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Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems.

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Abstract

We consider the normwise condition number and backward error of eigenvalues of matrix polynomials having $\star$-palindromic/antipalindromic and $\star$-even/odd structure with respect to structure preserving perturbations. Here $\star$ denotes either the transpose T or the conjugate transpose $*$. We show that when the polynomials are complex and $\star$ denotes complex conjugate, then to each of the structures there correspond portions of the complex plane so that simple eigenvalues of the polynomials lying in those portions have the same normwise condition number when subjected to both arbitrary and structure preserving perturbations. Similarly approximate eigenvalues of these polynomials belonging to such portions have the same backward error with respect to both structure preserving and arbitrary perturbations. Identical results hold when $*$ is replaced by the adjoint with respect to any sesquilinear scalar product induced by a Hermitian or skew-Hermitian unitary matrix. The eigenvalue symmetry of T-palindromic or T-antipalindromic polynomials, is with respect to the numbers 1 or -1 while that of T-even or T-odd polynomials is with respect to the origin. We show that except under certain conditions when 1, -1 and 0 are always eigenvalues of these polynomials, in all other cases their structured and unstructured condition numbers as simple eigenvalues of the corresponding polynomials are equal. The structured and unstructured backward error of these numbers as approximate eigenvalues of the corresponding polynomials are also shown to be equal. These results easily extend to the case when T is replaced by the transpose with respect to any bilinear scalar product that is induced by a symmetric or skew symmetric orthogonal matrix. In all cases the proofs provide appropriate structure preserving perturbations to the polynomials.

Keywords. Nonlinear eigenvalue problems, palindromic matrix polynomials, odd matrix polynomials, even matrix polynomials, structured eigenvalue condition number, structured backward error.

AMS subject classification. 65F15, 65F35, 65L15, 65L20, 15A18, 15A57

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1 Introduction

Given a matrix polynomial

$$P(\lambda) \;:=\; \Sigma_{k=0}^{m} \lambda^k A_k,\;\; A_1, A_2, \ldots, A_m \in \mathbb{F}^{n\times n} \tag{1}$$

where $\mathbb{F}$ denotes the field $\mathbb{C}$ or $\mathbb{R}$, the polynomial eigenvalue problem consists of finding a vector x and a scalar λ such that $P(\lambda)x = 0$. Such problems which arise frequently in the analysis and solution of higher order systems of differential equations find widespread application especially in the vibration analysis of machines, buildings and vehicles [7, 19, 40]. The classical approach towards solving such problems is to transform them into matrix pencils of the form $L(\lambda) = \lambda X + Y,\; X, Y \in \mathbb{F}^{mn\times mn}$, satisfying

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

where $E(\lambda)$ and $F(\lambda)$ are polynomials with constant non-zero determinants independent of λ [7]. The resulting generalized eigenvalue problem is then solved by using standard methods [8, 21, 25]. An appropriate choice of linearization is evidently very important for efficient and accurate computation of the eigenvalues. This topic has been dealt with in detail in [22, 24, 23, 14] and [12]. Very often matrix polynomials arising from various applications are attributed with additional structure which result in special symmetries in the distribution of their eigenvalues [24]. It is well known that numerical methods for solving structured eigenvalue problems that do not preserve their spectral symmetries may produce physically meaningless results in finite precision arithmetic [5, 40]. Moreover the use of structure preserving algorithms also results in greater efficiency as they require less computational time and can be more accurate ([8, 27, 41]). For an overview of various structure preserving algorithms see [18, 5] and references in them. These developments emphasize the need for a perturbation analysis of the structured eigenvalue problem that incorporates the effects of its structure. While structured perturbation analysis for eigenvalues of matrices and matrix pencils has been widely investigated (see for instance, [1, 3, 11, 17, 28, 34, 37] and references therein), much less is known about the effect of structured preserving perturbation on the eigenvalues of matrix polynomials ([39, 9]).

In this paper we consider real or complex polynomials of the form (1) having $\star$-palindromic, $\star$-antipalindromic, $\star$-even and $\star$-odd structure where $\star$ denotes the transpose T or the conjugate transpose $*$. The name 'palindromic' refers to a word or a phrase which remains unchanged upon writing the letters in the reverse order. In the context of polynomials, $\star$-palindromic structure implies that we get back the original polynomial on reversing the order of its coefficient matrices and applying the transpose $\star$. On the other hand $\star$-even polynomials are such that the original polynomial may be obtained upon replacing the coefficient matrices by their transposes and λ by $-\lambda$. We make more precise definitions of these polynomials in section 2 where we also display the associated spectral symmetries.

These polynomials came into focus in the work of Mackey, Mackey, Mehl and Mehrmann in [24] where they develop a systematic approach for constructing structure preserving linearizations of these polynomials and list some important applications. For instance, $*$-even polynomials arise in linear quadratic optimal control problems during the solution of the associated two point boundary value problems while $*$-palindromic polynomials arise in the context of solution of the discrete time optimal control problems. Complex T-palindromic polynomials arise in the vibration analysis of rail tracks excited by high speed trains [15, 16]. For some recent work on canonical forms of $\star$-palindromic polynomials, we refer to [35]. These developments underline the necessity of acquiring the tools to analyze the performance of any structure preserving algorithm that solves $\star$-palindromic/antipalindromic and $\star$-even/odd polynomial eigenvalue problems. With this aim in view we consider the normwise eigenvalue condition numbers and backward errors of approximate eigenvalues of these polynomials with respect to structure preserving perturbations.

The eigenvalues of complex $*$-palindromic/antipalindromic polynomials and $*$-even/odd polynomials are symmetrically placed with respect to the unit circle and the imaginary axis respectively. We show that simple eigenvalues of complex $*$-palindromic/antipalindromic polynomials lying on the unit circle have the same normwise condition number with respect to both structure preserving and arbitrary perturbations. The same also holds for simple eigenvalues of complex $*$-even/odd perturbations lying on the imaginary axis. We also show that structured and unstructured backward errors of all approximate eigenvalues of complex $*$-palindromic/antipalindromic polynomials lying on the unit circle are equal. The same is also true of approximate eigenvalues of complex $*$-even/odd polynomials lying on the imaginary axis. These results easily extend to the case when $*$ is replaced by the adjoint with respect to any sesquilinear scalar product induced by a Hermitian or skew Hermitian unitary matrix.

Structure preserving perturbations of smallest size that cause $*$-palindromic and $*$-even polynomials to have eigenvalues on the imaginary axis and the unit circle respectively are very important for control theory applications as these may result in loss of spectral symmetry and uniqueness in the choice of the deflating subspace associated with the eigenvalues in the left half plane (respectively the open unit disk) [6, 26, 31, 32, 33]. It is well known that such information may be read off from the structured ϵ-pseudospectra of polynomials [39]. The results for the backward error show that the same information may be obtained from the corresponding unstructured ϵ-pseudospectra. A similar conclusion holds for the $*$-even polynomials vis a vis the imaginary axis.

The results for the backward error when $\star$ denotes the adjoint with respect to the scalar product $\langle x, y\rangle_J := x^*Jy$, where $J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, imply that if a Hamiltonian matrix has no purely imaginary eigenvalue, then the size of the smallest perturbation which causes it to have such an eigenvalue is the same for both Hamiltonian and arbitrary perturbations. In terms of pseudospectra this means that the magnitude of the smallest Hamiltonian perturbation that causes a Hamiltonian matrix to have a purely imaginary eigenvalue is the smallest value of ϵ for which its unstructured ϵ-pseudospectrum touches the imaginary axis. This assumes significance in view of the fact that computation of the eigenvalues and stable invariant subspaces of Hamiltonian matrices is important for linear quadratic optimal control and H_∞ control problems as well as for the solution of continuous time algebraic Ricatti equations [26, 30, 20]. In [29] it was established that this is possible via structure preserving algorithms if there are no purely imaginary eigenvalues.

The eigenvalues of T-palindromic/antipalindromic polynomials are symmetrically placed either with respect to the number 1 or with respect to -1. We show that if 1 or -1 is a simple eigenvalue of such a polynomial, then it has the same structured and unstructured condition number except under certain conditions when the structured condition number is 0. We further show that the numbers 1 and -1 also have the same structured and unstructured backward error as approximate eigenvalues of such polynomials. These results are important because the presence of eigenvalues 1 and -1 may come in the way of finding structure preserving linearizations for T-palindromic/antipalindromic polynomials (see Section 6, [24]). The eigensymmetry of T-even and T-odd polynomials is with respect to the origin. We prove similar results for these polynomials vis-a-vis the number 0. Throughout the paper we consider the 2-norm and all proofs are established by constructing appropriate structure preserving perturbations to the polynomials.

This paper is organized as follows. In section 2, we make preliminary definitions and set some notations. All results pertaining to the structured condition number are contained in section 3 while those corresponding to the structured backward error are contained in section 4.

2 Preliminaries

We denote the space of all polynomials over real or complex square matrices of size n by $\mathcal{P}(\mathbb{F}_n)$. When the polynomials are structured we denote the corresponding space by $\mathcal{P}^S(\mathbb{F}_n)$ where S refers to any of the structures $\star$-palindromic, $\star$-antipalindromic, $\star$-even or $\star$-odd. The space of all square matrices with real or complex entries are denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ respectively while real and complex vectors of length are respectively denoted by $\mathbb{R}^n$ and $\mathbb{C}^n$. Finally we sometimes use the notation P to denote the polynomial $P(\lambda) \in \mathcal{P}(\mathbb{F}_n)$.

For a concise description of the $\star$-palindromic, $\star$-antipalindromic, $\star$-even and $\star$-odd polynomials, considered in this paper, we make the following definitions.

Definition 2.1 *Given* $Q(\lambda) = \Sigma_{k=0}^m \lambda^k B_k$, $B_1, \ldots, B_m \in \mathbb{F}^{n\times n}$, $B_k \neq 0$, *we define*

$$Q^\star(\lambda) := \Sigma_{k=0}^m \lambda^k B_k^\star \text{ and } \mathrm{rev}Q(\lambda) := \lambda^m Q(1/\lambda) = \Sigma_{i=0}^m \lambda^{m-i} B_i$$

and refer to $Q^\star(\lambda)$ *as the adjoint and* $\mathrm{rev}Q(\lambda)$ *as the reversal of* $Q(\lambda)$.

The following table now provides the basic definitions and associated spectral symmetries.

Basic structure	Property	Spectral symmetry
T-palindromic	$revP^T(\lambda) = P(\lambda)$	$(\lambda, 1/\lambda)$
T-antipalindromic	$revP^T(\lambda) = -P(\lambda)$	$(\lambda, -1/\lambda)$
$*$-palindromic	$revP^*(\lambda) = P(\lambda)$	$(\lambda, 1/\overline{\lambda})$
$*$-antipalindromic	$revP^*(\lambda) = -P(\lambda)$	$(\lambda, -1/\overline{\lambda})$
T-even	$P^T(-\lambda) = P(\lambda)$	$(\lambda, -\lambda)$
T-odd	$P^T(-\lambda) = -P(\lambda)$	$(\lambda, -\lambda)$
$*$-even	$P^*(-\lambda) = P(\lambda)$	$(\lambda, -\overline{\lambda})$
$*$-odd	$P^*(-\lambda) = -P(\lambda)$	$(\lambda, -\overline{\lambda})$

Table 1

Moreover algebraic, geometric and partial multiplicites of the two eigenvalues in each pair listed in the above table are equal (for a proof see [24]).

2.1 Condition number and backward error

The condition number of an eigenvalue measures its rate of change with respect to change in the initial data and indicates its sensitivity to perturbations in the data. The backward error of a complex number z is a measure of the perturbation of smallest magnitude in the presence of which, z becomes an eigenvalue of the perturbed problem. It gives a measure of the stability of a numerical method. These ideas are well developed in [43, 36, 10, 38]. The error in the computed solution is referred to as the forward error and it is related to the condition number and the backward error by the relation

$$\text{forward error} \leq \text{condition number} \times \text{backward error}$$

which is correct up to first order in the backward error. Thus if the perturbation in the data is of the order of the backward error, then the product of the condition number and the backward error gives a first order error bound on the computed solution. However in order to analyse the performance of a structure preserving algorithm and derive error bounds on the computed solution, it is more useful to consider the condition number and backward error under the restriction that the perturbations preserve the structure of the problem. For instance, if $P(\lambda)$ is a $*$-palindromic polynomial, we would like to measure the sensitivity of its eigenvalues and the backward error of its approximate eigenvalues only with respect to $*$-palindromic perturbations. We refer to the corresponding modified quantities as the structured condition number and the structured backward error respectively.

Given a simple eigenvalue λ of $P(\lambda) \in \mathcal{P}(\mathbb{F}_n)$, with corresponding right eigenvector x and left eigenvector y, the normwise condition number of λ is defined as

$$\begin{aligned}\kappa(\lambda, P) \quad := \quad & \limsup_{\epsilon \to 0} \{ \frac{|\Delta\lambda|}{\epsilon} : (P(\lambda + \Delta\lambda) + \Delta P(\lambda + \Delta\lambda))(x + \Delta x) = 0, \\ & \Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k \Delta A_k \in \mathcal{P}(\mathbb{F}^n), \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m \}. \end{aligned} \tag{2}$$

The above definition is also referred to as the absolute normwise condition number as it is a measure of the absolute change in λ under perturbation. For a simple eigenvalue λ, that is not zero or infinite, the relative change in λ may also be measured by dividing the ratio $\frac{|\Delta\lambda|}{\epsilon}$ by $|\lambda|$ in the above definition. However this does not matter for the purpose of this work as the main objective is to compare structured and unstructured condition numbers. Besides, our choice of the definition allows us to deal with the zero and infinite eigenvalue within the same framework without having to consider the condition number of the problem in homogeneous form as defined in [4] and [14]. A computable expression of the above condition number has been obtained in [38] as follows.

Theorem 2.1 *[38] The normwise condition number $\kappa(\lambda, P)$ is given by*

$$\kappa(\lambda, P) = \frac{\alpha \|y\|_2 \|x\|_2}{|y^* P'(\lambda) x|}, \ \textit{where } \alpha = \Sigma_{k=0}^{m} |\lambda|^k.$$

Given $P(\lambda) \in \mathcal{P}^S(\mathbb{F}^n)$ we modify Definition (2) in the following obvious manner to obtain a corresponding structured normwise condition number.

$$\begin{aligned}\kappa^S(\lambda, P) \quad := \quad & \limsup_{\epsilon \to 0} \left\{ \frac{|\Delta\lambda|}{\epsilon} : (P(\lambda + \Delta\lambda) + \Delta P(\lambda + \Delta\lambda))(x + \Delta x) = 0 \right. \\ & \left. \Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k \Delta A_k \in \mathcal{P}^S(\mathbb{F}^n) \text{ and } \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m \right\} \end{aligned} \tag{3}$$

Expanding the constraint $(P(\lambda + \Delta\lambda) + \Delta P(\lambda + \Delta\lambda))(x + \Delta x) = 0$ in the above definition, and neglecting second order terms we have,

$$\Delta\lambda P'(\lambda) x + P(\lambda)\Delta x + \Delta P(\lambda) x \quad = \quad \mathcal{O}(\epsilon^2)$$

If y be a left eigenvector of P corresponding to λ, then multiplying the above equation from the left by y^*, we have,

$$\Delta\lambda y^* P'(\lambda) x + y^* P(\lambda) \Delta x + y^* \Delta P(\lambda) x = \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \Delta\lambda = -\frac{y^* \Delta P(\lambda) x}{y^* P'(\lambda) x} + \mathcal{O}(\epsilon^2).$$

Therefore Definition (3) takes the form

$$\begin{aligned}\kappa^S(\lambda, P) \quad := \quad & \limsup_{\epsilon \to 0} \left\{ \frac{|y^* \Delta P(\lambda) x|}{\epsilon |y^* P'(\lambda) x|} : \Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k \Delta A_k \in \mathcal{P}^S(\mathbb{F}^n) \text{ and} \right. \\ & \left. \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m \right\}. \end{aligned} \tag{4}$$

It is evident from the definitions of $\kappa(\lambda, P)$ and $\kappa^S(\lambda, P)$ that in general we have

$$\kappa(\lambda, P) \geq \kappa^S(\lambda, P).$$

Given a complex number $\tilde{\lambda} \in \mathbb{C}$ and a vector $\tilde{x} \in \mathbb{C}^n$, the normwise backward error of the pair $(\tilde{\lambda}, \tilde{x})$ considered as an approximate eigenpair of $P(\lambda) \in \mathcal{P}(\mathbb{F}^n)$ is the size of the smallest perturbation which when applied to $P(\lambda)$ causes $\tilde{\lambda}$ to become an eigenvalue of the perturbed polynomial with $\tilde{x}$ as a corresponding eigenvector. It is defined as follows.

$$\begin{aligned} \eta(\tilde{\lambda}, \tilde{x}, P) \quad &:= \quad \min\{\epsilon : (P(\tilde{\lambda}) + \Delta P(\tilde{\lambda}))\tilde{x} = 0, \Delta P(\lambda) := \Sigma_{k=0}^m \Delta A_k \in \mathcal{P}(\mathbb{F}^n), \\ &\quad\quad \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m\} \end{aligned} \tag{5}$$

The following explicit formula for computing the backward error was obtained in [38].

Theorem 2.2 *The normwise backward error $\eta(\tilde{\lambda}, \tilde{x}, P)$ is given by*

$$\eta(\tilde{\lambda}, \tilde{x}, P) = \frac{\|P(\tilde{\lambda})\tilde{x}\|_2}{\tilde{\alpha}\|\tilde{x}\|_2}$$

where $\tilde{\alpha} := \Sigma_{k=0}^m |\tilde{\lambda}|^k$.

When the eigenvectors are not under consideration, the backward error may be computed from the above formula for an approximate eigenvalue only by taking the infimum over all non-zero $\tilde{x} \in \mathbb{C}^n$ in the above expression [38]. Thus if $\tilde{\lambda}$ is not an eigenvalue of $P(\lambda)$, then

$$\eta(\tilde{\lambda}, P) \quad = \quad \frac{1}{\tilde{\alpha}\|[P(\tilde{\lambda})]^{-1}\|_2} \tag{6}$$

However, if the polynomial $P(\lambda)$ has some additional structure then as is the case with the condition number, we modify Definition (5) as follows to obtain the structured backward error.

$$\begin{aligned} \eta^S(\tilde{\lambda}, \tilde{x}, P) \quad &:= \quad \min\{\epsilon : (P(\tilde{\lambda}) + \Delta P(\tilde{\lambda}))\tilde{x} = 0, \Delta P(\lambda) := \Sigma_{k=0}^m \Delta A_k \in \mathcal{P}^S(\mathbb{F}^n), \\ &\quad\quad \|\Delta A_k\|_2 \leq \epsilon, k = 0 : m\} \end{aligned} \tag{7}$$

Evidently, we have $\eta^S(\tilde{\lambda}, \tilde{x}, P) \geq \eta(\tilde{\lambda}, \tilde{x}, P)$.

3 The structured and unstructured condition number

Our aim is to show that certain eigenvalues of $\star$-palindromic, $\star$-antipalindromic, $\star$-even and $\star$-odd polynomials, have the same structured and unstructured condition number. In other words, such eigenvalues display equal sensitivity to any type of perturbation regardless of whether such a perturbation preserves structure or not.

3.1 The $\star$-palindromic and $\star$-antipalindromic polynomials

We first consider the case when P is a complex $*$-palindromic or $*$-antipalindromic polynomial. Recall that the eigenvalues of P occur in pairs $(\lambda, 1/\bar{\lambda})$, so that they are symmetrically placed with respect to the unit circle. For such polynomials we show that if an eigenvalue lying on the unit circle is simple, then it has the same condition number with respect to both structure preserving and arbitrary perturbations. To prove this we need the following result.

Lemma 3.1 *Let $P(\lambda)$ be a complex $*$-palindromic or $*$-antipalindromic polynomial having an eigenvalue λ_0 such that $|\lambda_0| = 1$. Then the left and right eigenvectors of λ_0 are equal.*

Proof: Let $P(\lambda)$ be a complex $*$-antipalindromic polynomial of degree m and let y be a left eigenvector of P corresponding to λ_0. Then
$y^*P(\lambda_0) = 0 \Rightarrow P(\lambda_0)^*y = 0 \Rightarrow -revP(\bar{\lambda_0})y = 0 \Rightarrow -\bar{\lambda_0}^m P(1/\bar{\lambda_0})y = 0 \Rightarrow P(\lambda_0)y = 0$.
Hence, y is also a right eigenvector of $P(\lambda)$ corresponding to λ_0. When $P(\lambda)$ is complex $*$-palindromic, we have $P(\lambda_0)^* = revP(\bar{\lambda_0})$ and the proof follows from identical arguments. $\square$

Theorem 3.1 *If a complex $*$-palindromic or $*$-antipalindromic polynomial has a simple eigenvalue on the unit circle, then it has the same condition number with respect to both arbitrary and structure preserving perturbations.*

Proof: Let $P(\lambda)$ be a complex $\star$-palindromic polynomial of degree m and let λ_0 be a simple eigenvalue of $P(\lambda)$ with $|\lambda_0| = 1$. Set $\omega_0 := \lambda_0^m$. Then $|\omega_0| = 1$. Let $z_0 := \sqrt{\frac{1+Re\omega_0}{2}} + i\sqrt{\frac{1-Re\omega_0}{2}}$. Then $|z_0| = 1$ and $\omega_0\bar{z_0} = z_0$. Choosing a right eigenvector x of $P(\lambda)$ corresponding to λ_0 and any positive number $\epsilon > 0$, let $H := z_0\dfrac{\epsilon xx^*}{\|x\|^2}$ and $\Delta A_k := \bar{\lambda_0}^k H$, $k = 0 : m$. Then for all $k = 0 : m$,

$$[\Delta A_{m-k}]^* = \lambda_0^{m-k}H^* = \omega_0\lambda_0^{-k}\bar{z_0}\frac{\epsilon xx^*}{\|x\|^2} = \bar{\lambda_0}^k z_0\frac{\epsilon xx^*}{\|x\|^2} = \bar{\lambda_0}^k H = \Delta A_k.$$

Hence the polynomial $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k = ((\lambda\bar{\lambda_0})^m + (\lambda\bar{\lambda_0})^{m-1} + \cdots + \lambda\bar{\lambda_0} + 1)H$ is $*$-palindromic with $\|\Delta A_k\| = \epsilon$ for all $k = 0 : m$ and $x^*\Delta P(\lambda_0)x = (m+1)z_0\epsilon\|x\|^2$. By Lemma 3.1 x is both a right and a left eigenvector of $P(\lambda)$ corresponding to λ_0. Therefore by Definition (4) we have

$$\kappa^{*-(pal)}(\lambda_0, P) \geq \lim_{\epsilon\to 0}\frac{|x^*\Delta P(\lambda_0)x|}{\epsilon|x^*P'(\lambda_0)x|} = \frac{(m+1)\|x\|^2}{|x^*P'(\lambda_0)x|}.$$

However by Theorem 2.1, $\kappa(\lambda_0, P) = \dfrac{(m+1)\|x\|^2}{|x^*P'(\lambda_0)x|}$. Hence, we must have that $\kappa^{*-(pal)}(\lambda_0, P) = \kappa(\lambda_0, P)$.

Now suppose that $P(\lambda)$ is $*$-antipalindromic and let λ_0 be a simple eigenvalue of $P(\lambda)$ such that $|\lambda_0| = 1$. As in the previous case, let $\omega_0 = \lambda_0^m$. Then $|\omega_0| = 1$. Let

$z_0 := -\sqrt{\frac{1-Re\omega_0}{2}} + i\sqrt{\frac{1+Re\omega_0}{2}}$. It can be easily verified that $\omega_0 \bar{z}_0 = -z_0$. Assuming that x is a right eigenvector of $P(\lambda)$ with respect to λ_0, and ϵ is an arbitrarily chosen positive quantity, let $H := z_0 \frac{\epsilon x x^*}{\|x\|^2}$ and $\Delta A_k := (\bar{\lambda_0})^k H$, $k = 0 : m$. Then for all $k = 0 : m$,

$$(\Delta A_{m-k})^* = \lambda_0^{m-k} H^* = \omega \bar{\lambda_0}^k \bar{z}_0 \frac{\epsilon x x^*}{\|x\|^2} = -z_0 \bar{\lambda_0}^k \epsilon \frac{x x^*}{\|x\|^2} = -\Delta A_k.$$

Therefore the polynomial $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k = ((\lambda\bar{\lambda_0})^m + (\lambda\bar{\lambda_0})^{m-1} + \cdots + \lambda\bar{\lambda_0} + 1)H$ is $*$-antipalindromic with $\|\Delta A_k\| = \epsilon$ and $x^* \Delta P(\lambda_0) x = -(m+1) z_0 \epsilon \|x\|^2$. Once again using the fact that x is both a left and right eigenvector of P with respect to λ_0, by Definition (4) and Theorem 2.1 we have,

$$\kappa^{*-(antipal)}(\lambda_0, P) \geq \lim_{\epsilon \to 0} \frac{|x^* \Delta P(\lambda_0) x|}{\epsilon |x^* P'(\lambda_0) x|} = \frac{(m+1)\|x\|^2}{|x^* P'(\lambda_0) x|} = \kappa(\lambda_0, P).$$

Hence $\kappa^{*-(antipal)}(\lambda_0, P) = \kappa(\lambda_0, P)$. $\square$

Since the eigenvalue symmetry of $*$-palindromic/antipalindromic polynomials remain unchanged when $*$ is replaced by the adjoint with respect to any sesquilinear scalar product induced by a Hermitian or skew-Hermitian nonsingular matrix, Theorem 3.1 may be easily generalised to such cases.

Corollary 3.1 *Let $\star$ denote the adjoint with respect a sesquilinear scalar product $\langle x, y \rangle_M := y^* M x$, $x, y \in \mathbb{C}^n$, where $M \in \mathbb{C}^{n \times n}$ is any unitary Hermitian or skew Hermitian matrix. If $P(\lambda)$ is a complex $\star$-palindromic or $\star$-antipalindromic polynomial, then any simple eigenvalue of $P(\lambda)$ lying on the unit circle has the same condition number with respect to both structure preserving and arbitrary perturbations.*

Proof: If $P(\lambda)$ is $\star$-palindromic, then $P^\star(\lambda) = revP(\lambda)$. Since, $P^\star(\lambda) = M^{-1} P^*(\lambda) M$ we have $P^*(\lambda) M = rev(MP(\lambda))$ which implies that $(MP)^*(\lambda) = rev(MP(\lambda))$ or $(MP)^*(\lambda) = -rev(MP(\lambda))$ according as whether M is Hermitian or skew Hermitian. Therefore if $P(\lambda)$ is $\star$-palindromic, then $MP(\lambda)$ is $*$-palindromic if M is Hermitian and $*$-antipalindromic if M is skew Hermitian. By the same logic it follows that if $P(\lambda)$ is $\star$-antipalindromic, then $MP(\lambda)$ is $*$-antipalindromic if M is Hermitian and $*$-palindromic if M is skew Hermitian. Also since M is unitary and the 2-norm is unitarily invariant for all simple eigenvalues λ of P, we have

$$\kappa(\lambda, P) = \kappa(\lambda, MP) \quad \text{and} \quad \kappa^S(\lambda, P) = \kappa^{S'}(\lambda, MP) \tag{8}$$

where S and S' denote the structures of the polynomials P and MP respectively.

Let P be a $\star$-palindromic polynomial and M be a Hermitian matrix. If λ_0 is a simple eigenvalue of P on the unit circle then the same holds for MP and by Theorem 3.1, we have

$$\kappa^{*-pal}(\lambda_0, MP) = \kappa(\lambda_0, MP). \tag{9}$$

Since MP is $*$-palindromic the equalities in (8) imply that

$$\kappa(\lambda_0, P) = \kappa(\lambda_0, MP) \text{ and } \kappa^{\star-pal}(\lambda_0, P) = \kappa^{*-pal}(\lambda_0, MP).$$

Using these, and the equality (9), we have $\kappa^{\star-pal}(\lambda_0, P) = \kappa(\lambda_0, P)$.

The proof in all other cases follows by using the same arguments since MP is either a $*$-palindromic or $*$-antipalindromic polynomial. □

Remark 3.1 *If $P(\lambda)$ has $\star$-palindromic or $\star$-antipalindromic structure, and λ_0 is a simple eigenvalue on the unit circle with corresponding right eigenvector x, then Mx is a corresponding left eigenvector. Since M is a unitary matrix, it is easy to see that the matrix $\tilde{H} := \dfrac{\epsilon M x x^*}{\|Mx\|\|x\|}$ is a $\star$-Hermitian or $\star$-skew Hermitian matrix for any arbitrary positive number ϵ. Thus if $\omega_0 := (\lambda_0)^m$, then the polynomial $\Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k \Delta A_k$, where $\Delta A_k := \bar{\lambda_0}^k z_0 \tilde{H}, |z_0| = 1$, $k = 0 : m$, is $\star$-palindromic if M is Hermitian and $\omega_0 \bar{z_0} = z_0$ or M is skew Hermitian and $\omega_0 \bar{z_0} = -z_0$ while it is $\star$-antipalindromic if M is Hermitian and $\omega_0 \bar{z_0} = -z_0$ or M is skew Hermitian and $\omega_0 \bar{z_0} = z_0$. This polynomial may therefore be used to provide an alternative proof of Corollary 3.1 on the lines of Theorem 3.1.*

Next we consider the real or complex T-palindromic and T-antipalindromic polynomials. From Table 2 we see that when the coefficient matrices of such polynomials are complex, then the eigenvalues pairing is $(\lambda, 1/\lambda)$. If the coefficient matrices are real then the eigenvalues occur in quadruples $(\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda})$. We observe that if the coefficient matrices are of odd dimension then 1 is always a simple eigenvalue of a T-antipalindromic polynomial while -1 is always an eigenvalue of a T-palindromic polynomial of odd degree and a T-antipalindromic polynomial of even degree.

Theorem 3.2 *Let $P(\lambda)$ be a polynomial over real or complex matrices of odd dimension.*

(i) The number 1 is always a simple eigenvalue of $P(\lambda)$ if it is a T-antipalindromic polynomial.

(ii) The number -1 is always a simple eigenvalue of $P(\lambda)$ if it is a T-palindromic polynomial of odd degree or a T-antipalindromic polynomial of even degree.

Proof: If $P(\lambda)$ is a T-antipalindromic polynomial then $P(1)$ is a skew symmetric matrix since $P(1)^T = -P(1)$ and it is of odd dimension if the coefficient matrices of $P(\lambda)$ are of odd dimension. Since 0 is always an eigenvalue of such a polynomial, there always exists a vector $x \neq 0$ such that $P(1)x = 0$. This implies that 1 is an eigenvalue of $P(\lambda)$ with corresponding eigenvector x. The proof of the second assertion follows by using the same arguments since the hypothesis implies that $P(-1)$ is again a skew symmetric matrix of odd dimension in each case. □

In the next result we show that if $P(\lambda)$ is T-palindromic then in view of Theorem 3.2, the structured and unstructured condition numbers of 1 and -1 are equal whenever they are simple eigenvalues of these polynomials except when the degree of $P(\lambda)$ is odd in which case the structured condition number of -1 is 0.

Theorem 3.3 *Let $P(\lambda)$ be a T-palindromic polynomial. If 1 is a simple eigenvalue of $P(\lambda)$, then $\kappa^{T-pal}(1, P) = \kappa(1, P)$. If -1 is a simple eigenvalue of $P(\lambda)$ then $\kappa^{T-pal}(-1, P) = \kappa(-1, P)$ if the degree of $P(\lambda)$ is even while $\kappa^{T-pal}(-1, P) = 0$ when the degree of $P(\lambda)$ is odd.*

Proof: Let $P(\lambda)$ be a T-palindromic polynomial. Then $P(-1)^T = rev(P(-1)) = (-1)^m P(-1)$ where m is the degree of P.

Let the degree m be an odd number. Then $P(-1)$ is a real or complex skew symmetric matrix depending upon whether P is a real or complex T-palindromic polynomial. If -1 is a simple eigenvalue of $P(\lambda)$ then 0 is a simple eigenvalue of $P(-1)$ which is possible only if $P(-1)$ is a matrix of odd dimension. Thus if -1 is a simple eigenvalue of P then the coefficient matrices are of odd dimension. But -1 is always a simple eigenvalue of $P(\lambda)$ in such cases as observed in Theorem 3.2. Therefore $\kappa^{T-pal}(-1, P) = 0$.

If the degree m is an even number then $P(-1)$ is a symmetric matrix. Let -1 be a simple eigenvalue of P. Assuming that the coefficient matrices are complex, if x is a right eigenvector of P with respect to -1 then $\bar{x}$ is a corresponding left eigenvector. Given an arbitrary positive number ϵ, let $H := \dfrac{\epsilon \bar{x} x^*}{\|x\|^2}$. Then $H^T = H$. Defining $\Delta A_k := (-1)^k H, k = 0 : m$ we have

$$(\Delta A_{m-k})^T = (-1)^{m-k} H^T = (-1)^k H = \Delta A_k, \; k = 0 : m$$

which shows that $\Delta P(\lambda) = \Sigma_{k=0}^m (\lambda^k (-1)^k) H$ is T-palindromic with $\|\Delta A_k\| = \epsilon$ for $k = 0 : m$. Also $\bar{x}^* \Delta P(-1) x = \Sigma_{k=0}^m (-1)^2 k \|x\|^2 = (m+1)\|x\|^2$. Therefore by Definition (4) and Theorem 2.1,

$$\kappa^{T-pal}(-1, P) \geq \lim_{\epsilon \to 0} \frac{\bar{x}^* \Delta P(-1) x}{\epsilon |\bar{x}^* P'(-1) x|} = \frac{(m+1)\|x\|^2}{|\bar{x}^* P'(-1) x|} = \kappa(-1, P).$$

Hence $\kappa^{T-pal}(-1, P) = \kappa(-1, P)$. If the coefficient matrices of $P(\lambda)$ are real then $P(\lambda)$ has the same left and right eigenvectors corresponding to the eigenvalue -1. Thus if x be such a vector then the equality $\kappa^{T-pal}(-1, P) = \kappa(-1, P)$ follows by replacing H in the above arguments by the symmetric matrix $S := \dfrac{\epsilon x x^T}{\|x\|^2}$ to construct a real T-palindromic perturbation of $P(\lambda)$.

Finally, consider the case when 1 is a simple eigenvalue of $P(\lambda)$. Let x be a right eigenvector of $P(\lambda)$ with respect to 1. Since $P(1)^T = P(1)$, if the coefficient matrices of $P(\lambda)$ are complex, then $\bar{x}$ is a left eigenvector of $P(\lambda)$ with respect to 1. Let $H := \dfrac{\epsilon \bar{x} x^*}{\|x\|^2}$ and set $\Delta A_k := H$, $k = 0 : m$. Then H is a complex symmetric matrix and it is easy to see that $\Delta P(\lambda) = \Sigma_{k=0}^m (\lambda^k \Delta A_k)$ is T-palindromic with $\|\Delta A_k\| = \epsilon$, $k = 0 : m$ and the proof follows on exactly the same lines as when -1 is a simple eigenvalue of $P(\lambda)$ and $P(\lambda)$ is of even degree. If $P(\lambda)$ is real T-palindromic then the left and right eigenvectors of $P(\lambda)$ with respect to 1 are equal. If x be such an eigenvector, then the proof follows by replacing H by the real symmetric matrix $S := \frac{\epsilon x x^T}{\|x\|^2}$ in the preceding arguments. $\square$

In the next result we consider the T-antipalindromic polynomials. In this case, the structured condition number of 1 is 0 for all such polynomials, while the same holds for -1 if the degree of P is even.

Theorem 3.4 *Let $P(\lambda)$ be a T-antipalindromic polynomial. If 1 is a simple eigenvalue of $P(\lambda)$, then $\kappa^{T-antipal}(1, P) = 0$. If -1 is a simple eigenvalue of $P(\lambda)$ then $\kappa^{T-antipal}(-1, P) = \kappa(-1, P)$ if the degree of $P(\lambda)$ is odd and $\kappa^{T-antipal}(-1, P) = 0$ if the degree of $P(\lambda)$ is even.*

Proof: Since $P(\lambda)$ is T-antipalindromic, we have $P(1)^T = -P(1)$. Thus, $P(1)$ is a skew symmetric matrix. Now 1 is a simple eigenvalue of $P(\lambda)$ if and only if 0 is a simple eigenvalue of $P(1)$. Since this is possible only when $P(1)$ is of odd dimension, it follows that 1 is a simple eigenvalue of P if and only if the coefficient matrices of $P(\lambda)$ are of odd dimension. But by Theorem 3.2, 1 is always an eigenvalue of $P(\lambda)$ in such cases. Hence $\kappa^{T-antipal}(1, P) = 0$.

If the degree say m of $P(\lambda)$ is even, then $P(-1)$ is a skew symmetric matrix since $P(-1)^T = -rev(P(-1)) = (-1)^m P(-1) = -P(-1)$. So by arguing exactly as in the preceding paragraph, it follows that if -1 is a simple eigenvalue of $P(\lambda)$, then $\kappa^{antipal}(-1, P) = 0$.

If m is an odd number, then $P(-1)$ is a symmetric matrix since $P(-1)^T = -revP(-1) = -(-1)^m P(-1) = P(-1)$. Let -1 be a simple eigenvalue of $P(\lambda)$ and x be a corresponding right eigenvector. Then either x or $\bar{x}$ is a corresponding left eigenvector depending upon whether $P(\lambda)$ is a real or complex polynomial. The proof in this case now follows by using arguments identical to those in the proof of Theorem 3.3 by constructing the T-antipalindomic polynomials $\Delta P(\lambda) := \lambda^k(-1)^k H$ and $\Delta P(\lambda) := \lambda^k(-1)^k S$ in the complex and real cases respectively where $H := \frac{\epsilon \bar{x} x^*}{\|x\|^2}$, $S := \frac{\epsilon x x^T}{\|x\|^2}$ and α is an arbitrary positive number. $\square$

All the preceding results hold with appropriate modifications if T is replaced by the transpose with respect to any bilinear scalar product satisfying some simple conditions. We begin by proving the counterpart of Theorem 3.2.

Theorem 3.5 *Let $P(\lambda)$ be a polynomial over real or complex matrices of odd dimension. and $\star$ denote the transpose with respect to the bilinear scalar product $\langle x, y\rangle_M := y^T M x, x, y \in \mathbb{F}^n$, where M is a real orthogonal matrix which is either symmetric or skew symmetric.*

If M is a symmetric matrix then the following hold.

(i) If $P(\lambda)$ is $\star$-antipalindromic, then 1 is always a simple eigenvalue.
(ii) If $P(\lambda)$ is a $\star$-palindromic polynomial of odd degree or a $\star$-antipalindromic polynomial of even degree, then -1 is always a simple eigenvalue.

If M is a skew symmetric matrix then the following hold.

(iii) If $P(\lambda)$ is $\star$-palindromic, then 1 is always a simple eigenvalue.
(iv) If $P(\lambda)$ is a $\star$-antipalindromic polynomial of odd degree or a $\star$-palindromic polynomial of even degree then -1 is always a simple eigenvalue.

Proof: Let $P(\lambda)$ be a $\star$-palindromic polynomial. Since we have, $P^\star(\lambda) = M^{-1}P^T(\lambda)M$, therefore,
$P^\star(\lambda) = revP(\lambda) \Longrightarrow M^{-1}P^T(\lambda)M = revP(\lambda) \Longrightarrow P^T(\lambda)M = rev(MP(\lambda))$.
Since $P^T(\lambda)M = (MP)^T(\lambda)$ if M is symmetric and $P^T(\lambda)M = -(MP)^T(\lambda)$ if M is skew symmetric, it follows from above that $MP(\lambda)$ is T-palindromic or T-antipalindromic according as whether M is symmetric or skew symmetric. Symilarly if $P(\lambda)$ is $\star$-antipalindromic, then $MP(\lambda)$ is T-antipalindromic or T-palindromic according as whether M is symmetric or skew symmetric. Using these observations the proofs of each of (i),(ii), (ii) and (iv) follow from the fact that $MP(1)$ is a skew symmetric matrix under the hypotheses of (i) and (iii) while the same holds for $MP(-1)$ under the hypotheses of (ii) and (iv). □

Theorem 3.6 *Let $\star$ denote the transpose with respect to the scalar product $\langle x, y\rangle_M := y^T Mx$, where M is a real orthogonal symmetric or skew symmetric matrix and $\kappa^S(\lambda, P)$ denote the structured condition number of λ where S denotes $\star$-palindromic or $\star$-antipalindromic structure according as whether $P(\lambda)$ is $\star$-palindromic or $\star$-antipalindromic.*

If $P(\lambda)$ is $\star$-palindromic and M is symmetric or $P(\lambda)$ is $\star$-antipalindromic and M is skew symmetric then the following hold.

(i) If 1 is a simple eigenvalue of $P(\lambda)$, then $\kappa^S(1, P) = \kappa(1, P)$.
(ii) If -1 is a simple eigenvalue of $P(\lambda)$ then $\kappa^S(-1, P) = \kappa(-1, P)$ if the degree of $P(\lambda)$ is even while $\kappa^S(-1, P) = 0$ when the degree of $P(\lambda)$ is odd.

If $P(\lambda)$ is $\star$-palindromic and M is skew symmetric or $P(\lambda)$ is $\star$-antipalindromic and M is symmetric then the following hold.

(iii) If 1 is a simple eigenvalue of $P(\lambda)$, then $\kappa^S(1, P) = 0$.
(iv) If -1 is a simple eigenvalue of $P(\lambda)$ then $\kappa^S(-1, P) = \kappa(-1, P)$ if the degree of $P(\lambda)$ is odd and $\kappa^S(-1, P) = 0$ if the degree of $P(\lambda)$ is even.

Proof: Since M is a real orthogonal matrix, the equalities in (8) hold under the given conditions also. As shown in the proof of Theorem 3.5, $MP(\lambda)$ is T-palindromic polynomial if $P(\lambda)$ is $\star$-palindromic and M is symmetric or $P(\lambda)$ is $\star$-antipalindromic and M is skew symmetric. Therefore the proofs of (i) and (ii) follow by using the equalities (8) and applying Theorem 3.3 to $MP(\lambda)$. On the other hand, $MP(\lambda)$ is T-antipalindromic if $P(\lambda)$ is $\star$-palindromic and M is skew Hermitian or $P(\lambda)$ is $\star$-antipalindromic and M is Hermitian. Hence the proof of (iii) and (iv) follow by using (8) and Theorem 3.4. □

Remark 3.2 *It may be easily observed that alternative proofs of the statements in the above Corollary that are similar to those of Theorem 3.3 and Theorem 3.4 may be provided by replacing the matrices H and S in these proofs by the complex matrix* $\tilde{H} := \dfrac{\epsilon M\bar{x}x^T}{\|M\bar{x}\|\|x\|}$ *and the real matrix* $S := \dfrac{\epsilon M x x^T}{\|Mx\|\|x\|}$ *respectively where $\epsilon > 0$ is arbitrarily chosen and x is a right eigenvector of either 1 or -1 as the case may be.*

Finally to end this subsection we note that if 0 is a simple eigenvalue of $P(\lambda)$, then imposition of any of the palindromic or antipalindromic structures considered in this subsection evidently have no effect on the condition number of 0. The same is the case for a simple infinite eigenvalue of $P(\lambda)$ in view of the facts that ∞ is a simple eigenvalue of $P(\lambda)$ if and only if 0 is a simple eigenvalue of $revP(\lambda)$ and $revP(\lambda)$ also has the same structure as $P(\lambda)$. Therefore we observe that

$$\kappa(0, P) = \kappa^S(0, P) \text{ and } \kappa(\infty, P) = \kappa^S(\infty, P)$$

where S implies $*$-palindromic/antipalindromic or T-palindromic/antipalindromic structure depending upon the structure of the polynomial $P(\lambda)$.

3.2 The $\star$-even and $\star$-odd polynomials

If $P(\lambda)$ is a complex $*$-even or $*$-odd polynomial then its eigenvalues occur in pairs as $(\lambda, -\bar{\lambda})$, so that they are symmetrically placed with respect to the imaginary axis. We show that if such a polynomial has a simple purely imaginary eigenvalue, then it has the same condition number with respect to arbitrary and structure preserving perturbations. The result uses the fact that a purely imaginary eigenvalue of a $*$-even or $*$-odd polynomial has the same left and right eigenvectors.

Lemma 3.2 *Let $P(\lambda)$ be a complex $*$-even or $*$-odd polynomial. Then every purely imaginary eigenvalue of $P(\lambda)$ has the same left and right eigenvectors.*

Proof: Let $P(\lambda)$ be a complex $*$-even and λ_0 be a purely imaginary eigenvalue of $P(\lambda)$ with left eigenvector y. Then $\bar{\lambda_0} = -\lambda_0$ and
$y^*P(\lambda_0) = 0 \Rightarrow P(\lambda_0)^*y = 0 \Rightarrow P^*(\bar{\lambda_0})y = 0 \Rightarrow P(-\bar{\lambda_0})y = 0 \Rightarrow P(\lambda_0)y = 0$.
Hence y is also a right eigenvector of $P(\lambda)$ corresponding to λ_0. The proof when $P(\lambda)$ is a complex $*$-odd polynomial follows identically by using the fact that $P^*(\bar{\lambda_0}) = -P(-\bar{\lambda_0})$. $\square$

Theorem 3.7 *Let $P(\lambda)$ be a complex $*$-even or $*$-odd polynomial. If $\lambda_0 \in \{z \in \mathbb{C} : Rez = 0\} \cup \{\infty\}$ is a simple eigenvalue, then it has the same condition number with respect to both arbitrary perturbations and perturbations that preserve structure.*

Proof: Let $P(\lambda)$ be a complex $*$-even polynomial of degree m and let λ_0 be a purely imaginary eigenvalue of $P(\lambda)$ which is simple. By Lemma 3.2 the left and right eigenvectors of $P(\lambda)$ corresponding to λ_0 are equal. Let $x \in \mathbb{C}^n$ be this eigenvector. Define $H := \epsilon\dfrac{xx^*}{\|x\|^2}$ where ϵ is any positive real number. Let $\Delta A_k :=$ $\left(\dfrac{\bar{\lambda}_0}{|\lambda_0|}\right)^k H,\ k = 0 : m$. Then for all odd values of k from 1 to m,

$$(\Delta A_k)^* = \left(\frac{\lambda_0}{|\lambda_0|}\right)^k H = \left(-\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k H = -\Delta A_k.$$

For $k = 0$ and all even values of k from 2 to m,

$$(\Delta A_k)^* = \left(\frac{\lambda_0}{|\lambda_0|}\right)^k H = \left(-\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k H = \Delta A_k.$$

Hence $\Delta P(\lambda) := \Sigma_{k=0}^m \left(\dfrac{\lambda\bar{\lambda}_0}{|\lambda_0|}\right)^k H$ is $*$-even with $\|\Delta A_k\| = \epsilon$ for $k = 0 : m$ and $x^*\Delta P(\lambda_0)x = \epsilon\left(\Sigma_{k=0}^m|\lambda_0|^k\right)\|x\|^2$. Let $\kappa^{*-even}(\lambda_0, P)$ denote the condition number of λ_0 with respect to $*$-even perturbations. Therefore by Definition 4

$$\kappa^{*-even}(\lambda_0, P) \geq \lim_{\epsilon\to 0}\frac{|x^*\Delta P(\lambda_0)x|}{\epsilon|x^*P'(\lambda_0)x|} = \frac{\Sigma_{k=0}^m|\lambda_0|^k\|x\|^2}{|x^*P'(\lambda_0)x|}.$$

But by Theorem 2.1 $\kappa(\lambda_0, P) = \dfrac{\Sigma_{k=0}^m|\lambda_0|^k\|x\|^2}{|x^*P'(\lambda_0)x|}$. Hence it follows that

$$\kappa^{*-odd}(\lambda_0, P) = \kappa(\lambda_0, P) = \frac{\Sigma_{k=0}^m|\lambda_0|^k\|x\|^2}{|x^*P'(\lambda_0)x||\lambda_0|}.$$

Next suppose that $P(\lambda)$ is a complex $*$-odd polynomial of degree m and λ_0 is a simple purely imaginary eigenvalue. Once again by Lemma 3.2 the left and right eigenvectors of $P(\lambda)$ are equal. Let x be both a left and right eigenvector of $P(\lambda)$ with respect to λ_0. We define $H := i\epsilon\dfrac{xx^*}{\|x\|^2}$ where ϵ is any arbitrary positive number. Let $\Delta A_k := \left(\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k H,\ k = 0 : m$. Then for all odd values of k from 1 to m,

$$(\Delta A_k)^* = \left(\frac{\lambda_0}{|\lambda_0|}\right)^k (-H) = -\left(\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k (-H) = \left(\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k H = \Delta A_k.$$

For $k = 0$ and all even values of k from 2 to m,

$$(\Delta A_k)^* = \left(\frac{\lambda_0}{|\lambda_0|}\right)^k (-H) = \left(\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k (-H) = -\left(\frac{\bar{\lambda}_0}{|\lambda_0|}\right)^k H = -\Delta A_k.$$

Therefore $\Delta P(\lambda) = \Sigma_{k=0}^m\lambda^k\Delta A_k = \Sigma_{k=0}^m\left(\dfrac{\lambda\bar{\lambda}_0}{|\lambda_0|}\right)^k H$ is $*$-odd with $\|\Delta A_k\| = \epsilon$ for $k = 0 : m$ and it is easy to see that $x^*\Delta P(\lambda_0)x = \epsilon\left(\Sigma_{k=0}^m|\lambda_0|^k\right)\|x\|^2$. Hence the proof follows as in the previous case.

If $P(\lambda)$ has a simple eigenvalue at ∞, then $revP(\lambda)$ has a simple eigenvalue at 0. Observe that if the degree of $P(\lambda)$ is even, then $revP(\lambda)$ has the same $*$-even or

$*$-odd structure as that of $P(\lambda)$. However if the degree of $P(\lambda)$ is odd, then $revP(\lambda)$ is $*$-even if $P(\lambda)$ is $*$-odd and vice versa. In all cases it follows from above that 0 has the same structured and unstructured condition number as a simple eigenvalue of $revP(\lambda)$. Therefore ∞ has the same structured and unstructured condition number as a simple eigenvalue of $P(\lambda)$. $\square$

As in the case of the palindromic polynomials, the eigenvalue symmetry of a $*$-even or $*$-odd polynomial remains unchanged if $*$ is replaced by any other adjoint $\star$ with respect to a sesquilinear scalar product defined via a Hermitian or skew Hermitian unitary matrix. Therefore the above result extends to such cases as well.

Corollary 3.2 *Let $\star$ denote the adjoint with respect a sesquilinear scalar product $\langle x, y\rangle_M := y^*Mx,\ x, y \in \mathbb{C}^n$, where $M \in \mathbb{C}^{n\times n}$ is any unitary Hermitian or skew Hermitian matrix. If $P(\lambda)$ is a $\star$-even or $\star$-odd polynomial, then any simple eigenvalue of $P(\lambda)$ lying on the imaginary axis or at ∞ has the same condition number with respect to both structure preserving and arbitrary perturbations.*

Proof: Let $P(\lambda)$ be a $\star$-even polynomial. Then $P^\star(-\lambda) = P(\lambda)$. Since $P^\star(\lambda) = M^{-1}P^*(\lambda)M$, we have $P^*(-\lambda)M = MP(\lambda)$. Therefore, either $(MP)^*(-\lambda) = MP(\lambda)$ if M is Hermitian or $(MP)^*(-\lambda) = -MP(\lambda)$ if M is skew Hermitian so that $MP(\lambda)$ is $*$-even or $*$-odd polynomial according as whether M is Hermitian or skew Hermitian. Similarly, if $P(\lambda)$ is $\star$-odd, then $MP(\lambda)$ is $*$-odd if M is Hermitian and $*$-even if M is skew Hermitian.

Since the equalities in (8) hold in this case also, the proof now follows by applying Theorem 3.7 to $MP(\lambda)$ and arguing as in the proof of Corollary 3.1. $\square$

Remark 3.3 *If $P(\lambda)$ is a $\star$-even or $\star$-odd polynomial and λ_0 is a simple purely imaginary eigenvalue with corresponding right eigenvector x, then Mx is a corresponding left eigenvector. Recalling that $\tilde{H} := \dfrac{\epsilon Mxx^*}{\|Mx\|\|x\|}$ is a $\star$-Hermitian or $\star$-skew Hermitian matrix for any given $\epsilon > 0$, it is easy to see that the polynomial $\Delta P(\lambda) := \Sigma_{k=0}^m \alpha \left(\frac{\lambda\lambda_0}{|\lambda_0|}\right)^k \tilde{H}$, is $\star$-even if M is Hermitian and $\alpha = 1$, or M is skew Hermitian and $\alpha = i$ while it is $\star$-odd if M is Hermitian and $\alpha = i$, or M is skew Hermitian and $\alpha = 1$. This polynomial may therefore be used to provide a proof of Corollary 3.2 which is similar to that of Theorem 3.7.*

If $P(\lambda)$ is a real or complex polynomial with T-even or T-odd structure, having 0 or ∞ as a simple eigenvalue, then the following result gives the relationship between its structured and unstructured condition numbers.

Theorem 3.8 *Let $P(\lambda)$ be a T-even polynomial with either real or complex coefficient matrices. If 0 or ∞ is a simple eigenvalue of $P(\lambda)$, then $\kappa^{T-even}(0, P) = \kappa(0, P)$ whereas $\kappa^{T-even}(\infty, P) = \kappa(\infty, P)$ if $P(\lambda)$ is of even degree while $\kappa^{T-even}(\infty, P) = 0$ if $P(\lambda)$ is of odd degree.*

If $P(\lambda)$ is T-odd with a simple eigenvalue at 0 or ∞, then, $\kappa^{T-odd}(0, P) = 0$ while $\kappa^{T-odd}(\infty, P) = 0$ if $P(\lambda)$ is of even degree while $\kappa^{T-odd}(\infty, P) = \kappa(\infty, P)$ if $P(\lambda)$ is of odd degree.

Proof: If $P(\lambda)$ is T-even then $P(0)$ is evidently a symmetric matrix. Let 0 be a simple eigenvalue of $P(\lambda)$ and x be a corresponding right eigenvector. If $P(\lambda)$ is a complex polynomial, then $\bar{x}$ is a corresponding left eigenvector. Let $H := \frac{\epsilon x\bar{x}^T}{\|x\|^2}$ where ϵ is an arbitrarily chosen positive number. Note that H is a complex symmetric matrix. Let Q and R be any two complex matrices such that $Q^T = Q$ and $R^T = -R$. If m be the degree of $P(\lambda)$, let $\Delta A_k := R$ for all odd indices k from 1 to m and $\Delta A_k := Q$ for all even indices k from 2 to m. Also let $\Delta A_0 := H$. Then $\Delta P(\lambda) := \Sigma_{k=0}^m \Delta A_k \lambda^k$ is a T-even polynomial and $x^*\Delta P(0)x = x^*Hx = \epsilon\|x\|^2$. Therefore, if $\kappa^{T-even}(0,P)$ be the condition number of 0 with respect to T-even perturbations of P, then,

$$\kappa^{T-even}(0,P) \geq \lim_{\epsilon\to 0} \frac{|x^*\Delta P(0)x|}{\epsilon|x^*P'(0)x|} = \frac{\|x\|^2}{|x^*P'(0)x|}.$$

But

$$\kappa^{T-even}(0,P) \leq \kappa(0,P) = \frac{\Sigma_{k=0}^m (0)^k\|x\|^2}{|x^*P'(0)x|} = \frac{\|x\|^2}{|x^*P'(0)x|}.$$

Hence it follows that $\kappa^{T-even}(0,P) = \kappa(0,P)$.

If $P(\lambda)$ is a real T-even polynomial then $P(0)$ is a real symmetric matrix. Thus if x is a right eigenvector of P with respect to the eigenvalue 0 then it is also a left eigenvector corresponding to 0. The proof in this case now follows by replacing H by the real symmetric matrix $S := \frac{\epsilon x x^T}{\|x\|^2}$ and Q and R by real symmetric and skew-symmetric matrices $\tilde{Q}$ and $\tilde{R}$ respectively in the preceding arguments.

Finally let $P(\lambda)$ be a real or complex T-odd polynomial. Thus if $P(\lambda) = \Sigma_{k=0}^m \lambda^k A_k$, then $A_0 = P(0)$ is a skew symmetric matrix. Now 0 is a simple eigenvalue of $P(\lambda)$ if and only if 0 is a simple eigenvalue of A_0. But this is possible only when A_0 is of odd dimension. Thus it follows that if 0 is a simple eigenvalue of $P(\lambda)$, then its coefficient matrices must be of odd dimension. But 0 always belongs to the set of eigenvalues of such polynomials. Hence $\kappa^{T-odd}(0,P) = 0$.

If $P(\lambda)$ has a simple eigenvalue at ∞ then $revP(\lambda)$ has a simple eigenvalue at 0. Moreover, $revP(\lambda)$ has the same structure as $P(\lambda)$, if it is of even degree while it has T-even structure if $P(\lambda)$ is T-odd and vice versa if $P(\lambda)$ is of odd degree. Therefore the proofs of the assertions for the structured and unstructured condition numbers of ∞, follow by the results proved above for the structured and unstructured condition numbers of 0 as a simple eigenvalue of $revP(\lambda)$. $\square$

The above result has the following obvious counterpart when T is replaced by the transpose with respect to a bilinear scalar product induced by an orthogonal symmetric or skew symmetric matrix.

Theorem 3.9 *Let $\star$ denote the transpose with respect to the scalar product $\langle x,y\rangle_M = y^TMx$ where M is a real orthogonal matrix which is either symmetric or skew symmetric. If either 0 or ∞ is a simple eigenvalue of $P(\lambda)$, then the following hold.*

(i) If $P(\lambda)$ is $\star$-even and M is symmetric or $P(\lambda)$ is $\star$-odd and M is skew symmetric then, $\kappa^S(0,P) = \kappa(0,P)$ while $\kappa^S(\infty,P) = \kappa(\infty,P)$ if the degree of $P(\lambda)$

is even whereas $\kappa^S(\infty, P) = 0$ if the degree of $P(\lambda)$ is odd. Here S denotes $\star$-even or $\star$-odd according as whether $P(\lambda)$ is $\star$-even or $\star$-odd.

(ii) If $P(\lambda)$ is $\star$-odd and M is symmetric or $P(\lambda)$ is $\star$-even and M is skew symmetric matrix then, $\kappa^S(0, P) = 0$. However, $\kappa^S(\infty, P) = 0$ if the degree of $P(\lambda)$ is even, while $\kappa^S(\infty, P) = \kappa(\infty, P)$ if the degree of $P(\lambda)$ is odd.

Proof: It is easy to see that $MP(\lambda)$ is a T-even polynomial if the hypothesis of part (i) holds while it is a T-odd polynomial if the hypothesis of part (ii) is satisfied. The proofs of the assertions in (i) and (ii) now follow by using the equalities in (8) and applying Theorem 3.8 to $MP(\lambda)$. $\square$

Remark 3.4 *A proof of the above corollary on the lines of the proof of Theorem 3.8 may be provided by replacing the matrices H and S by the matrices $\tilde{H} := \dfrac{\epsilon M\bar{x}x^T}{\|M\bar{x}\|\|x\|}$ and $\tilde{S} := \dfrac{\epsilon Mxx^T}{\|x\|^2}$ respectively and multiplying each of the matrices $Q, R, \tilde{Q}$ and $\tilde{R}$ by M.*

4 Structured and unstructured backward error

We show that for each of the structured polynomials under consideration, there exist sets of points in the complex plane such that the normwise backward error associated with such points is the same with respect to both structure preserving and arbitrary perturbations.

4.1 The $\star$-palindromic and $\star$-antipalindromic polynomials

We begin by considering the complex $*$-palindromic and $*$-antipalindromic polynomials and show that all approximate eigenpairs $(\tilde{\lambda}, \tilde{x}), \tilde{x} \in \mathbb{C}^n$, where $\tilde{\lambda}$ are points on the unit circle, have the same normwise backward errors with respect to structure preserving and arbitrary perturbations. The proof of this assertion is given by constructing appropriate structure preserving perturbations to $P(\lambda)$ such that the coefficient matrices of these perturbations have 2-norm equal to the value of the normwise backward error as given by the expression in Theorem 2.2 for arbitrary perturbations. It depends upon the following lemma.

Lemma 4.1 *Let $x, y \in \mathbb{C}^n$, such that $\|x\| = \|y\| = 1$. Also let $\omega, z \in \mathbb{C}$ such that $|\omega| = |z| = 1$ and $\omega\bar{z} = z$.*
(i) If $\langle x, y\rangle = \bar{\omega}\langle y, x\rangle$, then there exists a Householder reflector Q such that $Q(x) = \bar{z}y$.
(ii) If $\langle x, y\rangle + \bar{\omega}\langle y, x\rangle = 0$, then there exists a Householder reflector Q such that $Q(x) = i\bar{z}y$.

Proof: In view of the hypothesis, we have $\|\omega x\| = \|zy\|$. Now since $\omega \bar{z} = z$, and $\langle x, y\rangle = \bar{\omega}\langle y, x\rangle$, we have

$$\langle \omega x, zy\rangle = \omega \bar{z}\langle x, y\rangle = z\bar{\omega}\langle y, x\rangle = \langle zy, \omega x\rangle.$$

Therefore, $\langle \omega x, zy\rangle$ is real. So there exists a Householder reflector Q[42] such that

$$Q(\omega x) = zy \Rightarrow \omega Qx = zy \Rightarrow Qx = \bar{z}y.$$

This proves part (i).

If $\langle x, y\rangle = -\bar{\omega}\langle y, x\rangle$, then

$$\langle \omega x, izy\rangle = -i\omega \bar{z}\langle x, y\rangle = iz\bar{\omega}\langle y, x\rangle = \langle izy, \omega x\rangle.$$

Therefore, $\langle \omega x, izy\rangle$ is real. Hence once again there exists a Householder reflector Q such that

$$Q(\omega x) = izy \Rightarrow Qx = i\bar{z}y$$

and the proof of part (ii) follows. $\square$

Theorem 4.1 *Let $P(\lambda)$ be a complex $*$-palindromic or $*$-antipalindromic polynomial. Then all pairs $(\tilde{\lambda}, \tilde{x})$, where $\tilde{\lambda} \in \mathbb{C}$, $|\tilde{\lambda}| = 1$, and $\tilde{x} \in \mathbb{C}^n$, have the same normwise backward error with respect to both arbitrary and structure preserving perturbations.*

Proof: Let $P(\lambda)$ be a complex $*$-palindromic polynomial of degree m. Let $r := P(\tilde{\lambda})\tilde{x}$ and $\omega := (\tilde{\lambda})^m$. Then $|\omega| = 1$ and
$\overline{\tilde{x}^* r} = \tilde{x}^* P(\tilde{\lambda})^* \tilde{x} = \tilde{x}^* P^*(\bar{\tilde{\lambda}})\tilde{x} = \tilde{x}^*(\bar{\tilde{\lambda}})^m P(1/\bar{\tilde{\lambda}})\tilde{x} = \bar{\omega}\tilde{x}^* P(\tilde{\lambda})\tilde{x} = \bar{\omega}\tilde{x}^* r.$
Thus, $\overline{\langle r, \tilde{x}\rangle} = \bar{\omega}\langle r, \tilde{x}\rangle \Rightarrow \langle \tilde{x}, r\rangle = \bar{\omega}\langle r, \tilde{x}\rangle$. Let $z := \sqrt{\frac{1+Re\omega}{2}} + i\sqrt{\frac{1-Re\omega}{2}}$. Then $|z| = 1$ and $\omega\bar{z} = z$. Therefore by part (i) of Lemma 4.1, there exists a reflector Q such that $Q(\tilde{x}) = \bar{z}r\dfrac{\|\tilde{x}\|}{\|r\|}$. Let $H := z\dfrac{\|r\|}{\|\tilde{x}\|}Q$. Then $H\tilde{x} = r$ and $H^* = \bar{z}\dfrac{\|r\|}{\|\tilde{x}\|}Q$. Define $\Delta A_k := \dfrac{(\bar{\tilde{\lambda}})^k}{m+1}H$ for all $k = 0 : m$. Then
$(\Delta A_{m-k})^* = \dfrac{\tilde{\lambda}^{m-k}\bar{z}\|r\|}{\|\tilde{x}\|(m+1)}Q = \dfrac{\bar{\tilde{\lambda}}^k \omega\bar{z}\|r\|}{\|\tilde{x}\|(m+1)}Q = \dfrac{(\bar{\tilde{\lambda}})^k z\|r\|}{\|\tilde{x}\|(m+1)}Q = \dfrac{(\bar{\tilde{\lambda}})^k}{m+1}H = \Delta A_k$
for all $k = 0 : m$. Therefore, $\Delta P(\lambda) := \Sigma_{k=0}^m \dfrac{(\lambda\bar{\tilde{\lambda}})^k}{m+1}H$ is $*$-palindromic. Now $P(\tilde{\lambda})\tilde{x} - \Delta P(\tilde{\lambda})\tilde{x} = r - H\tilde{x} = 0$. Note that for all $k = 0 : m$

$$\|\Delta A_k\| = \frac{\|r\|}{\|\tilde{x}\|(m+1)} = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|\Sigma_{k=0}^m|\tilde{\lambda}|^k}.$$

Hence if $\eta^{*-pal}(\tilde{\lambda}, \tilde{x}, P)$ denotes the backward error with respect to $*$-palindromic perturbations, then $\eta^{*-pal}(\tilde{\lambda}, \tilde{x}, P) \leq \dfrac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|\Sigma_{k=0}^m|\tilde{\lambda}|^k} = \eta(\tilde{\lambda}, \tilde{x}, P)$. Since $\eta^{*-pal}(\tilde{\lambda}, \tilde{x}, P) \geq \eta(\tilde{\lambda}, \tilde{x}, P)$, always holds, it follows

$$\eta^{*-pal}(\tilde{\lambda}, \tilde{x}) = \eta(\tilde{\lambda}, \tilde{x}, P) = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|\Sigma_{k=0}^m|\tilde{\lambda}|^k}.$$

If $P(\lambda)$ is $*$-antipalindromic, then, $\overline{\tilde{x}^* r} = \tilde{x}^* P(\tilde{\lambda})^* \tilde{x} = -(\bar{\tilde{\lambda}})^m \tilde{x}^* P(1/\bar{\tilde{\lambda}})\tilde{x} = -\bar{\omega}\tilde{x}^* P(\tilde{\lambda})\tilde{x} = -\bar{\omega}\langle \tilde{x}, r\rangle$. Thus $\langle r, \tilde{x}\rangle = -\bar{\omega}\langle \tilde{x}, r\rangle$. By part (ii) of Lemma 4.1, there exists a Householder reflector Q such that $Q\tilde{x} = i\bar{z}r\frac{\|\tilde{x}\|}{\|r\|}$ where z is chosen as before so that $\omega\bar{z} = z$. Let $H := -iz\frac{\|r\|}{\|\tilde{x}\|}Q$ and $\Delta A_k := \frac{(\bar{\tilde{\lambda}})^k}{m+1}H$ for all $k = 0 : m$. Then $H\tilde{x} = r$, and

$$(\Delta A_{m-k})^* = \frac{\tilde{\lambda}^{m-k}}{m+1}\left(i\bar{z}\frac{\|r\|}{\|\tilde{x}\|}Q\right) = \frac{i\omega\bar{z}\bar{\tilde{\lambda}}^k\|r\|}{(m+1)\|\tilde{x}\|}Q = \frac{(\bar{\tilde{\lambda}})^k}{m+1}\left(iz\frac{\|r\|}{\|\tilde{x}\|}Q\right) = -\frac{(\bar{\tilde{\lambda}})^k}{m+1}H = -\Delta A_k$$

for all $k = 0 : m$. Therefore $\Delta P(\lambda) = \frac{\Sigma_{k=0}^m (\lambda\bar{\tilde{\lambda}})^k}{m+1}H$ is $*$-antipalindromic and $P(\tilde{\lambda})\tilde{x} - \Delta P(\tilde{\lambda})\tilde{x} = r - H\tilde{x} = 0$. Since $\|\Delta A_k\| = \frac{\|r\|}{\|\tilde{x}\|(m+1)} = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|(m+1)}$ for all $k = 0 : m$ it follows as in the previous case that

$$\eta^{*-antipal}(\tilde{\lambda}, \tilde{x}, P) = \eta(\tilde{\lambda}, \tilde{x}, P) = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|(m+1)} = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|\Sigma_{k=0}^m|\tilde{\lambda}|^k}. \quad \square$$

As already seen in the case of the corresponding results for the condition number in the previous section, the above Theorem also holds when $*$ is replaced by the adjoint say, $\star$, with respect to a sesquilinear scalar product defined via a Hermitian or skew Hermitian matrix that is also unitary.

Corollary 4.1 *Let $\star$ denote the adjoint with respect to the sesquilinear scalar product $\langle x, y\rangle_M = x^* M y$, where M is an unitary matrix which is either Hermitian or skew Hermitian. If P is a $\star$-palindromic or $\star$-antipalindromic polynomial then the backward error of the approximate eigenpair $(\tilde{\lambda}, \tilde{x})$ where $\tilde{\lambda} \in \mathbb{C}, |\tilde{\lambda}| = 1$, and $\tilde{x} \in \mathbb{C}^n$, is the same for both structure preserving and arbitrary perturbations.*

Proof: Since, M is a unitary matrix, and the 2-norm is unitarily invariant, it follows from the definition of backward error it that

$$\eta(\lambda, x, P) = \eta(\lambda, x, MP) \quad \text{and} \quad \eta^S(\lambda, x, P) = \eta^{S'}(\lambda, x, MP) \tag{10}$$

where S and S' denote the structures of the polynomials $P(\lambda)$ and $MP(\lambda)$ respectively. As already noted in the proof of Corollary 3.1, if $P(\lambda)$ is $\star$-palindromic then $MP(\lambda)$ is a $*$-palindromic polynomial if M is Hermitian and a $*$-antipalindromic polynomial if M is skew Hermitian. In either case, by Theorem 4.1 we have

$$\eta^S(\tilde{\lambda}, \tilde{x}, MP) \;=\; \eta(\tilde{\lambda}, \tilde{x}, MP) \tag{11}$$

where S denotes $*$-palindromic or $*$-antipalindromic according as whether MP is $*$-palindromic or $*$-antipalindromic. The second equality in (10) implies that

$$\text{either } \eta^{\star-pal}(\tilde{\lambda}, \tilde{x}, P) = \eta^{*-pal}(\tilde{\lambda}, \tilde{x}, MP) \text{ or } \eta^{\star-pal}(\tilde{\lambda}, \tilde{x}, MP) = \eta^{*-antipal}(\tilde{\lambda}, \tilde{x}, P).$$

Now this together with (11) and the first equality in (10), implies that

$$\eta^{\star-pal}(\tilde{\lambda}, \tilde{x}, P) = \eta(\tilde{\lambda}, \tilde{x}, P).$$

The proof of the equality $\eta^{\star-antipal}(\tilde{\lambda}, \tilde{x}, P) = \eta(\tilde{\lambda}, \tilde{x}, P)$ follows in a similar manner. $\square$

Remark 4.1 *If $P(\lambda)$ is $\star$-palindromic and M is Hermitian or $P(\lambda)$ is $\star$-antipalindromic and M is skew Hermitian, then $MP(\lambda)$ is $*$-palindromic, and as shown in the proof of Theorem 4.1, if $\tilde{\lambda} \in \mathbb{C}, |\tilde{\lambda}| = 1$ and $\tilde{x} \in \mathbb{C}^n$, then*

$$\langle \tilde{x}, MP(\tilde{\lambda})\tilde{x} \rangle = (\tilde{\lambda})^m \langle MP(\tilde{\lambda})\tilde{x}, \tilde{x} \rangle$$

where m is the degree of $P(\lambda)$. Since M is either Hermitian or skew Hermitian, we have

$$\langle M(\tilde{x}), P(\tilde{\lambda})\tilde{x} \rangle = (\tilde{\lambda})^m \langle P(\tilde{\lambda})\tilde{x}, M(\tilde{x}) \rangle.$$

Therefore applying Lemma 4.1 (i), there exists a Householder reflector Q such that $Q(M\tilde{x}) = \bar{\tilde{z}} \dfrac{\|M\tilde{x}\|}{\|P(\tilde{\lambda})\tilde{x}\|} P(\tilde{\lambda})\tilde{x}$ where $\tilde{z} \in \mathbb{C}$, such that $|\tilde{z}| = 1$ and $(\tilde{\lambda})^m \bar{\tilde{z}} = \tilde{z}$. Noting that $\|M\tilde{x}\| = \|\tilde{x}\|$, since the 2-norm is unitarily invariant and setting $\tilde{Q} := QM$ we have, $\tilde{Q}(\tilde{x}) = \bar{\tilde{z}} \dfrac{\|\tilde{x}\|}{\|P(\tilde{\lambda})\tilde{x}\|} P(\tilde{\lambda})\tilde{x}$. Evidently $\|\tilde{Q}\| = 1$ since M is a unitary matrix. Also $\tilde{Q}$ is $\star$-Hermitian or $\star$-skew Hermitian according as whether M is Hermitian or skew Hermitian. Therefore, setting $\Delta P(\lambda) := \Sigma_{k=0}^m \dfrac{(\bar{\tilde{\lambda}}\lambda)^k}{m+1} \tilde{H}$, where $\tilde{H} := \tilde{z} \dfrac{\|P(\tilde{\lambda})\tilde{x}\|}{\|\tilde{x}\|} \tilde{Q}$, we observe that $\tilde{H}\tilde{x} = P(\tilde{\lambda})\tilde{x}$ and $\Delta P(\lambda)$ is $\star$-palindromic if M is Hermitian and $\star$-antipalindromic if M is skew Hermitian. Also we have $(P(\tilde{\lambda}) - \Delta P(\tilde{\lambda}))\tilde{x} = P(\tilde{\lambda})\tilde{x} - H\tilde{x} = P(\tilde{\lambda})\tilde{x} - P(\tilde{\lambda})\tilde{x} = 0$ and $\|\Delta A_k\| = \dfrac{\|P(\tilde{\lambda})\tilde{x}\|}{(m+1)\|\tilde{x}\|} = \eta(\tilde{\lambda}, \tilde{x}, P)$, for all $k = 0 : m$. This construction suggests an alternative proof of Corollary 4.1 on the lines of Theorem 4.1.

Next we consider the T-palindromic and T-antipalindromic polynomials and show that for these polynomials, the numbers 1 and -1 have the same normwise backward errors as approximate eigenvalues of these polynomials with respect to both structure preserving and arbitrary perturbations.

Theorem 4.2 *Let $P(\lambda)$ be a real or complex polynomial which is either T-palindromic or T-antipalindromic. Then we have,*

$$\eta^S(1, P) = \eta(1, P) \text{ and } \eta^S(-1, P) = \eta(-1, P)$$

where S indicates either T-palindromic or T-antipalindromic structure according as whether $P(\lambda)$ is T-palindromic or T-antipalindromic. In particular suppose that the coefficient matrices of $P(\lambda)$ are of odd dimension. Then we have the following:

(i) If $P(\lambda)$ is T-palindromic of odd degree, $\eta^{T-pal}(-1, P) = \eta(-1, P) = 0$.

(ii) If $P(\lambda)$ is T-antipalindromic of even degree, $\eta^{T-antipal}(-1, P) = \eta(-1, P) = 0$.

(iii) Finally if $P(\lambda)$ is T-antipalindromic, $\eta^{T-antipal}(1, P) = \eta(1, P) = 0$.

Proof: As has been already observed in Theorem 3.2 if the coefficient matrices of $P(\lambda)$ are of odd dimension, then -1 is always an eigenvalue of a T-palindromic polynomial of odd degree or a T-antipalindromic polynomial of even degree while 1 is always an eigenvalue of a T-antipalindromic polynomial. The proofs of (i), (ii) and (iii) follow immediately in view of these facts.

Let $P(\lambda)$ be a T-palindromic polynomial of odd degree say m, whose coefficient matrices are of even dimension such that 0 is not an eigenvalue of $P(\lambda)$. Then $P(-1)$ is a skew symmetric matrix of even dimension and rank say $2r$. By Theorem 3.5 of [44] $P(-1)$ has the decomposition

$$P(-1) \;=\; U\Sigma_{2r}U^T,\ \Sigma_{2r} = \mathrm{diag}(S_1, \ldots, S_r), S_i = \begin{pmatrix} 0 & s_i \\ -s_i & 0 \end{pmatrix}, \qquad (12)$$

where U is unitary and $s_i > 0$, $i = 1, 2, \ldots, r$, are the singular values of $P(-1)$. Thus if $u_1, u_2, \ldots, u_i, u_{i+1}, \ldots u_r, u_{r+1}$ be the columns of the matrix U, then $P(-1) = \Sigma_{i=1}^{r} s_i(u_i u_{i+1}^T - u_{i+1}u_i^T)$. Suppose without loss of generality that s_r is the smallest singular value of $P(-1)$ and let $E := -s_r(u_r u_{r+1}^T - u_{r+1}u_r^T)$. Then evidently $E^T = -E$ and $\|E\| = s_r$. Also $P(-1) + E = U \begin{pmatrix} \Sigma_{2r-2} & 0 \\ 0 & 0 \end{pmatrix} U^T$ where the 0 in the lower right corner is a zero matrix of size 2. Thus E is a skew symmetric perturbation to $P(-1)$ which induces a zero eigenvalue. Further, in view of the decomposition (12), E is real if the coefficient matrices of P are real. Let $x \in \mathbb{C}^n$ be such that $(P(-1) + E)x = 0$. If $\Delta A_k := (-1)^k E/(m+1)$, $k = 0 : m$, then

$$\Delta A_{m-k}^T = (-1)^{m-k}E^T/(m+1) = (-1)^k E/(m+1) = \Delta A_k$$

which implies that $\Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k \Delta A_k$ is a T-palindromic polynomial with $\|\Delta A_k\| = s_r/(m+1), k = 0 : m$, which is real or complex depending upon whether the coefficient matrices of $P(\lambda)$ are real or complex. Now, $(P(-1) + \Delta P(-1))x = (P(-1) + E)x = 0$. Therefore x is an eigenvector of the perturbed polynomial $(P + \Delta P)(\lambda)$ corresponding to the eigenvalue -1 and it follows that

$$\eta^{T-pal}(-1, P) \le \|E\|/(m+1) = s_r/(m+1) = \|P(-1)^{-1}\|^{-1}/(m+1) = \eta(-1, P).$$

Since the reverse inequality always holds, we have $\eta^{T-pal}(-1, P) = \eta(-1, P)$.

Next suppose that $P(\lambda)$ is a T-antipalindromic polynomial with coefficient matrices of even dimension. Then $P(1)$ is a skew -symmetric matrix of even dimension while the same holds for $P(-1)$ if $P(\lambda)$ is of even degree. In both cases, the decomposition (12) may be used to construct a skew symmetric matrix E such that $\|E\|$ is the smallest singular value of $P(1)$ or $P(-1)$ as the case may be. The proofs of the equalities $\eta^{T-antipal}(-1, P) = \eta(-1, P)$ and $\eta^{T-antipal}(1, P) = \eta(1, P)$ follow by defining $\Delta P(\lambda) := \Sigma_{k=0}^{m} \lambda^k (-1)^k E/(m+1)$ in the former case and $\Delta P(\lambda) := \lambda^k E/(m+1)$ in the latter so that both $P(-1) + \Delta P(-1)$ and $P(1) + \Delta P(1)$ are singular matrices and the inequalities $\eta^{T-antipal}(-1, P) \le \eta(-1, P)$ and $\eta^{T-antipal}(1, P) \le \eta(1, P)$ hold.

If $P(\lambda)$ is a T-palindromic polynomial then $P(1)$ is evidently a symmetric matrix. Suppose that 1 is not an eigenvalue of $P(\lambda)$. Then $P(1)$ has the symmetric singular value decomposition (see, Theorem 2.1 of [2])

$$P(1) \quad = \quad U\Sigma U^T,\, \Sigma = \mathrm{diag}(s_1, s_2, \ldots, s_n) \tag{13}$$

where U is an unitary matrix and $s_1, s_2, \ldots, s_n$ are the singular values of $P(1)$. Without loss of generality assume that s_n is the smallest singular value of $P(1)$. Thus if u_n be the last column of U, then it is a normalised left as well right singular vectors of s_n. Let $E = -s_n u_n u_n^T$. Then E is such that $E^T = E$, $\|E\| = s_n$, and $P(1)+E$ is a singular matrix. Let $x \in \mathbb{C}^n$ such that $(P(1)+E)x = 0$. For $k = 0 : m$, let $\Delta A_k := E/(m+1)$. Then $(\Delta A_{m-k})^T = E^T/(m+1) = E/(m+1) = \Delta A_k$ for all $m = 0 : k$. This implies that $\Delta P(\lambda) := \left(\Sigma_{k=0}^m \lambda^k\right) E/(m+1)$ is T-palindromic. Now $(P(1) + \Delta P(1))x = (P(1) + E)x = 0$. Thus x is a right eigenvector of the perturbed polynomial $(P+\Delta P)(\lambda)$ corresponding to the eigenvalue 1. Since $\Delta P(\lambda)$ is T-palindromic, it follows that

$$\eta^{T-pal}(1, P) \leq \|E\|/(m+1) = s_n/(m+1) = \frac{1}{(m+1)\|(P(1))^{-1}\|} = \eta(1, P).$$

Since the reverse inequality holds from definition of $\eta^{T-pal}(\lambda, P)$, we have $\eta^{T-pal}(1, P) = \eta(1, P)$.

If $P(\lambda)$ is a real T-palindromic polynomial, then in view of the decomposition (13), u_n is a real vector. If E is defined in exactly the same way as above, then E is a real symmetric matrix and the proof follows by using exactly the same arguments as above.

Now it remains to show that $\eta^{T-antipal}(-1, P) = \eta(-1, P)$ when $P(\lambda)$ is a T-antipalindromic polynomial of odd degree and $\eta^{T-pal}(-1, P) = \eta(-1, P)$ if $P(\lambda)$ is a T-palindromic polynomial of even degree. In both cases $P(-1)$ is evidently a symmetric matrix. So once again choosing s_n to be the smallest singular value of $P(-1)$ we use the decomposition (13) of $P(-1)$ to construct the symmetric matrix $E = -s_n u_n u_n^T$ with norm s_n so that $P(-1) + E$ is a singular matrix. Let $x \in \mathbb{C}^n$ such that $(P(-1)+E)x = 0$. In particular if $P(\lambda)$ is a polynomial over real matrices, then E is a real symmetric matrix. Now defining $\Delta A_k := (-1)^k E/(m+1)$, we have

$$(\Delta A_{m-k})^T = (-1)^{(m-k)}\frac{E^T}{m+1} = \begin{cases} -\Delta A_k & \text{if m is odd} \\ \Delta A_k & \text{if m is even} \end{cases}$$

which implies that $\Delta P(\lambda) := \left(\Sigma_{k=0}^m \lambda^k\right) E/(m+1)$ is T-antipalindromic if m is odd and T-palindromic if m is even. Also in both cases, $(P + \Delta P)(-1)x = P(-1)x + \Delta P(-1)x = P(-1)x + Ex = 0$. Therefore in both cases, there exists a structure preserving perturbation ΔP which when applied to P, makes -1 an eigenvalue of the perturbed polynomial. Since, $\|\Delta A_k\| = \dfrac{\|E\|}{m+1}$, for all $k = 0 : m$, we have,

$$\eta^S(-1, P) \leq \frac{\|E\|}{m+1} = \frac{s_n}{m+1} = \frac{1}{\|P(-1)^{-1}\|(m+1)} = \eta(-1, P).$$

Hence the equalities $\eta^{T-pal}(-1,P) = \eta(-1,P)$ and $\eta^{T-antipal}(-1,P) = \eta(-1,P)$ follow. $\square$

Finally we show that the previous result holds with appropriate modifications when T is replaced by the transpose with respect to a bilinear scalar product induced by a symmetric or skew symmetric orthogonal matrix.

Theorem 4.3 *Let $\star$ denote the transpose with respect to the scalar product $\langle x, y\rangle_M := x^T M y$ $x.y \in \mathbb{F}^n$, where M is an orthogonal matrix which is either symmetric or skew symmetric. If $P(\lambda)$ is a $\star$-palindromic or $\star$-antipalindromic polynomial then*

$$\eta^S(1,P) = \eta(1,P) \text{ and } \eta^S(-1,P) = \eta(-1,P)$$

where S denotes $\star$-palindromic or $\star$-antipalindromic structure depending upon the structure of $P(\lambda)$. In particular if the coefficient matrices of $P(\lambda)$ are of odd dimension then we have the following.
(i) If M is a Hermitian matrix then $\eta^{\star-antipal}(1,P) = \eta(1,P)$ and $\eta^S(-1,P) = \eta(-1,P)$ if $P(\lambda)$ is either a $\star$-palindromic polynomial of odd degree or a $\star$-antipalindromic polynomial of even degree.
(ii) If M is a skew-Hermitian matrix then $\eta^{\star-pal}(1,P) = \eta(1,P)$ and $\eta^S(-1,P) = \eta(-1,P)$ if $P(\lambda)$ is either a $\star$-antipalindromic polynomial of odd degree or a $\star$-palindromic polynomial of even degree.

Proof: The proofs of (i) and (ii) follow from Theorem 3.5. In all other cases the proofs of the equalities $\eta^S(1,P) = \eta(1,P)$ and $\eta^S(-1,P) = \eta(-1,P)$ follow either by applying Theorem 4.2 to the polynomial $MP(\lambda)$ and using the equalities in (10) or by replacing the matrix E by the matrix ME in all parts of the proof of Theorem 4.2 to construct appropriate structure preserving perturbations to the polynomials. $\square$

To end this section, we note that for all the structured polynomials considered in this section, the structured and unstructured backward errors of the numbers 0 and ∞ as approximate eigenvalues of the corresponding polynomials must be equal.

4.2 The $\star$-even and $\star$-odd polynomials

Next we show that if $P(\lambda)$ is a complex $*$-odd or $*$-even polynomial, then all approximate eigenpairs $(\tilde{\lambda}, \tilde{x})$ where $\tilde{\lambda}$ is a purely imaginary number have the same backward error with respect to structure preserving and arbitrary perturbations. Once again in each case the assertion is proved by constructing an appropriate structure preserving perturbation to $P(\lambda)$, so that the 2-norm of its coefficient matrices are all equal to the corresponding value of the normwise backward error $\eta(\tilde{\lambda}, \tilde{x}, P)$ for arbitrary perturbations as given by Theorem 2.2.

Theorem 4.4 *Let $P(\lambda)$ be a $*$-even or $*$-odd polynomial. Then any approximate eigenpair $(\tilde{\lambda}, \tilde{x})$ where $\tilde{\lambda} \in \mathbb{C}, Re\tilde{\lambda} = 0$ or $\tilde{\lambda} = \infty$, and $\tilde{x} \in \mathbb{C}^n$, has the same normwise backward error with respect to structure preserving and arbitrary perturbations.*

Proof: We first prove the result for the case when $\tilde{\lambda} \in \mathbb{C}, Re\tilde{\lambda} = 0$. Let $P(\lambda)$ be a $*$-even polynomial of degree m and $r := P(\tilde{\lambda})\tilde{x}$. Then, $\overline{\langle r, \tilde{x}\rangle} = \overline{\tilde{x}^* P(\tilde{\lambda})\tilde{x}} = \tilde{x}^*(P(\tilde{\lambda}))^*\tilde{x} = \tilde{x}^* P^*(\bar{\tilde{\lambda}})\tilde{x} = \tilde{x}^* P^*(-\tilde{\lambda})\tilde{x} = \tilde{x}^* P(\tilde{\lambda})\tilde{x} = \tilde{x}^* r = \langle r, \tilde{x}\rangle$. Applying Lemma 4.1, there exists a Householder reflector Q such that $Q\tilde{x} = r\dfrac{\|\tilde{x}\|}{\|r\|}$. Let $H := \dfrac{\|r\|}{\|\tilde{x}\|}Q$. Then $H^* = H$ and $H\tilde{x} = r$. Define $\Delta A_k := \left(\dfrac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \dfrac{H}{\alpha}$ for all $k = 0 : m$ where $\alpha = \Sigma_{k=0}^m |\tilde{\lambda}|^k$. Then for all odd values of k from 1 to m,

$$(\Delta A_k)^* = \left(\frac{\tilde{\lambda}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha} = -\left(\frac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha} = -\Delta A_k$$

and for $k = 0$ as well as for all even values of k from 2 to m,

$$(\Delta A_k)^* = \left(\frac{\tilde{\lambda}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha} = \left(\frac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha} = \Delta A_k.$$

Hence $\Delta P(\lambda) := \Sigma_{k=0}^m \left(\dfrac{\lambda\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \dfrac{H}{\alpha}$ is $*$-even. Also $P(\tilde{\lambda})\tilde{x} - \Delta P(\tilde{\lambda})\tilde{x} = r - H\tilde{x} = 0$. Now for $k = 0 : m$,

$$\|\Delta A_k\| = \frac{\|r\|}{\alpha\|\tilde{x}\|} = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\alpha\|\tilde{x}\|}.$$

Therefore, if $\eta^{*-even}(\tilde{\lambda}, \tilde{x}, P)$ denotes the normwise backward error with respect to perturbations that are also $*$-even, then

$$\eta^{*-even}(\tilde{\lambda}, \tilde{x}, P) \leq \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\alpha\|\tilde{x}\|} = \eta(\tilde{\lambda}, \tilde{x}, P).$$

Since the reverse of the above inequality already holds, it follows that $\eta^{*-even}(\tilde{\lambda}, \tilde{x}, P) = \eta(\tilde{\lambda}, \tilde{x}, P)$.

Now suppose that $P(\lambda)$ is a $*$-odd polynomial. Once again if $(\tilde{\lambda}, \tilde{x})$ be an approximate eigenpair where $Re\tilde{\lambda} = 0$, then, $\overline{\tilde{x}^* r} = \tilde{x}^* P(\tilde{\lambda})^*\tilde{x} = \tilde{x}^* P^*(\bar{\tilde{\lambda}})\tilde{x} = \tilde{x}^* P^*(-\tilde{\lambda})\tilde{x} = -\tilde{x}^* P(\tilde{\lambda})\tilde{x} = -\tilde{x}^* r$. Thus $< \tilde{x}, r > = - < r, \tilde{x} >$. Since, $-\bar{i} = i$, we apply Lemma 4.1(i) with $\omega = -1$ and $z = i$, to obtain a Householder reflector Q such that $Q\tilde{x} = -ir\dfrac{\|\tilde{x}\|}{\|r\|}$. Let $H := i\dfrac{\|r\|}{\|\tilde{x}\|}Q$. Then $H^* = -H$, and $H\tilde{x} = r$. Define $\Delta A_k := \left(\frac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \dfrac{H}{\alpha}$ for $k = 0 : m$, where $\alpha := \Sigma_{k=0}^m |\tilde{\lambda}|^k$. Then for all odd values of k from 1 to m,

$$(\Delta A_k)^* = \left(\frac{\tilde{\lambda}}{|\tilde{\lambda}|}\right)^k \frac{H^*}{\alpha} = -\left(\frac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \frac{-H}{\alpha} = \left(\frac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha} = \Delta A_k$$

and for $k = 0$ and all even values of k from 2 to m,

$$(\Delta A_k)^* = \left(\frac{\tilde{\lambda}}{|\tilde{\lambda}|}\right)^k \frac{H^*}{\alpha} = -\left(\frac{\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha} = -\Delta A_k.$$

Therefore, $\Delta P(\lambda) = \Sigma_{k=0}^{m} \left(\frac{\lambda\bar{\tilde{\lambda}}}{|\tilde{\lambda}|}\right)^k \frac{H}{\alpha}$ is a $*$-odd polynomial and $P(\tilde{\lambda})\tilde{x} - \Delta P(\tilde{\lambda})\tilde{x} = r - H\tilde{x} = 0$. Now for all $k = 0 : m$, $\|\Delta A_k\| = \frac{\|r\|}{\|\tilde{x}\|\alpha} = \frac{\|P(\tilde{\lambda})\tilde{x}\|}{\alpha\|\tilde{x}\|}$. Therefore, if $\eta^{*-odd}(\tilde{\lambda}, \tilde{x}, P)$ denotes the normwise backward error with respect to perturbations that are also $*$-odd polynomials, then

$$\eta^{*-odd}(\tilde{\lambda}, \tilde{x}, P) \leq \frac{P(\tilde{\lambda})\tilde{x}\|}{\alpha\|\tilde{x}\|} = \eta(\tilde{\lambda}, \tilde{x}, P)$$

and the proof follows since the reverse of the above inequality already holds.

Observe that the backward error both structured and unstructured of $(\infty, \tilde{x})$ as an approximate eigenpair of $P(\lambda)$ is equal to the backward error of $(0, \tilde{x})$ as an approximate eigenpair of $revP(\lambda)$. The proof for the case when $\tilde{\lambda} = \infty$, now follows from above by recalling that $P(\lambda)$ and $revP(\lambda)$ have the same $*$-even or $*$-odd structure if the degree of $P(\lambda)$ is even while $revP(\lambda)$ has $*$-even structure if $P(\lambda)$ has $*$-odd structure and vice versa if the degree of $P(\lambda)$ is odd. Note that an appropriate structure preserving perturbation $\Delta P(\lambda)$ that makes $(\infty, \tilde{x})$ an exact eigenpair of the perturbed polynomial $(P+\Delta P)(\lambda)$ may be constructed by following the above proof to construct a structure preserving perturbation to $revP(\lambda)$ that makes $(0, \tilde{x})$ an exact eigenpair of the perturbed polynomial and then taking the reversal of the perturbation polynomial. $\square$

We observe that the above result also holds if $*$ is replaced by the adjoint with respect to a sesquilinear scalar product induced by a Hermitian or skew Hermitian matrix.

Corollary 4.2 *If $P(\lambda)$ is $\star$-even or $\star$-odd, where $\star$ denotes the adjoint with respect to a sesquilinear scalar product $\langle x, y\rangle_M := x^* M y, x, y \in \mathbb{C}^n$, then an approximate eigenpair $(\tilde{\lambda}, \tilde{x})$, where $\tilde{\lambda} \in \mathbb{C}, Re\tilde{\lambda} = 0$, or $\tilde{\lambda} = \infty$, and $\tilde{x} \in \mathbb{C}^n$, has the same backward error with respect to both structure preserving and arbitrary perturbations if M is a Hermitian or skew Hermitian unitary matrix.*

Proof: Recall that the polynomial $MP(\lambda)$ is $*$-even if either $P(\lambda)$ is $\star$-even and M is Hermitian or $P(\lambda)$ is $\star$-odd and M is skew Hermitian. On the other hand MP is a $*$-odd polynomial if either $P(\lambda)$ is $\star$-even and M is skew Hermitian or $P(\lambda)$ is $\star$-odd and M is Hermitian. Therefore the proof in all the cases follows by applying Theorem 4.4 for the polynomial $MP(\lambda)$ and using the equalities in (10). $\square$

Remark 4.2 *An alternative proof of the above Corollary which proceeds by explicitly constructing structure preserving perturbations in all the cases may be obtained by replacing the Householder reflector Q in the proof of Theorem 4.4 by MQ As in the case of Corollary 4.1*

Note 4.1 *We note that if the matrix M in Corollary 4.2 is replaced by $J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, then given any Hamiltonian matrix A, $A - \lambda I$ is a $\star$-odd polynomial* *and it follows that all approximate eigenvalues of A lying on the imaginary axis have the same backward error with respect to both structured and arbitrary perturbations.*

Finally we consider the T-even and T-odd polynomials and show that the number 0 and ∞ have the same backward error as an approximate eigenvalue of these polynomials with respect to both structured and arbitrary perturbations.

Theorem 4.5 *Let $P(\lambda)$ be a real or complex polynomial of degree m which is either T-even or T-odd. Then*

$$\eta^S(0,P) = \eta(0,P) \text{ and } \eta^S(\infty,P) = \eta(\infty,P).$$

In particular if the coefficient matrices of $P(\lambda)$ are of odd dimension, then $\eta^{T-odd}(0,P) = 0 = \eta(0,P)$ if $P(\lambda)$ is a T-odd polynomial whereas $\eta^S(\infty,P) = 0 = \eta(\infty,P)$ if $P(\lambda)$ is either a T-odd polynomial of even degree or a T-even polynomial of odd degree.

Proof: If $P(\lambda)$ is T-odd and the coefficient matrices are of odd dimension, then the proof of $\eta^{T-odd}(0,P) = 0 = \eta(0,P)$ follows immediately from the fact that if $P(\lambda)$ is T-odd then $P(0)$ is a skew symmetric matrix of odd dimension and hence 0 is always an eigenvalue of $P(0)$.

Let $P(\lambda)$ be a T-odd polynomial with coefficient matrices of even dimension say $2r$. Then evidently $P(0)$ is skew symmetric. Suppose that 0 is not an eigenvalue of $P(0)$. Hence $P(0)$ has a decomposition similar to (12) say,

$$P(0) = \tilde{U}\tilde{\Sigma}_{2r}\tilde{U}^T,\ \tilde{\Sigma}_{2r} = \mathrm{diag}(\tilde{S}_1,\ldots,\tilde{S}_r), \tilde{S}_i = \begin{pmatrix} 0 & \tilde{s}_i \\ -\tilde{s}_i & 0 \end{pmatrix},$$

where $\tilde{U}$ is unitary and $\tilde{s}_i > 0$, $i = 1,2,\ldots,r$, are the singular values of $P(0)$. Suppose without loss of generality that $\tilde{s}_r$ is the smallest singular value. Thus if $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_i, \tilde{u}_{i+1}, \ldots \tilde{u}_r, \tilde{u}_{r+1}$ be the columns of the matrix $\tilde{U}$, then applying the same arguments as used in the second paragraph of the proof of Theorem 4.2, we construct the skew symmetric matrix $E := -\tilde{s}_r(\tilde{u}_r\tilde{u}_{r+1}^T - \tilde{u}_{r+1}\tilde{u}_r^T)$ such that $\|E\| = \tilde{s}_r$ and $P(0)+E = \tilde{U}\begin{pmatrix} \tilde{\Sigma}_{2r-2} & 0 \\ 0 & 0 \end{pmatrix}\tilde{U}^T$ where the 0 in the lower right corner is a zero matrix of size 2. Evidently, E is real if the coefficient matrices of $P(\lambda)$ are real. Let $x \in \mathbb{C}^n$ be such that $(P(0)+E)x = 0$. Now given any symmetric matrix F of size $2r$ which we choose to be real if $P(\lambda)$ is real, let $\Delta A_k := E$ if $k = 0$ or k is any even number from 2 to m and $\Delta A_k := \tilde{s}_r F/\|F\|$ for all odd indices k from 1 to m. Then $\Delta A_k^T = -\Delta A_k$ for $k = 0$ and all even numbers k from 2 to m while $\Delta A_k^T = \Delta A_k$ for all odd values of k from 1 to m. Thus if $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k$, then $\Delta P(\lambda)$ is a T-odd polynomial with $\Delta A_k = \tilde{s}_r, k = 0 : m$, which is real or complex according as whether P is real or complex and $(P(0) + \Delta P(0))x = (P(0)+E)x = 0$. Thus 0 is an eigenvalue of the perturbed polynomial, $(P + \Delta P)(\lambda)$ with corresponding right eigenvector x. This implies that

$$\eta^{T-odd}(0,P) \le \tilde{s}_r = \|P(0)^{-1}\|^{-1} = \eta(0,P)$$

and the proof follows.

If $P(\lambda)$ is a T-even polynomial, then evidently $P(0)$ is a symmetric matrix. We use the symmetric SVD of $P(0)$ similar to (13) and apply arguments identical to those used in the proof of Theorem 4.2 to construct a symmetric matrix $E = -\tilde{s}_n\tilde{u}\tilde{u}^T$ where $\tilde{s}_n$ is the smallest singular value of $P(0)$ and $\tilde{u}$ is a corresponding left as well as right normalized singular vector. Evidently E is real if the coefficients of $P(\lambda)$ are real matrices. Now $\|E\| = \tilde{s}_n$ and $P(0) + E$ is a singular matrix. Let $x \in \mathbb{C}^n$ such that $(P(0) + E)x = 0$. Now taking any skew symmetric matrix F, which may be chosen to be real if P is real, let $\Delta A_k := \tilde{s}_n F/\|F\|$ for all odd values of k from 1 to m and $\Delta A_k := E$ for $k = 0$ and all even values of k from 2 to m. Then it follows that $(\Delta A_k)^T = -\Delta A_k$ if k is odd and $(\Delta A_k)^T = \Delta A_k$ if $k = 0$ or if k is even. Therefore $\Delta P(\lambda) := \Sigma_{k=0}^m \lambda^k \Delta A_k$ is a T-even polynomial with $\|\Delta A_k\| = \tilde{s}_n = \|P(0)^{-1}\|^{-1}$ for all $k = 0 : m$ and $(P(0) + \Delta P(0))x = (P(0) + E)x = 0$. Therefore $\eta^{T-even}(0, P) \leq \|P(0)^{-1}\|^{-1} = \eta(0, P)$.

Since $P(\lambda)$ has an eigenvalue at ∞ if and only if 0 is an eigenvalue of $revP(\lambda)$, the proofs of the equalities for the backward error of ∞ follow from those proved above for the backward error of 0 in view of the fact $revP(\lambda)$ has the same structure as $P(\lambda)$ if the degree of $P(\lambda)$ is even while it has T-even structure if $P(\lambda)$ is T-odd and vice versa if $P(\lambda)$ is of odd degree. Hence the proof. □

As in all previous cases, the above results with appropriate modifications extend to the case when T is replaced by the transpose with respect to a bilinear scalar product induced by an orthogonal symmetric or skew symmetric matrix.

Theorem 4.6 *Let $P(\lambda)$ be a $\star$-even or $\star$-odd polynomial where $\star$ denotes the transpose with respect to the scalar product $\langle x, y\rangle_M = x^T M y$, M being an orthogonal symmetric or skew symmetric matrix. Then,*

$$\eta^S(0, P) = \eta(0, P) \text{ and } \eta^S(\infty, P) = \eta(\infty, P).$$

In particular if the coefficient matrices of $P(\lambda)$ are of odd dimension, we have the following.
i) If $P(\lambda)$ is $\star$-odd and M is symmetric or $P(\lambda)$ is $\star$-even and M is skew symmetric then $\eta^S(0, P) = 0 = \eta(0, P)$ whereas $\eta^S(\infty, P) = 0 = \eta(\infty, P)$ if the degree of $P(\lambda)$ is even.
ii) If $P(\lambda)$ is a $\star$-even polynomial and M is symmetric or $P(\lambda)$ is a $\star$-odd polynomial and M is skew symmetric, then, $\eta^S(\infty, P) = 0 = \eta(\infty, P)$ if $P(\lambda)$ is of odd degree.

Proof: Since $MP(\lambda)$ is either a $*$-even or $*$-odd polynomial under the given hypothesis, the proofs follow by applying Theorem 4.5 to the polynomial $MP(\lambda)$ and applying the equalities in (10).

An alternative proof of the assertions also follow by replacing the matrices E and F in the proof of Theorem 4.5 by ME and MF respectively to construct structure preserving perturbations to the polynomials and applying identical arguments. □

Conclusion We have shown that certain simple eigenvalues of polynomials having $\star$-palindromic, $\star$-antipalindromic, $\star$-even or $\star$-odd structure have the same normwise condition number with respect to structure preserving and arbitrary perturbations. We have also established that for each of the structures under consideration, their exist sets of complex numbers that have the same backward error with respect to both structure preserving and arbitrary perturbations. All proofs are given by constructing appropriate structure preserving perturbations to the polynomials. Our conjecture is that the equality of the two condition numbers does not hold in general for all simple eigenvalues of these polynomials. Similarly, we conjecture that the subsets of the complex plane for which we prove the equality of the structured and unstructured backward error are optimal in each case in the sense that there exists no larger set where such an equality holds. However a general formula for structured condition number as well as backward error for each of the structures under consideration remains to be formulated. We would like to address these issues in future work.

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