathcal{K}$ already includes appropriate boundary conditions.

4.1. Mild setting.

Consider the PDE with delay (3.2) with initial data $u_0 \in \mathcal{H}$ and a history function $\Phi \in L^1(-\tau, 0; \mathcal{H})$. Let the right-hand side moreover satisfy $f \in L^1(0,T;\mathcal{H})$. In this subsection we show that the mild solution behaves, in general, neutrally, i.e., the regularity of the solution remains unchanged. Only in the case of $-\mathcal{K}$ generating an analytic semigroup we are able to prove a retarded solution behaviour as in the corresponding ODE case.

In order to analyse the regularity of the solution, we first consider equation (3.2) with given initial value and history function on the interval $\bar{I}_1 = [0, \tau]$. Here we have

$$\dot{u} + \mathcal{K}u = f + \Phi_\tau, \qquad u(0) = u_0.$$

Since the right-hand side $f + \Phi_\tau$ is in $L^1(0,\tau;\mathcal{H})$, there exists a unique mild solution $u_1 \in C(\bar{I}_1, \mathcal{H})$, cf. [Paz83, Ch. 4.2]. This solution is given by

$$u_1(t) = S(t)u_0 + \int_0^t S(t-s)\big[f(s) + \Phi(s-\tau)\big]\,\mathrm{d}s.$$

On the subsequent interval $\bar{I}_2 = [\tau, 2\tau]$ equation (3.2) has the form

$$\dot{u} + \mathcal{K}u = f + u_{1,\tau}, \qquad u(\tau) = u_{2,0} := u_1(\tau).$$

Note that the right-hand side is now in $L^1(\tau, 2\tau; \mathcal{H})$, since we have $u_1 \in L^1(0,\tau;\mathcal{H})$ from the previous step. Thus, again by [Paz83, Ch. 4.2], we obtain a unique mild solution $u_2 \in C(\bar{I}_2, \mathcal{H})$ given by

$$u_2(t) = S(t-\tau)u_{2,0} + \int_\tau^t S(t-s)\big[f(s) + u_1(s-\tau)\big]\,\mathrm{d}s.$$

Advancing with this procedure, we obtain in each subinterval a solution u_k for $k = 1, \dots, N$. In summary, the function

$$u(t) = \begin{cases} u_0, & t = 0 \\ u_k(t), & t \in I_k \end{cases} \tag{4.1}$$

is continuous because of the chosen initial condition in each step and thus, is the mild solution of the delay equation (3.2). We summarize this result in form of a theorem.

Theorem 4.1 (mild solution)**.** *Consider $f \in L^1(0,T;\mathcal{H})$, $u_0 \in \mathcal{H}$, and a history function $\Phi \in L^1(-\tau,0;\mathcal{H})$. Furthermore, let $-\mathcal{K}$ generate a C_0-semigroup. Then, the constructed function $u \in C([0,T],\mathcal{H})$ from* (4.1) *is the unique mild solution of equation* (3.2)*.*

Note that this implies that the delay PDE behaves at least *neutrally*, since no regularity is lost over time. It remains to analyse whether the system may even behave retardedly, i.e., whether the solution may gain regularity over time. For this, we now assume $f \in C^\infty([0,T],\mathcal{H})$.

4.1.1. *Strongly continuous semigroup.* We show that the mild solution does, in general, not behave retardedly if $-\mathcal{K}$ generates a strongly continuous but not an analytic semigroup. For this, we first discuss by means of the equations that a retarded behaviour is not to be expected and second, we consider the transport equation as a counterexample.

We differentiate equation (3.2), which (formally) yields an evolution equation for $v := \dot{u}$, namely

$$\dot{v} + \mathcal{K}v = \dot{f} + v_\tau, \qquad v(t_0) = \dot{u}(t_0). \tag{4.2}$$

Now the question is whether there exists a $t_0 \geq \tau$ such that this equation has a mild solution. In order to use the existence result for mild solutions, we need an initial value $v(t_0) \in \mathcal{H}$ and an integrable right-hand side, meaning in particular $\dot{u} \in L^1(t_0-\tau, T-\tau;\mathcal{H})$. The evolution equation in (3.2) implies

$$v(t_0) = \dot{u}(t_0) = f(t_0) + u_\tau(t_0) - \mathcal{K}u(t_0),$$

where $f(t_0) \in \mathcal{H}$ by assumption and $u_\tau(t_0) = u(t_0 - \tau) \in \mathcal{H}$, since u is the mild solution of the original delay system. The remaining term $\mathcal{K}u(t_0)$ is an element of $\mathcal{H}$ if and only if $u(t_0) \in D(\mathcal{K})$. Thus, in order to gain regularity in time, the solution has to gain regularity in space as well. Since such a smoothing property is not to be expected in this case, we cannot guarantee that there exists a time t_0 such that the initial value $v(t_0)$ is smooth enough for the existence of a mild solution. Thus, in the case that $-\mathcal{K}$ does not satisfy the analytic smoothing property, we are lead to assume that the solution of the evolution equation behaves neutrally. Finally, we give an example, which illustrates the missing smoothing property.

Example 4.2. Consider $\Omega = \mathbb{R}$ and the semigroup $S(t)\colon L^2(\Omega) \to L^2(\Omega)$ given by the (left) shift operator $S(t)u(x) := u(x+t)$. The corresponding generator is given by

$$\mathcal{A}u(x) = \lim_{t\to 0} \frac{S(t)u(x) - u(x)}{t} = \lim_{t\to 0} \frac{u(x+t) - u(x)}{t} = \frac{\partial u(x)}{\partial x} =: \partial_x u(x).$$

The domain of $\mathcal{A}$ is given by $H^1(\Omega)$. The semigroup is strongly continuous [BFR17, Ex. 9.11] but not analytic, since $\mathcal{A}e^{\mathcal{A}t} = \mathcal{A}S(t)$ is not bounded. In other words, no smoothness is gained by a shift. If we consider the delay equation

$$\dot{u} + \partial_x u = 0 + u_\tau, \qquad u(t) = \Phi(t) \text{ on } t \in [-\tau, 0), \qquad u(0) = u_0, \tag{4.3}$$

then the solution is given by

$$u(t,x) = S(t)u_0(x) + \int_0^t S(t-s)u_\tau(s,x)\,\mathrm{d}s = u_0(x+t) + \int_0^t u(s-\tau, x+t-s)\,\mathrm{d}s. \tag{4.4}$$

Note that u has to be replaced by Φ in the integral for $s < \tau$. Let us now consider an initial function u_0 with a discontinuity at $x = 0$. Since the integral term in (4.4) vanishes for $t \to 0$, there exists a constant $a_0 \leq \tau$ such that

$$u(t, -t^+) - u(t, -t^-) = u_0(0^+) - u_0(0^-) + \int_0^t \Phi(s-\tau, -s^+) - \Phi(s-\tau, -s^-) \,\mathrm{d}s \neq 0$$

for almost every $t \in (0, a_0)$. Accordingly, we obtain for $t > \tau$,

$$u(t, (\tau - t)^\pm) = u_0(\tau^\pm) + \int_0^\tau \Phi(s-\tau, -(s-\tau)^\pm) \,\mathrm{d}s + \int_\tau^t u\big(s-\tau, -(s-\tau)^\pm\big) \,\mathrm{d}s.$$

Note that the second integral differs for τ^+ and τ^- for all $t \in (\tau, a_0 + \tau)$. Hence, there exists a positive constant $a_1 \leq a_0$ such that there is a jump at $x = \tau - t$ for almost every time point $t \in (\tau, a_1 + \tau)$. Repeating these arguments for all $k \in \mathbb{N}$, we observe a discontinuity at $x = k\tau - t$ for almost all $t \in (k\tau, a_k + k\tau)$, $0 < a_k \leq a_{k-1}$. As a result, there is no gain of regularity for the given example, which implies that the delay system is not retarded in terms of the smoothing property.

The numerical example discussed in Section 5 is a slightly modified version of (4.3) but shows the same behaviour.

Remark 4.3. The shift operator from Example 4.2 can also be formulated on the bounded domain $\Omega = (0, 1)$ with appropriate boundary conditions, cf. [BFCO12, Sect. 2.3]. This again results in a C_0-semigroup, which is not analytic.

4.1.2. *Analytic semigroup.* In this subsection, we discuss the case where $-\mathcal{K}$ generates an analytical semigroup. This means that the mapping $t \mapsto S(t)$ is analytic as a mapping from $(0, T]$ to $\mathcal{L}(\mathcal{H})$, i.e., the space of linear and bounded maps from $\mathcal{H}$ to $\mathcal{H}$, cf. [Lun95]. Furthermore, this implies the *parabolic smoothing property*, i.e., the existence of a constant $C > 0$ such that

$$\|\mathcal{K}S(t)\|_{\mathcal{L}(\mathcal{H})} \leq C\, t^{-1}, \qquad 0 < t \leq T.$$

This means, in particular, that $u_0 \in \mathcal{H}$ implies $S(t)u_0 \in D(\mathcal{K})$ for all $t > 0$.

Further we assume in this subsection that $\Phi \in C([-\tau, 0], \mathcal{H})$ with the continuity condition $u_0 = u(0) = \Phi(0)$. We address the general case $\Phi \in L^1(-\tau, 0; \mathcal{H})$ in Remark 4.5 below. Because of Theorem 4.1 and the assumptions on Φ we know that the right-hand side of (3.2) satisfies $f + u_\tau \in C([0, T], \mathcal{H}) \hookrightarrow L^p(0, T; \mathcal{H})$ for all $p \geq 1$. The theory of abstract Cauchy problems, cf. [Paz83, Ch. 4, Th. 3.1], then implies that $u \in C^{0,r}([\frac{\varepsilon}{2}, T], \mathcal{H})$ for all exponents $r \in (0, 1)$ and $\varepsilon > 0$. Here, $C^{0,r}([a, b], \mathcal{H})$ denotes the space of Hölder continuous functions with exponent r.

For the right-hand side of (3.2) this in turn implies $f + u_\tau \in C^{0,r}([\tau + \frac{\varepsilon}{2}, T], \mathcal{H})$. Following [Paz83, Ch. 4, Th. 3.5], we obtain for the mild solution

$$\mathcal{K}u, \dot{u} \in C^{0,r}([\tau + \varepsilon, T], \mathcal{H}).$$

This means that the solution has gained regularity in time after one step of size τ (plus some arbitrary small ε). Thus, in contrast to the finite-dimensional case, the gain of regularity does not appear immediately at $t = \tau$. We consider once more the evolution equation, which we obtain by differentiation, i.e., we consider equation (4.2) with $v = \dot{u}$. As starting value we take $v(\tau + \varepsilon) \in \mathcal{H}$. The right-hand side of this equation satisfies $\dot{f} + v_\tau = \dot{f} + \dot{u}_\tau \in C^{0,r}([2\tau + \varepsilon, T], \mathcal{H})$ such that [Paz83, Ch. 4, Th. 3.5] implies for the unique mild solution,

$$\mathcal{K}v, \dot{v} = \ddot{u} \in C^{0,r}([2\tau + 2\varepsilon, T], \mathcal{H}).$$

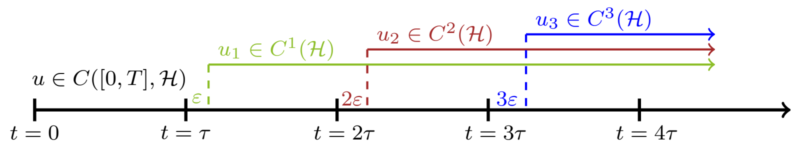

FIGURE 4.1. Illustration of the gain of regularity in the case of $-\mathcal{K}$ generating an analytic semigroup. For $u_k := u|_{[k\tau+k\varepsilon,T]}$ we obtain that $u_k \in C^k(\mathcal{H}) := C^k([k\tau + k\varepsilon, T], \mathcal{H})$.

This procedure can now be repeated for the following time intervals, cf. the illustration in Figure 4.1. This means that we consider the derivative of (4.2) on the time interval $[2\tau + 2\varepsilon, T]$ and so on.

As a result, the delay PDE (3.2) with $-\mathcal{K}$ generating a C_0-semigroup behaves neutrally, whereas a generator of an analytic semigroup leads to a – more or less – retarded behaviour. For the limit case $\varepsilon \to 0$ the considerations above lead to the following theorem.

Theorem 4.4. *Consider $f \in C^\infty([0,T],\mathcal{H})$ and a history function $\Phi \in C([-\tau,0],\mathcal{H})$ with $\Phi(0) = u_0$. Furthermore, let $-\mathcal{K}$ generate an analytic semigroup. Then, the mild solution $u \in C([0,T],\mathcal{H})$ of (3.2) satisfies*

$$u|_{(k\tau,T]} \in C^k((k\tau,T],\mathcal{H}) \quad \textit{for } k = 0,\dots,N-1.$$

Remark 4.5. In the general case $\Phi \in L^1(-\tau,0;\mathcal{H})$ we need one time step of size τ to obtain a right-hand side, which is continuous and consistent. Thus, we obtain the same result as in Theorem 4.4 but shifted by one time step. More precisely, we obtain in this case

$$u|_{(k\tau,T]} \in C^{k-1}((k\tau,T],\mathcal{H}) \quad \text{for } k = 1,\dots,N-1.$$

Remark 4.6. Theorem 4.4 remains valid if we insert a linear and bounded operator $\mathcal{B}\colon \mathcal{H} \to \mathcal{H}$ in front of the delay term, i.e., if we consider the delay equation

$$\dot{u} + \mathcal{K}u = f + \mathcal{B}u_\tau, \qquad u(t) = \Phi(t) \text{ on } t \in [-\tau,0), \qquad u(0) = u_0.$$

4.2. **Classical setting.** We switch to the setting of classical solutions. For this, we consider the delay system (3.2) with initial data $u_0 \in D(\mathcal{K})$ and a history function $\Phi \in C([-\tau,0],\mathcal{H})$ satisfying $\Phi(t) \in D(\mathcal{K})$ for all t, $\Phi(0) = u_0$, and $\mathcal{K}\Phi \in L^1(-\tau,0;\mathcal{H})$. For the applied force we assume similarly $f \in C([0,T],\mathcal{H})$ with $f(t) \in D(\mathcal{K})$ for all t and $\mathcal{K}f \in L^1(0,T;\mathcal{H})$. We first show, that these assumptions imply the existence of a classical solution u. Afterwards, we analyse the smoothing property of the solution in the case of a strongly continuous as well as an analytic semigroup.

As for the mild solution we use Bellman's method of steps and consider first equation (3.2) on the interval $\bar{I}_1 = [0,\tau]$. The right-hand side is then given by $f + \Phi_\tau$ and satisfies the assumptions of Theorem 3.2. Thus, there exists a unique classical solution $u_1 \in C(\bar{I}_1,\mathcal{H})$, which is differentiable for $t > 0$ and satisfies $u_1(t) \in D(\mathcal{K})$ for all $t \in \bar{I}_1$ and $\mathcal{K}u_1 \in C(\bar{I}_1,\mathcal{H})$. Equation (3.2) on the interval $\bar{I}_2 = [\tau,2\tau]$ has then the right-hand side $f + u_{1,\tau}$ and as initial value $u_1(\tau) \in D(\mathcal{K})$. A consecutive application of Theorem 3.2 on the intevals $\bar{I}_k$, $k = 2,\dots,N$ then implies the existence of a classical solution on the entire interval $[0,T]$. We summarize this result in the following theorem.

Theorem 4.7 (classical solution)**.** *Consider the delay PDE (3.2) with $-\mathcal{K}$ generating a C_0-semigroup, initial data $u_0 \in D(\mathcal{K})$, and $f \in C([0,T],\mathcal{H})$ satisfying $f(t) \in D(\mathcal{K})$ for all t and $\mathcal{K}f \in L^1(0,T;\mathcal{H})$. Let the history function $\Phi \in C([-\tau,0],\mathcal{H})$ satisfy in addition $\Phi(s) \in D(\mathcal{K})$, $\Phi(0) = u_0$, and $\mathcal{K}\Phi \in L^1(-\tau,0;\mathcal{H})$. Then, there exists a unique classical solution $u \in C([0,T],D(\mathcal{K}))$.*

Note that this again implies that the solution does not lose regularity and thus, behaves at least *neutrally*. In order to see whether the solution gains regularity over time, we distinguish once more the cases of $-\mathcal{K}$ generating a strongly continuous or even analytic semigroup. In any case, we assume that the applied force is given by a smooth function in the sense that $f \in C^\infty([0,T], D(\mathcal{K}))$.

4.2.1. *Strongly continuous semigroup.* Let $-\mathcal{K}$ generate a strongly continuous but not analytic semigroup. As in Section 4.1 we consider the derivative of the evolution equation with $v := \dot{u}$, cf. equation (4.2). We first show that there exists a mild solution v for $t \geq \tau$. Second, we check whether this v could even be a classical solution.

For $t = \tau$ the initial value is given by $v(\tau) = \dot{u}(\tau) = f(\tau) + u(0) - \mathcal{K}u(\tau) \in \mathcal{H}$. Furthermore, the right-hand side $\dot{f} + \dot{u}_\tau$ of (4.2) is bounded in $L^1(\tau, T; \mathcal{H})$, since f is smooth and u is the classical solution with $\mathcal{K}u \in L^1(0,T;\mathcal{H})$. Thus, we have

$$\|\dot{u}_\tau\|_{L^1(\tau,T;\mathcal{H})} = \|\dot{u}\|_{L^1(0,T-\tau;\mathcal{H})} = \int_0^{T-\tau} \big| f(t) + u(t-\tau) - \mathcal{K}u(t) \big| \,\mathrm{d}t < \infty.$$

As a result, Section 3.1 implies that there exists a unique mild solution $v \in C([\tau,T],\mathcal{H})$. The question is whether v is a classical solution as well, at least from some point in time $t_0 \geq \tau$ on. This requires that for t_0 we attain $v(t) \in D(\mathcal{K})$ for $t \geq t_0$. This in turn means that we obtain some kind of smoothing property for mild solutions, which is not to be expected in the considered case, cf. Section 4.1.1.

4.2.2. *Analytic semigroup.* Let $-\mathcal{K}$ now generate an analytic semigroup and v denote the mild solution of (4.2). Aim of this subsection is to show that v is indeed a classical solution for $t > 2\tau$.

Theorem 4.8. *Consider the delay PDE* (3.2) *with $-\mathcal{K}$ generating an analytic semigroup. Further assume $f \in C^\infty([0,T], D(\mathcal{K}))$ and a history function $\Phi \in C([-\tau,0],\mathcal{H})$ with $\Phi(s) \in D(\mathcal{K})$, $\Phi(0) = u_0$, and $\mathcal{K}\Phi \in L^1(-\tau,0;\mathcal{H})$. Then, the classical solution $u \in C([0,T], D(\mathcal{K}))$ satisfies for $k = 0,\dots,N-1$ that*

$$u|_{(k\tau,T]} \in C^k\big((k\tau,T],\mathcal{H}\big), \quad u|_{((k+1)\tau,T]} \in C^k\big(((k+1)\tau,T], D(\mathcal{K})\big).$$

Proof. Since u is the classical solution of (3.2) we know that $v := \dot{u} \in C([0,T],\mathcal{H})$. For $t \geq \tau$ we consider the delay equation

$$\dot{v} + \mathcal{K}v = \dot{f} + v_\tau \tag{4.5}$$

with initial value $v(\tau) := \dot{u}(\tau) = f(\tau) + u(0) - \mathcal{K}u(\tau) \in \mathcal{H}$ and corresponding history function

$$\Phi_v := v|_{[0,\tau]} \in C([0,\tau],\mathcal{H}).$$

Theorem 4.4 implies that the mild solution $v \in C([\tau,T],\mathcal{H})$ satisfies (for $k = 1$) that

$$\dot{u}|_{(2\tau,T]} = v|_{(2\tau,T]} \in C^1((2\tau,T],\mathcal{H}).$$

Note that equation (4.5) then implies that $\mathcal{K}\dot{u} \in C((2\tau,T],\mathcal{H})$. Thus, v is a classical solution of (4.5) for $t > 2\tau$. This procedure can be repeated consecutively, which then yields the assertion. □

Remark 4.9. With additional assumptions on the history function, we can achieve that

$$u|_{(k\tau,T]} \in C^{k+1}((k\tau,T],\mathcal{H}) \quad \text{for } k = 0,\dots,N-1.$$

A sufficient condition for this is $\Phi \in C^{0,r}([-\tau,0],\mathcal{H})$ for any $r \in (0,1]$, cf. Theorem 4.4.

Remark 4.10. As in the mild setting, we may include a linear and bounded operator $\mathcal{B}\colon \mathcal{H} \to \mathcal{H}$ in front of the delay term u_τ without changing the result of Theorem 4.8.

4.3. **Weak setting.** Let us now consider the weak solution concept, cf. Definition 3.6. We use Belman's method of steps as in Section 4.1, i.e., we split the interval $[0,T]$ into equidistant intervals of length τ. For a given right-hand side $f \in L^+(0,T)$, a history function $\Phi \in L^+(-\tau,0)$, and an initial value $u_0 \in \mathcal{H}$ we first consider system (3.2) on $\bar{I}_1 = [0,\tau]$, i.e.,

$$\dot{u} + \mathcal{K}u = f + u_\tau = f + \Phi_\tau.$$

This is an evolution equation without delay and with a right-hand side in $L^+(0,\tau)$. Thus, following the existence result in Theorem 3.5, we obtain a unique solution $u_1 \in L^2(0,\tau;\mathcal{V}) \cap C([0,\tau],\mathcal{H})$ with $\dot{u}_1 \in L^+(0,\tau)$. By an iterative application we obtain a weak solution on the entire interval $[0,T]$.

Theorem 4.11 (weak solution)**.** *Assume that the linear, bounded operator $\mathcal{K}\colon \mathcal{V} \to \mathcal{V}^*$ satisfies a Gårding inequality (3.4) and that the initial data fulfills $u_0 \in \mathcal{H}$. Suppose that the right-hand side is given by $f = f_1 + f_2 \in L^+(0,T)$ with $f_1 \in L^2(0,T;\mathcal{V}^*)$, $f_2 \in L^1(0,T;\mathcal{H})$. Accordingly, let the history function be given by $\Phi = \Phi_1 + \Phi_2 \in L^+(-\tau,0)$ with $\Phi_1 \in L^2(-\tau,0;\mathcal{V}^*)$ and $\Phi_2 \in L^1(-\tau,0;\mathcal{H})$. Then, system (3.2) has a unique weak solution $u \in C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$ with $\dot{u} \in L^+(0,T)$. Furthermore, the solution is bounded by*

$$\|u\|^2_{C([0,t],\mathcal{H})} + \mu\|u\|^2_{L^2(0,t;\mathcal{V})} \leq z(t),$$

where the function $z \in C([0,T])$ is defined by

$$z(t) = \bigg[\Big(\|u_0\|^2_\mathcal{H} + \frac{2}{\mu}\int_0^t \|f_1\|^2_{\mathcal{V}^*} + \|\Phi_{1,\tau}\chi_{[0,\tau]}\|^2_{\mathcal{V}^*}\, ds\Big)^{1/2} + \int_0^t e^{-(\kappa+1)s}\big(\|f_2\|_\mathcal{H} + \|\Phi_{2,\tau}\chi_{[0,\tau]}\|_\mathcal{H}\big)\, ds\bigg]^2 e^{2(\kappa+1)t}.$$

Proof. The existence and uniqueness of the weak solution u was already discussed in the beginning of this subsection. For the bounds we observe that

$$\begin{aligned} 2\int_0^t \langle u_\tau, u\rangle \,\mathrm{d}s &\leq \int_0^\tau \tfrac{2}{\mu}\|\Phi_1\|^2_{\mathcal{V}^*} + \tfrac{\mu}{2}\|u\|^2_\mathcal{V} + \|\Phi_2\|_\mathcal{H}\|u\|_\mathcal{H}\,\mathrm{d}s + \int_\tau^t \|u_\tau\|^2_\mathcal{H} + \|u\|^2_\mathcal{H}\,\mathrm{d}s \\ &\leq \int_0^t \tfrac{2}{\mu}\|\Phi_{1,\tau}\chi_{[0,\tau]}\|^2_{\mathcal{V}^*} + \tfrac{\mu}{2}\|u\|^2_\mathcal{V} + \|\Phi_{2,\tau}\chi_{[0,\tau]}\|_\mathcal{H}\|u\|_\mathcal{H}\,\mathrm{d}s + 2\|u\|^2_\mathcal{H}\,\mathrm{d}s \end{aligned} \tag{4.6}$$

for all $t > \tau$. Now we can follow the steps within the proof of Theorem 3.5 and replace inequality (3.6) by

$$\begin{aligned} \|u(t)\|^2_\mathcal{H} + \mu\int_0^t \|u\|^2_\mathcal{V}\,\mathrm{d}s \overset{(3.4),(4.6)}{\leq} \|u_0\|^2_\mathcal{H} + \int_0^t \tfrac{2}{\mu}\big(\|f_1\|^2_{\mathcal{V}^*} + \|\Phi_{1,\tau}\chi_{[0,\tau]}\|^2_{\mathcal{V}^*}\big) + 2(\kappa+1)\|u\|^2_\mathcal{H} \\ + 2\big(\|f_2\|_\mathcal{H} + \|\Phi_{2,\tau}\chi_{[0,\tau]}\|_\mathcal{H}\big)\|u\|_\mathcal{H}\,\mathrm{d}s. \end{aligned}$$

□

For the investigation of the smoothing effect we note that $\dot{u}_1$ can be used as history function if we consider the derivative of equation (3.2). The choice of the initial value for this equation is, in general, not as obvious. We apply well-known tools of the weak solution concept, which includes the consideration of the function $t\dot{u}$. This has the advantage that the initial value equals zero. For the proof of the smoothing property we define for an interval $(a,b]$ the space

$$H^k_{\mathrm{loc}}(a,b;\mathcal{V}) := \bigcap_{\varepsilon\in(0,b-a)} H^k(a+\varepsilon,b;\mathcal{V}).$$

The following theorem shows that also in the weak setting the smoothing property is translated to the delay system (3.2), i.e., the system behaves retardedly.

Theorem 4.12. *Consider a right-hand side $f \in C^\infty(0,T;\mathcal{V}^*)$, a history function $\Phi \in L^+(-\tau,0)$, and $u_0 \in \mathcal{H}$. Then, the solution u of (3.2) satisfies for $k = 0,\dots,N-1$ that*

$$u|_{(k\tau,T]} \in H^k_{\text{loc}}\big(k\tau,T;\mathcal{V}\big) \cap C^k\big((k\tau,T],\mathcal{H}\big). \tag{4.7}$$

Proof. Let $\varepsilon \in (0,\tau/N)$ be arbitrary. We show by mathematical induction that

$$u|_{[k(\tau+\varepsilon),T]} \in H^k\big(k(\tau+\varepsilon),T;\mathcal{V}\big) \cap C^k\big([k(\tau+\varepsilon),T],\mathcal{H}\big), \tag{4.8a}$$

$$u^{(k+1)}|_{[k(\tau+\varepsilon),T]} \in L^+(k(\tau+\varepsilon),T) \tag{4.8b}$$

for $k = 0,\dots,N-1$. The case $k=0$ corresponds to the existence of a weak solution of the delay PDE (3.2) and is part of Theorem 4.11.

Now consider $k \ge 1$ and assume that (4.8) has been proven for $k-1$. Then, there exist $g_1 \in L^2\big((k-1)(\tau+\varepsilon),T;\mathcal{V}^*\big)$ and $g_2 \in L^1\big((k-1)(\tau+\varepsilon),T;\mathcal{H}\big)$ such that

$$u^{(k)}|_{[(k-1)(\tau+\varepsilon),T]} = g_1 + g_2.$$

Note that we also have $u^{(k-1)}((k-1)(\tau+\varepsilon)) \in \mathcal{H}$. We define $a_{k,\varepsilon} := (k-1)(\tau+\varepsilon)+\tau$ and with this $\tilde f_1 \in H^1(a_{k,\varepsilon},T;\mathcal{V}^*)$, $\tilde f_2 \in W^{1,1}(a_{k,\varepsilon},T;\mathcal{H})$ by

$$\tilde f_1(t) := f^{(k-1)}(t) + \int_{a_{k,\varepsilon}}^t g_1(s-\tau)\,\mathrm{d}s, \quad \tilde f_2(t) := u^{(k-1)}((k-1)(\tau+\varepsilon)) + \int_{a_{k,\varepsilon}}^t g_2(s-\tau)\,\mathrm{d}s.$$

Then, in the time interval $[a_{k,\varepsilon},T]$ the function $u^{(k-1)}$ is obviously the solution of the problem

$$\dot w + \mathcal{K}w = \tilde f_1 + \tilde f_2 = f^{(k-1)} + u_\tau^{(k-1)}$$

with initial value $w_0 = u^{(k-1)}(a_{k,\varepsilon})$. By [Tar06, p. 115] it holds that $(t-a_{k,\varepsilon})u^{(k)} \in L^2(a_{k,\varepsilon},T;\mathcal{V}) \cap C([a_{k,\varepsilon},T],\mathcal{H})$. Its derivative satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(t-a_{k,\varepsilon})u^{(k)} \in L^+(a_{k,\varepsilon},T).$$

If we restrict the solution to the interval $[a_{k,\varepsilon}+\varepsilon,T] = [k(\tau+\varepsilon),T]$ and divide by $t-a_{k,\varepsilon}$, then we obtain the inclusions in (4.8). Finally, statement (4.7) follows by considering the limit $\varepsilon \searrow 0$ in (4.8a). □

Remark 4.13. In Theorem 4.12 we restrict ourselves to $f \in C^\infty([0,T],\mathcal{V}^*)$. This is reasonable, since for every $k \in \mathbb{N}$ we have the embedding $C^k([0,T],\mathcal{H}) \hookrightarrow C^k([0,T],\mathcal{V}^*)$. Nevertheless, the assumption on the right-hand side can be weakened to

$$f \in C^{N-2}([0,T];\mathcal{V}^*) \quad \text{with} \quad f^{(N-1)} \in L^+(0,T).$$

Remark 4.14. We consider once more the situation with an additional linear and bounded operator $\mathcal{B}\colon \mathcal{H} \to \mathcal{H}$ in front of the delay term. Following Remark 3.4 and [Bal77], we can translate the smoothing properties from the mild setting in Remark 4.6 to the weak setting if f and Φ take values in $\mathcal{H}$. Note furthermore that $u^{(k+1)}, u_\tau^{(k)} \in L^2(t_0,T;\mathcal{H})$ implies $u^{(k)} \in L^2(t_0,T;\mathcal{V})$ by equation (3.2) and the Gårding inequality (3.4). In Appendix A we prove the smoothing property under slightly more restrictive assumptions. This restriction will lead to a smoothing of the solution after every time step of size τ, without the assumption of Φ being continuous.

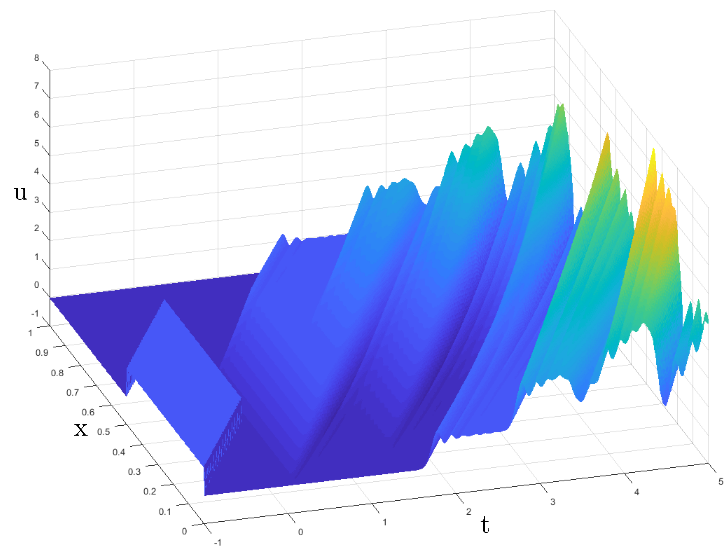

FIGURE 5.1. Numerical solution of the spatial discretized delay PDE (5.1).

5. Numerical example

In Example 4.2 we have mentioned that the solution of the delay PDE

$$\dot{u} + \partial_x u = u_\tau$$

does not gain smoothness over time, since the included operator $\mathcal{K} = \partial_x$ only generates a strongly continuous semigroup. If we consider the same problem on a bounded domain with periodic boundary conditions and assume that the history function Φ has jumps in space, then the number of jumps can even increase over time. To illustrate this, we consider the problem

$$\dot{u}(t;x) = -0.3\,\partial_x u(t;x) + u(t-1;x), \quad (t,x) \in (0,T] \times (0,1) \tag{5.1a}$$

$$u(t;0) = u(t;1), \quad t \in [0,T] \tag{5.1b}$$

$$u(0;x) = 0, \quad x \in (0,1) \tag{5.1c}$$

$$\Phi(t;x) = \chi_{(0.3,0.8]}(x - 0.3t), \quad (t,x) \in [-1,0) \times (0,1). \tag{5.1d}$$

Note that we consider $\tau = 1$ in this example. If we discretize this delay PDE in space by central differences on a grid of 250 equidistant points, we obtain the solution shown in Figure 5.1. Within this figure one can observe that the solution becomes smoother in time and space. This, however, is only caused by *dispersion*. This means that the velocities of the traveling waves depend on the corresponding wave numbers, in contrast to the shift operator generated by $\mathcal{K} = \partial_x$, cf. Example 4.2. Hence, jumps in space will be smoothened, since the waves building sharp fronts move with different speeds [Tre82].

To obtain a more accurate approximate solution of the delay PDE (5.1) we introduce the function $w(t;x) := u(t;x+0.3t)$ for a fixed $x \in [0,1)$. For this function it holds that

$$\dot{w}(t;x) = \dot{u}(t;x+0.3t) + 0.3\,\partial_x u(t;x+0.3t) \overset{(5.1a)}{=} u(t-1;x+0.3t) = w(t-1;x+0.3).$$

Because of the assumed periodic boundary conditions, for a fixed $\tilde{x} \in [0,0.1)$ we can describe the time evolution of the function $w_i(t;\tilde{x}) := w(t;\tilde{x}+0.1(i-1))$ with $i = 1,\ldots,10$

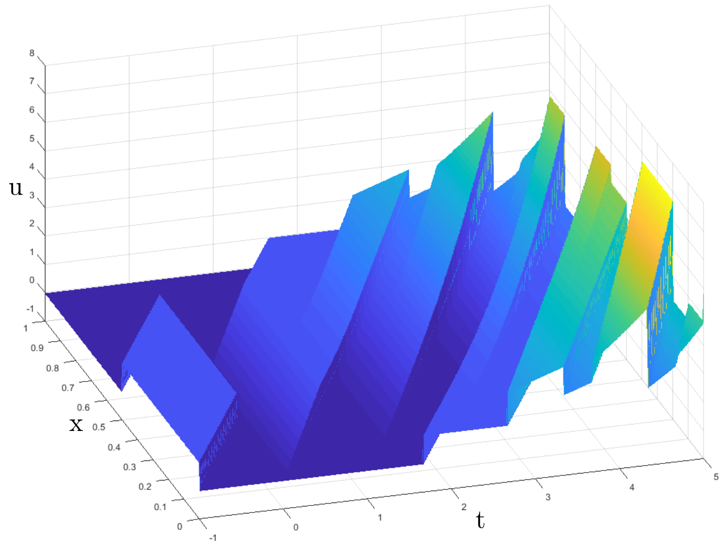

FIGURE 5.2. Solution of delay PDE (5.1) using the reformulation as delay ODE.

by a delay ODE of dimension 10, namely

$$\dot{w}_i(t;\tilde{x}) = w_{(i+2 \bmod 10)+1}(t-1;\tilde{x}), \qquad t \in (0,T] \tag{5.2a}$$

$$w_i(0;\tilde{x}) = 0, \tag{5.2b}$$

$$w_i(t;\tilde{x}) = \Phi\big(t; (\tilde{x} + 0.1(i-1) + 0.3\,t) \bmod 1\big), \quad t \in [-1,0). \tag{5.2c}$$

Note that with the initial condition given in (5.1c) every $\tilde{x} \in [0, 0.1)$ generates the same initial value problem (5.2). Thus, it is sufficient to solve (5.2) in order to describe the entire dynamics of the delay PDE (5.1). The results shown in Figure 5.2 clearly indicate that the jumps of the history function indeed do not vanish over time and that the number of discontinuities even increases in space. Roughly speaking, the number discontinuities increases, since the jumps are shifted in space by the action of the semigroup and duplicated by the delay term. As a result, the solution neither gains regularity in space nor in time, cf. Section 4.1.

6. Conclusion

The classification of ODEs with delay into retarded, neutral, or advanced type can be made by means of a structural decision. This classification is then directly related to smoothness properties of the solution. For a retarded ODE it is well-known that the solution gains smoothness in each step of size τ.

Although linear PDEs with delay are often classified in the same structural manner, we have shown in this paper that the solution behaviour does also depend on the semigroup generated by the considered differential operator. More precisely, we have discussed under which conditions a linear delay PDE, which leads to a retarded delay ODE under spatial discretization, behaves retardedly in terms of the smoothing property. In this context we have considered mild, classical, as well as weak solutions. Furthermore, we have seen that the smoothing in the retarded case may occur slightly later than in the ODE setting.

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Appendix A. Weak solutions with an operator $\mathcal{B}$

In this section, we investigate delay PDEs of the form

$$\dot{u} + \mathcal{K}u = f + \mathcal{B}u_\tau \tag{A.1}$$

with a linear and bounded operator $\mathcal{B}\colon \mathcal{H} \to \mathcal{H}$ in the weak setting. In contrast to Remark 4.14, in which we have used the smoothing property of the mild setting, we apply here tools from the theory of weak solutions. For this, we make additional assumptions on the right-hand side and the operator $\mathcal{K}$. To be more precise, we consider $f \in L^2(0,T;\mathcal{H})$ and $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, where $\mathcal{K}_1 \colon \mathcal{V} \to \mathcal{V}^*$ is elliptic and symmetric and $\mathcal{K}_2$ maps from $\mathcal{V}$ to $\mathcal{H}$. Before we show that under these assumptions the solution becomes smoother over time, we recap well-known solution properties for the given setting.

Lemma A.1. *Let $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ be decomposed into the two linear and bounded operators $\mathcal{K}_1 \colon \mathcal{V} \to \mathcal{V}^*$ and $\mathcal{K}_2 \colon \mathcal{V} \to \mathcal{H}$, where $\mathcal{K}_1$ is elliptic and symmetric. Then, for every right-hand side $f \in L^2(0,T;\mathcal{H})$ and initial value $u_0 \in \mathcal{H}$ the operator equation* (3.1) *has a unique solution*

$$u \in C([0,T],\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \cap C((0,T],\mathcal{V}) \cap H^1_{\mathrm{loc}}(0,T;\mathcal{H}).$$

Furthermore, the solution depends continuously on f and u_0.

Proof. Since $\mathcal{H} \cong \mathcal{H}^*$ is densely embedded in $\mathcal{V}^*$, there exists a unique solution $u \in L^2(0,T;\mathcal{V})$ of (3.1) by Theorem 3.5. Furthermore, we can rewrite (3.1) as

$$\dot{w} + \mathcal{K}_1 w = f - \mathcal{K}_2 u \in L^2(0,T;\mathcal{H}),$$

where we now search for a solution w. Obviously, it holds that $w = u$. The stated properties then follow by [Tar06, Lem. 21.1]. □

We can now prove the smoothing property for the delay equation (A.1).

Theorem A.2. *Assume that the operator $\mathcal{K}$ can be decomposed as in Lemma A.1, that $\mathcal{B} \colon \mathcal{H} \to \mathcal{H}$ is linear and bounded, and that $f \in C^\infty([0,T];\mathcal{H})$. Furthermore, let the initial data and the history function satisfy $u_0 \in \mathcal{H}$ and $\Phi \in L^2(-\tau,0;\mathcal{H})$. Then, the solution u of the delay equation* (A.1) *satisfies*

$$u|_{(k\tau,T]} \in C^k((k\tau,T],\mathcal{V}) \cap H^{k+1}_{\mathrm{loc}}(k\tau,T;\mathcal{H}) \quad \textit{for } k = 0,\dots,N-1.$$

Proof. The existence and uniqueness of a solution u is given by Theorem 4.11. For the smoothing property let $\varepsilon > 0$ be small enough such that $(N-1)\varepsilon < \tau$. We will show iteratively that u restricted to the interval $[k(\tau+\varepsilon),T]$ has weak derivatives in $\mathcal{V}$ up to the order k with

$$u^{(k)}|_{(k(\tau+\varepsilon),T]} \in C((k(\tau+\varepsilon),T],\mathcal{V}) \cap H^1_{\mathrm{loc}}(k(\tau+\varepsilon),T;\mathcal{H}) \tag{A.2}$$

for $k = 0,\dots,N-1$. Taking the limit $\varepsilon \to 0$ then completes the proof.

For $k = 0$ statement (A.2) follows by Bellman's method of steps in combination with Lemma A.1. Now consider $k > 0$ and assume that (A.2) is proven for $k-1$. In this case, the restriction of $f^{(k)} + \mathcal{B}u_\tau^{(k)}$ to the interval $[k\tau + (k-\frac{1}{2})\varepsilon, T]$ is an element of L^2 with values in $\mathcal{H}$ and thus, we also have

$$\big(t - k\tau - (k-\tfrac{1}{2})\varepsilon\big)\big(f^{(k)} + \mathcal{B}u_\tau^{(k)}\big)\big|_{\left[k\tau+(k-\frac{1}{2})\varepsilon,T\right]} \in L^2(k\tau + (k-\tfrac{1}{2})\varepsilon, T;\mathcal{H}).$$

Furthermore, we obtain

$$u^{(k-1)}\big(k\tau + (k-\tfrac{1}{2})\varepsilon\big) \in \mathcal{V}.$$

By [Emm04, Th. 8.5.3] we conclude that $u^{(k)}|_{[k\tau+k\varepsilon,T]}$ is continuous in $\mathcal{H}$. Therefore, $u^{(k)}(k(\tau+\varepsilon))$ is well-defined in $\mathcal{H}$. In addition, it holds that

$$\sqrt{t - k(\tau+\varepsilon)}(f^{(k)} + \mathcal{B}u_\tau^{(k)})|_{[k(\tau+\varepsilon),T]} \in L^2(k(\tau+\varepsilon),T;\mathcal{H})$$

by (A.2) from the previous step. Finally, the inclusion (A.2) follows from the results in [Tar06, Lem. 21.1]. □

Note that in contrast to Remark 4.5, where we could only show that the mild solution is k-times continuously differentiable for $t > (k+1)\tau$, the weak solution is k-times continuously differentiable (with values in $\mathcal{V}$) already for $t > k\tau$.

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