ON THE SPECTRAL THEORY OF SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS

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Abstract. We consider a singular Sturm-Liouville differential expression with an indefinite weight function and we show that the corresponding self-adjoint differential operator in a Krein space locally has the same spectral properties as a definitizable operator.

1. Introduction

In this paper we investigate the spectral properties of a Sturm-Liouville operator associated to the differential expression

$$\frac{1}{r}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right), \qquad p^{-1},q,r\in L^1_{\mathrm{loc}}(\mathbb{R}). \tag{1}$$

In contrast to standart Sturm-Liouville theory we deal with the case where the weight function r changes its sign. If (1) is in the limit point case at both singular endpoints ∞ and $-\infty$ and the functions p,q,r are real, $r\neq 0$ a.e., then the usual maximal operator A associated to (1) is self-adjoint in the Krein space $(L^2_{|r|}(\mathbb{R}),[\cdot,\cdot])$, where the indefinite inner product is defined by

$$[f,g]:=\int_{\mathbb{R}} f(x)\overline{g(x)}\,r(x)\,dx, \qquad f,g\in L^2_{|r|}(\mathbb{R}).$$

Spectral problems for such singular indefinite differential operators have been considered in, e.g. [4, 7, 8, 9, 10, 11, 14, 20]. Under suitable assumptions on the indefinite weight r and the functions p and q the operator A turns out to be definitizable in the sense of H. Langer and the well developed spectral theory for these operators can be used for further investigations, see [21].

Here we are in interested in more general indefinite differential expressions of the form (1) and our main goal is to show that under suitable assumptions the self-adjoint differential operator A in $(L^2_{|r|}(\mathbb{R}),[\cdot,\cdot])$ at least locally has the same spectral properties as a definitizable operator. For this we assume that the weight function r is negative on an interval $(-\infty,a)$ and positive on an interval (b,∞). With the help of Glazmans decomposition method A can be regarded as a finite-dimensional perturbation in resolvent sense of the direct sum of three self-adjoint differential operators A_-, A_{ab} and A_+ which correspond to restrictions of (1) onto the intervals $(-\infty,a)$, (a,b) and (b,∞) and are subject to suitable boundary conditions. The singular differential operators A_+ and A_- act in Hilbert spaces (or anti-Hilbert spaces) and A_{ab} is a regular indefinite Sturm-Liouville expression which is known to be definitizable, cf. [7]. Under the assumption that A_+ and A_- are semibounded the direct sum of A_-, A_{ab} and A_+ becomes a locally definitizable operator. Making use of a recent perturbation result from [2] we show in Theorem 3.2 that A is also locally definitizable and the region of definitizability will be expressed in terms of the essential spectra of A_+ and A_-. A typical difficulty is to ensure that the resolvent set $\rho(A)$ of A is nonempty; here we will impose a (rather weak) condition on the absolutely continuous spectrum of one of the singular differential operators A_+ or A_- and make use of Titchmarsh-Weyl theory.

The paper is organized as follows. In Section 2 we briefly recall the definitions and some important properties of definitizable and locally definitizable self-adjoint operators. Furthermore we provide the reader with a very short introduction into extension and spectral theory of symmetric and self-adjoint operators in Krein spaces with the help of boundary triples and Weyl functions. Section 3 is devoted to the analysis of the spectral properties of the self-adjoint operator A associated to (1) and contains our main result on local definitizability of A.

2. Locally definitizable self-adjoint operators in Krein spaces

We briefly recall the definitions and basic properties of definitizable and locally definitizable self-adjoint operators in Krein spaces. For a detailed exposition we refer the reader to the fundamental papers [18, 21].

Let in the following $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let A be a self-adjoint operator in $(\mathcal{K}, [\cdot,\cdot])$. A point $\lambda \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{ap}(A)$ of A if there exists a sequence $(x_n) \subset \operatorname{dom} A$ with $\|x_n\| = 1$, $n = 1, 2, \dots$, and $\|(A-\lambda)x_n\| \to 0$ if $n \to \infty$. If $\lambda \in \sigma_{ap}(A)$ and each sequence $(x_n) \subset \operatorname{dom} A$ with $\|x_n\| = 1$, $n = 1, 2, \dots$, and $\|(A-\lambda)x_n\| \to 0$ for $n \to \infty$, satisfies

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \big(\limsup_{n\to\infty} [x_n, x_n] < 0\big),$$

then λ is called a *spectral point of positive* (*resp. negative*) *type of* A, cf. [18, 22]. The self-adjointness of A implies that the spectral points of positive and negative type are real. An open set $\Delta \subset \mathbb{R}$ is said to be of *positive* (*negative*) *type* with respect to A if $\Delta \cap \sigma(A)$ consists of spectral points of positive (resp. negative) type. We say that $\Delta \subset \mathbb{R}$ is of *definite type* with respect to A if Δ is either of positive or negative type with respect to A. The following definition can be found in a more general form in, e.g. [17].

Definition 2.1. *Let $I \subset \mathbb{R}$ be a closed interval and let A be a self-adjoint operator in $(\mathcal{K}, [\cdot,\cdot])$ such $\sigma(A) \cap (\mathbb{C}\backslash\mathbb{R})$ consists of isolated points which are poles of the resolvent of A, and no point of $\overline{\mathbb{R}}\backslash I$ is an accumulation point of the non-real spectrum of A. Then A is said to be* definitizable over $\overline{\mathbb{C}}\backslash I$, *if the following holds.*

- (i) *Every point $\mu \in \overline{\mathbb{R}}\backslash I$ has an open connected neighborhood $\mathcal{U}_\mu$ in $\overline{\mathbb{R}}$ such that both components of $\mathcal{U}_\mu\backslash\{\mu\}$ are of definite type with respect to A.*
- (ii) *For every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \overline{\mathbb{R}}\backslash I$, there exists $m \geq 1$, $M > 0$ and an open neighborhood $\mathcal{O}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that*

$$\|(A-\lambda)^{-1}\| \leq M(1+|\lambda|)^{2m-2}\,|\operatorname{Im}\lambda|^{-m} \quad \textit{for } \lambda \in \mathcal{O}\backslash\overline{\mathbb{R}}.$$

If A is a self-adjoint operator in $\mathcal{K}$ such that $\sigma(A) \cap (\mathbb{C}\backslash\mathbb{R})$ consists of at most finitely many poles of the resolvent of A, and (i) and (ii) in Definition 2.1 hold with $\overline{\mathbb{R}}\backslash I$ replaced by $\overline{\mathbb{R}}$, then A is said to be *definitizable*. This is equivalent to the fact that there exists a polynomial p such that $[p(A)x, x] \geq 0$ holds for all $x \in \operatorname{dom} p(A)$ and $\rho(A) \neq \emptyset$, cf. [18, Theorem 4.7] and [21]. Roughly speaking, a self-adjoint operator A which is locally definitizable over $\overline{\mathbb{C}}\backslash I$ can be regarded as the direct sum of a definitizable operator and an operator with spectrum in a neighborhood of I, see [18, Theorem 4.8].

Let $I \subset \mathbb{R}$ be a closed interval and let A be a self-adjoint operator in $\mathcal{K}$ which is definitizable over $\overline{\mathbb{C}}\backslash I$. Then A possesses a local spectral function $\delta \mapsto E(\delta)$ on $\overline{\mathbb{R}}\backslash I$ which is defined for all finite unions δ of connected subsets of $\overline{\mathbb{R}}\backslash I$ the endpoints of which belong to $\overline{\mathbb{R}}\backslash I$ and are of definite type, see [18, Section 3.4 and Remark 4.9]. We note that an open set $\Delta \subset \mathbb{R}\backslash I$ is of positive (negative) type with respect to A if and only if for every finite union δ of open intervals, $\overline{\delta} \subset \Delta$, such that the boundary points of δ in $\mathbb{R}$ are of definite type, the spectral subspace $(E(\delta)\mathcal{K}, [\cdot,\cdot])$ (resp. $(E(\delta)\mathcal{K}, -[\cdot,\cdot])$) is a Hilbert space. As a generalization of open sets of positive and negative type we introduce open sets of type π_+ and type π_- in the next definition, cf. [17].

Definition 2.2. *Let $I \subset \mathbb{R}$ be a closed interval and let A be a self-adjoint operator in $\mathcal{K}$ which is definitizable over $\overline{\mathbb{C}}\backslash I$ An open set $\Delta \subset \mathbb{R}\backslash I$ is called of* type π_+ *(*type π_-*) with respect to A if for every finite union δ of open intervals, $\overline{\delta} \subset \Delta$, such that the boundary points of δ in $\mathbb{R}$ are of definite type, the spectral subspace $(E(\delta)\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (resp. positivity).*

We remark that spectral points in sets of type π_+ and type π_- can also be characterized with the help of approximative eigensequences, see [1].

In the proof of our main result in the next section locally definitizable operators will arise as self-adjoint extensions of a symmetric operator. We use the notion of so-called boundary triples and associated Weyl functions for the description of the closed extensions of a symmetric operator in a Krein space, see [11] and, e.g. [12].

Definition 2.3. *Let S be a densely defined closed symmetric operator in $(\mathcal{K}, [\cdot,\cdot])$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a* boundary triple *for the adjoint operator S^+, if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} S^+ \to \mathcal{G}$ are linear mappings such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : \operatorname{dom} S^+ \to \mathcal{G}^2$ is surjective, and the "abstract Lagrange identity"*

$$[S^+f, g] - [f, S^+g] = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g)$$

holds for all $f, g \in \operatorname{dom} S^+$.

Let S be a densely defined closed symmetric operator in $\mathcal{K}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triple for S^+. Then $A_0 := S^+ \upharpoonright \ker \Gamma_0$ and $A_1 := S^+ \upharpoonright \ker \Gamma_1$ are self-adjoint extensions of S in $\mathcal{K}$. Furthermore, if Θ is self-adjoint in $\mathcal{G}$, then the extension $A_\Theta := S^+ \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$ is a self-adjoint operator in the Krein space $\mathcal{K}$.

Assume that the self-adjoint operator $A_0 = S^+ \upharpoonright \ker \Gamma_0$ has a nonempty resolvent set. Then for each $\lambda \in \rho(A_0)$ we have $\operatorname{dom} S^+ = \operatorname{dom} A_0 \dot{+} \ker(S^+ - \lambda)$ and hence the operator

$$M(\lambda) = \Gamma_1 \big(\Gamma_0 \upharpoonright \ker(S^+ - \lambda)\big)^{-1} \in \mathcal{L}(\mathcal{G})$$

is well-defined. Here $\mathcal{L}(\mathcal{G})$ denotes the space of everywhere defined bounded linear operators in $\mathcal{G}$. The $\mathcal{L}(\mathcal{G})$-valued function $\lambda \mapsto M(\lambda)$ is called the *Weyl function* of the boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$, M is holomorphic on $\rho(A_0)$ and symmetric with respect to the real axis, i.e. $M(\overline{\lambda}) = M(\lambda)^*$ holds for all $\lambda \in \rho(A_0)$. The Weyl function can be used to describe the spectral properties of the closed extensions of S, see [11] for details. We will later in particular use the fact that a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$, $A_\Theta = S^+ \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$, if and only if $0 \in \rho(M(\lambda) - \Theta)$.

Finally we remark that if $\mathcal{K}$ is a Hilbert space, S is a closed densely defined symmetric operator in $\mathcal{K}$ and M is the Weyl function of a boundary triple for the adjoint operator S^*, then M is a Nevanlinna function with the additional property $0 \in \rho(\operatorname{Im} M(\lambda))$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$, cf. [12].

3. Spectral properties of a class of singular indefinite Sturm-Liouville operators

In this section we investigate the spectral properties of an operator associated to the Sturm-Liouville differential expression

$$\frac{1}{r}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right), \tag{2}$$

where $p^{-1}, q, r \in L^1_{\text{loc}}(\mathbb{R})$ are assumed to be real valued functions such that $p > 0$ and $r \neq 0$ for a.e. $x \in \mathbb{R}$. Here we are interested in the case that the weight function r has different signs at ∞ and $-\infty$, more precisely, we will assume that the following condition (I) holds.

(I) There exist $a, b \in \mathbb{R}$, $a < b$, such that the restrictions $r_+ := r \upharpoonright (b, \infty)$ and $r_- := r \upharpoonright (-\infty, a)$ satisfy $r_+ > 0$ and $r_- < 0$ for a.e. $\lambda \in (-\infty, a) \cup (b, \infty)$.

We note that the case $r_+ < 0$ and $r_- > 0$ can be treated analogously and the case that r_+ and r_- have the same signs at ∞ and $-\infty$ is not of special interest to us, cf. Remark 3.3. In the following we agree to choose $a, b \in \mathbb{R}$ in such a way that the sets $\{x \in (a, b) \,|\, r(x) > 0\}$ and $\{x \in (a, b) \,|\, r(x) < 0\}$ have positive Lebesgue measure. This is no restriction.

Let $L^2_{|r|}(\mathbb{R})$ be the Hilbert space of all equivalence classes of measurable functions f defined on $\mathbb{R}$ for which $\int_{\mathbb{R}} |f|^2 |r|$ is finite. We equip $L^2_{|r|}(\mathbb{R})$ with the inner product

$$[f, g] := \int_{\mathbb{R}} f(x)\overline{g(x)}r(x)dx, \quad f, g \in L^2_{|r|}(\mathbb{R}),$$

and denote the corresponding Krein space $(L^2_{|r|}(\mathbb{R}), [\cdot,\cdot])$ by $L^2_r(\mathbb{R})$. As a fundamental symmetry in $L^2_r(\mathbb{R})$ we choose $(Jf)(x) := (\operatorname{sgn} r(x))f(x)$, $f \in L^2_r(\mathbb{R})$, then $[J\cdot,\cdot]$ coincides with the usual Hilbert scalar product $(f, g) = \int_{\mathbb{R}} f\overline{g}|r|$ on $L^2_{|r|}(\mathbb{R})$.

Let us assume that the Sturm-Liouville differential expression

$$\ell := \frac{1}{|r|}\left(-\frac{d}{dx}\left(p\frac{d}{dx}\right)+q\right) \tag{3}$$

is in the limit point case at both singular endpoints ∞ and $-\infty$. Then it is well-known that the operator $By = \ell(y)$ defined on the usual maximal domain

$$\mathcal{D}_{\max} = \left\{y \in L^2_{|r|}(\mathbb{R}) : y, py' \in AC_{\text{loc}}(\mathbb{R}),\ \ell(y) \in L^2_{|r|}(\mathbb{R})\right\}, \tag{4}$$

is self-adjoint in the Hilbert space $L^2_{|r|}(\mathbb{R})$, see e.g. [13, 23, 24, 25].

In the following we are interested in the spectral properties of the indefinite Sturm-Liouville operator

$$Ay := JBy = \frac{1}{r}\big(-(py')' + qy\big), \qquad \operatorname{dom} A = \operatorname{dom} JB = \mathcal{D}_{\max}, \tag{5}$$

which is self-adjoint in the Krein space $L^2_r(\mathbb{R})$. We shall interpret the operator A as a finite rank perturbation in resolvent sense of the direct sum of three differential operators A_-, A_{ab} and A_+ defined in the sequel. Let us denote the restrictions of p and q onto the intervals $(-\infty, a)$ and (b, ∞) by p_-, p_+ and q_-, q_+, respectively.

Moreover we denote the restriction of r, p and q onto the finite interval (a, b) by r_{ab}, p_{ab} and q_{ab}. Besides the differential expression ℓ in (3) we shall deal with the differential expressions ℓ_-, ℓ_+ and ℓ_{ab} defined by

$$\ell_- := \frac{1}{-r_-}\left(\frac{d}{dx}\left(p_-\frac{d}{dx}\right) - q_-\right), \quad \ell_+ := \frac{1}{r_+}\left(-\frac{d}{dx}\left(p_+\frac{d}{dx}\right) + q_+\right), \tag{6}$$

and

$$\ell_{ab} := \frac{1}{|r_{ab}|}\left(-\frac{d}{dx}\left(p_{ab}\frac{d}{dx}\right) + q_{ab}\right), \tag{7}$$

respectively, and operators associated to them. Note that ℓ_+ and ℓ_- are in the limit point case at ∞ and $-\infty$ and regular at the endpoints b and a, respectively, whereas ℓ_{ab} is regular at both endpoints a and b. By $\mathcal{D}_{\max,+}$ ($\mathcal{D}_{\max,-}$ and $\mathcal{D}_{\max,ab}$) we denote the set in (4) if r, $\mathbb{R}$ and ℓ are replaced by r_+, (b, ∞) and ℓ_+ (resp. r_-, $(-\infty, a)$, ℓ_- and r_{ab}, (a, b), ℓ_{ab}). We shall in particular make use of the differential operators

$$\begin{aligned} A_+ g &= \ell_+(g), \qquad \operatorname{dom} A_+ = \{g \in \mathcal{D}_{\max,+} : g(b) = 0\}, \\ A_- f &= \ell_-(f), \qquad \operatorname{dom} A_- = \{f \in \mathcal{D}_{\max,-} : f(a) = 0\}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} A_{ab} h &= \frac{1}{r_{ab}}\big(-(p_{ab}h')' + q_{ab}h\big), \\ \operatorname{dom} A_{ab} &= \{h \in \mathcal{D}_{\max,ab} : h(a) = h(b) = 0\}. \end{aligned}$$

Here A_+ is self-adjoint in the Hilbert space $L^2_{|r_+|}((b, \infty)) = L^2_{r_+}((b, \infty))$ and A_- is self-adjoint in the Hilbert space $L^2_{|r_-|}((-\infty, a)) = L^2_{-r_-}((-\infty, a))$ as well as in the anti-Hilbert space $L^2_{r_-}((-\infty, a)) = (L^2_{|r_-|}((-\infty, a)), -(\cdot, \cdot))$. Moreover A_{ab} is self-adjoint in the Krein space $L^2_{r_{ab}}((a, b))$ and the spectrum $\sigma(A_{ab})$ is discrete and consists of eigenvalues of multiplicity one which accumulate to ∞ and $-\infty$, see [7, Propositions 1.8 and 2.2].

Besides condition (I) we will assume that the following condition (II) is satisfied.

(II) The operator A_+ is semibounded from below and the operator A_- is semibounded from above.

Sufficient criteria on r_+, p_+ and q_+ or r_-, p_- and q_- such that A_+ or A_- are semibounded can be found in, e.g. [13, 23, 24, 25] (see also Corollary 3.4), and the essential spectra $\sigma_{\text{ess}}(A_+)$ and $\sigma_{\text{ess}}(A_-)$ can be described. The next lemma states that condition (II) is independent of the choice of the finite interval (a, b) and the self-adjoint realizations A_+ and A_- in (8).

Lemma 3.1. *Let a, b and A_+, A_- be as above and assume that $\tilde{a}, \tilde{b} \in \mathbb{R}$ satisfy condition* (I). *Let $\tilde{\ell}_+$ and $\tilde{\ell}_-$ be the differential expressions on $(\tilde{b}, \infty)$ and $(-\infty, \tilde{a})$ defined analogously to ℓ_+ and ℓ_- in* (6), *and let $\widetilde{A}_+$ and $\widetilde{A}_-$ be arbitrary self-adjoint realizations of $\tilde{\ell}_+$ and $\tilde{\ell}_-$ in the Hilbert spaces $L^2_{\tilde{r}_+}((\tilde{b}, \infty))$ and $L^2_{-\tilde{r}_-}((-\infty, \tilde{a}))$, respectively. Then $\widetilde{A}_+$ is semibounded from below, $\widetilde{A}_-$ is semibounded from above and we have*

$$\sigma_{\text{ess}}(A_+) = \sigma_{\text{ess}}(\widetilde{A}_+) \quad \textit{and} \quad \sigma_{\text{ess}}(A_-) = \sigma_{\text{ess}}(\widetilde{A}_-).$$

Proof. We make use of Glazmans decomposition method (see [16]) and consider only the endpoints $b, \tilde b$ for the case $\tilde b < b$. Let $\ell_{\tilde b b}$ be the restriction of the differential expression ℓ onto $(\tilde b, b)$ defined analogously to (7). Then $r_{\tilde b b} = r \upharpoonright (\tilde b, b) > 0$ a.e. and as $\ell_{\tilde b b}$ is regular and $p \upharpoonright (\tilde b, b) > 0$ it follows that any self-adjoint realization $A_{\tilde b b}$ of $\ell_{\tilde b b}$ in the Hilbert space $L^2_{r_{\tilde b b}}((\tilde b, b))$ is semibounded from below and $\sigma(A_{\tilde b b})$ consists of eigenvalues with ∞ as only accumulation point. The operators $\widetilde A_+$ and $A_{\tilde b b} \times A_+$ are self-adjoint extensions of the direct sum $S_{\tilde b b} \times S_+$ of the minimal operators associated to $\ell_{\tilde b b}$ and ℓ_+. Since the deficiency indices of $S_{\tilde b b} \times S_+$ are finite the resolvents of $\widetilde A_+$ and $A_{\tilde b b} \times A_+$ differ by a finite rank operator and hence well-known perturbation results imply that $\widetilde A_+$ is also semibounded from below and $\sigma_{\rm ess}(\widetilde A_+) = \sigma_{\rm ess}(A_+)$. □

Assume that condition (II) holds and let $\eta_+, \eta_- \in \mathbb{R}$ be lower and upper bounds for the essential spectra of A_+ and A_-, respectively, that is

$$\sigma_{\rm ess}(A_+) \subseteq [\eta_+, \infty) \quad \text{and} \quad \sigma_{\rm ess}(A_-) \subseteq (-\infty, \eta_-]. \tag{9}$$

The following theorem is the main result in our paper. In terms of the bounds η_+ and η_- of the essential spectra of A_+ and A_- we characterize the regions where the indefinite Sturm-Liouville operator A from (5) is definitizable. In order to ensure that the resolvent set $\rho(A)$ of A is nonempty we assume that there exists a point in the absolutely continuous spectrum σ_{ac} of A_+ or A_- which is an eigenvalue of A_{ab}. The emphasis in Theorem 3.2 is on cases (i) and (ii) where the essential spectra of A_+ and A_- overlap, i.e. $\eta_+ \le \eta_-$. If the essential spectra of A_+ and A_- are separated (case (iii)), then $\rho(A)$ is automatically nonempty and A turns out to be definitizable (over $\overline{\mathbb{C}}$). This can also be deduced from [7] (cf. [5, Proposition 6.2]).

Theorem 3.2. *Let A be the self-adjoint indefinite Sturm-Liouville operator in the Krein space $L^2_r(\mathbb{R})$ from (5) and assume that conditions (I) and (II) are satisfied. Choose $\eta_+, \eta_- \in \mathbb{R}$ as in (9) and suppose that there exists a point $\mu \in \sigma(A_{ab})$, such that*

$$\mu \in \sigma_{\rm ac}(A_-) \cap \rho(A_+) \quad \textit{or} \quad \mu \in \sigma_{\rm ac}(A_+) \cap \rho(A_-).$$

Then the following holds.

(i) *If $\eta_+ < \eta_-$, then A is definitizable over $\overline{\mathbb{C}} \setminus [\eta_+, \eta_-]$.*

(ii) *If $\eta_+ = \eta_-$, then A is definitizable over $\overline{\mathbb{C}} \setminus \{\eta_+\}$. If, in addition,*

$$\sigma_p(A_+) \cap (\eta_+ - \varepsilon, \eta_+) = \emptyset \quad \textit{and} \quad \sigma_p(A_-) \cap (\eta_-, \eta_- + \varepsilon) = \emptyset$$

for some $\varepsilon > 0$, then A is definitizable.

(iii) *If $\eta_- < \eta_+$, then A is definitizable and $\sigma(A) \cap (\eta_-, \eta_+)$ consists of eigenvalues of A with η_+ and η_- as only possible accumulation points.*

Furthermore, the interval $(-\infty, \eta_+)$ is of type π_+ with respect to A and the interval (η_-, ∞) is of type π_- with respect to A.

Proof. The regular indefinite Sturm-Liouville operator

$$S_{ab}h = \frac{1}{r_{ab}}\big(-(p_{ab}h')' + q_{ab}h\big),$$
$$\operatorname{dom} S_{ab} = \big\{h \in \mathcal{D}_{\max,ab} : h(a) = h(b) = (p_{ab}h')(a) = (p_{ab}h')(b) = 0\big\},$$

is a densely defined closed symmetric operator in the Krein space $L^2_{r_{ab}}((a,b))$ and has defect two, its adjoint S_{ab}^+ is given by

$$S_{ab}^+ h = \frac{1}{r_{ab}}\big(-(p_{ab}h')' + q_{ab}h\big), \qquad \operatorname{dom} S_{ab}^+ = \mathcal{D}_{\max,ab}.$$

We leave it to the reader to check that $\{\mathbb{C}^2, \Gamma_0^{ab}, \Gamma_1^{ab}\}$, where

$$\Gamma_0^{ab} h = \begin{pmatrix} -(p_{ab}h')(a) \\ (p_{ab}h')(b) \end{pmatrix} \quad \text{and} \quad \Gamma_1^{ab} h = \begin{pmatrix} h(a) \\ h(b) \end{pmatrix},$$

is a boundary triple for S_{ab}^+. Note that the self-adjoint operator A_{ab} coincides with $A_{ab,1} = S_{ab}^+ \upharpoonright \ker \Gamma_1$ and that the Weyl function m_{ab} of $\{\mathbb{C}^2, \Gamma_0^{ab}, \Gamma_1^{ab}\}$ is a two-by-two matrix-valued holomorphic function on $\mathbb{C}\backslash\sigma_p(A_{ab,0})$, where $A_{ab,0} = S_{ab}^+ \upharpoonright \ker \Gamma_0$. Let $\varphi_\lambda, \psi_\lambda \in L^2_{r_{ab}}((a,b))$ be the fundamental solutions of $-(p_{ab}h')' + q_{ab}h = \lambda r_{ab} h$, $\lambda \in \mathbb{C}$, satisfying the boundary conditions

$$\varphi_\lambda(a) = 1,\ (p_{ab}\varphi_\lambda')(a) = 0 \quad \text{and} \quad \psi_\lambda(a) = 0,\ (p_{ab}\psi_\lambda')(a) = 1.$$

Since $\ker(S_{ab}^+ - \lambda) = \operatorname{sp}\{\varphi_\lambda, \psi_\lambda\}$ and $x \mapsto \varphi_\lambda(x)(p_{ab}\psi_\lambda')(x) - (p_{ab}\varphi_\lambda')(x)\psi_\lambda(x)$ has the constant value 1 we find that the Weyl function m_{ab} is given by

$$m_{ab}(\lambda) = \frac{1}{(p_{ab}\varphi_\lambda')(b)} \begin{pmatrix} (p_{ab}\psi_\lambda')(b) & 1 \\ 1 & \varphi_\lambda(b) \end{pmatrix}.$$

Next we define the singular Sturm-Liouville operators

$$\begin{aligned} S_- f &= \ell_-(f), \quad \operatorname{dom} S_- = \big\{ f \in \mathcal{D}_{\max,-} : f(a) = (p_- f')(a) = 0 \big\}, \\ S_+ g &= \ell_+(g), \quad \operatorname{dom} S_+ = \big\{ g \in \mathcal{D}_{\max,+} : g(b) = (p_+ g')(b) = 0 \big\}, \end{aligned}$$

which are closed densely defined symmetric operators of defect one in the Hilbert spaces $L^2_{-r_-}((-\infty,a))$ and $L^2_{r_+}((b,\infty))$, respectively. We will regard S_- in the following as a symmetric operator in the anti-Hilbert space $L^2_{r_-}((-\infty,a)) = (L^2_{-r_-}((-\infty,a)), -(\cdot,\cdot))$. Then $\{\mathbb{C}, \Gamma_{0,-}, \Gamma_{1,-}\}$, where

$$\Gamma_{0,-} f := f(a), \qquad \Gamma_{1,-} f := -(p_- f')(a), \quad f \in \operatorname{dom} S_-^+ = \mathcal{D}_{\max,-},$$

is a boundary triple for the adjoint $S_-^+ f = \ell_-(f)$ in $L^2_{r_-}((-\infty,a))$ and $\{\mathbb{C}, \Gamma_{0,+}, \Gamma_{1,+}\}$,

$$\Gamma_{0,+} g := g(b), \qquad \Gamma_{1,+} b := (p_+ g')(b), \quad g \in \operatorname{dom} S_+^* = \mathcal{D}_{\max,+},$$

is a boundary triple for the adjoint $S_+^*(g) = \ell_+(g)$ in $L^2_{r_+}((b,\infty))$. The Weyl functions corresponding to $\{\mathbb{C}, \Gamma_{0,-}, \Gamma_{1,-}\}$ and $\{\mathbb{C}, \Gamma_{0,+}, \Gamma_{1,+}\}$ will be denoted by m_- and m_+. Note that m_+ and $-m_-$ are scalar Nevanlinna functions holomorphic on $\rho(A_+)$ and $\rho(A_-)$, respectively, so that for $\lambda \in \mathbb{C}^+$ we have $\operatorname{Im} m_+(\lambda) > 0$ and $\operatorname{Im} m_-(\lambda) < 0$.

The operator $S_- \times S_+ \times S_{ab}$ is a closed densely defined symmetric operator of defect 4 in the Krein space $L^2_{r_-}((-\infty,a))[\dot{+}]L^2_{r_+}((b,\infty))[\dot{+}]L^2_{r_{ab}}((a,b))$ and it is straightforward to check that $\{\mathbb{C}^4, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where

$$\widetilde{\Gamma}_0\{f,g,h\} := \begin{pmatrix} \Gamma_{0,-} f \\ \Gamma_{0,+} g \\ \Gamma_0^{ab} h \end{pmatrix}, \quad \widetilde{\Gamma}_1\{f,g,h\} := \begin{pmatrix} \Gamma_{1,-} f \\ \Gamma_{1,+} g \\ \Gamma_1^{ab} h \end{pmatrix},$$

$\{f,g,h\} \in \operatorname{dom} S_-^+ \times \operatorname{dom} S_+^* \times \operatorname{dom} S_{ab}$ is a boundary triple for the adjoint operator $S_-^+ \times S_+^* \times S_{ab}^+$. Note that $\mathbb{C}\backslash\mathbb{R}$ belongs to the resolvent set of the self-adjoint operators $S_-^+ \times S_+^* \times S_{ab}^+ \upharpoonright \ker \widetilde{\Gamma}_i,\ i = 0,1$. The Weyl function corresponding to $\{\mathbb{C}^4, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is given by

$$M(\lambda) = \begin{pmatrix} m_-(\lambda) & 0 & 0 & 0 \\ 0 & m_+(\lambda) & 0 & 0 \\ 0 & 0 & \frac{(p_{ab}\psi_\lambda')(b)}{(p_{ab}\varphi_\lambda')(b)} & \frac{1}{(p_{ab}\varphi_\lambda')(b)} \\ 0 & 0 & \frac{1}{(p_{ab}\varphi_\lambda')(b)} & \frac{\varphi_\lambda(b)}{(p_{ab}\varphi_\lambda')(b)} \end{pmatrix}, \quad \lambda \in \rho(A_-)\cap\rho(A_+)\cap\rho(A_{ab,0}).$$

If we identify $L^2_{r_-}((-\infty,a))[\dot{+}]L^2_{r_+}((b,\infty))[\dot{+}]L^2_{r_{ab}}((a,b))$ with the Krein space $L^2_r(\mathbb{R})$ then the self-adjoint operator $S_-^+ \times S_+^* \times S_{ab}^+ \upharpoonright \ker(\widetilde{\Gamma}_1 - \widetilde{\Theta}\widetilde{\Gamma}_0)$, where

$$\widetilde{\Theta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

coincides with the self-adjoint operator A from (5). In fact, an element $\{f,g,h\} \in \operatorname{dom} S_-^+ \times \operatorname{dom} S_+^* \times \operatorname{dom} S_{ab}$ belongs to $\ker(\widetilde{\Gamma}_1 - \widetilde{\Theta}\widetilde{\Gamma}_0)$ if and only if

$$f(a) = h(a), \quad (p_- f')(a) = (p_{ab}h')(a)$$

and

$$g(b) = h(b), \quad (p_+ g')(b) = (p_{ab}h')(b)$$

holds, that is, $\{f,g,h\} \in \mathcal{D}_{\max} = \operatorname{dom} A$.

We claim that $\rho(A)$ is nonempty. For this it suffices to show that $\ker(\widetilde{M}(\lambda) - \widetilde{\Theta})$ is nontrivial for some $\lambda \in \mathbb{C}\backslash\mathbb{R}$, see the end of Section 2. Assume that

$$\begin{aligned} \det(\widetilde{M}(\lambda) - \widetilde{\Theta}) =& m_-(\lambda)\left(m_+(\lambda)\det m_{ab}(\lambda) - \frac{(p_{ab}\psi_\lambda')(b)}{(p_{ab}\varphi_\lambda')(b)}\right) \\ & - m_+(\lambda)\frac{\varphi_\lambda(b)}{(p_{ab}\varphi_\lambda')(b)} + 1 \end{aligned}$$

vanishes identically for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Let $\mu \in \sigma_p(A_{ab})$ be a real point as in the assumptions of the theorem and assume e.g. that $\mu \in \sigma_{\mathrm{ac}}(A_-) \cap \rho(A_+)$ holds. Then the functions m_{ab} and m_+ are holomorphic in an open neighborhood $\mathcal{O}_\mu$ of μ and take real values in $\mathcal{O}_\mu \cap \mathbb{R}$, since $A_{ab} = S_{ab}^+ \upharpoonright \ker \Gamma_1$ and $\mu \in \rho(A_+)$. By standart Titchmarsh-Weyl theory the limit $m_-(\lambda + i0) = \lim_{\delta\to+0} m_-(\lambda + i\delta)$ from the upper half-plane exists for a.e. $\lambda \in \mathbb{R}$ and by [6, Proposition 4.2] (see also [15]) the Lebesgue measure of the set

$$(\mu - \varepsilon, \mu + \varepsilon) \cap \{x \in \mathbb{R} : \operatorname{Im} m_-(\lambda + i0) < 0\}$$

is positive for every $\varepsilon > 0$. As the imaginary part of $\det(\widetilde{M}(\lambda) - \widetilde{\Theta})$ vanishes for each $\lambda \in \mathbb{C}\backslash\mathbb{R}$ it follows that

$$m_+(\lambda) = \frac{(p_{ab}\psi_\lambda')(b)}{(p_{ab}\varphi_\lambda')(b)\det m_{ab}(\lambda)} = \frac{(p_{ab}\psi_\lambda')(b)}{\psi_\lambda(b)}$$

holds for all real λ in a neighborhood of μ, $\lambda \neq \mu$, with $\operatorname{Im} m_-(\lambda + i0) < 0$. But the expression on the right hand side has a pole at μ, which contradicts the holomorphy of m_+. Therefore $\rho(A) \neq \emptyset$ holds.

The operator $A' := A_- \times A_{ab} \times A_+$ is self-adjoint in the Krein space $L^2_r(\mathbb{R})$ and $\mathbb{C}\backslash\mathbb{R}$ belongs to $\rho(A')$. Here we regard A_- as a self-adjoint operator in the anti-Hilbert space $L^2_{r_-}((-\infty,a))$. Since $\rho(A)\cap\rho(A')\neq\emptyset$ and both A and A' are self-adjoint extensions of a symmetric operator of defect 4 we conclude

$$\dim\bigl(\operatorname{ran}\left((A-\lambda)^{-1}-(A'-\lambda)^{-1}\right)\bigr)\leq 4,\qquad \lambda\in\rho(A)\cap\rho(A'). \tag{10}$$

The interval $(-\infty,\eta_+)$ consists of eigenvalues of A_+ with η_+ as only possible accumulation point and each point in $\sigma(A_+)$ is a spectral point of positive type. By [7] A_{ab} is a definitizable operator with the additional property that the hermitian form $[A_{ab}\cdot,\cdot]$ has a finite number of negative squares. Therefore the eigenvalues of A_{ab} in $(-\infty,\eta_+)$ are, with the exception of finitely many, of negative type in the Krein space $L^2_{r_{ab}}((a,b))$. Moreover $\sigma(A_-)$ consists only of negative spectral points and this implies that the interval $(-\infty,\eta_+)$ is of type π_- with respect to A' and that for some ν, $-\infty<\nu<\eta_+$, the interval $(-\infty,\nu)$ is of negative type with respect to A'. A similar argument shows that (η_-,∞) is of type π_+ with respect to A' and that for some ζ, $\eta_-<\zeta<\infty$, the interval (ζ,∞) is of positive type with respect to A'.

Therefore, if e.g. $\eta_+<\eta_-$, then A' is definitizable over $\overline{\mathbb{C}}\backslash[\eta_+,\eta_-]$ and [2, Theorem 2.2] on finite rank perturbations of locally definitizable operators together with (10) implies that the indefinite Sturm-Liouville operator A is definitizable over $\overline{\mathbb{C}}\backslash[\eta_+,\eta_-]$. This proves assertion (i). An analogous argument proves the first assertion in (ii). Note that under the additional conditions $\sigma_p(A_+)\cap(\eta_+-\varepsilon,\eta_+)=\emptyset$ and $\sigma_p(A_-)\cap(\eta_-,\eta_-+\epsilon)=\emptyset$ the operator A' is definitizable and hence so is A, cf. [19, Theorem 1]. Assertion (iii) can be deduced from [7] or follows in a similar manner as (i) and (ii), here it is again sufficient to use the result on finite rank perturbations of definitizable operators from [19]. Finally, since $(-\infty,\eta_+)$ $((\eta_-,\infty))$ is of type π_- (resp. type π_+) with respect to A' it follows from [1, 3] (see also [2, Theorem 2.1]) and (10) that the interval $(-\infty,\eta_+)$ $((\eta_-,\infty))$ is also of type π_- (resp. type π_+) with respect to A. □

Remark 3.3. *We note that if condition* (I) *is replaced by an analogous condition where $r\restriction(-\infty,a)$ is positive, $r\restriction(b,\infty)$ is negative and η_+ is defined to be the lower bound of $\sigma_{\text{ess}}(A_-)$ and η_- is defined to be the upper bound of $\sigma_{\text{ess}}(A_+)$, then the statements in Theorem* 3.2 *remain true. The case that r has the same sign on $(-\infty,a)$ and (b,∞) leads automatically to a definitizable operator A.*

In the next corollary we impose some extra conditions on r, p and q such that conditions (I) and (II) are met and

$$(\eta_+,\infty)\subset\sigma_{\text{ac}}(A_+)\quad\text{and}\quad(-\infty,\eta_-)\subset\sigma_{\text{ac}}(A_-)$$

hold (see [25, Satz 14.25]), that is, the assumptions in Theorem 3.2 are fulfilled.

Corollary 3.4. *Let $r(x)=\operatorname{sgn}x$ and $p(x)=1$ for $x\in(-\infty,a)\cup(b,\infty)$ and some $a,b\in\mathbb{R}$, $a\leq 0\leq b$. Suppose that the limits*

$$q_\infty:=\lim_{x\to\infty}q_+(x)\quad\textit{and}\quad q_{-\infty}:=\lim_{x\to-\infty}q_-(x)$$

exist and that the functions $x\mapsto q_+(x)-q_\infty$ and $x\mapsto q_-(x)-q_{-\infty}$ belong to $L^1((b,\infty))$ and $L^1((-\infty,a))$, respectively. Then the statements (i)-(iii) *in Theorem* 3.2 *hold with $\eta_+=q_\infty$ and $\eta_-=-q_{-\infty}$.*

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