An Analytic Characterization of the Eigenvalues of Self-adjoint Extensions

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Abstract. Let $\widetilde{A}$ be a self-adjoint extension in $\widetilde{\mathcal{K}}$ of a fixed symmetric operator A in $\mathcal{K} \subseteq \widetilde{\mathcal{K}}$. An analytic characterization of the eigenvalues of $\widetilde{A}$ is given in terms of the Q-function and the parameter function in the Krein-Naimark formula. Here $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are Krein spaces and it is assumed that $\widetilde{A}$ locally has the same spectral properties as a self-adjoint operator in a Pontryagin space. The general results are applied to a class of boundary value problems with λ-dependent boundary conditions.

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1. Introduction

Let A be a densely defined simple symmetric operator in a Hilbert space $\mathcal{K}$ and let A_0 be a self-adjoint extension of A in $\mathcal{K}$. We assume first for simplicity that the deficiency indices of A are $(1,1)$. It is well known that to the pair (A, A_0) there corresponds a function m holomorphic on the resolvent set $\rho(A_0)$ of A_0, a so-called Q-function or Weyl function, which in this case is a scalar Nevanlinna function, that is, it maps the upper half plane holomorphically into itself and is symmetric with respect to the real axis. Then the classical Krein-Naimark formula

$$P_{\mathcal{K}}\big(\widetilde{A}-\lambda\big)^{-1}|_{\mathcal{K}} = (A_0-\lambda)^{-1} - \frac{1}{m(\lambda)+\tau(\lambda)}\big(\cdot, \varphi_{\overline{\lambda}}\big)\varphi_\lambda \tag{1.1}$$

establishes a bijective correspondence between the class of Nevanlinna functions τ including the constant ∞ and the compressed resolvents of self-adjoint extensions $\widetilde{A}$ of A which act in Hilbert spaces $\widetilde{\mathcal{K}} \supseteq \mathcal{K}$ and fulfill a certain minimality condition, cf. [12, 28, 32, 36]. Here $\varphi_\lambda \in \ker(A^* - \lambda)$ denotes the defect element of A at the point λ.

The Nevanlinna function τ in (1.1) is equal to a real constant or ∞ if and only if the self-adjoint extension $\widetilde{A}$ is a canonical extension of A, i.e., $\widetilde{A}$ acts in $\mathcal{K} = \widetilde{\mathcal{K}}$. In this case the Krein-Naimark formula reduces to

$$(\widetilde{A}-\lambda)^{-1} = (A_0-\lambda)^{-1} - \frac{1}{m(\lambda)+\tau}\big(\cdot, \varphi_{\overline{\lambda}}\big)\varphi_\lambda. \tag{1.2}$$

We emphasize that here the spectral properties of the operator $\widetilde{A}$ can be described with the help of the function $\lambda \mapsto -(m(\lambda)+\tau)^{-1}$ on the right hand side of (1.2). This follows immediately from the fact that in this case $\widetilde{A}$ is a minimal representing operator of this function. In particular, a point $w_0 \in \mathbb{C}$ is an eigenvalue of $\widetilde{A}$ if and only if it is a generalized zero of the function $\lambda \mapsto m(\lambda)+\tau$, that is, the limit $\lim_{\lambda \hat{\to} w_0}(\lambda - w_0)^{-1}(m(\lambda)+\tau)$ exists, see [31]. Note that this analytic characterization holds also for eigenvalues of $\widetilde{A}$ which lie in the essential spectrum of A_0.

In the present paper we generalize this analytic characterization of eigenvalues to the case that $\widetilde{A}$ is a self-adjoint extension of A which acts in a larger space $\widetilde{\mathcal{K}} \supseteq \mathcal{K}$ and corresponds to the function $\lambda \mapsto \tau(\lambda)$ via (1.1). One might guess that the generalized zeros of the function $\lambda \mapsto m(\lambda) + \tau(\lambda)$ on the right hand side of (1.1) coincide with the eigenvalues of $\widetilde{A}$ as it is obvious that the generalized zeros belonging to $\rho(A_0)$ are eigenvalues of $\widetilde{A}$. However, due to the fact that $\widetilde{A}$ is (in general) not a minimal representing operator of the function $-(m+\tau)^{-1}$, it turns out that such a correspondence does not hold in general, but an analytic characterization of the eigenvalues can still be given, cf. Theorem 4.1.

We do not restrict our investigations to Hilbert spaces $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ and the case of a symmetric operator of defect one. Here we allow $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ to be Krein spaces and A to be a (not necessarily densely defined) symmetric operator of finite defect. It will be assumed that A possesses a canonical self-adjoint extension A_0 which is locally of type π_+, that is, it has locally the same spectral properties as a self-adjoint operator or relation in a Pontryagin space, see e.g. [2, 6, 26]. Furthermore, we assume that also $\widetilde{A}$ is locally of type π_+ and τ behaves locally like a matrix-valued generalized Nevanlinna function. In the case that the the symmetric operator A is of defect one we show in Theorem 4.1 that w_0 is an eigenvalue of $\widetilde{A}$ if and only if w_0 is either a generalized zero of $m+\tau$ or w_0 is a generalized pole of both m and τ. For higher (but finite) defect one has to require an additional property. Namely, if τ assumes a so-called generalized value (see Definition 3.9) at some point w_0, then w_0 is an eigenvalue of $\widetilde{A}$ if and only if w_0 is a generalized zero of the function $m + \tau$.

Our second objective in this paper is a class of boundary value problems with boundary conditions depending on the spectral parameter which is closely connected with the self-adjoint extensions $\widetilde{A}$ of a symmetric operator A described by (1.1). If e.g. τ is a scalar Nevanlinna function and A is a singular Sturm-Liouville operator in $L^2(0,\infty)$,

$$Af = -(pf')' + qf,$$
$$\operatorname{dom} A = \{f \in \mathcal{D}_{\max} \mid f(0) = (pf')(0) = 0\},$$

with real valued functions $p^{-1}, q \in L^1(0,\infty)$, $p > 0$, and the usual maximal domain $\mathcal{D}_{\max}$, such that the differential expression is limit point at ∞, then a solution $f \in L^2(0,\infty)$ of the boundary value problem

$$(A^* - \lambda)f = -(pf')' + qf - \lambda f = g, \quad \tau(\lambda)f(0) + f'(0) = 0, \tag{1.3}$$

is given by

$$P_{L^2}(\widetilde{A} - \lambda)^{-1}|_{L^2}\, g = (A_0 - \lambda)^{-1} g - \frac{1}{m(\lambda) + \tau(\lambda)}(g, \varphi_{\overline{\lambda}})\varphi_\lambda.$$

Here A_0 is the self-adjoint extension of A corresponding to Dirichlet boundary conditions at the left endpoint, m is the classical Titchmarsh-Weyl function and φ_λ is a solution of $-(pf')' + qf = \lambda f$ which belongs to $L^2(0,\infty)$.

Boundary value problems with λ-dependent boundary conditions have extensively been studied in a more or less abstract framework in the last decades, see e.g. [1, 3, 6, 7, 8, 12, 16, 18, 19, 20, 37]. The spectral properties of $\widetilde{A}$ and in particular the eigenvalues and eigenvectors of $\widetilde{A}$ are closely connected with the solvability and the nontrivial solutions of the (homogeneous) boundary value problem. With the help of our general results we show in Section 5 how the solvability of the homogeneous boundary value problem is connected with the generalized zeros of the function $m + \tau$ and the eigenvalues of $\widetilde{A}$.

The paper is organized as follows. In Section 2 we recall the definitions and basic properties of self-adjoint operators and relations which are locally of type π_+ and the class of local generalized Nevanlinna functions. In the next section the notion of generalized poles and zeros of generalized Nevanlinna functions is recalled and extended to the local classes considered here. Moreover, we introduce the concept of generalized values of local generalized Nevanlinna functions and we study the behaviour of these functions at such points in Theorem 3.13. Section 4 contains some of our main results. Under the assumption that $\widetilde{A}$ is a self-adjoint extension of A in possibly larger Krein space which is locally of type π_+ and connected with a local generalized Nevanlinna function τ in a similar form as in (1.1) we give an analytic characterization of the eigenvalues of $\widetilde{A}$ in Theorem 4.1 and discuss their sign types in Proposition 4.9. The notion of boundary value spaces and associated Weyl functions is briefly recalled in the beginning of Section 5. It will be shown that a local generalized Nevanlinna function satisfying an additional condition can be realized as a Weyl function and the properties of the Weyl function are investigated at points where it assumes a generalized value, cf. Proposition 5.4. Next we investigate a class of abstract boundary value problems with local generalized Nevanlinna functions in the boundary condition. Finally, as an application we study a singular Sturm-Liouville operator with the indefinite weight sgn x and a λ-dependent interface condition in Section 5.3.

2. Self-adjoint relations locally of type π_+ and local generalized Nevanlinna functions

In this section we first fix some basic notations, we recall the notions of local generalized Nevanlinna functions and self-adjoint relations in Krein spaces which are locally of type π_+, and we show how these objects are connected via (minimal) π_+-realizations.

2.1. **Notations.** Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces, then the linear space of all bounded linear operators defined on $\mathcal{K}_1$ with values in $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K})$. Besides bounded and unbounded operators we will also study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K} \times \mathcal{K}$. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators in $\mathcal{K}$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [21]. The direct sum of subspaces in $\mathcal{K}$ will be denoted by $\widehat{+}$.

Let in the following $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space and let S be a closed linear relation in $\mathcal{K}$. The *resolvent set* $\rho(S)$ of S is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the *spectrum* $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The *extended spectrum* $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. We shall say that $\lambda \in \mathbb{C}$ is a *point of regular type* of S, $\lambda \in r(S)$, if $(S-\lambda)^{-1}$ is a (not necessarily everywhere defined) bounded operator. A point $\lambda \in \mathbb{C}$ is an *eigenvalue* of S if $\ker(S-\lambda) \neq \{0\}$; we write $\lambda \in \sigma_p(S)$. If the *multivalued part* $\operatorname{mul} S = \left\{ f' \,|\, \left(\begin{smallmatrix} 0 \\ f' \end{smallmatrix}\right) \in S \right\}$ of S is not trivial, that is, S is not an operator, we shall say that ∞ is an eigenvalue of S and each element $f' \in \operatorname{mul} S$ with $f' \neq 0$ is called a corresponding *eigenvector*. The continuous spectrum of S is denoted by $\sigma_c(S)$.

The *adjoint* $S^+ \in \widetilde{\mathcal{C}}(\mathcal{K})$ of a linear relation S in $\mathcal{K}$ is defined by

$$S^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} \,\middle|\, [h, f'] = [h', f] \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}$$

and S is said to be *symmetric* (*self-adjoint*) if $S \subset S^+$ (resp. $S = S^+$). We say that a closed symmetric relation $S \in \widetilde{\mathcal{C}}(\mathcal{K})$ has *defect* $n \in \mathbb{N} \cup \{\infty\}$ if there exists a self-adjoint extension S_0 of S in $\mathcal{K}$ such that $\dim\big(S_0/S\big) = n$.

2.2. **Self-adjoint relations locally of type π_+.** We recall the definition of a class of self-adjoint relations in $\mathcal{K}$ which locally have the same spectral properties as self-adjoint relations in Pontryagin spaces, cf. [26].

Let Ω be a domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the open upper half plane $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \,|\, \operatorname{Im} \lambda > 0\}$ and the open lower half plane $\mathbb{C}^- = \{\lambda \in \mathbb{C} \,|\, \operatorname{Im} \lambda < 0\}$ are simply connected. Whenever not explicitly mentioned we tacitly assume that a domain Ω has these properties.

Definition 2.1. Let Ω be a domain as above and let T_0 be a self-adjoint relation in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. T_0 is said to be of *type* π_+ *over* Ω if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a self-adjoint projection E in $\mathcal{K}$ such that T_0 can be decomposed as

$$T_0 = \big(T_0 \cap (E\mathcal{K})^2\big) \,\widehat{+}\, \big(T_0 \cap ((1-E)\mathcal{K})^2\big)$$

and the following holds:

(i) $(E\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity and $\rho(T_0 \cap (E\mathcal{K})^2)$ is nonempty,

(ii) $\widetilde{\sigma}\big(T_0 \cap ((1-E)\mathcal{K})^2\big) \cap \Omega' = \emptyset$.

Let T_0 be a self-adjoint relation in $\mathcal{K}$ which is of type π_+ over Ω. Then the set $\sigma(T_0) \cap (\Omega \backslash \overline{\mathbb{R}})$ is discrete and the nonreal spectrum of T_0 in Ω can only accumulate to the boundary of Ω. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let E be a self-adjoint projection with the properties as in Definition 2.1. If E' is the spectral function of the self-adjoint relation $T_0 \cap (E\mathcal{K})^2$ in the Pontryagin space $E\mathcal{K}$, then the mapping

$$\Delta \mapsto E'(\Delta)E =: E_{T_0}(\Delta)$$

defined for all finite unions Δ of connected subsets of $\Omega' \cap \overline{\mathbb{R}}$ the endpoints of which belong to $\Omega' \cap \overline{\mathbb{R}}$ and are not critical points of $T_0 \cap (E\mathcal{K})^2$, is the *local spectral function* of T_0 on $\Omega' \cap \overline{\mathbb{R}}$ (see [26, Section 3.4, Remark 4.9]).

2.3. **Generalized Nevanlinna functions.** Recall that an $n \times n$-matrix valued function G belongs by definition to the *generalized Nevanlinna class* $\mathcal{N}_\kappa^{n\times n}$ if it is meromorphic in $\mathbb{C}\backslash\mathbb{R}$, symmetric with respect to the real axis, that is $G(\lambda) = G(\overline{\lambda})^*$ for all points λ of holomorphy of G, and the so-called Nevanlinna kernel

$$K_G(\lambda, \mu) := \frac{G(\lambda) - G(\mu)^*}{\lambda - \overline{\mu}}$$

has κ negative squares. The set consisting of the points of holomorphy of G in $\mathbb{C}\backslash\mathbb{R}$ and all points $\mu \in \mathbb{R}$ such that G can be analytically continued to μ and the continuations from $\mathbb{C}^+$ and $\mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(G)$.

It is well known (see [24, 30]) that generalized Nevanlinna functions can also be characterized by their operator representations. Namely, G belongs to the class $\mathcal{N}_\kappa^{n\times n}$ if and only if G can be represented with a self-adjoint linear relation A_0 in a Pontryagin space Π_κ with negative index κ in the form

$$\begin{aligned} G(\lambda) = \operatorname{Re} G(\lambda_0) + \gamma^+\big((\lambda - \operatorname{Re}\lambda_0) \\ + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma, \end{aligned}$$

$\lambda \in \mathfrak{h}(G)$, where $\gamma \in \mathcal{L}(\mathbb{C}^n, \Pi_\kappa)$, $\lambda_0 \in \mathfrak{h}(G)$, and the minimality condition

$$\Pi_\kappa = \overline{\operatorname{span}}\big\{(1 + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma x \,|\, \lambda \in \rho(A_0),\, x \in \mathbb{C}^n\big\}$$

holds. We shall say that the triple $(\Pi_\kappa, A_0, \gamma(\lambda))$, where

$$\gamma(\lambda) := (1 + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma,$$

is a *minimal realization of* G, cf. Definition 2.4.

The class $\mathcal{N}_0^{n\times n}$ coincides with the class of $n\times n$-matrix valued Nevanlinna functions. In particular, a function $G \in \mathcal{N}_0^{n\times n}$ admits also an integral representation

$$G(\lambda) = A + \lambda B + \int_{-\infty}^{\infty} \Big(\frac{1}{t - \lambda} - \frac{t}{1 + t^2}\Big)\, d\Sigma(t),$$

where A and B are self-adjoint $n \times n$-matrices, $B \geq 0$ and Σ is a nondecreasing, left-continuous $n \times n$-matrix function on $\mathbb{R}$ such that $\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma(t)$ exists.

2.4. **Local generalized Nevanlinna functions.** Next we recall the definition of the class of local generalized Nevanlinna functions, which is a subclass of the so-called locally definitizable functions, see [27].

Definition 2.2. Let Ω be a domain as in the beginning of this section and let τ be an $n \times n$-matrix valued function which is meromorphic in $\Omega\backslash\overline{\mathbb{R}}$ and symmetric with respect to the real axis. Then τ is said to be a *local generalized Nevanlinna function in* Ω if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, τ can be written as a sum $\tau = \tau_0 + \tau_1$ of a generalized Nevanlinna function $\tau_0 \in \mathcal{N}_\kappa^{n\times n}$ and an $n \times n$-matrix valued function τ_1 which is holomorphic on $\overline{\Omega'}$.

The class of $n \times n$-matrix valued local generalized Nevanlinna function in Ω will be denoted by $\mathcal{N}^{n\times n}(\Omega)$. In the case $n = 1$ we write $\mathcal{N}(\Omega)$ instead of $\mathcal{N}^{1\times 1}(\Omega)$.

We note that τ belongs to $\mathcal{N}^{n\times n}(\overline{\mathbb{C}})$ if and only if $\tau \in \mathcal{N}_\kappa^{n\times n}$ for some $\kappa \in \mathbb{N}_0$ (see [27]). However, in general, for $\tau \in \mathcal{N}^{n\times n}(\Omega)$ the functions τ_0 and τ_1 (and, in particular, the negative index of τ_0) depend on the chosen subdomain Ω'. The next lemma is a direct consequence of Definition 2.1.

Lemma 2.3. *Let T_0 be a self-adjoint relation of type π_+ over Ω in a Krein space $\mathcal{H}$, let $S_0 = S_0^*$ be an $n \times n$-matrix, $\gamma \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ and fix some $\lambda_0 \in \rho(T_0) \cap \Omega$. Then the function*

$$\tau(\lambda) := S_0 + \gamma^+\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(T_0 - \lambda)^{-1}\big)\gamma, \tag{2.1}$$

$\lambda \in \rho(T_0) \cap \Omega$, belongs to the class $\mathcal{N}^{n\times n}(\Omega)$.

In order to simplify the formulations in the following we introduce the notion of (minimal) π_+-realizations of local generalized Nevanlinna functions, cf. [17] for functions from the class $\mathcal{N}_\kappa^{n\times n}$.

Definition 2.4. Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ and let Λ be a domain with the same properties as Ω, $\Lambda \subseteq \Omega$. Let $\mathcal{H}$ be a Krein space, let T_0 be a self-adjoint linear relation in $\mathcal{H}$ which is of type π_+ over Λ and let $\gamma'(\lambda) \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$, $\lambda \in \rho(T_0)$, be a family of mappings which satisfy

$$\gamma'(\lambda) = (1 + (\lambda - \mu)(T_0 - \lambda)^{-1})\gamma'(\mu), \qquad \lambda,\, \mu \in \rho(T_0) \cap \Lambda. \tag{2.2}$$

Then the triple $(\mathcal{H}, T_0, \gamma'(\lambda))$ is called a *π_+-realization of τ over Λ* if for all $\lambda \in \Lambda \cap \rho(T_0)$ and some fixed $\lambda_0 \in \Lambda \cap \rho(T_0)$ the representation

$$\tau(\lambda) = \tau(\overline{\lambda}_0) + (\lambda - \overline{\lambda}_0)\gamma'(\lambda_0)^+\gamma'(\lambda),$$

or, equivalently,

$$\begin{aligned}\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + \gamma'(\lambda_0)^+\big(&(\lambda - \operatorname{Re}\lambda_0)\\ &+ (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(T_0 - \lambda)^{-1}\big)\gamma'(\lambda_0)\end{aligned} \tag{2.3}$$

holds. Furthermore, a π_+-realization of τ over Λ is called *minimal* if the condition

$$\mathcal{K} = \overline{\operatorname{span}}\big\{\gamma'(\lambda)x \,|\, \lambda \in \rho(T_0) \cap \Lambda,\ x \in \mathbb{C}^n\big\}$$

is fulfilled.

Sometimes we also say simply *realization* instead of a π_+-realization and we call the relation T_0 *representing relation.* We note that a family of mappings $\gamma'(\lambda)$ satisfying (2.2) is often obtained from a fixed mapping $\gamma' = \gamma'(\lambda_0) \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ by defining $\gamma'(\lambda)$ as in (2.2), where $\mu = \lambda_0$. If e.g. $\mathcal{H}$, T_0, Ω and γ are as in the assumptions of Lemma 2.3 and $\gamma(\lambda)$ is defined as mentioned above, then $(\mathcal{H}, T_0, \gamma(\lambda))$ is a π_+-realization of the function τ in (2.1) over Ω. The following theorem gives an inverse statement. For its proof we refer to [27].

Theorem 2.5. *Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ be given. Then for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a minimal π_+-realization of τ over Ω'.*

A function $\tau \in \mathcal{N}^{n\times n}(\Omega)$ is said to be *regular* if $\det \tau(\lambda_0) \neq 0$ for some $\lambda_0 \in \mathfrak{h}(\tau) \cap \Omega$. It was shown in [1, Proposition 2.6] that for a regular function $\tau \in \mathcal{N}^{n\times n}(\Omega)$ the function $\lambda \mapsto \widehat{\tau}(\lambda) := -\tau(\lambda)^{-1}$ also belongs to the class $\mathcal{N}^{n\times n}(\Omega)$ of local generalized Nevanlinna functions over Ω. In the following proposition a realization of $\widehat{\tau}$ is given in terms of the realization of τ. The proof is essentially a consequence of [33, Proposition 2.1] and [5, Theorem 2.4].

Proposition 2.6. *Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ be regular, let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let $(\mathcal{H}, T_0, \gamma'(\lambda))$ be a (minimal) π_+-realization of τ over Ω' such that* $\det \tau(\lambda_0) \neq 0$, $\lambda_0 \in \Omega'$. *Define $\widehat{T}_0$ by*

$$(\widehat{T}_0 - \lambda_0)^{-1} := (T_0 - \lambda_0)^{-1} - \gamma'(\lambda_0)\tau(\lambda_0)^{-1}\gamma'(\overline{\lambda}_0)^+$$

and $\widehat{\gamma}'(\lambda) \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ by

$$\widehat{\gamma}'(\lambda) = \bigl(1 + (\lambda - \lambda_0)(\widehat{T}_0 - \lambda)^{-1}\bigr)\widehat{\gamma}'(\lambda_0), \quad \widehat{\gamma}'(\lambda_0) := -\gamma'(\lambda_0)\tau(\lambda_0)^{-1}.$$

Then the triple $(\mathcal{H}, \widehat{T}_0, \widehat{\gamma}'(\lambda))$ is a (minimal) π_+-realization of $\widehat{\tau}$ over Ω'. Moreover, for all $\lambda \in \mathfrak{h}(\tau) \cap \mathfrak{h}(\widehat{\tau}) \cap \Omega'$ it holds

$$(\widehat{T}_0 - \lambda)^{-1} = (T_0 - \lambda)^{-1} - \gamma'(\lambda)\tau(\lambda)^{-1}\gamma'(\overline{\lambda})^+ \quad \text{and} \quad \widehat{\gamma}'(\lambda) = -\gamma'(\lambda)\tau(\lambda)^{-1}.$$

3. Generalized poles and generalized values of local generalized Nevanlinna functions

The concept of generalized poles and zeros is important in the investigation of (global) generalized Nevanlinna functions. In this section we generalize these notions to functions from the local classes $\mathcal{N}^{n\times n}(\Omega)$. Furthermore we introduce so-called generalized values for functions in $\mathcal{N}^{n\times n}(\Omega)$ and we investigate the properties of these functions at such points.

3.1. Generalized poles and generalized zeros.

Recall first the definitions of generalized poles and generalized zeros for generalized Nevanlinna functions.

Definition 3.1. Let $G \in \mathcal{N}_\kappa^{n\times n}$ be a generalized Nevanlinna function with a minimal realization $\bigl(\Pi_\kappa, A_0, \gamma(\lambda)\bigr)$. Then the eigenvalues of the representing relation A_0 are called the *generalized poles* of G. Furthermore, if G is regular a point $\beta \in \mathbb{C} \cup \{\infty\}$ is called a *generalized zero* of G if it is a generalized pole of the reciprocal function $\lambda \mapsto \widehat{G}(\lambda) = -G(\lambda)^{-1}$.

The following extension to local generalized Nevanlinna functions is natural.

Definition 3.2. Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ and let $\alpha \in \Omega$. If for some domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and $\alpha \in \Omega'$ there exists a generalized Nevanlinna function τ_0 and a function τ_1 holomorphic in $\overline{\Omega'}$ such that $\tau = \tau_0 + \tau_1$ and α is a generalized pole of τ_0, then α is called a *generalized pole of* τ. Furthermore, if τ is regular a generalized pole of $\widehat{\tau}$ is called a *generalized zero* of τ.

Remark 3.3. If $\bigl(\mathcal{K}, T_0, \gamma'(\lambda)\bigr)$ is a minimal π_+-realization of τ over Ω', then $\alpha \in \Omega'$ is a generalized pole of τ if and only if it is an eigenvalue of T_0.

Generalized poles that are isolated eigenvalues of the representing relation are just ordinary poles of τ. But we will need also analytic characterizations of those generalized poles, which are not isolated singularities of τ. To this end one introduces so-called pole-cancellation functions, cf. [9, 35]. Let $\alpha \in \Omega$, and let $\mathcal{U}_\alpha$ be an open neighborhood of the point α. By $\lambda \hat{\to} \alpha$ we denote the usual limit if $\alpha \in \mathbb{C}\backslash\mathbb{R}$ and the nontangential limit in $\mathbb{C}^+$ if $\alpha \in \overline{\mathbb{R}}$.

Definition 3.4. A holomorphic function $\eta : \mathcal{U}_\alpha \cap \Omega \cap \mathbb{C}^+ \to \mathbb{C}^n$ is called a *pole-cancellation function of* $\tau \in \mathcal{N}^{n\times n}(\Omega)$ *at* $\alpha \in \Omega$ if $\lim_{\lambda \hat{\to} \alpha} \eta(\lambda) = 0$, $\lim_{\lambda \hat{\to} \alpha} \tau(\lambda)\eta(\lambda) \neq 0$ and, furthermore, the limit

$$\lim_{\lambda,\mu \hat{\to} \alpha} \left(\frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}} \eta(\lambda), \eta(\mu) \right)$$
$$\left(\lim_{\lambda,\mu \hat{\to} \infty} \left(\frac{\lambda\overline{\mu}}{\lambda - \overline{\mu}} \Big(\tau(\lambda) - \tau(\overline{\mu})\Big) \eta(\lambda), \eta(\mu) \right) \right)$$

exists if $\alpha \neq \infty$ (resp. if $\alpha = \infty$). The vector $\eta_0 := \lim_{\lambda \hat{\to} \alpha} \tau(\lambda)\eta(\lambda)$ is called *pole vector.*

Then the following characterization holds.

Lemma 3.5. *Let* $\tau \in \mathcal{N}^{n\times n}(\Omega)$ *be given. The point* $\alpha \in \Omega$ *is a generalized pole of* τ *if and only if there exists a pole-cancellation function of* τ *at* α.

Proof. We choose some suitable domain Ω', $\overline{\Omega'} \subset \Omega$, with $\alpha \in \Omega'$ and consider the corresponding decomposition

$$\tau(\lambda) = \tau_0(\lambda) + \tau_1(\lambda), \qquad \lambda \in \mathfrak{h}(\tau) \cap \Omega',$$

where τ_0 is a generalized Nevanlinna function and τ_1 is holomorphic on $\overline{\Omega'}$. Hence a function η is a pole-cancellation function of τ at α if and only if it is a pole-cancellation function of τ_0 at α. If $\alpha \in \Omega\backslash\{\infty\}$ is a generalized pole of τ_0, then according to [35, Theorem 5.1 and Section 5.3] there exists a pole-cancellation function of τ_0 at α (which even has an additional property). Conversely, as in the proof of [35, Theorem 3.3] the existence of a pole-cancellation function of τ_0 at α implies that α is a generalized pole of τ_0. For the case $\alpha = \infty$, note that τ_0 has a generalized pole at ∞ if and only if the function $\widetilde{\tau}_0(\lambda) := \tau_0(-\lambda^{-1})$ has a generalized pole at 0 (for details on the corresponding realizations see e.g. [23]). □

The following characterization of generalized zeros of local generalized Nevanlinna functions is an immediate consequence of Lemma 3.5.

Corollary 3.6. *Let* $\tau \in \mathcal{N}^{n\times n}(\Omega)$ *be regular. A point* $\beta \in \Omega$ *is a generalized zero of the function* τ *if and only if there exists a holomorphic function* $\xi : \mathcal{U}_\beta \cap \Omega \cap \mathbb{C}^+ \to \mathbb{C}^n$ *such that* $\lim_{\lambda \hat{\to} \beta} \xi(\lambda) \neq 0$, $\lim_{\lambda \hat{\to} \beta} \tau(\lambda)\xi(\lambda) = 0$ *and, furthermore,*

$$\lim_{\lambda,\mu \hat{\to} \beta} \left(\frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}} \xi(\lambda), \xi(\mu) \right) \tag{3.1}$$
$$\left(\lim_{\lambda,\mu \hat{\to} \infty} \left(\frac{\lambda\overline{\mu}}{\lambda - \overline{\mu}} \Big(\tau(\lambda) - \tau(\overline{\mu})\Big) \xi(\lambda), \xi(\mu) \right) \right)$$

exists if $\beta \neq \infty$ *(resp.* $\beta = \infty$*). The function* $\lambda \mapsto \xi(\lambda)$ *is said to be a* root function of τ at β *and the vector* $\xi_0 := \lim_{\lambda \hat{\to} \beta} \xi(\lambda)$ *is called* root vector.

Proof. Consider the function $\lambda \mapsto \xi(\lambda) := \widehat{\tau}(\lambda)\eta(\lambda)$, where η is a pole cancellation function for $\widehat{\tau}$ at β. $\square$

The type of a generalized pole of a generalized Nevanlinna function is defined as the type of the eigenspace of a minimal representing relation, cf. [9, 35]. In the next definition this notion is extended to local generalized Nevanlinna functions.

Definition 3.7. Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$, let the point $\alpha \in \Omega$ be a generalized pole of τ and let $\big(\mathcal{H}, T_0, \gamma'(\lambda)\big)$ be a minimal π_+-realization of τ over Ω', $\overline{\Omega'} \subset \Omega$, such that $\alpha \in \Omega'$. We say that α is a generalized pole of *positive* (*negative, nonpositive, nonnegative*) *type* of τ if the eigenspace of T_0 at α is positive (resp. negative, nonpositive, nonnegative). Correspondingly the type of a generalized zero $\beta \in \Omega$ of τ is defined as the type of β as a generalized pole of $\widehat{\tau}$.

The following technical remark details the connection between a root function and the type of a generalized zero.

Remark 3.8. Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ and let $\big(\mathcal{K}, T_0, \gamma'(\lambda)\big)$ be a minimal π_+-realization of τ over some domain Ω'. If $\beta \in \Omega'$ is a generalized zero of τ, then (as in [35, Theorem 3.3]) for every root function ξ (from Corollary 3.6) $\gamma'(\lambda)\xi(\lambda)$ converges to an element $\widehat{x}_\beta \in \mathcal{K}$ as $\lambda \hat{\to} \beta$. Here $\widehat{x}_\beta$ is an eigenvector of the minimal representing relation $\widehat{T_0}$ of $\widehat{\tau}$ (cf. Proposition 2.6) and, in particular, $[\widehat{x}_\beta, \widehat{x}_\beta]$ coincides with the limit in (3.1). Note also that root functions with linearly independent root vectors induce linearly independent eigenvectors (see [35, Theorem 3.3 (iii) and (iv)]).

Applying Remark 3.8 to the reciprocal function $\widehat{\tau}$ yields the corresponding statement for generalized poles and pole-cancellation functions.

3.2. **Generalized values.** In the next definition we introduce the notion of a generalized value of a local generalized Nevanlinna function.

Definition 3.9. Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ be a local generalized Nevanlinna function and let $w_0 \in \Omega$. We say that τ *assumes a generalized value* at w_0 if $w_0 \neq \infty$ ($w_0 = \infty$) and the limit

$$(3.2) \qquad \lim_{\lambda,\mu \hat{\to} w_0} \frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}} \qquad \left(\text{resp.} \lim_{\lambda,\mu \hat{\to} \infty} \frac{\lambda\overline{\mu}}{\lambda - \overline{\mu}} \Big(\tau(\lambda) - \tau(\overline{\mu})\Big)\right)$$

exists. In this case $\tau(w_0) := \lim_{\lambda \hat{\to} w_0} \tau(\lambda)$ is called the *generalized value of* τ *at* w_0.

We emphasize that the existence of the limit (3.2) implies the existence of the generalized value $\tau(w_0)$. Indeed, the assumption that $\lim_{\lambda \hat{\to} w_0} \tau(\lambda)$ does not exist contradicts $\tau(\lambda) - \tau(\overline{\mu}) \to 0$ as $\lambda, \mu \hat{\to} w_0$

If w_0 belongs to the domain of holomorphy of τ then the limit in (3.2) obviously exists. In particular, for $w_0 \notin \mathbb{R}$ the existence of $\lim_{\lambda \hat{\to} w_0} \tau(\lambda)$ already implies the existence of the limit in (3.2).

Example 3.10. Let $\tau(\lambda) := \sqrt{\lambda}$, where $\sqrt{\cdot}$ denotes the branch of $\sqrt{\cdot}$ defined in $\mathbb{C}$ with a cut along $(-\infty, 0]$ and fixed by $\operatorname{Re}\sqrt{\lambda} > 0$ for $\lambda \notin (-\infty, 0]$ and $\operatorname{Im}\sqrt{\lambda} \geq 0$ for $\lambda \in (-\infty, 0]$. Then τ belongs to the class $\mathcal{N}_0$ and we have

$\lim_{\lambda \hat{\to} 0} \tau(\lambda) = 0$ but τ does not assume a generalized value at $w_0 = 0$ since the limit in (3.2) does not exist.

If $n = 1$, then τ assumes the generalized value $\tau(w_0)$ at $w_0 \in \Omega$ if and only if w_0 is a generalized zero of the function $\lambda \mapsto \tau(\lambda) - \tau(w_0)$. For $n > 1$ the notation of a generalized zero, roughly speaking, only means "assuming the value 0 in a certain direction" as the following example shows.

Example 3.11. The function $\tau(\lambda) := \left(\begin{smallmatrix} \lambda-1 & 0 \\ 0 & \lambda \end{smallmatrix}\right) \in \mathcal{N}_0^{2\times 2}$ has a generalized zero at $\beta = 1$, but it assumes the generalized value $\tau(1) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Conversely, τ does not need to assume a generalized value at a generalized zero.

Example 3.12. The function $\tau(\lambda) := \left(\begin{smallmatrix} -\lambda^{-1} & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{N}_0^{2\times 2}$ has a generalized zero at $\beta = 0$ since $\widehat{\tau} = -\tau^{-1}$ has a pole at $\beta = 0$, but evidently τ does not assume a generalized value at this point, since also τ itself has a pole.

In the following proposition we collect some properties of τ that follow from assuming a generalized value.

Theorem 3.13. *Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ be given. Then the following holds.*

(i) *Suppose that the function τ assumes a generalized value at the point $w_0 \in \Omega$. If $w_0 \in \mathbb{C}\backslash\mathbb{R}$ then τ is holomorphic at w_0, if $w_0 \in \mathbb{R} \cup \{\infty\}$ then $\tau(w_0)^* = \tau(w_0)$.*

(ii) *Suppose that function τ assumes a generalized value at $w_0 \in \Omega \setminus \{\infty\}$ and let $(\mathcal{K}, T_0, \gamma'(\lambda))$ be a minimal π_+-realization of τ over some domain Ω', $\overline{\Omega'} \subset \Omega$, such that $w_0 \in \Omega'$. Then the representation* (2.3) *holds even for $\lambda = w_0$:*

$$\tau(w_0) = \operatorname{Re}\tau(\lambda_0) + \gamma'(\lambda_0)^+\big((w_0 - \operatorname{Re}\lambda_0) + (w_0 - \lambda_0)(w_0 - \overline{\lambda}_0)(T_0 - w_0)^{-1}\big)\gamma'(\lambda_0).$$

In particular, $T_0 - w_0$ is injective and $\operatorname{ran}\gamma'(\lambda_0) \subseteq \operatorname{ran}(T_0 - w_0)$.

(iii) *The function τ assumes a generalized value at $w_0 \in \Omega \cap \mathbb{R}$ if and only if there exists an open interval Δ, $\overline{\Delta} \subset \Omega \cap \mathbb{R}$, such that $w_0 \in \Delta$ and τ can be written in the form*

$$\tau(\lambda) = \int_\Delta \frac{1}{t-\lambda}\, d\Sigma(t) + H_\Delta(\lambda),$$

where Σ is a nondecreasing, left-continuous $n \times n$-matrix function on Δ such that $\int_\Delta \frac{1}{(t-w_0)^2}\, d\Sigma(t)$ exists and H_Δ is holomorphic in Δ.

Proof. (i) is immediately clear from the definition and implies also (ii) for non-real w_0. In order to prove (ii) for $w_0 \in \Omega \cap \mathbb{R}$ we follow the lines of [34, Theorem 3.3]. Let $(\mathcal{K}, T_0, \gamma'(\lambda))$ be a minimal π_+-realization of τ over Ω', $\overline{\Omega'} \subset \Omega$, such that $w_0 \in \Omega'$. Note first that relation (3.2) and Lemma 3.5 imply that w_0 is not a generalized pole of τ and hence $w_0 \notin \sigma_p(T_0)$. Let $(\lambda_k)_{k\in\mathbb{N}} \subset \mathfrak{h}(\tau) \cap \Omega' \cap \mathbb{C}^+$ be a sequence converging nontangentially to $w_0 \in \Omega' \cap \mathbb{R}$. First we show that for every $x \in \mathbb{C}^n$ the strong limit

$$\lim_{k\to\infty} \gamma'(\lambda_k)x =: \gamma'(w_0)x$$

exists. Let E be a self-adjoint projection in $\mathcal{H}$ as in Definition 2.1 and define

$$\gamma_0'(\lambda) := \big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big) E\gamma'(\lambda_0), \qquad \lambda \in \mathfrak{h}(\tau) \cap \Omega',$$

and

$$\gamma_1'(\lambda) := \big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)(1 - E)\gamma'(\lambda_0), \qquad \lambda \in \mathfrak{h}(\tau) \cap \Omega'.$$

Then $\gamma' = \gamma_0' + \gamma_1'$ and $\lim_{k\to\infty} \gamma_1'(\lambda_k)x$ exists, since γ_1' is holomorphic at w_0. As $(E\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space the strong limit $\lim_{k\to\infty} \gamma_0'(\lambda_k)x$ exists if and only if the limits

$$\lim_{k\to\infty} [\gamma_0'(\lambda_k)x, u] \quad \text{and} \quad \lim_{k,l\to\infty} [\gamma_0'(\lambda_k)x, \gamma_0'(\lambda_l)x]$$

exist for all u in a dense subset of $E\mathcal{H}$ (see [25, Theorem 2.4]). But this follows from the identity

$$[\gamma_0'(\lambda)x, \gamma_0'(\mu)y] = \left(\frac{\tau_0(\lambda) - \tau_0(\overline{\mu})}{\lambda - \overline{\mu}} x, y\right), \quad \lambda, \mu \in \mathfrak{h}(\tau) \cap \Omega',\ x, y \in \mathbb{C}^n,$$

and $E\mathcal{H} = \overline{\operatorname{span}}\,\{\gamma_0'(\mu)y \mid \mu \in \mathfrak{h}(\tau) \cap \Omega',\ y \in \mathbb{C}^n\}$, which is a direct consequence of the minimality of the π_+-realization $(\mathcal{K}, T_0, \gamma'(\lambda))$. Furthermore, it holds

$$\begin{aligned}&\big(1 + (\lambda_0 - w_0)(T_0 - \lambda_0)^{-1}\big)\gamma'(w_0)x \\ &\qquad = \lim_{\lambda \hat{\to} w_0} \big(1 + (\lambda_0 - \lambda)(T_0 - \lambda_0)^{-1}\big)\gamma'(\lambda)x = \gamma'(\lambda_0)x.\end{aligned}$$

and hence $\gamma'(\lambda_0)x \in \operatorname{ran}(T_0 - w_0)$ and

$$\gamma'(w_0)x = \big(1 + (w_0 - \lambda_0)(T_0 - w_0)^{-1}\big)\gamma'(\lambda_0)x.$$

Now the representation of $\tau(w_0)$ follows from

$$\tau(w_0) = \lim_{\lambda \hat{\to} w_0} \tau(\lambda) = \tau(\overline{\lambda}_0) + \lim_{\lambda \hat{\to} w_0} \big((\lambda - \overline{\lambda}_0)\gamma'(\lambda_0)^+\gamma'(\lambda)\big).$$

In order to show (iii) we choose a domain Ω', $\overline{\Omega'} \subset \Omega$, such that $w_0 \in \Omega'$ and $\tau = \tau_0 + \tau_1$, where τ_0 is a generalized Nevanlinna function and τ_1 is holomorphic on $\overline{\Omega'}$. As τ_0 has no generalized pole at w_0 we can choose an open interval Δ, $\overline{\Delta} \subset \Omega' \cap \mathbb{R}$, such that $w_0 \in \Delta$ and Δ contains no generalized poles of nonpositive type of τ_0. Hence τ_0 can be written as the sum of the function

$$\lambda \mapsto \int_\Delta \frac{1}{t - \lambda}\, d\Sigma(t),$$

where Σ is a nondecreasing, left-continuous $n \times n$-matrix function on Δ, and a function which is holomorphic in Δ. Note that for every $x \in \mathbb{C}^n$ it holds

$$\Big(\frac{\tau_0(\lambda) - \tau_0(\overline{\mu})}{\lambda - \overline{\mu}} x, x\Big) = \int_\Delta \frac{1}{(t - \lambda)(t - \overline{\mu})}\, d(\Sigma(t)x, x) + H(\lambda, \overline{\mu}), \tag{3.3}$$

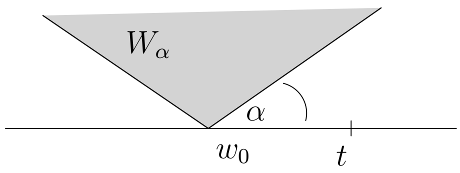
where H is holomorphic in both variables on Δ.

Suppose now that τ assumes a generalized value at w_0 and hence the limit of the left hand side of (3.3) exists for $\lambda, \mu \hat{\to} w_0$. Setting $\lambda = \mu = w_0 + i\varepsilon$ we conclude from the monotone convergence theorem that the integral

$\int_\Delta \frac{1}{|t-w_0|^2}\, d(\Sigma(t)x,x)$, $x \in \mathbb{C}^n$, exists and the polarization identity implies that

$$\int_\Delta \frac{1}{(t-w_0)^2}\, d\Sigma(t)$$

exists. Conversely, we have to show that the nontangential limit in (3.2) exists. Assume that $\lambda, \mu \in W_\alpha$, where W_α denotes the symmetric angular domain with angle $\alpha \in (0, \frac{\pi}{2})$ as in the following figure.



Then the estimate

$$\left|\frac{1}{(t-\lambda)(t-\overline{\mu})}\right| \leq \frac{1}{\sin^2\alpha} \cdot \frac{1}{|t-w_0|^2}$$

holds and by assumption the right hand side is integrable with respect to the measures $d(\Sigma(t)x,x)$, $x \in \mathbb{C}^n$. Then the dominated convergence theorem implies the existence of the limit of (3.3) for $\lambda, \mu \hat{\to} w_0$ and, again with the polarization identity, hence also the limit in (3.2). □

4. Eigenvalues of self-adjoint extensions which are locally of type π_+

This section contains the main result, namely, for a fixed symmetric operator A in a Krein space $\mathcal{K}$ we give an analytic characterization of the eigenvalues of self-adjoint extensions $\widetilde{A}$ in $\widetilde{\mathcal{K}}$, $\mathcal{K} \subset \widetilde{\mathcal{K}}$, in terms of a so-called Q-function of A and the parameter $\tau(\lambda)$ in the Krein-Naimark formula.

First let us fix the setting. Within this section let Ω be a symmetric domain in $\overline{\mathbb{C}}$ as in Section 2 and let A be a symmetric operator of finite defect n in some Krein space $\mathcal{K}$. In the following we assume that there exists a self-adjoint extension A_0 of A which is of type π_+ over Ω. By $\gamma(\lambda)$, $\lambda \in \rho(A_0) \cap \Omega$, denote a corresponding defect function, that is

$$\gamma(\lambda) := (1 + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma,$$

where γ is a fixed bijection $\gamma : \mathbb{C}^n \to \mathcal{N}_{\lambda_0} = \ker(A^+ - \lambda_0)$ and $\lambda_0 \in \rho(A_0) \cap \Omega$. And, furthermore, we assume that the minimality condition

$$\mathcal{K} = \overline{\operatorname{span}}\left\{\gamma(\lambda)x \,|\, \lambda \in \rho(A_0) \cap \Omega,\ x \in \mathbb{C}^n\right\} \tag{4.1}$$

is satisfied. Note, that this implies $\sigma_p(A) = \emptyset$, sometimes in this case A is said to be *simple*. By the relation

$$\frac{m(\lambda) - m(w)^*}{\lambda - \overline{w}} = \gamma(w)^+\gamma(\lambda), \qquad \lambda, w \in \rho(A_0) \cap \Omega,$$

a function m is determined uniquely up to a self-adjoint constant. Let S be a self-adjoint $n \times n$-matrix, then we fix m by

$$m(\lambda) := S + \gamma^+\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma, \tag{4.2}$$

$\lambda \in \rho(A_0) \cap \Omega$. Note that the triple $(\mathcal{K}, A_0, \gamma(\lambda))$ is a minimal π_+-realization of m over Ω and hence $m \in \mathcal{N}^{n\times n}(\Omega)$, cf. Section 2.4. We note that in the Pontryagin or Hilbert space setting m is often called the *Q-function* corresponding to the pair (A, A_0) (see e.g. [29, 32]).

From $\ker \gamma(\lambda) = \{0\}$ for all $\lambda \in \rho(A_0) \cap \Omega$ and (4.1) it follows that

$$\bigcap_{\lambda \in \mathfrak{h}(m) \cap \Omega} \ker \frac{m(\lambda) - m(w)^*}{\lambda - \overline{w}} = \{0\} \tag{4.3}$$

holds. A local generalized Nevanlinna function which fulfils this condition for one (and hence for all) $w \in \Omega$ is called *strict*. Note that, conversely, this property is sufficient for a local generalized Nevanlinna function to be the Q-function of a pair (A, A_0) as above (cf. Proposition 5.3).

Let $\widetilde{A}$ be another self-adjoint extension of A in some larger Krein space $\widetilde{\mathcal{K}} \supset \mathcal{K}$, which contains $\mathcal{K}$ as a Krein-subspace, and denote the bounded self-adjoint projection onto $\mathcal{K}$ by $\mathcal{P}_{\mathcal{K}}$. We assume that $\widetilde{A}$ is also of type π_+ over Ω, $\lambda_0 \in \rho(\widetilde{A})$, and that $\widetilde{A}$ is $\mathcal{K}$-*minimal*, that is

$$\widetilde{\mathcal{K}} = \overline{\operatorname{span}}\big\{(1 + (\lambda - \lambda_0)(\widetilde{A} - \lambda)^{-1})\mathcal{K} \mid \lambda \in \rho(\widetilde{A}) \cap \Omega\big\}.$$

The following theorem is the main result of this section.

Theorem 4.1. *Let $\widetilde{A}$ be a $\mathcal{K}$-minimal self-adjoint extension of A in $\widetilde{\mathcal{K}}$ which is of type π_+ over Ω, let A_0 and $m \in \mathcal{N}^{n\times n}(\Omega)$ be as above and assume that*

$$P_{\mathcal{K}}\big(\widetilde{A} - \lambda\big)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(m(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{4.4}$$

holds for all $\lambda \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((m + \tau)^{-1}) \cap \Omega$ and some function $\tau \in \mathcal{N}^{n\times n}(\Omega)$. Then the following is true.

(i) *If τ assumes a generalized value at $w_0 \in \Omega$, then w_0 is an eigenvalue of $\widetilde{A}$ if and only if w_0 is a generalized zero of $m + \tau$.*
(ii) *If A is of defect one, then $w_0 \in \Omega$ is an eigenvalue of $\widetilde{A}$ if and only if w_0 is either a generalized zero of $m + \tau$ or a generalized pole of both m and τ.*

In a similar way also the sign type of the eigenvalue will be characterized in terms of the functions m and τ, see Proposition 4.9.

Remark 4.2. Note that (4.4) is a natural assumption since it is well-known to hold in several important special cases. It was shown by V. Derkach in [10] and [11] that for Pontryagin spaces $\mathcal{K}$ and $\widetilde{\mathcal{K}}$, $\Omega = \overline{\mathbb{C}}$ and $n \geq 1$ formula (4.4) establishes a bijective correspondence between the compressed resolvents of $\mathcal{K}$-minimal self-adjoint exit space extensions of A and the so-called $\mathcal{N}_\kappa$-families, a class of relation-valued functions which includes the generalized Nevanlinna functions (over $\overline{\mathbb{C}}$). In the special case of Hilbert spaces (4.4) is well known as the Krein-Naimark formula, cf. [12, 28, 32, 36]. Here τ belongs to the class of Nevanlinna families. If, in addition, $\widetilde{A} \cap \mathcal{K}^2 = A$ holds, then τ is a usual Nevanlinna function. Moreover, it is shown in [4] that in the case $n = 1$ the compressed resolvents of an exit space extension $\widetilde{A}$ of A which is of type π_+ over Ω can be written in the form (4.4) with some function $\tau \in \mathcal{N}(\Omega)$.

Remark 4.3. If $w_0 = \infty$ is not an eigenvalue of $\widetilde{A}$, then obviously $\widetilde{A}$ is an operator. In the special case of Hilbert spaces $\mathcal{K}$, $\widetilde{\mathcal{K}}$ and $\Omega = \overline{\mathbb{C}}$ this condition on $\widetilde{A}$ is called *admissibility* and has also been characterized by m and τ with different methods, see e.g. [12].

In the special case that $\widetilde{A}$ is a canonical self-adjoint extension of A and $\mathcal{K}\,(= \widetilde{\mathcal{K}})$ is a Hilbert or Pontryagin space the following statement is well known. Here it is an immediate consequence of Theorem 4.1 and [5, Theorem 2.4].

Corollary 4.4. *Let $\widetilde{A}$ be a self-adjoint extension of A in $\mathcal{K}$, $\rho(\widetilde{A}) \cap \Omega \neq \emptyset$, let A_0 and $m \in \mathcal{N}^{n\times n}(\Omega)$ be as above and assume that*

$$(\widetilde{A} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(m(\lambda) + \tau\big)^{-1}\gamma(\overline{\lambda})^{+}$$

holds for all $\lambda \in \rho(A_0) \cap \mathfrak{h}((m+\tau)^{-1}) \cap \Omega$ and some self-adjoint $n \times n$-matrix τ. Then $w_0 \in \Omega$ is an eigenvalue of $\widetilde{A}$ if and only if w_0 is a generalized zero of $\lambda \mapsto m(\lambda) + \tau$.

For the proof of Theorem 4.1 we will show two propositions which are also of interest for their own. The idea of the proof is, roughly speaking, the following: we first construct a $\mathcal{K}$-minimal self-adjoint extension $\widehat{\mathbb{A}}$ of A which is the representing relation in a minimal π_+-realization over Ω of the function

$$\widetilde{M}(\lambda) := -\begin{pmatrix} m(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}^{-1} \tag{4.5}$$

such that the compressed resolvents of $\widetilde{A}$ and $\widehat{\mathbb{A}}$ coincide. Then the $\mathcal{K}$-minimality of the extensions $\widetilde{A}$ and $\widehat{\mathbb{A}}$ yields that locally, that is, restricted to certain spectral subspaces which are Pontryagin spaces, these two relations are unitarily equivalent. Hence (locally) the eigenvalues of $\widetilde{A}$ are the generalized poles of $\widetilde{M}$, and it is shown that then the characterizations in the theorem hold.

Remark 4.5. The function $m+\tau$ also has a realization with $\widetilde{A}$ as representing relation. It is clear from Theorem 4.1 (ii) that in general this realization cannot be minimal. However, due to the special structure of the $2n \times 2n$-matrix function $\widetilde{M}$, at least in special cases (see e.g. [18] where τ is a scalar rational function) there exists also an $n \times n$-matrix function for which $\widetilde{A}$ is a minimal representing relation.

We start with an easy observation. If $\lambda \in \mathfrak{h}(m) \cap \mathfrak{h}(\tau) \cap \Omega$ and $\det \tau(\lambda) \neq 0$, then $\widetilde{M}(\lambda)$ in (4.5) exists if and only if $(m(\lambda) + \tau(\lambda))^{-1}$ exists. In this case we have

$$\widetilde{M}(\lambda) = \begin{pmatrix} -(m(\lambda) + \tau(\lambda))^{-1} & (m(\lambda) + \tau(\lambda))^{-1}\tau(\lambda) \\ \tau(\lambda)(m(\lambda) + \tau(\lambda))^{-1} & m(\lambda)(m(\lambda) + \tau(\lambda))^{-1}\tau(\lambda) \end{pmatrix}. \tag{4.6}$$

Proposition 4.6. *Let $(\mathcal{K}, A_0, \gamma(\lambda))$ be a minimal π_+-realization over Ω of the strict function $m \in \mathcal{N}^{n\times n}(\Omega)$ and let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ be given such that τ and $m + \tau$ are regular. Then the following holds.*

(i) *The function $\widetilde{M}$ in (4.5) belongs to the class $\mathcal{N}^{2n\times 2n}(\Omega)$.*

(ii) *For every domain* Ω' *with the same properties as* Ω, $\overline{\Omega'} \subset \Omega$, *there exists a* $\mathcal{K}$*-minimal self-adjoint extension* $\widehat{\mathbb{A}}$ *of* A *such that*

$$P_{\mathcal{K}}\big(\widehat{\mathbb{A}} - \lambda\big)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(m(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{4.7}$$

holds for all $\lambda \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((m+\tau)^{-1}) \cap \Omega'$ *and the function* $\widetilde{M}$ *has a minimal* π_+*-realization over* Ω' *with representing relation* $\widehat{\mathbb{A}}$.

Proof. (i) From the assumption that $\tau \in \mathcal{N}^{n\times n}(\Omega)$ is regular it follows $-\tau^{-1} \in \mathcal{N}^{n\times n}(\Omega)$ and therefore the function

$$\lambda \mapsto \begin{pmatrix} m(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}, \tag{4.8}$$

$\lambda \in \mathfrak{h}(m) \cap \mathfrak{h}(\tau^{-1}) \cap \Omega$, and hence also $\widetilde{M}$ belong to the class $\mathcal{N}^{2n\times 2n}(\Omega)$.

In order to verify assertion (ii), let, as in Theorem 2.5 and Proposition 2.6, $(\mathcal{H}, T_0, \gamma'(\lambda))$ and $(\mathcal{H}, \widehat{T}_0, \widehat{\gamma}'(\lambda))$ be minimal π_+-realizations for the functions τ and $-\tau^{-1}$ over Ω', respectively. Then the triple $(\mathcal{K} \times \mathcal{H}, \mathbb{A}, \gamma_{\mathbb{A}}(\lambda))$ is a minimal π_+-realization for the function in (4.8) over Ω', where $\mathbb{A} := A_0 \times \widehat{T}_0$ and $\gamma_{\mathbb{A}} := \gamma \times \widehat{\gamma}'$. Once more applying Proposition 2.6 gives a minimal π_+-realization $(\mathcal{K} \times \mathcal{H}, \widehat{\mathbb{A}}, \widehat{\gamma}_{\mathbb{A}}(\lambda))$ for the function $\widetilde{M}$, where

$$\widehat{\gamma}_{\mathbb{A}}(\lambda) = \gamma_{\mathbb{A}}(\lambda)\widetilde{M}(\lambda) = \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & -\gamma'(\lambda)\tau(\lambda)^{-1} \end{pmatrix} \widetilde{M}(\lambda) \tag{4.9}$$

and

$$\big(\widehat{\mathbb{A}} - \lambda\big)^{-1} = \begin{pmatrix} (A_0 - \lambda)^{-1} & 0 \\ 0 & (\widehat{T}_0 - \lambda)^{-1} \end{pmatrix} + \gamma_{\mathbb{A}}(\lambda)\widetilde{M}(\lambda)\gamma_{\mathbb{A}}(\overline{\lambda})^+ \tag{4.10}$$

hold for all $\lambda \in \mathfrak{h}(m) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}) \cap \mathfrak{h}((m+\tau)^{-1}) \cap \Omega'$. Making use of (4.6), (4.9) and (4.10) it is easy to see that the compressed resolvent $P_{\mathcal{K}}(\widehat{\mathbb{A}} - \lambda)^{-1}|_{\mathcal{K}}$ has the form (4.7). It remains to show that $\widehat{\mathbb{A}}$ is $\mathcal{K}$-minimal, i.e. the condition

$$\mathcal{K} \times \mathcal{H} = \overline{\operatorname{span}}\left\{\big(1 + (\lambda - \lambda_0)(\widehat{\mathbb{A}} - \lambda)^{-1}\big)\mathcal{K} \,|\, \lambda \in \rho(\widehat{\mathbb{A}}) \cap \Omega'\right\} \tag{4.11}$$

is fulfilled. Note that the set $\rho(\widehat{\mathbb{A}}) \cap \Omega'$ in (4.11) can be replaced by any nonempty open subset of $\rho(\widehat{\mathbb{A}}) \cap \Omega'$ which is symmetric with respect to the real axis. The relations (4.10), (4.9) and (4.6) imply

$$P_{\mathcal{H}}(\widehat{\mathbb{A}} - \lambda)^{-1}|_{\mathcal{K}} = -\gamma'(\lambda)\big(m(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{4.12}$$

for $\lambda \in \mathfrak{h}(m) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}(\tau^{-1}) \cap \mathfrak{h}((m+\tau)^{-1}) \cap \Omega'$. From the simplicity of m, that is, $\operatorname{ran} \gamma(\overline{\lambda})^+ = (\ker \gamma(\overline{\lambda}))^\perp = \mathbb{C}^n$ and the minimality of the π_+-realization $(\mathcal{H}, T_0, \gamma'(\lambda))$ we conclude that the ranges of the operators in (4.12) span $\mathcal{H}$ and hence (4.11) holds. □

We are now turning to the generalized poles of $\widetilde{M}$.

Proposition 4.7. *Let* $\tau, m \in \mathcal{N}^{n\times n}(\Omega)$ *be given such that* τ *and* $m + \tau$ *are regular and let*

$$\widetilde{M}(\lambda) = -\begin{pmatrix} m(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}^{-1}.$$

Then the following holds.

(i) *If τ assumes a generalized value at $w_0 \in \Omega$, then w_0 is a generalized pole of $\widetilde{M}$ if and only if w_0 is a generalized zero of $m+\tau$.*

(ii) *If $n=1$, then $w_0 \in \Omega$ is a generalized pole of $\widetilde{M}$ if and only if w_0 is either a generalized zero of $m+\tau$ or a generalized pole of both m and τ.*

The following example shows that the assumption on τ assuming a generalized value can be dropped only in the scalar case and the second statement in the proposition does not hold for $n>1$.

Example 4.8. Consider the functions

$$m(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\frac{1}{\lambda} \end{pmatrix}, \quad \tau_1(\lambda) = \begin{pmatrix} -\lambda & 1 \\ 1 & \frac{1}{\lambda} \end{pmatrix} \quad \text{and} \quad \tau_2(\lambda) = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda-1} \end{pmatrix}.$$

Then the point $w_0 = 0$ is not a generalized zero of the functions $m+\tau_i$, $i=1,2$. However, it is easy to check that the function

$$\widetilde{M}_i(\lambda) = -\begin{pmatrix} m(\lambda) & -I \\ -I & -\tau_i(\lambda)^{-1} \end{pmatrix}^{-1}$$

has a generalized pole at $w_0 = 0$ for $i=1$ (choose e.g. $\xi(\lambda) = (1, 2\lambda, 0, -2)^\top$ as a root function for $-\widetilde{M}^{-1}$ at 0) but not for $i=2$.

Proof of Proposition 4.7. Recall that w_0 is a generalized pole of the function $\widetilde{M}$ if and only if it is a generalized zero of the function

$$-\widetilde{M}(\lambda)^{-1} = \begin{pmatrix} m(\lambda) & -1 \\ -1 & -\tau(\lambda)^{-1} \end{pmatrix}.$$

In what follows we assume that $w_0 \in \Omega \cap \mathbb{C}$, since the case $w_0 = \infty$ can be deduced from this by using the transformation $z = -\frac{1}{\lambda}$.

(i) Suppose that τ assumes a generalized value at w_0 and assume first that w_0 is a generalized zero of $-\widetilde{M}^{-1}$. Then by Corollary 3.6 there exists a root function $\lambda \mapsto \xi(\lambda) = (x(\lambda), y(\lambda))^\top$, that is,

$$\lim_{\lambda \hat{\to} w_0} \begin{pmatrix} x(\lambda) \\ y(\lambda) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.13}$$

and

$$\lim_{\lambda \hat{\to} w_0} \begin{pmatrix} m(\lambda)x(\lambda) \\ -\tau(\lambda)^{-1}y(\lambda) \end{pmatrix} = \begin{pmatrix} y_0 \\ x_0 \end{pmatrix} \tag{4.14}$$

hold and the limit

$$\begin{aligned} \lim_{\lambda, w \hat{\to} w_0} \Bigg[&\left(\frac{m(\lambda) - m(\overline{w})}{\lambda - \overline{w}} x(\lambda), x(w) \right) \\ &+ \left(\frac{-\tau(\lambda)^{-1} + \tau(\overline{w})^{-1}}{\lambda - \overline{w}} y(\lambda), y(w) \right) \Bigg] \end{aligned} \tag{4.15}$$

exists. Setting $v(\lambda) := -\tau(\lambda)^{-1} y(\lambda)$ we also have

$$\lim_{\lambda \hat{\to} w_0} v(\lambda) = x_0 \quad \text{and} \quad \lim_{\lambda \hat{\to} w_0} \tau(\lambda) v(\lambda) = -y_0 \tag{4.16}$$

and the limit

$$\lim_{\lambda, w \hat{\to} w_0} \left[\left(\frac{m(\lambda) - m(\overline{w})}{\lambda - \overline{w}} x(\lambda), x(w) \right) + \left(\frac{\tau(\lambda) - \tau(\overline{w})}{\lambda - \overline{w}} v(\lambda), v(w) \right) \right] \tag{4.17}$$

exists and coincides with the one in (4.15). Since τ assumes a generalized value at w_0 the limit of the second summand in (4.17) exists and hence this implies also the existence for the first summand.

We claim that $\lambda \mapsto x(\lambda)$ is a root function for $m + \tau$. In fact, first of all we have $\lim_{\lambda \hat{\to} w_0} x(\lambda) = x_0 \neq 0$, as otherwise the existence of $\lim_{\lambda \hat{\to} w_0} \tau(\lambda)$ and (4.16) would imply also $y_0 = 0$; a contradiction to (4.13). From $\lim_{\lambda \hat{\to} w_0} \tau(\lambda)x(\lambda) = -y_0$ we obtain $\lim_{\lambda \hat{\to} w_0}(m(\lambda) + \tau(\lambda))x(\lambda) = 0$. Moreover, also the limit of

$$\left(\frac{m(\lambda) - m(\overline{w})}{\lambda - \overline{w}}\, x(\lambda), x(w)\right) + \left(\frac{\tau(\lambda) - \tau(\overline{w})}{\lambda - \overline{w}}\, x(\lambda), x(w)\right)$$

exists, for the first summand by the argument above and for the second by the assumption that τ assumes a generalized value at w_0.

Conversely, if $w_0 \in \Omega \cap \mathbb{C}$ is a generalized zero of $m + \tau$ and $\lambda \mapsto x(\lambda)$ is a corresponding root function, then the existence of $\lim_{\lambda \hat{\to} w_0} \tau(\lambda)$ implies that

$$\lambda \mapsto \xi(\lambda) := \begin{pmatrix} x(\lambda) \\ -\tau(\lambda)x(\lambda) \end{pmatrix}$$

is a root function for $-\widetilde{M}^{-1}$ at w_0.

(ii) Without the assumption that w_0 is a generalized value of τ more careful considerations are necessary. Assume first that $w_0 \in \Omega \cap \mathbb{C}$ is a generalized pole of $\widetilde{M}$ and let us choose a root function $\lambda \mapsto \xi(\lambda) = (x(\lambda), y(\lambda))^\top$ for $-\widetilde{M}^{-1}$ at w_0, that is, it has the properties (4.13), (4.14) and (4.15).

We claim that in this case w_0 is a generalized pole of τ if and only if w_0 is generalized pole of m. In fact, if w_0 is a generalized pole of τ we have $x_0 = 0$ and $y_0 \neq 0$ by (4.13). As w_0 is a generalized zero of $-\tau^{-1}$ the limit of the second summand in (4.15) exists and hence also the limit of the first summand in (4.15) exists. Together with $\lim_{\lambda \hat{\to} w_0} x(\lambda) = 0$ and $\lim_{\lambda \hat{\to} w_0} m(\lambda)x(\lambda) = y_0 \neq 0$ this implies that $\lambda \mapsto x(\lambda)$ is a pole cancellation function of m at w_0, i.e. w_0 is a generalized pole of m. For the converse assume that w_0 is a generalized pole of m but not a generalized pole of τ. From (4.13) and (4.14) we obtain $x_0 = 0$ and $y_0 \neq 0$. Let, as in part (i) of the proof, $v(\lambda) = -\tau(\lambda)^{-1}y(\lambda)$. Then the limit of the second summand of (4.17) does not exist as otherwise v would be a pole cancellation function of τ at w_0. But then also the first limit in (4.17) cannot exist which (in the scalar case) is a contradiction to w_0 being a generalized pole of m.

Therefore we can assume in the following that w_0 is not a generalized pole of the functions m and τ. Then there exist functions m_1 and τ_1 holomorphic in a neighborhood of w_0 such that

$$m(\lambda) = m_0(\lambda) + m_1(\lambda) \quad \text{and} \quad \tau(\lambda) = \tau_0(\lambda) + \tau_1(\lambda)$$

holds, where $m_0(\lambda) = \int_\Delta \frac{d\sigma_m(t)}{t-\lambda}$ and $\tau_0(\lambda) = \int_\Delta \frac{d\sigma_\tau(t)}{t-\lambda}$ are Nevanlinna functions, Δ is an open interval containing w_0 and σ_m and σ_τ are finite measures. In particular, then the existence of the limit (4.17) implies also the existence of

$$\lim_{\lambda \hat{\to} w_0} \left(\frac{m_0(\lambda) - m_0(\overline{\lambda})}{\lambda - \overline{\lambda}}|x(\lambda)|^2 + \frac{\tau_0(\lambda) - \tau_0(\overline{\lambda})}{\lambda - \overline{\lambda}}|v(\lambda)|^2\right). \tag{4.18}$$

But since both summands in (4.18) are either convergent or divergent to $+\infty$ it follows that the limits

$$\lim_{\lambda \hat{\to} w_0} \frac{m_0(\lambda) - m_0(\overline{\lambda})}{\lambda - \overline{\lambda}} |x(\lambda)|^2 \quad \text{and} \quad \lim_{\lambda \hat{\to} w_0} \frac{\tau_0(\lambda) - \tau_0(\overline{\lambda})}{\lambda - \overline{\lambda}} |v(\lambda)|^2 \tag{4.19}$$

exist separately. Similarly as in the proof of Theorem 3.13 one verifies that $\lim_{\lambda \hat{\to} w_0}$ and $\overline{\lambda}$ in (4.19) can be replaced by $\lim_{\lambda, w \hat{\to} w_0}$ and $\overline{w}$, respectively, and therefore

$$\lim_{\lambda, w \hat{\to} w_0} \frac{m(\lambda) - m(\overline{w})}{\lambda - \overline{w}} x(\lambda)\overline{x(w)} \tag{4.20}$$

and

$$\lim_{\lambda, w \hat{\to} w_0} \frac{\tau(\lambda) - \tau(\overline{w})}{\lambda - \overline{w}} v(\lambda)\overline{v(w)}$$

exist.

Note, that $x_0 \neq 0$ as otherwise $\lim_{\lambda \hat{\to} w_0} m(\lambda)x(\lambda) = y_0 \neq 0$ and the existence of the limit in (4.20) would imply that $\lambda \mapsto x(\lambda)$ is a pole cancellation function for m. Hence also

$$\lim_{\lambda, w \hat{\to} w_0} \frac{\tau(\lambda) - \tau(\overline{w})}{\lambda - \overline{w}}$$

exists, that is, τ assumes a generalized value at w_0. Therefore we can apply part (i) of the proposition and it follows that w_0 is a generalized zero of $m + \tau$.

Let us, conversely, first assume that $w_0 \in \Omega$ is a generalized zero of $m+\tau$ and w_0 is not a generalized pole of τ. Hence w_0 can also not be a generalized pole of m, since the same arguments as above show that the existence of

$$\lim_{\lambda, w \hat{\to} w_0} \left(\frac{m(\lambda) - m(\overline{w})}{\lambda - \overline{w}} + \frac{\tau(\lambda) - \tau(\overline{w})}{\lambda - \overline{w}} \right)$$

implies even the existence of both limits separately. Hence m and τ assume a generalized value at w_0. Therefore the first statement implies that w_0 is a generalized pole of $\widetilde{M}$. Finally, if w_0 is a generalized pole of both functions m and τ, then $\lambda \mapsto \xi(\lambda) = (m(\lambda)^{-1}, 1)^\top$ is a root function of $-\widetilde{M}^{-1}$ at w_0. □

Proof of Theorem 4.1. Since the relations A and A_0 determine the function m in (4.2) only up to a self-adjoint $n \times n$-matrix it is no restriction to assume that m is such that τ is regular. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, such that $w_0 \in \Omega'$ and choose a minimal π_+-realization $(\mathcal{K} \times \mathcal{H}, \widehat{\mathbb{A}}, \widehat{\gamma}_{\mathbb{A}})$ for the function $\widetilde{M}$ in (4.5) over Ω' as in Proposition 4.6 (ii). If $E(\cdot, \widetilde{A})$ and $E(\cdot, \widehat{\mathbb{A}})$ denote the local spectral functions of $\widetilde{A}$ and $\widehat{\mathbb{A}}$ in Ω and Ω', respectively, and Δ, $\overline{\Delta} \subset \Omega' \cap \overline{\mathbb{R}}$, is an open connected set, then the $\mathcal{K}$-minimality of $\widetilde{A}$ and $\widehat{\mathbb{A}}$ and similar arguments as in [27, §3] imply that $E(\Delta, \widetilde{A})$ is defined if and only if $E(\Delta, \widehat{\mathbb{A}})$ is defined, and in this case the Pontryagin spaces $E(\Delta, \widetilde{A})(\widetilde{\mathcal{K}})$ and $E(\Delta, \widehat{\mathbb{A}})(\mathcal{K} \times \mathcal{H})$ have the same finite rank of negativity and the self-adjoint relations

$$\widetilde{A}_\Delta := \widetilde{A} \cap \big(E(\Delta, \widetilde{A})(\widetilde{\mathcal{K}})\big)^2 \quad \text{and} \quad \widehat{\mathbb{A}}_\Delta := \widehat{\mathbb{A}} \cap \big(E(\Delta, \widehat{\mathbb{A}})(\mathcal{K} \times \mathcal{H})\big)^2$$

are unitarily equivalent, that is, there exists an isometric isomorphism V which maps $E(\Delta,\widetilde{A})(\widetilde{\mathcal{K}})$ onto $E(\Delta,\widehat{\mathbb{A}})(\mathcal{K}\times\mathcal{H})$ such that

$$\left\{\begin{pmatrix} V\{k,h\} \\ V\{k',h'\}\end{pmatrix} \,\Big|\, \begin{pmatrix} \{k,h\} \\ \{k',h'\}\end{pmatrix} \in \widetilde{A}_\Delta\right\} = \widehat{\mathbb{A}}_\Delta$$

holds. Therefore w_0 is an eigenvalue of $\widetilde{A}$ if and only if w_0 is an eigenvalue of $\widehat{\mathbb{A}}$. As the generalized poles of $\widetilde{M}$ in Ω' coincide with the eigenvalues of $\widehat{\mathbb{A}}$ the statement of Theorem 4.1 follows by applying Proposition 4.7. □

In the next proposition we characterize the sign type of the eigenvalues of $\widetilde{A}$ with the help of the function $m+\tau$. For simplicity in the presentation we exclude the case $w_0=\infty$.

Proposition 4.9. *Let the relation $\widetilde{A}$ and the functions m, $\tau\in\mathcal{N}^{n\times n}(\Omega)$ be given as in Theorem* 4.1 *and assume that $w_0\in\Omega\cap\mathbb{R}$ is an eigenvalue of $\widetilde{A}$. Then the following holds.*

(i) *If the function τ assumes a generalized value at the point w_0 then the dimension of the geometric eigenspace of $\widetilde{A}$ at w_0 is at most n.*

(ii) *Suppose that τ assumes a generalized value at w_0 and let $\lambda\mapsto x(\lambda)$ be a root function of $m+\tau$ at w_0. Then $\widetilde{A}$ has an eigenvector x_0 at w_0 such that*

$$\begin{aligned}[x_0,x_0] = \lim_{\lambda,w\hat{\to} w_0}\Bigg[&\left(\frac{m(\lambda)-m(\overline{w})}{\lambda-\overline{w}}x(\lambda),x(w)\right)\\ &+\left(\frac{\tau(\lambda)-\tau(\overline{w})}{\lambda-\overline{w}}x(\lambda),x(w)\right)\Bigg].\end{aligned}\tag{4.21}$$

Conversely, for every eigenvector x_0 at w_0 there exists a root function $\lambda\mapsto x(\lambda)$ of $m+\tau$ at w_0 such (4.21) *holds.*

(iii) *In the case $n=1$ the geometric eigenspace of $\widetilde{A}$ at w_0 is one-dimensional and its type is given by the the sign of*

$$\lim_{\lambda\hat{\to} w_0}\frac{m(\lambda)+\tau(\lambda)}{\lambda-w_0}\qquad\left(\lim_{\lambda\hat{\to} w_0}\frac{-m(\lambda)^{-1}-\tau(\lambda)^{-1}}{\lambda-w_0}\right)$$

if w_0 is not a generalized pole of τ (resp. if w_0 is a generalized pole of τ).

Proof. In the proof of Theorem 4.1 and Proposition 4.7 we have seen that to each eigenvector of $\widetilde{A}$ at w_0 there exists a root function of $-\widetilde{M}^{-1}$ at w_0 and conversely. If we identify root functions which have equal root vectors then this correspondence is even one-to-one (cf. Remark 3.8). Then relation (4.14) shows that there exist at most n linearly independent root vectors for $-\widetilde{M}^{-1}$, which proves (i).

(ii) Let us now assume that $\lambda\mapsto x(\lambda)$ is a root function of $m+\tau$. Then, according to the proof of Proposition 4.7, the function $\lambda\mapsto\xi(\lambda)=(x(\lambda),-\tau(\lambda)x(\lambda))^\top$ is a pole-cancellation function for $\widetilde{M}$ and hence a root function for $-\widetilde{M}^{-1}$ at w_0. Thus (again with Remark 3.8) for the corresponding eigenvector x_0

$$[x_0, x_0] = \lim_{\lambda, w \hat{\to} w_0} \left(\frac{-\widetilde{M}(\lambda)^{-1} + \widetilde{M}(\overline{w})^{-1}}{\lambda - \overline{w}} \xi(\lambda), \xi(w) \right)$$

holds which implies statement (ii).

If $n = 1$ then according to Theorem 4.1 either τ assumes a generalized value at w_0 or this point is a generalized pole of τ. In the first case the above considerations hold with $x(\lambda) = 1$. In the second case, as in the proof of Proposition 4.7 one can choose $\lambda \mapsto \xi(\lambda) = (m(\lambda)^{-1}, 1)^\top$ as a root function for $-\widetilde{M}^{-1}$ at w_0. □

As a direct consequence of Theorem 4.1 and Theorem 3.13 the following necessary condition for embedded eigenvalues of $\widetilde{A}$ can be given. Although Corollary 4.10 below can be formulated with the help of the local spectral function in a more general setting we restrict ourselves to the case of Hilbert spaces $\mathcal{K}$ and $\widetilde{\mathcal{K}}$. Recall, that if A is a simple operator of defect 1 in a Hilbert space $\mathcal{K}$, then every canonical self-adjoint extension A_0 of A in $\mathcal{K}$ is unitarily equivalent to the operator of multiplication in a space L^2_σ, where σ is called *spectral measure* of A_0.

Corollary 4.10. *Let A be a simple symmetric operator with deficiency indices $(1,1)$ in the Hilbert space $\mathcal{K}$ and fix a self-adjoint extension A_0 of A with spectral measure σ. If $w_0 \in \mathbb{R} \setminus \sigma_p(A_0)$ is an eigenvalue of some $\mathcal{K}$-minimal self-adjoint extension $\widetilde{A}$ of A in a Hilbert space $\widetilde{\mathcal{K}} \supseteq \mathcal{K}$, then*

$$\int_{\mathbb{R}} \frac{1}{|t - w_0|^2}\, d\sigma(t) < \infty.$$

5. A class of abstract λ-dependent boundary value problems

As an application of the results in the foregoing sections we study a class of abstract eigenvalue dependent boundary value problems. Here the so-called *linearization* (cf. Theorem 5.5) plays an important role for questions of solvability. First we recall the notion of boundary value spaces and associated Weyl functions and show that the above mentioned linearization is a self-adjoint linear relation of the type considered before.

In fact, there appear also a few repetitions of what has already been obtained, but now in the language of boundary value spaces. However, we want to point out that the first approach in Section 4 is more general, since τ was not supposed to be strict.

5.1. **Boundary value spaces and associated Weyl functions.** We use the so-called boundary value spaces for the description of the closed extensions of a symmetric operator. The following definition can be found in e.g. [10].

Definition 5.1. Let A be a (not necessarily densely defined) closed symmetric operator in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. The triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a *boundary value space for A^+* if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist linear mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G} \times \mathcal{G}$ is surjective and

$$[f, g'] - [f', g] = (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) - (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})$$

holds for all $\hat{f} = \binom{f}{f'}, \hat{g} = \binom{g}{g'} \in A^+$.

In the following we briefly recall some basic facts on boundary value spaces which can be found in e.g. [10] and [11]. For the Hilbert space case we refer to [22], [14] and [15]. Let A be a closed symmetric operator in $\mathcal{K}$, define for the points of regular type $\lambda \in r(A)$ the defect subspace of A by $\mathcal{N}_{\lambda,A^+} := \ker(A^+ - \lambda) = \operatorname{ran}(A - \overline{\lambda})^{[\perp]}$ and let

$$\hat{\mathcal{N}}_{\lambda,A^+} = \left\{ \binom{f_\lambda}{\lambda f_\lambda} \,\middle|\, f_\lambda \in \mathcal{N}_{\lambda,A^+} \right\}. \tag{5.1}$$

When no confusion can arise we will simply write $\mathcal{N}_\lambda$ and $\hat{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_{\lambda,A^+}$ and $\hat{\mathcal{N}}_{\lambda,A^+}$. If there exists a self-adjoint extension $\hat{A}$ of A in $\mathcal{K}$ such that $\rho(\hat{A}) \neq \emptyset$, then we have

$$A^+ = \hat{A} \,\widehat{+}\, \hat{\mathcal{N}}_\lambda \tag{5.2}$$

for all $\lambda \in \rho(\hat{A})$ and there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that $\ker \Gamma_0 = \hat{A}$, see e.g. [11].

Let in the following A, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and Γ be as in Definition 5.1. Then $A = \ker \Gamma$, the mappings Γ_0 and Γ_1 are continuous and

$$A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1$$

are self-adjoint extensions of A. The mapping Γ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \left\{ \hat{f} \in A^+ \,|\, \Gamma \hat{f} \in \Theta \right\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{5.3}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between the set of closed linear relations $\widetilde{\mathcal{C}}(\mathcal{G})$ in $\mathcal{G}$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (5.3) gives a one-to-one correspondence between the closed symmetric (self-adjoint) extensions of A and the closed symmetric (resp. self-adjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker\big(\Gamma_1 - \Theta \Gamma_0\big). \tag{5.4}$$

Let $\rho(A_0) \neq \emptyset$ and denote by π_1 the orthogonal projection onto the first component of $\mathcal{K} \times \mathcal{K}$. For every $\lambda \in \rho(A_0)$ we define the operators

$$\gamma(\lambda) := \pi_1 (\Gamma_0 | \hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}, \mathcal{K})$$

and

$$m(\lambda) := \Gamma_1 (\Gamma_0 | \hat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}).$$

The functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto m(\lambda)$ are called the *γ-field* and the *Weyl function* corresponding to A and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. Then γ and m are holomorphic on $\rho(A_0)$ and

$$\gamma(w) = (1 + (w - \lambda)(A_0 - w)^{-1})\gamma(\lambda) \tag{5.5}$$

and

$$m(\lambda) - m(w)^* = (\lambda - \overline{w})\gamma(w)^+ \gamma(\lambda) \tag{5.6}$$

hold for $\lambda, w \in \rho(A_0)$. Making use of (5.6) and (5.5) one verifies

$$m(\lambda) = \operatorname{Re} m(\lambda_0) + \gamma(\lambda_0)^+\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A_0-\lambda)^{-1}\big)\gamma(\lambda_0) \tag{5.7}$$

for a fixed $\lambda_0 \in \rho(A_0)$ and all $\lambda \in \rho(A_0)$. If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ if and only if 0 belongs to $\rho(\Theta - m(\lambda))$. For $\lambda \in \rho(A_\Theta)\cap\rho(A_0)$ the well-known resolvent formula

$$(A_\Theta - \lambda)^{-1} = (A_0-\lambda)^{-1} + \gamma(\lambda)\big(\Theta - m(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{5.8}$$

holds (for a proof see e.g. [11]).

We are now turning to the case that A_0 is locally of type π_+. Let Ω be a domain as in Section 2. The following proposition is a direct consequence of the considerations in Subsection 2.4, the relations (5.7), (5.8) and [5, Theorem 2.4].

Proposition 5.2. *Let A be a closed symmetric operator of finite defect in the Krein space $\mathcal{K}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ with corresponding γ-field γ and Weyl function m, respectively, and assume that $A_0 = \ker\Gamma_0$ is of type π_+ over Ω. Then the following holds.*

(i) *The Weyl function m belongs to the class $\mathcal{N}^{n\times n}(\Omega)$ and $(\mathcal{K}, A_0, \gamma(\lambda))$ is π_+-realization of m over Ω.*

(ii) *If the condition $\mathcal{K} = \overline{\operatorname{span}}\,\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(A_0)\cap\Omega\}$ is fulfilled, then m is strict and the π_+-realization $(\mathcal{K}, A_0, \gamma(\lambda))$ is minimal.*

(iii) *If A_Θ is a self-adjoint extension of A in $\mathcal{K}$ and $\rho(A_\Theta)\cap\Omega$ is nonempty, then A_Θ is also of type π_+ over Ω.*

In the next proposition we show that every strict function $\tau \in \mathcal{N}^{n\times n}(\Omega)$ can be realized as the Weyl function corresponding to a symmetric operator T of defect n and a suitable boundary value space $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$. For strict generalized Nevanlinna functions, i.e. the case $\Omega = \overline{\mathbb{C}}$, Proposition 5.3 reduces to [13, Proposition 3.1] and for scalar functions $\tau \in \mathcal{N}(\Omega)$ it was proven in [6]. The proof of Proposition 5.3 is very similar to the proof of [6, Theorem 3.3]. For the convenience of the reader we sketch the proof.

Proposition 5.3. *Let $\tau \in \mathcal{N}^{n\times n}(\Omega)$ be strict, let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let $(\mathcal{H}, T_0, \gamma'(\lambda))$ be a minimal π_+-realization of τ over Ω'. Then there exists a symmetric operator $T \subset T_0$ of defect n in $\mathcal{H}$ and a boundary value space $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ for T^+ such that $T_0 = \ker\Gamma_0'$ and τ and γ' coincide with the corresponding Weyl function and γ-field in Ω', respectively.*

Proof. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let $(\mathcal{H}, T_0, \gamma'(\lambda))$ be a minimal π_+-realization of τ over Ω'. From

$$\frac{\tau(\lambda) - \tau(w)^*}{\lambda - \overline{w}} = \gamma'(w)^+\gamma'(\lambda), \quad \lambda, w \in \mathfrak{h}(\tau)\cap\Omega',$$

and the assumption that τ is strict (cf. (4.3)) we conclude that the mappings $\gamma'(\lambda)$, $\lambda \in \mathfrak{h}(\tau)\cap\Omega'$, are injective.

For some $\mu \in \mathfrak{h}(\tau) \cap \Omega'$ we define

$$T := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in T_0 \,\middle|\, [g - \overline{\mu} f, \gamma'(\mu) h] = 0 \text{ for all } h \in \mathbb{C}^n \right\}.$$

Then T is a closed symmetric operator of defect n in $\mathcal{H}$ which does not depend on the choice of $\mu \in \mathfrak{h}(\tau) \cap \Omega'$. Moreover we have

$$\mathcal{N}_{\lambda,T^+} = \ker(T^+ - \lambda) = \operatorname{ran} \gamma'(\lambda), \quad \lambda \in \mathfrak{h}(\tau) \cap \Omega'.$$

The mapping $\gamma'(\lambda)$, $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$, is an isomorphism of $\mathbb{C}^n$ onto $\mathcal{N}_{\lambda,T^+}$. The inverse of this mapping is denoted by $\gamma'(\lambda)^{(-1)}$.

For some fixed $\mu \in \mathfrak{h}(\tau) \cap \Omega'$ we write the elements $\hat{f} \in T^+$ in the form

$$\hat{f} = \begin{pmatrix} f_0 \\ f_0' \end{pmatrix} + \begin{pmatrix} f_\mu \\ \mu f_\mu \end{pmatrix},$$

where $\left(\begin{smallmatrix} f_0 \\ f_0' \end{smallmatrix}\right) \in T_0$ and $f_\mu \in \mathcal{N}_{\mu,T^+}$ (see (5.1), (5.2)). Let $\Gamma_0', \Gamma_1' : T^+ \to \mathbb{C}^n$ be the linear mappings defined by

$$\begin{aligned} \Gamma_0' \hat{f} &:= \gamma'(\mu)^{(-1)} f_\mu, \\ \Gamma_1' \hat{f} &:= \gamma'(\mu)^+ (f_0' - \overline{\mu} f_0) + \tau(\mu) \gamma'(\mu)^{(-1)} f_\mu. \end{aligned}$$

Then we have $T_0 = \ker \Gamma_0'$ and the same calculation as in the proof of [6, Theorem 3.3] shows that $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ is a boundary value space for T^+ and the corresponding Weyl function and γ-field coincide with τ and γ' in Ω'. □

If $\tau \in \mathcal{N}^{n \times n}(\Omega)$ is the Weyl function corresponding to T and a boundary value space $\{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ we have $\tau(\lambda)\Gamma_0' \hat{f}_\lambda = \Gamma_1' \hat{f}_\lambda$ for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$ and $\hat{f}_\lambda \in \hat{\mathcal{N}}_{\lambda,T^+}$. In the next proposition we show that this property remains true for points w_0 where τ assumes a generalized value. Note that if w_0 does not belong to $\mathfrak{h}(\tau)$ then by Theorem 3.13 we have $w_0 \in \sigma_c(T_0)$ and therefore ran $(T - w_0)$ can not be closed, i.e. w_0 is not a point of regular type, $w_0 \notin r(T)$. We agree to extend the definition of the defect spaces spaces $\mathcal{N}_{w_0,T^+} = \ker(T^+ - w_0)$ to points w_0 where τ assumes a generalized value and we set

$$\hat{\mathcal{N}}_{w_0,T^+} := \left\{ \left(\begin{smallmatrix} f_{w_0} \\ w_0 f_{w_0} \end{smallmatrix}\right) \mid f_{w_0} \in \ker(T^+ - w_0) \right\}.$$

Proposition 5.4. *Let $\tau \in \mathcal{N}^{n \times n}(\Omega)$ be strict and suppose that τ assumes a generalized value at some point $w_0 \in \Omega \cap \mathbb{R}$. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and choose a boundary value space $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ such that τ is the corresponding Weyl function. Then the following holds.*

(i) *The point w_0 is an eigenvalue of the self-adjoint extension*

$$T_{\tau(w_0)} = \ker(\Gamma_1' - \tau(w_0)\Gamma_0')$$

of T_0 and $\ker(T_{\tau(w_0)} - w_0)$ has dimension n.

(ii) *The mapping $\Gamma_0' : \hat{\mathcal{N}}_{w_0,T^+} \to \mathbb{C}^n$ is bijective and*

$$\tau(w_0)\Gamma_0' \hat{f}_{w_0} = \Gamma_1' \hat{f}_{w_0}$$

holds for all $\hat{f}_{w_0} \in \hat{\mathcal{N}}_{w_0,T^+}$.

We remark that if $\lambda \mapsto \tau(\lambda) - \tau(w_0)$ is regular assertion (i) follows from the fact that $\lambda \mapsto -(\tau(\lambda) - \tau(w_0))^{-1}$ is the Weyl function corresponding to T and the boundary value space $\{\mathbb{C}^n, \Gamma_1' - \tau(w_0)\Gamma_0', -\Gamma_0'\}$.

Proof. Note, that assertions (i) and (ii) are obvious if the point w_0 belongs to $\mathfrak{h}(\tau) \cap \Omega'$. (i) Let γ' be the γ-field corresponding to the boundary value space $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ and let $(\lambda_k) \subset \mathfrak{h}(\tau) \cap \Omega' \cap \mathbb{C}^+$ be a sequence converging nontangentially to $w_0 \in \Omega' \cap \mathbb{R}$. As in the proof of Theorem 3.13 (ii) one shows that for every $x \in \mathbb{C}^n$ the strong limit

$$\lim_{k\to\infty} \gamma'(\lambda_k)x =: \gamma'(w_0)x$$

exists. Since T^+ is closed we conclude

$$\hat{\gamma}'(w_0)x := \begin{pmatrix} \gamma'(w_0)x \\ w_0\gamma'(w_0)x \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} \gamma'(\lambda_k)x \\ \lambda_k\gamma'(\lambda_k)x \end{pmatrix} \in \hat{\mathcal{N}}_{w_0,T^+} \subset T^+.$$

We claim that $\hat{\gamma}'(w_0)x \in T_{\tau(w_0)}$, i.e. $\gamma'(w_0)x$ is an eigenvector of $T_{\tau(w_0)}$ corresponding to the eigenvalue w_0. In fact, since τ assumes a generalized value at w_0 and the mappings Γ_0', Γ_1' are continuous

$$\begin{aligned}\tau(w_0)\Gamma_0'\hat{\gamma}'(w_0)x &= \lim_{k\to\infty} \tau(\lambda_k)\Gamma_0'\hat{\gamma}'(\lambda_k)x = \lim_{k\to\infty} \Gamma_1'\hat{\gamma}'(\lambda_k)x \\ &= \Gamma_1'\hat{\gamma}'(w_0)x\end{aligned}$$

implies $\hat{\gamma}'(w_0)x \in T_{\tau(w_0)}$. In order to see that the dimension of the eigenspace is n, we show that the elements $\gamma'(w_0)x_i$, $i = 1, \dots, n$, are linearly independent if the $x_i \in \mathbb{C}^n$ are linearly independent. Assume $\sum_{i=1}^n \mu_i\gamma'(w_0)x_i = 0$. Since $\gamma'(\mu)^+$, $\mu \in \rho(T_0)\cap\Omega'$, is continuous and τ assumes a generalized value at w_0 this implies

$$\begin{aligned}0 &= \lim_{k\to\infty} \sum_{i=1}^n \mu_i\gamma'(\mu)^+\gamma'(\lambda_k)x_i = \lim_{k\to\infty} \sum_{i=1}^n \mu_i \frac{\tau(\lambda_k) - \tau(\mu)^*}{\lambda_k - \overline{\mu}} x_i \\ &= \frac{\tau(w_0) - \tau(\mu)^*}{w_0 - \overline{\mu}} \sum_{i=1}^n \mu_i x_i\end{aligned}$$

and hence

$$\sum_{i=1}^n \mu_i x_i \in \ker \frac{\tau(\lambda) - \tau(\mu)^*}{\lambda - \overline{\mu}}, \qquad \lambda, \mu \in \rho(T_0) \cap \Omega'.$$

As τ is assumed to be strict we conclude $\sum_{i=1}^n \mu_i x_i = 0$ and since the x_i are linearly independent this finally gives $\mu_i = 0$ for $i = 1, \dots, n$, hence $\dim(\ker(T_{\tau(w_0)} - w_0)) = n$.

(ii) As w_0 is not a generalized pole of τ it is no eigenvalue of the relation T_0 and therefore the mapping $\Gamma_0' : \hat{\mathcal{N}}_{w_0,T^+} \to \mathbb{C}^n$ is injective and hence with (i) also bijective. That is, for every $x \in \mathbb{C}^n$ there exists an element $\hat{h} \in \hat{\mathcal{N}}_{w_0,T^+} \subset T_{\tau(w_0)} = \ker(\Gamma_1' - \tau(w_0)\Gamma_0')$ with $\Gamma_0'\hat{h} = x$ and hence with this notation we find

$$\Gamma_1'\big(\Gamma_0'|_{\hat{\mathcal{N}}_{w_0,T^+}}\big)^{-1}x = \Gamma_1'\hat{h} = \tau(w_0)\Gamma_0'\hat{h} = \tau(w_0)x,$$

which finishes the proof. ☐

5.2. **Boundary value problems with local generalized Nevanlinna functions in the boundary condition.** Now we can formulate the abstract boundary value problem. Let A be a closed symmetric operator of finite defect n in the Krein space $\mathcal{K}$ and assume that there exists a self-adjoint extension A_0 of A which is of type π_+ over Ω and the minimality condition

$$\mathcal{K} = \overline{\operatorname{span}}\,\{\mathcal{N}_{\lambda,A^+} \mid \lambda \in \rho(A_0) \cap \Omega\}$$

holds, cf. (4.1). Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary value space for A^+ such that $A_0 = \ker \Gamma_0$ and denote by γ and m the corresponding γ-field and Weyl function, respectively.

Let $\widetilde{\Omega}$ be a domain with the same properties as Ω, $\overline{\Omega} \subset \widetilde{\Omega}$, and let $\tau \in \mathcal{N}^{n\times n}(\widetilde{\Omega})$ be a strict local generalized Nevanlinna function over $\widetilde{\Omega}$. In the sequel we consider the following boundary value problem: For a given $g \in \mathcal{K}$ find an element $\hat{f} = \binom{f}{f'} \in A^+$ such that

$$f' - \lambda f = g \quad \text{and} \quad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0 \tag{5.9}$$

holds. If $g \neq 0$ we shall refer to (5.9) as the *inhomogeneous* boundary value problem and as the *homogeneous* boundary value problem otherwise. The points $\lambda \in \mathbb{C}$ where the homogeneous boundary value problem has a nontrivial solution $\hat{f} \in A^+$ are said to be the *eigenvalues* of the homogeneous boundary value problem. A priori (5.9) is stated for $\lambda \in \mathfrak{h}(\tau)$ and then it is – at least in special cases – well known that the linearization $\widetilde{A}$ (see below) provides information about the solvability and the solutions of this problem, see e.g. [3, 6, 7, 11, 12, 18, 19, 20]. However, we shall see, that this still holds true in the larger set of points where τ assumes a generalized value.

The following theorem is a generalization of [6, Theorem 4.1] where the boundary value problem (5.9) was considered only for scalar functions $\tau \in \mathcal{N}(\widetilde{\Omega})$ in the points of holomorphy of τ.

Theorem 5.5. *Let A, $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$, γ and m be as above, let $\tau \in \mathcal{N}^{n\times n}(\widetilde{\Omega})$ be a strict function and assume that $m+\tau$ is regular. Fix a symmetric operator T of defect n in a Krein space $\mathcal{H}$ and a boundary value space $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function and $T_0 = \ker \Gamma_0'$ is of type π_+ over Ω. Then the following holds.*

(i) *The relation*

$$\widetilde{A} = \{\{\hat{f}, \hat{h}\} \in A^+ \times T^+ \mid \Gamma_1 \hat{f} - \Gamma_1' \hat{h} = \Gamma_0 \hat{f} + \Gamma_0' \hat{h} = 0\} \tag{5.10}$$

in $\mathcal{K}\times\mathcal{H}$ is a $\mathcal{K}$-minimal self-adjoint extension of A which is of type π_+ over Ω. Every $\lambda \in \rho(A_0) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((m+\tau)^{-1}) \cap \Omega$ belongs to $\rho(\widetilde{A})$ and it holds

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(m(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+. \tag{5.11}$$

(ii) *If τ assumes a generalized value at $\lambda \in \rho(\widetilde{A}) \cap \Omega$, then a solution of the inhomogeneous boundary value problem (5.9) is given by*

$$f = P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}\, g \quad \textit{and} \quad f' = \lambda f + g. \tag{5.12}$$

(iii) *If m and τ assume a generalized value at $\lambda \in \rho(\widetilde{A}) \cap \Omega$ and $\det(m(\lambda) + \tau(\lambda)) \neq 0$, then the solution (5.12) of (5.9) is unique.*

Proof. (i) It is easy to see that $\{\mathbb{C}^{2n},\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$, where

$$\widetilde{\Gamma}_0\{\hat f,\hat h\}:=\begin{pmatrix}\Gamma_0\hat f\\ \Gamma_0'\hat h\end{pmatrix},\quad \widetilde{\Gamma}_1\{\hat f,\hat h\}:=\begin{pmatrix}\Gamma_1\hat f\\ \Gamma_1'\hat h\end{pmatrix},\quad \hat f\in A^+,\ \hat h\in T^+,$$

is a boundary value space for $A^+\times T^+$ with corresponding γ-field

$$\lambda\mapsto\widetilde{\gamma}(\lambda)=\begin{pmatrix}\gamma(\lambda)&0\\0&\gamma'(\lambda)\end{pmatrix},\qquad \lambda\in\rho(A_0)\cap\rho(T_0)\cap\Omega,$$

and Weyl function

$$\lambda\mapsto\widetilde{m}(\lambda)=\begin{pmatrix}m(\lambda)&0\\0&\tau(\lambda)\end{pmatrix},\qquad \lambda\in\rho(A_0)\cap\rho(T_0)\cap\Omega.$$

The relation

$$\widetilde{\Theta}:=\left\{\begin{pmatrix}\{u,-u\}\\ \{v,v\}\end{pmatrix}\;\middle|\;u,v\in\mathbb{C}^n\right\}$$

is self-adjoint and the corresponding self-adjoint extension $\widetilde{\Gamma}^{-1}\Theta$ via (5.3) has the form (5.10). We leave it to the reader to verify that a point $\lambda\in\rho(A_0)\cap\rho(T_0)\cap\Omega$ belongs to $\mathfrak{h}((m+\tau)^{-1})$ if and only if $0\in\rho(\widetilde{\Theta}-\widetilde{m}(\lambda))$. From

$$\big(\widetilde{\Theta}-\widetilde{m}(\lambda)\big)^{-1}=\begin{pmatrix}-(m(\lambda)+\tau(\lambda))^{-1}&(m(\lambda)+\tau(\lambda))^{-1}\\(m(\lambda)+\tau(\lambda))^{-1}&-(m(\lambda)+\tau(\lambda))^{-1}\end{pmatrix}$$

and

(5.13)
$$\big(\widetilde{A}-\lambda\big)^{-1}=\begin{pmatrix}(A_0-\lambda)^{-1}&0\\0&(T_0-\lambda)^{-1}\end{pmatrix}+\widetilde{\gamma}(\lambda)\big(\widetilde{\Theta}-\widetilde{m}(\lambda)\big)^{-1}\widetilde{\gamma}(\overline{\lambda})^+,$$

$\lambda\in\rho(A_0)\cap\mathfrak{h}(\tau)\cap\mathfrak{h}((m+\tau)^{-1})\cap\Omega$, we conclude that the compressed resolvent of $\widetilde{A}$ has the form (5.11). Moreover, relation (5.13), the fact that $A_0\times T_0$ is of type π_+ over Ω, and [5, Theorem 2.4] imply that $\widetilde{A}$ is also of type π_+ over Ω.

(ii) Let $\lambda\in\rho(\widetilde{A})\cap\Omega$ and suppose that τ assumes a generalized value at λ. Let

$$f:=P_{\mathcal{K}}\big(\widetilde{A}-\lambda\big)^{-1}\{g,0\}\quad\text{and}\quad h:=P_{\mathcal{H}}\big(\widetilde{A}-\lambda\big)^{-1}\{g,0\}.$$

Then

$$\begin{pmatrix}\{f,h\}\\ \{g+\lambda f,\lambda h\}\end{pmatrix}\in\widetilde{A}\subset A^+\times T^+,$$

where $\hat f=\binom{f}{g+\lambda f}\in A^+$ and $\hat h=\binom{h}{\lambda h}\in\hat{\mathcal{N}}_{\lambda,T^+}$, and Proposition 5.4 (ii) and (5.10) imply

$$\tau(\lambda)\Gamma_0\hat f=-\tau(\lambda)\Gamma_0'\hat h=-\Gamma_1'\hat h=-\Gamma_1\hat f,$$

hence $\hat f=\binom{f}{g+\lambda f}\in A^+$ is a solution of (5.9).

(iii) Let us assume that $\hat f=\binom{f}{g+\lambda f}$ and $\hat k=\binom{k}{g+\lambda k}$ are both solutions of (5.9). Then $\hat f-\hat k=\binom{f-k}{\lambda(f-k)}\in\hat{\mathcal{N}}_{\lambda,A^+}$ and

(5.14)
$$\tau(\lambda)\Gamma_0(\hat f-\hat k)+\Gamma_1(\hat f-\hat k)=0$$

holds. By assumption m assumes a generalized value at the point λ and therefore $\lambda \notin \sigma_p(A_0)$ and $m(\lambda)\Gamma_0(\hat{f} - \hat{k}) = \Gamma_1(\hat{f} - \hat{k})$, cf. Proposition 5.4. From (5.14) we conclude

$$\big(m(\lambda) + \tau(\lambda)\big)\Gamma_0(\hat{f} - \hat{k}) = 0$$

and $\det(m(\lambda)+\tau(\lambda)) \neq 0$ yields $\Gamma_0(\hat{f}-\hat{k}) = 0$. But then $\hat{f}-\hat{k} \in A_0 \cap \mathcal{N}_{\lambda,A^+}$ and since λ is not an eigenvalue of A_0 we conclude $\hat{f} = \hat{k}$, that is, the solution (5.12) is unique. □

In the next proposition we show how the eigenvalues of $\widetilde{A}$ are connected with the eigenvalues of the homogeneous boundary value problem (5.9).

Proposition 5.6. *Let A, $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$, m, τ and $\widetilde{A}$ be as in Theorem* 5.5 *and suppose that τ assume a generalized value at $w_0 \in \Omega$.*

Then w_0 is an eigenvalue of the homogeneous boundary value problem (5.9) *if and only if w_0 is an eigenvalue of $\widetilde{A}$. In this case a solution f is given by the first component of the eigenvector $\{f, h\} \in \mathcal{K} \times \mathcal{H}$ of $\widetilde{A}$.*

Proof. Let us first assume that $\hat{f} := \binom{f}{f'} \in A^+$ is a nontrivial solution of the boundary value problem

$$f' - w_0 f = 0, \qquad \tau(w_0)\Gamma_0\hat{f} + \Gamma_1\hat{f} = 0. \tag{5.15}$$

Since τ assumes a generalized value at w_0 by Proposition 5.4 the mapping $\Gamma_0' : \hat{\mathcal{N}}_{w_0,T^+} \to \mathbb{C}^n$ is bijective and hence there exists $\hat{h} \in \hat{\mathcal{N}}_{w_0,T^+}$ such that

$$-\Gamma_0\hat{f} = \Gamma_0'\hat{h} \tag{5.16}$$

holds. Making use of (5.15) and Proposition 5.4 we obtain

$$\Gamma_1\hat{f} = -\tau(w_0)\Gamma_0\hat{f} = \tau(w_0)\Gamma_0'\hat{h} = \Gamma_1'\hat{h}. \tag{5.17}$$

Relations (5.16) and (5.17) show $\{\hat{f}, \hat{h}\} \in \widetilde{A}$. Conversely, if w_0 is an eigenvalue of $\widetilde{A}$ and $\{f, h\} \in \mathcal{K} \times \mathcal{H}$ is a corresponding eigenvector, then

$$\hat{f} = \begin{pmatrix} f \\ w_0 f \end{pmatrix} \in A^+, \quad \hat{h} = \begin{pmatrix} h \\ w_0 h \end{pmatrix} \in T^+$$

and

$$\Gamma_1\hat{f} - \Gamma_1'\hat{h} = \Gamma_0\hat{f} + \Gamma_0'\hat{h} = 0 \tag{5.18}$$

holds. In particular $f \neq 0$ as otherwise (5.18) would imply $\hat{h} \in T$, but T has no eigenvalues. From Proposition 5.4 (ii) and (5.18) we obtain

$$\tau(w_0)\Gamma_0\hat{f} = -\tau(w_0)\Gamma_0'\hat{h} = -\Gamma_1'\hat{h} = -\Gamma_1\hat{f},$$

hence $\hat{f}$ is a nontrivial solution of the homogeneous boundary value problem (5.15). □

The following example shows that this theorem does not remain true if we drop the condition that τ assumes a generalized value at w_0.

Example 5.7. The homogeneous problem $-\frac{d^2}{dx^2} f - \lambda f = 0$ in $L^2(0,\infty)$ with boundary condition $\tau(\lambda) f'(0) = f(0)$, where $\tau(\lambda) := -\sqrt{-\lambda+1} - 1 \in \mathcal{N}_0$, can be written in the form (5.9), cf. Section 5.3. Here the function m is a Titchmarsh-Weyl function of the singular Sturm-Liouville differential expression $-\frac{d^2}{dx^2}$ in $L^2(0,\infty)$.

If we set $\tau(-1) := \lim_{\lambda \hat{\to} -1} \tau(\lambda) = -1$ the problem can be stated for $\lambda = -1$ and it has the nontrivial solution $f(x) = e^{-x}$. However, the corresponding linearization $\widetilde{A}$ has no eigenvalues. In particular, it is easy to see that -1 cannot be an eigenvalue, since then (according to Theorem 4.1(ii)) it should be either a generalized pole of τ, or a generalized zero of $m + \tau$. The latter would imply that τ assumes a generalized value at $\lambda = -1$, which is not the case.

The above considerations show that the results from Section 4 can be applied to the boundary value problem of the form (5.9), this is formulated in the following corollary.

Corollary 5.8. *Let the boundary value problem* (5.9) *be given.*

(i) *If τ assumes a generalized value at $w_0 \in \Omega$, then w_0 is an eigenvalue of the homogeneous boundary value problem if and only if w_0 is a generalized zero of $m + \tau$. In this case there exist at most n linearly independent solutions.*

(ii) *If $n = 1$, then $w_0 \in \Omega$ is an eigenvalue of the homogeneous boundary value problem if and only if w_0 is either a generalized zero of $m + \tau$ or w_0 is a generalized pole of both m and τ.*

Moreover, the type of the solution is given by the type of the generalized zero w_0 of $m + \tau$ (or of $\widehat{m} + \widehat{\tau}$ if $n = 1$ and w_0 is a generalized pole of τ).

5.3. **An example.** We study a singular Sturm-Liouville operator with the signum function as indefinite weight in the Krein space

$$L^2(\mathbb{R}, \mathrm{sgn}) := (L^2(\mathbb{R}), [\cdot,\cdot]),$$

where $[\cdot,\cdot]$ is defined by

$$[f,g] := \int_{-\infty}^{\infty} f(x)\overline{g(x)} \,\mathrm{sgn}\, x \, dx, \qquad f, g \in L^2(\mathbb{R}).$$

Denote by J the fundamental symmetry of $L^2(\mathbb{R}, \mathrm{sgn})$ defined by

$$(Jf)(x) := (\mathrm{sgn}\, x) f(x), \qquad x \in \mathbb{R}.$$

Then $[J\cdot,\cdot] =: (\cdot,\cdot)$ is the usual scalar product of $L^2(\mathbb{R})$. In the following the elements f of $L^2(\mathbb{R})$ will often be identified with the elements $\langle f_+, f_- \rangle$, $f_\pm := f|_{\mathbb{R}^\pm}$, of $L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty, 0)$, $\mathbb{R}^+ := (0,\infty)$.

We consider the following problem: Find $\lambda \in \mathbb{C}$ for which there exists a nontrivial $f = \langle f_+, f_- \rangle \in W^{2,2}(\mathbb{R}^+) \times W^{2,2}(\mathbb{R}^-)$ such that

$$-(\mathrm{sgn}\, x) f''(x) = \lambda f(x), \qquad x \in \mathbb{R}^+ \cup \mathbb{R}^- \tag{5.19}$$

and the boundary conditions

$$\frac{1}{\lambda^k} f_+'(0+) = f_+(0+) \quad \text{and} \quad \frac{1}{\lambda^l} f_-(0-) = f_-'(0-) \tag{5.20}$$

are satisfied for some $k, l \in \mathbb{N}$.

In the next lemma we choose a symmetric differential operator A in $L^2(\mathbb{R},\mathrm{sgn}\,)$ and a boundary value space $\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$ for A^+ such that problem (5.19)-(5.20) can be written in the form (5.9). In order to apply the results of the foregoing section we calculate the Weyl function m of $\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$. As in Example 3.10 we denote by $\sqrt{\cdot}$ the branch of $\sqrt{\cdot}$ defined in $\mathbb{C}$ with a cut along $(-\infty,0]$ and fixed by $\mathrm{Re}\,\sqrt{\lambda}>0$ for $\lambda\notin(-\infty,0]$ and $\mathrm{Im}\,\sqrt{\lambda}\geq 0$ for $\lambda\in(-\infty,0]$.

Lemma 5.9. *The operator*

$$\begin{aligned}(Af)(x) &:= -(\mathrm{sgn}\;x)f''(x),\\ \mathrm{dom}\;A &:= \bigl\{f\in W^{2,2}(\mathbb{R})\,|\,f(0)=f'(0)=0\bigr\},\end{aligned}$$

is a densely defined closed symmetric operator of defect two in the Krein space $L^2(\mathbb{R},\mathrm{sgn}\,)$. The adjoint operator A^+ is given by

$$\begin{aligned}\bigl(A^+\langle f_+,f_-\rangle\bigr)(x) &= \langle -f_+'',f_-''\rangle(x),\\ \mathrm{dom}\;A^+ &= \bigl\{\langle f_+,f_-\rangle\in W^{2,2}(\mathbb{R}^+)\times W^{2,2}(\mathbb{R}^-)\bigr\},\end{aligned}\tag{5.21}$$

and the minimality condition $L^2(\mathbb{R},\mathrm{sgn}\,)=\overline{\mathrm{span}}\,\{\ker(A^+-\lambda)\,|\,\lambda\in\mathbb{C}\backslash\mathbb{R}\}$ is satisfied. The triple $\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$, where

$$\Gamma_0\hat f := \begin{pmatrix} f_+'(0+)\\ -f_-(0-)\end{pmatrix}\quad\textit{and}\quad \Gamma_1\hat f := \begin{pmatrix} -f_+(0+)\\ f_-'(0-)\end{pmatrix},\quad \hat f := \begin{pmatrix} f\\ A^+f\end{pmatrix},$$

is a boundary value space for A^+ and the operator $A_0=\ker\Gamma_0$ is of type π_+ over the domain $\mathbb{C}\backslash(-\infty,0]$. The Weyl function corresponding to $\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$ is given by

$$\lambda\mapsto m(\lambda)=\begin{pmatrix}\frac{1}{\sqrt{-\lambda}} & 0\\ 0 & -\sqrt{\lambda}\end{pmatrix},\qquad \lambda\in\mathbb{C}\backslash\mathbb{R}.\tag{5.22}$$

Remark 5.10. The self-adjoint extension A_Θ of A corresponding to the self-adjoint 2×2-matrix $\Theta=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ via (5.3) is the usual self-adjoint second order differential operator in $L^2(\mathbb{R},\mathrm{sgn}\,)$ associated with $-\mathrm{sgn}\,x\frac{d^2}{dx^2}$, that is,

$$(A_\Theta f)(x)=-(\mathrm{sgn}\;x)f''(x),\quad \mathrm{dom}\;A_\Theta=W^{2,2}(\mathbb{R}).$$

Proof. The operators $S_+f_+=-f_+''$ and $S_-f_-=f_-''$ with

$$\mathrm{dom}\;S_\pm=\bigl\{f_\pm\in W^{2,2}(\mathbb{R}^\pm)\,|\,f_\pm(0\pm)=f_\pm'(0\pm)=0\bigr\}$$

in $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^-)$, respectively, are closed, densely defined, and have both deficiency indices $(1,1)$. Since $\mathrm{dom}\;JA=\mathrm{dom}\;A$, $JAf=-f''$, and A is the orthogonal sum of S_+ and S_- we conclude that A is a closed densely defined symmetric operator of defect two in $L^2(\mathbb{R},\mathrm{sgn}\,)$. This gives (5.21) and as the operators $S_\pm$ are simple we have

$$L^2(\mathbb{R}^\pm)=\overline{\mathrm{span}}\,\bigl\{\ker(S_\pm^*-\lambda)\,|\,\lambda\in\mathbb{C}\backslash\mathbb{R}\bigr\}.$$

Now $\ker(A^+-\lambda)=\ker(S_+^*-\lambda)\times\ker(S_-^*-\lambda)$ implies

$$L^2(\mathbb{R},\mathrm{sgn}\,)=\overline{\mathrm{span}}\,\bigl\{\ker(A^+-\lambda)\,|\,\lambda\in\mathbb{C}\backslash\mathbb{R}\bigr\}.$$

It is straightforward to check that $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is a boundary value space for A^+ and that $\ker(A^+ - \lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, is the span of

$$f_{\lambda,+}(x) = \begin{cases} \exp(-\sqrt{-\lambda}\, x), & x \in \mathbb{R}^+ \\ 0, & x \in \mathbb{R}^- \end{cases}$$

and

$$f_{\lambda,-}(x) = \begin{cases} 0, & x \in \mathbb{R}^+ \\ \exp(\sqrt{\lambda} x), & x \in \mathbb{R}^- \end{cases}.$$

From $m(\lambda)\Gamma_0 \hat{f}_{\lambda,\pm} = \Gamma_1 \hat{f}_{\lambda,\pm}$, where $\hat{f}_{\lambda,\pm} = \left(\begin{smallmatrix} f_{\lambda,\pm} \\ \lambda f_{\lambda,\pm} \end{smallmatrix} \right)$, we obtain that the Weyl function m corresponding to $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ has the form (5.22). It remains to check that

$$\begin{aligned} (A_0 \langle f_+, f_- \rangle)(x) &= \langle -f_+'', f_-'' \rangle (x), \\ \operatorname{dom} A_0 &= \big\{ \langle f_+, f_- \rangle \in W^{2,2}(\mathbb{R}^+) \times W^{2,2}(\mathbb{R}^-) \,| \\ &\qquad\qquad f_+'(0+) = f_-(0-) = 0 \big\}, \end{aligned}$$

is of type π_+ over $\Omega = \mathbb{C}\backslash(-\infty, 0]$. Note that $\sigma(A_0) = \mathbb{R}$ since m is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and no point of $\mathbb{R}$ belongs to $\mathfrak{h}(m)$. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let $\Delta \subset \mathbb{R}^+$ be an open interval such that $\overline{\Omega'} \cap \mathbb{R} \subset \overline{\Delta}$ and $\overline{\Delta} \subset \Omega \cap \mathbb{R}$ holds. If $E_+(\Delta)$ denotes the spectral projection of the self-adjoint operator $A_{0,+} f_+ = -f_+''$, $\operatorname{dom} A_{0,+} = \{ f_+ \in W^{2,2}(\mathbb{R}^+) \,|\, f_+'(0+) = 0\}$, in the Hilbert space $L^2(\mathbb{R}^+)$ corresponding to the interval Δ, then

$$E := E_+(\Delta) P_+, \qquad P_+ f := f_+, \quad f \in L^2(\mathbb{R}),$$

is a self-adjoint projection in $L^2(\mathbb{R}, \operatorname{sgn})$ such that $EL^2(\mathbb{R}, \operatorname{sgn})$ is a Hilbert space and properties (i) and (ii) of Definition 2.1 are fulfilled. □

With the help of the operator $A \subset A^+$ from Lemma 5.9, the boundary value space $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ and the generalized Nevanlinna function

$$\tau(\lambda) := \begin{pmatrix} \lambda^{-k} & 0 \\ 0 & \lambda^{-l} \end{pmatrix}$$

the boundary value problem (5.19)-(5.20) can now be written in the form

$$(5.23) \qquad (A^+ - \lambda) f = 0, \qquad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \quad \hat{f} \in A^+.$$

By Corollary 5.8 the homogeneous boundary value problem (5.23) has a nontrivial solution for $\lambda \in \mathbb{C}\backslash(-\infty, 0]$ (and in a similar manner for $\lambda \in \mathbb{C}\backslash[0, \infty)$) if and only if λ is a generalized zero of the function

$$\lambda \mapsto M(\lambda) + \tau(\lambda) = \begin{pmatrix} \frac{1}{\sqrt{-\lambda}} + \lambda^{-k} & 0 \\ 0 & \lambda^{-l} - \sqrt{\lambda} \end{pmatrix}.$$

Here the k generalized zeros of the function $\lambda \mapsto \frac{1}{\sqrt{-\lambda}} + \lambda^{-k}$ are given by -1 if $k = 1$,

$$\Big\{ -1, \exp\big(\pm \tfrac{\pi i}{2k-1}\big) \; \exp\big(\pm \tfrac{5\pi i}{2k-1}\big), \exp\big(\pm \tfrac{9\pi i}{2k-1}\big), \ldots, \exp\big(\pm \tfrac{(2k-5)\pi i}{2k-1}\big) \Big\}$$

if k is odd and $k \geq 3$ and

$$\Big\{\exp\big(\pm\tfrac{\pi i}{2k-1}\big)\ \exp\big(\pm\tfrac{5\pi i}{2k-1}\big),\ \exp\big(\pm\tfrac{9\pi i}{2k-1}\big),\ \ldots,\ \exp\big(\pm\tfrac{(2k-3)\pi i}{2k-1}\big)\Big\}$$

if k is even. The l generalized zeros of the function $\lambda \mapsto \lambda^{-l} - \sqrt{\lambda}$ are

$$\Big\{1, \exp\big(\pm\tfrac{4i\pi}{2l+1}\big), \ldots, \exp\big(\pm\tfrac{4i\pi}{2l+1}\big(\tfrac{l-1}{2} - 1\big)\big), \exp\big(\pm\tfrac{4i\pi}{2l+1}\big(\tfrac{l-1}{2}\big)\big)\Big\}$$

if l is odd or we have $l+1$ generalized zeros

$$\Big\{1, \exp\big(\pm\tfrac{4i\pi}{2l+1}\big), \ldots, \exp\big(\pm\tfrac{4i\pi}{2l+1}\big(\tfrac{l}{2} - 1\big)\big), \exp\big(\pm\tfrac{4i\pi}{2l+1}\big(\tfrac{l}{2}\big)\big)\Big\}$$

if l is even. Since for $\beta = 1$ the limit in (3.1) equals $-l - \frac{1}{2}$ it follows that the eigenvalue 1 is of negative type.

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