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# Regularization of linear and nonlinear descriptor systems

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**Abstract**

Differential-algebraic equations (DAEs) present today the state-of-the-art in dynamical systems arising from automated modularized modeling in almost all areas of science and engineering. While the modeling becomes more and more convenient, the resulting models are typically not easy to treat with current numerical simulation, control and optimization methods. In many cases a reformulation of the models or even a regularization is necessary to avoid failure of the computational methods. In this contribution we will discuss general DAE control problems and how they can be systematically reformulated and regularized so that the resulting system can be used in control and optimization procedures without much further difficulties.

**Keywords:** differential-algebraic equation, solvability, consistency, regularity, regularization, derivative array, differentiation-index, strangeness-index

**AMS(MOS) subject classification:** 65L80, 49K15, 34H05, 93C10, 93C15,

## 1 Introduction

Modern modeling and simulation packages such as MODELICA (`https://www.modelica.org/`) or MATLAB/SIMULINK (`http://www.mathworks.com`) offer excellent tools for the automated generation of dynamic system models. Modeling is done in a modularized way, based on a network of subsystems which again consists of simple standardized sub-components. Such approaches have been the industrial standard in circuit simulation for decades and have also made their way into industrially used packages in other areas such as multi-physics problems from different physical domains including mechanical, mechatronic, fluidic, thermic, hydraulic, pneumatic, elastic, plastic or electric components [1, 7, 9, 21, 26, 27, 28, 29, 30, 31], These automatically generated models are today used for the simulation, control and optimization of complex technological systems, in particular in the design phase.

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This concept of modeling dynamical systems leads to systems with differential-algebraic character, it is a differential-algebraic equation (DAE) when the components are semi-discretized (i. e., discretized with respect to all variables but the time variable). Such systems contain differential equations that model the dynamical behavior and algebraic equations that model constraints, interface and boundary conditions, conservation laws, as well as balance equations.

When the system arises from automated modeling, then typically all the mathematical difficulties arising in the numerical simulation, control and optimization are pushed onto the numerical methods. A reformulation, remodeling or regularization may be required to deal with these difficulties.

A classical and nowadays again popular approach to overcome these difficulties in numerical simulation methods is to resolve all algebraic constraints and interface conditions (often with computer algebra methods [8]) and to turn the system into an explicit differential equation. For this system then one applies standard integration or control techniques. As simple as it looks, this approach, however, has several drawbacks. First of all, it is known that computer algebra based methods cannot be applied to large scale problems and that they often produce formulas with bad numerical properties. Secondly, this procedure leads to variables that loose their interpretation as physical quantities and furthermore, since the algebraic constraints are not explicitly available any more, often the numerical methods deviate from the constraints and interface conditions and produce physically meaningless results.

A very successful alternative approach [4, 16] uses the system and some of its derivatives to produce an equivalent (so called strangeness-free) system with the same solution set, but where all all explicit and implicit constraints and therefore the complete constraint manifold are available. This approach keeps the physical meaning of all the variables and allows easy initialization. Furthermore, since the constraint manifolds are available, it is easier to guarantee that the numerical solution stays on these manifolds. Related approaches have been developed using dummy variables [23, 24]. The terms minimally extended system or index reduction are also sometimes used. In the context of control and optimization methods, however, further difficulties arise requiring the reformulation, remodeling and regularization of the given problem. It is the purpose of this paper to present these ideas and to discuss their advantages and disadvantages.

In the general nonlinear case the resulting DAE mathematical model can be written in the form

$$F(t, x, \dot{x}, u, y) = 0,\ x(\underline{t}) = \underline{x}. \tag{1}$$

Denoting by $C^0(\mathbb{I}, \mathbb{R}^m)$ the set of continuous functions from a compact time interval $\mathbb{I} = [\underline{t}, \overline{t}] \subseteq \mathbb{R}$ to $\mathbb{R}^m$, here $x$ represents the state, $u$ the input and $y$ the output of the system. Although more general function spaces can be considered, we assume that $F \in C^0(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \times \mathbb{D}_u \times \mathbb{D}_y, \mathbb{R}^\ell)$ is sufficiently smooth, and that $\mathbb{D}_x, \mathbb{D}_{\dot{x}} \subseteq \mathbb{R}^n$, $\mathbb{D}_u \subseteq \mathbb{R}^m$, $\mathbb{D}_y \subseteq \mathbb{R}^p$ are open sets.

In many practical applications the output is given explicitly, i. e., the system has the form

$$\begin{aligned} F(t, x, \dot{x}, u) &= 0,\ x(\underline{t}) = \underline{x}, && \text{(2a)} \\ y &= G(t, x, u). && \text{(2b)} \end{aligned}$$

Then we assume that $F \in C^0(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \times \mathbb{D}_u, \mathbb{R}^\ell)$ and $G \in C^0(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_u, \mathbb{R}^p)$.

To avoid confusion, in the following we use $L$ for the number of equations, where $L = \ell$ if as in (2) the output equation is included in the system of equations or if the output equation is kept separately and $L = \ell + p$ if the output equation is given as a separate equation but included in the total system.

Throughout the paper we assume that all functions are sufficiently smooth, i. e., sufficiently often continuously differentiable, whenever this is needed.

In the following we make use of a *behavior approach* [25], which introduces a *descriptor vector* $z$ that includes the state variables, the input variables and also the output variables in (2). If the output equation is given explicitly as in (2b) then it often suffices to build the descriptor vector just from the state and input variables in (2a) and adding the output equation to the system later again.

The behavior approach does not distinguish the meaning of variables and is ideally suited for the analysis and also for some of the numerical methods. In some cases, however, it is necessary (and we will do so) to analyze the representation in terms of the original variables $x, u, y$.

Whether an output equation is included or not, the resulting behavior system has the form

$$\mathcal{F}(t, z, \dot{z}) = 0, \tag{3}$$

with $\mathcal{F} \in C^0(\mathbb{I} \times \mathbb{D}_z \times \mathbb{D}_{\dot{z}}, \mathbb{R}^L)$ sufficiently smooth, together with a set of equations for the initial conditions

$$Hz(\underline{t}) = \underline{z}, \tag{4}$$

with a projection matrix $H$, that are compiled from the given initial conditions. Here $z \in C^0(\mathbb{I}, \mathbb{R}^N)$ with $N = n + m + p$ if the output variables are included or $N = n + m$ if not.

We use the following solution concept in the behavior context. A function $z : \mathbb{I} \to \mathbb{R}^N$ is called a *solution* of (3), if $z \in C^0(\mathbb{I}, \mathbb{R}^N)$ is differentiable wherever this is needed and if $z$ satisfies the equation pointwise. It is called a *solution of the initial value problem* if it furthermore satisfies the initial conditions (4), which are then called *consistent*. We say that the system is *regular* if it has a unique solution for every sufficiently smooth input function $u$ and every initial value that is consistent for the system with this $u$.

**Example 1** A simple example of such a descriptor system is the model of a two-dimensional, three-link mobile manipulator from [10] (see Figure 1), see also [3]. The Lagrangian equations of motion take the form

$$\begin{aligned} \tilde{M}(\Theta)\ddot{\Theta} + \tilde{C}(\Theta, \dot{\Theta}) + \tilde{G}(\Theta) &= \tilde{u} + \Psi^T \lambda, \\ \psi(\Theta) &= 0, \end{aligned}$$

where $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix}^T$ is the vector of joint displacements, $\tilde{u}$ is the three component vector of control torques applied at the joints, $\tilde{M}$ is a $3 \times 3$ mass matrix, $\tilde{C}$ is the vector of centrifugal and Coriolis forces and $\tilde{G}$ is the gravity vector. The constraint function $\psi$ is given by

$$\psi(\Theta) = \begin{bmatrix} l_1 \cos(\Theta_1) + l_2 \cos(\Theta_1 + \Theta_2) + l_3 \cos(\Theta_1 + \Theta_2 + \Theta_3) l_3 - l \\ \Theta_1 + \Theta_2 + \Theta_3 \end{bmatrix}.$$

Here $\Psi = \frac{\partial \psi}{\partial \Theta}$, the Lagrange multiplier $\lambda$ has two components and $\Psi^T \lambda$ is the generalized constraint force. Typically a tracking output $y = \tilde{C}(\Theta)$ is used for the system as in [10].

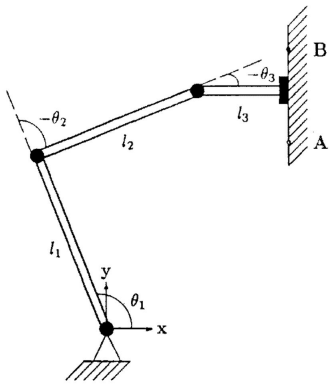


Figure 1: Three link mobile manipulator model [10].

**Example 2** Another simple example from the mathematical modeling of chemical reactions [24] is a first order isomerization reaction which is externally cooled.

With $c_0$ the given feed reactant concentration, $T_0$ the initial temperature, $c(t)$ and $T(t)$ the concentration and temperature at time $t$, and by $R$ the reaction rate per unit volume, the model takes the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{T} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} k_1(c_0 - c) - R \\ k_1(T_0 - T) + k_2 R - k_3(T - T_C) \\ R - k_3 \exp(-\frac{k_4}{T})c \end{bmatrix}. \tag{5}$$

Here the cooling temperature $T_C$ may be used as input and $k_1, k_2, k_3, k_4$ are constants.

We will also discuss in detail the case of linear systems, which are either the original model equations or are obtained after linearization along trajectories, see [5] for details. Then the control system takes the form

$$E(t)\dot{x} = A(t)x + B(t)u + f(t),\ x(\underline{t}) = \underline{x}, \tag{6a}$$
$$y = C(t)x + D(t)u + g(t), \tag{6b}$$

with $E, A \in C^0(\mathbb{I}, \mathbb{R}^{\ell,n})$, $B \in C^0(\mathbb{I}, \mathbb{R}^{\ell,m})$, $C \in C^0(\mathbb{I}, \mathbb{R}^{p,n})$ $D \in C^0(\mathbb{I}, \mathbb{R}^{p,m})$, $f \in C^0(\mathbb{I}, \mathbb{R}^{\ell})$ and $g \in C^0(\mathbb{I}, \mathbb{R}^{p})$. For a better readability we will omit in the following the argument $t$ of the involved coefficient functions.

In the linear case the behavior system takes the form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \phi, \tag{7}$$

where $\mathcal{E}, \mathcal{A} \in C^0(\mathbb{I}, \mathbb{R}^{L,N})$, $\phi \in C^0(\mathbb{I}, \mathbb{R}^{L})$ and $L = \ell + p$ if the output equation is included or $L = \ell$ otherwise. Again $z \in C^0(\mathbb{I}, \mathbb{R}^{N})$ with $N = n + m + p$ if the output variables are included or $N = n + m$ if not. In either case we have an initial condition of the form (4).

**Example 3** Considering Example 1, rewriting the joint displacements in Cartesian coordinates for positions $p$, and linearizing around a non-stationary solution yields a linear model

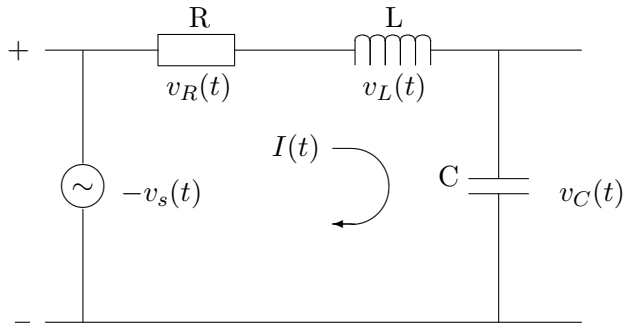


Figure 2: A simple $RLC$ circuit.

of the form

$$\begin{aligned} M(t)\delta\ddot{\tilde{p}} + D(t)\delta\dot{\tilde{p}} + K(t)\delta\tilde{p} &= S\delta\tilde{u} + G(t)^T\delta\lambda \\ G(t)\delta p &= 0. \end{aligned}$$

Transforming to first order and letting $x = \begin{bmatrix} \delta\tilde{p} & \delta\dot{\tilde{p}} & \delta\lambda \end{bmatrix}^T$ and $u = \delta\tilde{u}$, one obtains a linear descriptor system of the form $E\dot{x} = Ax + Bu, y = Cx$ with

$$E = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0 & I_3 & 0 \\ -K & -D & G^T \\ G & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ S \\ 0 \end{bmatrix} u.$$

**Example 4** A simple example that comes directly as a constant coefficient linear system is the simple $RLC$ electrical circuit from [6] in Figure 2. Here $R$, $L$ and $C$ are the resistance, inductance and capacitance, respectively. The corresponding voltage drops are denoted by $v_R(t)$, $v_L(t)$ and $v_C(t)$, respectively, and $I(t)$ denotes the current. Typically one would consider the voltage source $v_s(t)$ as the control input.

Applying Kirchhoff's laws we obtain the following circuit equation.

$$\begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I} \\ \dot{v}_L \\ \dot{v}_C \\ \dot{v}_R \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I \\ v_L \\ v_C \\ v_R \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} v_s.$$

If we measure the voltage at the capacitor as output, we have

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ v_L \\ v_C \\ v_R \end{bmatrix}.$$

## 2 Regularization of nonlinear systems

As we have already noted, the general formulation of the system as it arises in automated modeling is typically not adequate for numerical simulation, control or optimization. For this

reason, the first step in any treatment of descriptor systems is a reformulation of the system that turns the system into a so-called *strangeness-free formulation* of the system [16].

For this, consider the general nonlinear system (3) and construct a derivative array, as introduced in [4], of the form

$$\mathcal{F}_\mu(t, z, \dot{z}, \ldots, z^{(\mu+1)}) = 0, \tag{8}$$

which stacks the original equation and all its derivatives up to level $\mu$ in one large system. Here, partial derivatives of $F_\mu$ with respect to selected variables $\zeta$ from $(t, z, \dot{z}, \ldots, z^{(\mu+1)})$ are denoted by $F_{\mu;\zeta}$. The solution set of the nonlinear algebraic equation associated with the derivative array $F_\mu$ for some integer $\mu$ (considering the variables as algebraic variables) is denoted by $\mathbb{L}_\mu = \{z_\mu \in \mathbb{I} \times \mathbb{R}^N \times \ldots \times \mathbb{R}^N \mid \mathcal{F}_\mu(z_\mu) = 0\}$.

We will need the following hypothesis, see [16].

**Hypothesis 1** *Consider the general system of nonlinear differential-algebraic equations (3). There exist integers $\mu$, $r$, $a$, $d$, and $v$ such that $\mathbb{L}_\mu$ is not empty and such that for every $z_\mu^0 = (t_0, z_0, \dot{z}_0, \ldots, z_0^{(\mu+1)}) \in \mathbb{L}_\mu$ there exists a (sufficiently small) neighborhood in which the following properties hold:*

1. *The set $\mathbb{L}_\mu \subseteq \mathbb{R}^{(\mu+2)(N)+1}$ forms a manifold of dimension $(\mu+2)N + 1 - r$.*
2. *We have* $\operatorname{rank} \mathcal{F}_{\mu;z,\dot{z},\ldots,z^{(\mu+1)}} = r$ *on $\mathbb{L}_\mu$.*
3. *We have* $\operatorname{corank} \mathcal{F}_{\mu;z,\dot{z},\ldots,z^{(\mu+1)}} - \operatorname{corank} F_{\mu-1;z,\dot{z},\ldots,z^{(\mu)}} = v$ *on $\mathbb{L}_\mu$, where the corank is the dimension of the corange and the convention is used that* corank *of $\mathcal{F}_{-1;z}$ is* 0.
4. *We have* $\operatorname{rank} \mathcal{F}_{\mu;\dot{z},\ldots,z^{(\mu+1)}} = r - a$ *on $\mathbb{L}_\mu$ such that there exist smooth full rank matrix functions $Z_2$ and $T_2$ of size $(\mu+1)l \times a$ and $N \times (N-a)$, respectively, satisfying $Z_2^T \mathcal{F}_{\mu;\dot{z},\ldots,z^{(\mu+1)}} = 0$,* $\operatorname{rank} Z_2^T \mathcal{F}_{\mu;z} = a$, *and $Z_2^T \mathcal{F}_{\mu;z} T_2 = 0$ on $\mathbb{L}_\mu$.*
5. *We have* $\operatorname{rank} \mathcal{F}_{\dot{z}} T_2 = d = L - a - v$ *on $\mathbb{L}_\mu$ such that there exists a smooth full rank matrix function $Z_1$ of size $N \times d$ satisfying* $\operatorname{rank} Z_1^T \mathcal{F}_{\dot{z}} T_2 = d$.

The quantity $v$ measures the number of equations in the original system that give rise to trivial equations $0 = 0$, i. e., it counts the number of redundancies in the system. Of course, trivial equations can be simply removed without altering the solution set and we assume from now on that this has been done.

The smallest possible $\mu$ for which Hypothesis 1 holds is called the *strangeness-index* of (3), see [16]. It generalizes the concept of differentiation index [2] to arbitrary over- and underdetermined systems.

It has been shown in [14] that Hypothesis 1 implies locally (via the implicit function theorem) the existence of a *reduced system* given by

$$\hat{\mathcal{F}}_1(t, z, \dot{z}) = 0, \tag{9a}$$
$$\hat{\mathcal{F}}_2(t, z) = 0, \tag{9b}$$

with $\hat{\mathcal{F}}_1 = Z_1^T \mathcal{F}$ describing the dynamics of the system, while $\hat{\mathcal{F}}_2(t, z) = 0$ contains all the algebraic constraints and can be used to parameterize the solution manifold. Consider the following example from [15].

**Example 5** Example 1 (written in Cartesian coordinates and first order form) is a special case of a control problem for a multibody system of the form.

$$\begin{aligned} \dot{p} &= v, \\ M(p)\dot{v} &= f(p,q,u) + g_p(p)^T\lambda, \\ g(p) &= 0, \\ y &= C(p), \end{aligned}$$

with position variable $p$, velocity $v$, mass matrix $M(p)$, forcing function $f$, constraint $g(p)$, Lagrange multiplier $\lambda$ and output $y$. Here the coupling between the constraint and the dynamical equation leads to a strangeness index 2 (differentiation index 3). Instead of using the whole derivative array to compute the reduced strangeness-free form, using the structure of the system it suffices here to differentiate only the constraint $g(p) = 0$ twice and to add these equations to the system to obtain a *reduced* derivative array, [15, 16].

Assuming that $g_p(p)$ has full row rank and that $M(p)$ is symmetric and positive definite one can (locally) determine a permutation matrix $\Pi = [\Pi_1\ \Pi_2]$ such that for the Jacobian matrix $g_p(p)$ we have $g_p(p)[\Pi_1\ \Pi_2] = [G_1\ G_2]$, with $G_2$ being square and nonsingular. Partitioning

$$[\Pi_1\ \Pi_2]^T p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad [\Pi_1\ \Pi_2]^T v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{10}$$

conformally and replacing every occurrence of $\dot{p}_2$ by the auxiliary variable $w_1$ and every occurrence of $\dot{v}_2$ by the auxiliary variable $w_2$ gives the system

$$\begin{aligned} \dot{p}_1 &= v_1, \\ w_1 &= v_2, \\ M\Pi \begin{bmatrix} \dot{v}_1 \\ w_2 \end{bmatrix} &= f(p,v,u) + g_p(p)^T\lambda, \\ 0 &= g(p), \\ 0 &= g_p(p)v, \\ 0 &= g_{pp}(p)(v,v) + g_p(p)\Pi \begin{bmatrix} \dot{v}_1 \\ w_2 \end{bmatrix} \\ y &= C(p), \end{aligned}$$

from which then we can (by inverting $M$) remove the auxiliary variables $w_1, w_2$ and obtain the system

$$\begin{aligned} \dot{p}_1 &= v_1, \\ \dot{v}_1 &= \Pi_1^T M(p)^{-1}(f(p,v,u) + g_p(p)^T\lambda), \\ 0 &= g(p), \\ 0 &= g_p(p)v, \\ 0 &= g_{pp}(v,v) + g_p(p)\Pi_1^T M(p)^{-1}(f(p,v,u) + g_p(p)^T\lambda), \\ y &= C(p), \end{aligned}$$

together with (10). This system is regular and strangeness-free for given $u$ near the initial value [15].

Based on the construction of reduced strangeness-free systems we are now able to characterize consistency and solvability of the system and also perform, if necessary, a regularization.

## 2.1 Regularization of initial conditions

It is clear that for an initial condition to be consistent, equation (9b) has to be compatible with the initial condition (4), i. e. the combined system

$$\hat{\mathcal{F}}_2(\underline{t}, z(\underline{t})) = 0, \ Hz(\underline{t}) = \underline{z} \tag{11}$$

must be uniquely solvable to obtain a unique solution for a given input $u$. If this is not the case, then a modification of the initial conditions is necessary, so that (11) has a unique solution. This typically can be done in many different ways, e. g. by changing the initial value $\underline{x}$ and with this $\underline{z}$, so that (11) becomes solvable. A good choice should certainly also be justified from the underlying physical problem.

**Example 6** Consider Example 5 with initial conditions $p(0) = \underline{p}$ and $v(0) = \underline{v}$. Then due to the two extra constraints that are obtained from the derivative array the following conditions have to be satisfied for the initial value to be consistent

$$\begin{aligned} g(\underline{p}) &= 0, \\ g_p(\underline{p})\underline{v} &= 0, \\ g_{pp}(\underline{v},\underline{v}) + g_p(\underline{p})M(\underline{p})^{-1}(f(\underline{p},\underline{v},u(0)) + g_p(\underline{p})^T\lambda(0)) &= 0. \end{aligned}$$

Note that due to the assumptions, the third condition simply fixes the initial value $\lambda(0)$ of the Lagrange parameter.

## 2.2 Reinterpretation of variables

In some applications it turns out that the original choice of input and state variables was not appropriate. In the general behavior approach [25] such a interpretation of variables should be made after the analysis.

In order to (locally) get an interpretation of the variables, we use a suitable splitting of the unknown $z$ as $z = (z_1, z_2, z_3) \in \mathbb{R}^d \times \mathbb{R}^{N-d-a} \times \mathbb{R}^a$ in such a way that Part 4 of Hypothesis 1 guarantees that equation (9b) can be solved for $z_3$ according to $z_3 = \mathcal{R}(t, z_1, z_2)$. Eliminating $z_3$ and $\dot{z}_3$ in (9a) with the help of this relation and its derivative then leads to

$$\hat{\mathcal{F}}_1(t, z_1, z_2, \mathcal{R}(t, z_1, z_2), \dot{z}_1, \dot{z}_2, \mathcal{R}_t(t, z_1, z_2) + \mathcal{R}_{z_1}(t, z_1, z_2)\dot{z}_1 + \mathcal{R}_{z_2}(t, z_1, z_2)\dot{z}_2) = 0. \tag{12}$$

By Part 5 of Hypothesis 1, we may assume without loss of generality that this system can (locally) be solved for $\dot{z}_1$ leading to a system

$$\begin{aligned} \dot{z}_1 &= \mathcal{L}(t, z_1, z_2, \dot{z}_2), && \text{(13a)} \\ z_3 &= \mathcal{R}(t, z_1, z_2). && \text{(13b)} \end{aligned}$$

Obviously, in this system, interpreted as a differential-algebraic equation, $z_2 \in C^1(\mathbb{I}, \mathbb{R}^{N-d-a})$ can be chosen arbitrarily (at least when staying in the domain of definition of $\mathcal{R}$ and $\mathcal{L}$). This means that the variables in $z_2$ can be considered as controls, while the resulting system has locally a unique solution for $z_1$ and $z_3$, provided a consistent initial condition as in (11) is given.

The variables in $z_3$ represent algebraic variables and if output variables were included, then they are part of $z_3$. In the general behavior framework we are now free to choose output

variables, but clearly this should be done according to the underlying physical model, in particular if the output variables are meant to be variables that are measured.

In summary, by appropriately renaming the variables, it follows that after this step we have obtained a regularization of the system in the form of a strangeness-free system

$$\begin{aligned} \hat{\mathcal{F}}_1(t, \hat{x}, \dot{\hat{x}}, \hat{u}, \dot{\hat{u}}, \hat{y}) &= 0, && \text{(14a)} \\ \hat{\mathcal{F}}_2(t, \hat{x}, \hat{u}, \hat{y}) &= 0, && \text{(14b)} \end{aligned}$$

together with initial conditions that are modified so that they satisfy (11).

If the output equation was kept as a separate equation, then (by appropriately renaming the variables and functions) we obtain a system

$$\begin{aligned} \hat{\mathcal{F}}_1(t, \hat{x}, \dot{\hat{x}}, \hat{u}, \dot{\hat{u}}) &= 0, && \text{(15a)} \\ \hat{\mathcal{F}}_2(t, \hat{x}, \hat{u}) &= 0, && \text{(15b)} \\ \hat{y} &= \hat{G}(t, \hat{x}, \hat{u}). && \text{(15c)} \end{aligned}$$

Note that in both formulations derivatives of the chosen input variables may arise. If the original system was formulated in an appropriate way this should not happen.

However, if there is an explicit dependency on $\dot{\hat{u}}$, then one can introduce $\tilde{u} = \dot{\hat{u}}$ and add this differential equation for $u$ as new dynamical equation to the system. In this case appropriate initial conditions for $u$ have to be added as well. Again we can rename the variables appropriately and can assume in the following that the state equations do not contain explicit derivatives of the input equations, i. e. $\dot{\hat{u}}$ does not occur.

If the variables of the original system are given as states $x$ and inputs $u$, and if we keep these variables, then we obtain a system

$$\begin{aligned} \hat{\mathcal{F}}_1(t, x, \dot{x}, u) &= 0, && \text{(16a)} \\ \hat{\mathcal{F}}_2(t, x, u) &= 0, && \text{(16b)} \\ \hat{y} &= \hat{G}(t, x, u), && \text{(16c)} \end{aligned}$$

with $L = d + a$ state equations and unknowns of size $n$ for $x$ and $m$ for $u$.

Then several possibilities can arise. If $d + a \neq n$, then for given $u$ we cannot expect a unique solution, i. e., the system is not regular. In a general behavior setting this does not matter, since the variables are not distinguished and one can make a decision which variables one wants to consider as inputs and states at this point, see [11, 12, 25]. If, however, the application clearly defines which variables are input or state variables, then a reinterpretation of these variables is necessary. If $d+a > n$, then we just introduce a new vector $\tilde{x}$ by attaching $d + a - n$ of the input variables in $u$ to the vector $x$ and considering a new input vector $\tilde{u}$ which contains the remaining variables. If $d + a < n$ then we attach $n - d + a$ variables from $x$ to $u$ to obtain $\hat{u}$ and $\hat{x}$.

This renaming of variables yields a system

$$\begin{aligned} \tilde{\mathcal{F}}_1(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}) &= 0, && \text{(17a)} \\ \tilde{\mathcal{F}}_2(t, \tilde{x}, \tilde{u}) &= 0, && \text{(17b)} \\ \tilde{y} &= \tilde{G}(t, \tilde{x}, \tilde{u}), && \text{(17c)} \end{aligned}$$

There is a lot of freedom in the choice of these new variables. From a theoretical point of view any choice is just fine, but from a practical point of view there may be preferred choices and also different choices may lead to different scaling and stability properties of the resulting system and it may be necessary to take this into consideration.

### 2.3 Feedback regularization

In general, the system given by (17) is not strangeness-free as a free system, i. e., when $\tilde{u} = 0$ is chosen as an input, although as we have seen, there always is a possibility of reinterpreting the variables so that is the case. If, however, the original variables $x$ and $u$ were kept, $d + a = n$, and the system is not strangeness-free as a free system, then a classical approach [3, 19] is to choose a feedback such that the resulting reduced problem with is regular and strangeness-free.

In the nonlinear case, a state feedback has the form

$$\tilde{u} = K(t, \tilde{x}) + w, \tag{18}$$

leading to a closed loop reduced problem

$$\begin{aligned} \tilde{\mathcal{F}}_1(t, \tilde{x}, \dot{\tilde{x}}, K(t, \tilde{x}) + w) &= 0, && \text{(19a)} \\ \tilde{\mathcal{F}}_2(t, \tilde{x}, K(t, \tilde{x}) + w) &= 0, && \text{(19b)} \\ \tilde{y} &= \tilde{G}_1(t, \tilde{x}, K(t, \tilde{x}) + w), && \text{(19c)} \end{aligned}$$

This system is regular and strangeness-free if

$$\begin{bmatrix} \tilde{\mathcal{F}}_{1;\dot{\tilde{x}}} \\ \tilde{\mathcal{F}}_{2;\tilde{x}} + \tilde{\mathcal{F}}_{2;\tilde{u}} K_{\tilde{x}} \end{bmatrix}$$

is nonsingular. Since the reduced system is only defined locally, it is sufficient to satisfy this condition only locally. Thus, one can use linear feedbacks and it has been shown in [19] that then there (locally) exists a state feedback $\tilde{u} = K(t, \tilde{x}) + w$ satisfying $\tilde{u}(\underline{t}) = K(\underline{t}, \underline{x})$ and $\dot{u}(\underline{t}) = K_t(\underline{t}, \underline{x}) + K_x(\underline{t}, \underline{x})\dot{x}(\underline{t}) + \dot{w}$ such that the closed loop reduced problem is regular and strangeness-free.

It also has been discussed in [16, 19] under which conditions such a regularization can be achieved via output control. To determine such an output feedback, however, usually requires changes of variables.

It should be noted, however, that it depends on the application whether this feedback is necessary. For example in optimal control problems it suffices if the system is strangeness-free in the behavior sense, see [18].

**Example 7** Consider the control problem

$$F(t, x, u, \dot{x}) = \begin{bmatrix} \dot{x}_2 \\ \log x_2 + \sin u \end{bmatrix} = 0,$$

with $n = 2$ and $\ell = 1$. The corresponding behavior system reads

$$F(t, z, \dot{z}) = \begin{bmatrix} \dot{z}_2 \\ \log z_2 + \sin z_3 \end{bmatrix} = 0.$$

This system satisfies Hypothesis 1 with $\mu = 0$, $v = 0$, $a = 1$, and $d = 1$ and is already in strangeness-free form. The free system with $u = 0$, however, is not strangeness-free.

To get a regular and strangeness-free closed-loop reduced problem, we may choose $u = x_1 + w$ and obtain the system $\dot{x}_2 = 0$, $\log x_2 + \sin x_1 + \sin w = 0$, which is strangeness-free as a behavior as well as a free system with $w = 0$. For more details see [16].

### 2.4 General procedure

Considering the system with separate output equation (2), we can summarize the procedure as follows:

$$\boxed{\begin{aligned} F(t,x,\dot{x},u) &= 0,\ x(\underline{t}) = \underline{x},\\ y &= G(t,x,u)\end{aligned}}$$

$\mu \neq 0$ ↓ index reduction in behavior

$$\boxed{\begin{aligned} \hat{\mathcal{F}}_1(t,z,\dot{z}) &= 0,\ Hz(\underline{t}) = \underline{z},\\ \hat{\mathcal{F}}_2(t,z) &= 0,\\ y &= \hat{G}(t,z)\end{aligned}}$$

$\hat{\mathcal{F}}_2(\underline{t}, z(\underline{t})) = 0,\ Hz(\underline{t}) = \underline{z}$ not solvable ↓ force consistency

$a + d \neq N$ ↓ reinterpret variables

$\dot{u}$ occurs ↓ transform to remove $\dot{u}$

$$\boxed{\begin{aligned} \tilde{\mathcal{F}}_1(t,\tilde{x},\dot{\tilde{x}},\tilde{u}) &= 0,\\ \tilde{\mathcal{F}}_2(t,\tilde{x},\tilde{u}) &= 0,\\ \tilde{y} &= \tilde{G}(t,\tilde{x},\tilde{u})\end{aligned}}$$

not strangeness-free for $u = 0$ ↓ feedback $\tilde{u} = K(t,\tilde{x}) + w$

$$\boxed{\begin{aligned} \tilde{\mathcal{F}}_1(t,\tilde{x},\dot{\tilde{x}},K(t,\tilde{x})+w) &= 0,\\ \tilde{\mathcal{F}}_2(t,\tilde{x},K(t,\tilde{x})+w) &= 0,\\ \tilde{y} &= \tilde{G}_1(t,\tilde{x},K(t,\tilde{x})+w).\end{aligned}}$$

In the case where the equations have some structure, such as with multibody systems, the procedure can be simplified substantially by making use of the structure available in the equations.

If we want to keep the original variables and no reinterpretation of variables is necessary then the procedure looks the same except that the "reinterpret variables" and "transform" lines are removed along with the third block. In all cases, the final system is strangeness-free as a free system and as a behavior system and by the implicit function theorem it can locally be expressed as

$$\dot{\tilde{x}}_1 = \mathcal{L}(t,\tilde{x}_1,w), \tag{20a}$$
$$\tilde{x}_2 = \mathcal{R}(t,\tilde{x}_1,w), \tag{20b}$$
$$\tilde{y} = \tilde{G}(t,\tilde{x}_1,\tilde{x}_2). \tag{20c}$$

This system is in the right form to carry out the tasks in numerical simulation [16], stability analysis [17, 22], or optimal control [18].

## 3 Regularization of linear descriptor systems

In the linear case we can be more specific and also consider a more general situation, since we do not have make the assumption that the solution set is non-empty. In this case the derivative array of (7) satisfies the following.

$$\mathcal{M}_\mu \dot{z}_\mu = \mathcal{N}_\mu z_k + \phi_\mu, \tag{21}$$

where

$$
\begin{aligned}
(\mathcal{M}_\mu)_{i,j} &= \tbinom{i}{j}\mathcal{E}^{(i-j)} - \tbinom{i}{j+1}\mathcal{A}^{(i-j-1)},\ i,j = 0,\ldots,\mu,\\
(\mathcal{N}_\mu)_{i,j} &= \begin{cases} \mathcal{A}^{(i)} & \text{for } i = 0,\ldots,\mu,\ j = 0,\\ 0 & \text{otherwise,}\end{cases}\\
(z_\mu)_j &= z^{(j)},\ j = 0,\ldots,\mu,\\
(\phi_\mu)_i &= \phi^{(i)},\ i = 0,\ldots,\mu.
\end{aligned}
$$

In the linear case Hypothesis 1 can be proved under some constant rank assumptions, see [13, 16]. It then has the form

**Hypothesis 2** *There exist integers $\mu$, $d$, $a$, and $v$ such that the pair $(\mathcal{M}_\mu, \mathcal{N}_\mu)$ associated with (21) has the following properties:*

1. *For all $t \in \mathbb{I}$ we have* $\operatorname{rank} \mathcal{M}_\mu(t) = (\mu+1)L - a - v$. *This implies the existence of a smooth matrix function $Z$ of size $(\mu+1)L \times (a+v)$ and pointwise maximal rank satisfying $Z^T\mathcal{M}_\mu = 0$.*

2. *For all $t \in \mathbb{I}$ we have* $\operatorname{rank} Z^T\mathcal{N}_\mu[I_N\ 0\ \cdots\ 0]^T = a$. *This implies that without loss of generality $Z$ can be partitioned as $Z = [\ Z_2\ \ Z_3\ ]$, with $Z_2$ of size $(\mu+1)L \times a$ and $Z_3$ of size $(\mu+1)L \times v$, such that $\hat{A}_2 = Z_2^T\mathcal{N}_\mu[I_N\ 0\ \cdots\ 0]^T$ has full row rank $a$ and $Z_3^T\mathcal{N}_\mu[I_n\ 0\ \cdots\ 0]^T = 0$. Furthermore, there exists a smooth matrix function $T_2$ of pointwise maximal rank satisfying $\hat{A}_2T_2 = 0$.*

3. *For all $t \in \mathbb{I}$ we have* $\operatorname{rank} \mathcal{E}(t)T_2(t) = d$. *This implies the existence of a smooth matrix function $Z_1$ of size $L \times d$ and pointwise maximal rank satisfying* $\operatorname{rank} \hat{E}_1 = d$ *with $\hat{E}_1 = Z_1^T\mathcal{E}$.*

The following theorem adapted to the setting in this paper is from [19].

**Theorem 8** *If Hypothesis 2 holds for the inflated pair $(\mathcal{M}_\mu, \mathcal{N}_\mu)$ associated with $(\mathcal{E}, \mathcal{A})$, then system (7) has the same solution set as the system*

$$
\begin{bmatrix} \hat{E}_1 \\ 0 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \\ 0 \end{bmatrix} z + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix}, \tag{22}
$$

*where $\hat{E}_1 = Z_1^T\mathcal{E}$, $\hat{A}_1 = Z_1^T\mathcal{A}$, $\hat{f}_1 = Z_1^T\phi$, $\hat{f}_i = Z_i^T\phi_\mu$ for $i = 2, 3$.*

The coefficients of the differential-algebraic system (22) can either be computed analytically or numerically at every time point $t$ by using three nullspace computations, as long as this is feasible in the available computing environment. It is a *reformulation* of (7) (using the original model and its derivatives) without changing the solution set. Note that no transformation or splitting of the vector $z$ into parts has been made, which means that the variables still have their original physical meaning.

If the original system was in the form (6a) and if we ignore (for the moment) the fact that the vector $z$ in (7) is composed of parts that may have quite different orders of differentiability, then the constructed submatrices $\hat{A}_1$ and $\hat{A}_2$ have been obtained from the block matrix

$$
\begin{bmatrix} A & B \\ \dot{A} & \dot{B} \\ \vdots & \vdots \\ A^{(\mu)} & B^{(\mu)} \end{bmatrix}
$$

by transformations from the left. This has two immediate consequences [19]. First, this means that derivatives of the input function $u$ are nowhere needed, just derivatives of the coefficient matrices, i. e., although formally the derivatives of $u$ occur in the derivative array, they are not used for the form (22), and hence, we do not need any additional smoothness requirements for the input function $u$.

Second, it follows from the construction of $\hat{A}_1$ and $\hat{A}_2$ that the partitioning into the part stemming from the original states $x$ and the original controls $u$ is not mixed up. Including again the output equation (6b), we then have obtained a system of the form

$$\begin{aligned} E_1\dot{x} &= A_1x + B_1u + \hat{f}_1, && (23a)\\ 0 &= A_2 + B_2u + \hat{f}_2, && (23b)\\ 0 &= \hat{f}_3, && (23c)\\ y &= Cx + Du + g, && (23d) \end{aligned}$$

where

$$E_1 = \hat{E}_1 \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad A_i = \hat{A}_i \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad B_i = \hat{A}_i \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad i = 1, 2.$$

Here $E_1, A_1$ have size $d \times n$, while $E_2, A_2$ are of size $a \times n$.

### 3.1 Regularization of equations and initial values

In the following we continue with the general form (22). In contrast to the nonlinear case where we have assumed solvability, here this can be checked by considering the function $\hat{f}_3$. If $\hat{f}_3$ is identically 0 in $\mathbb{I}$, then we have solvability and we can just leave off the third equation in (22). If $\hat{f}_3$ is nonzero, then the model is not consistent. This actually happens frequently in applications due to modeling errors or when overdetermined systems are considered which are consistent in exact computation but due to modeling errors, measurement errors or round-off errors one has nonzero $\hat{f}_3$.

If $\hat{f}_3$ is nonzero, then we can regularize the system by setting $\hat{f}_3 \equiv 0$, and by removing the third equation of (22) which is then a redundant equation. We then continue with the modified model of $\tilde{L} = d + a$ equations

$$\begin{bmatrix} \hat{E}_1 \\ 0 \end{bmatrix} \dot{z} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} z + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix}, \tag{24}$$

together with initial conditions (4).

Consistency of the initial values can again be easily checked, since these have to satisfy the equation

$$\hat{A}_2(\underline{t})z(\underline{t}) + \hat{f}_2(\underline{t}) = 0, \quad Hz(\underline{t}) = \underline{z}. \tag{25}$$

If the initial values are not consistent then there are again several possibilities to regularize the system. One can, e. g., compute a consistent least squares approximation $\hat{\underline{z}}$ of $\underline{z}$ or one can choose a consistent initial value $\hat{\underline{z}}$ by making sure that important user requirements on the initial value are satisfied exactly, see [20] for a detailed discussion and also software implementations of this regularization step.

### 3.2 Reinterpretation of variables

After having made the system and the initial conditions consistent, we can analyze the uniqueness of the resulting solution of the strangeness-free behavior system. With $\tilde{L} = d+a$ equations and $N$ unknowns, it is clear that to obtain uniqueness, it is necessary that $N = \tilde{L} = d+a$. If $N > d+a$, then we have free variables in the system which can be considered as controls or inputs.

If $N = \tilde{L}$, then as in the nonlinear case, an appropriate transformation of the variables $\hat{x} = Tx$ can be performed so that we obtain a set of $a+d$ (in the whole interval $\mathbb{I}$) linearly independent columns of the matrix function $\tilde{E} := \begin{bmatrix} \hat{E}_1 T \\ \hat{A}_2 T \end{bmatrix}$ and so that in $\hat{A}_2 T$ the columns numbered $d+1, \ldots, d+a$ are linearly independent (in the whole interval $\mathbb{I}$). Denoting the variables associated with these columns in $\hat{A}_2 T$ as $x_2$ and the variables associates with the first $d$ columns of $\tilde{E}$ as $x_1$ and the remaining $N-d-a$ variables as $u = x_3$, we obtain a new system

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix}, \tag{26}$$

where $\tilde{A}_{22}$ is invertible. This choice of variables makes sure that the free system obtained by setting $u = x_3 = 0$ is itself strangeness-free.

Typically, there is a lot of freedom in the choice of the variables $x_1, x_2, x_3$ and this is a place where the physical background of the system has to come into play. If for example in the original system the variables had a clear definition as input and state variables and if the number of state variables was $d+a$, then it seems a reasonable choice to pick this partitioning. Another motivation for the choice of variables may come from the fact that we may not want derivatives of input functions to appear in the model. In this case (if possible) the choice of the state variables should be made in such away that the block $\tilde{E}_{13}$ is zero, because then no derivative of $u = x_3$ occurs.

**Example 9** In the electrical circuit of example 4 we have used the voltage source as input but from a mathematical point of view we may as well choose the variable $v_L$ as input.

In the case that the reduced system has the form (23), we now have $d+a$ equations in (23a) and (23b) and still $n$ variables in $x$ and $m$ variables in $u$. In order for this system to be regular, i. e., uniquely solvable for all sufficiently smooth inputs $u$, all sufficiently smooth inhomogeneities $\hat{f}_1, \hat{f}_2$, and all consistent initial conditions $\underline{x}$, we would need that $d+a = n$.

If $d+a < n$, then for given $u, \hat{f}_1, \hat{f}_2$ we cannot expect a unique solution, i. e., the system is not regular. In a general behavior setting this does not matter, since the variables are not distinguished and one can make a decision which variables one wants to consider as inputs and states [11, 12, 25]. If, however, the application clearly defines which variables are input or state variables, then a reinterpretation of these variables is necessary. If $d+a > n$, then we just introduce a new vector $\hat{x}$ by attaching $d+a-n$ of the input variables in $u$ to the vector $x$ and considering a new input vector $\hat{u}$ which contains the remaining variables and if $d+a < n$ we attach $n-d+a$ variables from $x$ to $u$ to obtain $\hat{u}$ and $\hat{x}$. There is freedom in the choice of these variables. As already discussed in the nonlinear case, different choices are possible and the physical problem as well as scaling and stability properties should be considered.

When performing this reinterpretation of variables, we must also change the output equation by moving appropriate columns from $D$ to $C$ or vice versa.

In both the general and the special case after an appropriate renaming of coefficients we obtain a new system

$$\begin{aligned} \tilde{E}_1\dot{\tilde{x}} &= \tilde{A}_1\tilde{x} + \tilde{B}_1\tilde{u} + \tilde{f}_1, && \text{(27a)} \\ y &= \tilde{C}\tilde{x} + \tilde{D}\tilde{u} + \tilde{g}, && \text{(27b)} \end{aligned}$$

where now the matrix functions $\begin{bmatrix} \tilde{E}_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{bmatrix}$ are square of size $\tilde{n} = d + a$ and $\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$ is of size $\tilde{n} \times \tilde{m}$ with $\tilde{m} = n + m - \tilde{n}$.

### 3.3 Removing feed-through terms

In the linear case a feed-through can be easily removed by increasing the size of the state vector. Suppose that $\tilde{D}$ has constant rank $p_1$ in the interval under consideration, then we can find, see e. g., [Thm 3.9, [16]] orthogonal matrix functions $P, Q$ with the same smoothness as $\tilde{D}$ such that $P^T\tilde{D}Q = \begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & 0 \end{bmatrix}$ with $\tilde{D}_1$ being $p_1 \times p_1$ and pointwise invertible. Setting (partitioned accordingly)

$$P^T y = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix},\ P^T\tilde{C} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix},\ P^T g = \begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{bmatrix},\ \tilde{u} = Qu = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

with $\tilde{C}_1$ of size $p_1 \times d + a$ and $\tilde{u}_2$ of size $m - p_1$, then we obtain a new system without feed-through term of the form

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u} + \bar{f}, && \text{(28a)} \\ \bar{y} &= \bar{C}\bar{x} + \bar{g}, && \text{(28b)} \end{aligned}$$

with data

$$\begin{aligned} \bar{x} &= \begin{bmatrix} \tilde{x} \\ \tilde{y}_1 \end{bmatrix},\ \bar{y} = \tilde{y}_2,\ \bar{u} = \tilde{u}_2,\ \bar{f} = \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{g}_1 \end{bmatrix},\ \bar{g} = \tilde{g}_2, \\ \bar{E} &= \begin{bmatrix} \tilde{E}_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ \bar{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ \tilde{A}_2 & 0 \\ \tilde{C}_1 & -I_{p_1} \end{bmatrix},\ \bar{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{D}_1 \end{bmatrix},\ \bar{C} = \tilde{C}_2. \end{aligned}$$

### 3.4 Feedback regularization

As in the nonlinear case, if no reinterpretation of variables was made, i. e., the system is still in terms of the original $x$ and $u$ and if the output equation was kept separately, we may have that the resulting system is not strangeness-free as a free system with $\bar{u} = 0$. It has been shown in [19], that then there exists a linear feedback $\bar{u} = K\bar{x} + w$, with $K \in C^0(\mathbb{I}, \mathbb{R}^{\bar{m},\bar{n}})$ such that in the closed loop system

$$\bar{E}\dot{\bar{x}} = (\bar{A} + \bar{B}K)\bar{x} + \bar{B}w + \bar{f}, \quad \bar{x}(\underline{t}) = \underline{\bar{x}}, \tag{29}$$

the matrix function $(\bar{A}_2 + \bar{B}_2K)\bar{T}_2'$ is pointwise nonsingular, where $\bar{T}_2'$ is a matrix valued function that spans the kernel of $\bar{E}_1$. This implies that the DAE in (29) is regular and strangeness-free as a free system with $w = 0$. The output equation then changes to $\bar{y} = \bar{C}\bar{x} + \bar{g}$.

### 3.5 General procedure

We can summarize the whole procedure in the following table.

$$\boxed{\begin{aligned} E\dot{x} &= Ax + Bu + f,\ x(\underline{t}) = \underline{x},\\ y &= Cx + Du + g \end{aligned}}$$

$\mu \neq 0$ ↓ index reduction in behavior

$$\boxed{\begin{aligned} E_1\dot{x} &= A_1x + B_1u + \hat{f}_1,\\ 0 &= A_2x + B_2u + \hat{f}_2,\\ 0 &= \hat{f}_3,\\ y &= Cx + Du + g \end{aligned}}$$

$\hat{f}_3 \neq 0$ ↓ force $\hat{f}_3 = 0$

$0 \neq A_2(\underline{t})\underline{x} + B_2(\underline{t})u(\underline{t}) + \hat{f}_2(\underline{t})$ ↓ force consistency

$a + d \neq n$ ↓ reinterpret variables

$$\boxed{\begin{aligned} \tilde{E}_1\dot{\tilde{x}} &= \tilde{A}_1\tilde{x} + \tilde{B}_1\tilde{u} + \tilde{f}_1,\\ 0 &= \tilde{A}_2\tilde{x} + \tilde{B}_2\tilde{u} + \tilde{f}_2,\\ \tilde{y} &= \tilde{C}\tilde{x} + \tilde{D}\tilde{u} + \tilde{g} \end{aligned}}$$

$\tilde{D} \neq 0$ ↓ remove feed-through

$$\boxed{\begin{aligned} \bar{E}_1\dot{\bar{x}} &= \bar{A}_1\bar{x} + \bar{B}_1\bar{u} + \bar{f}_1,\\ 0 &= \bar{A}_2\bar{x} + \bar{B}_2\bar{u} + \bar{f}_2,\\ \bar{y} &= \bar{C}\bar{x} + \bar{g} \end{aligned}}$$

not strangeness-free for $u = 0$ ↓ feedback $\bar{u} = K\bar{x} + w$

$$\boxed{\begin{aligned} \bar{E}_1\dot{\bar{x}} &= (\bar{A}_1 + \bar{B}_1K)\bar{x} + \bar{B}_1w + \bar{f}_1,\\ 0 &= (\bar{A}_2 + \bar{B}_2K)\bar{x} + \bar{B}_2w + \bar{f}_2,\\ \bar{y} &= \bar{C}\bar{x} + \bar{g}. \end{aligned}}$$

If no reinterpretation of variables is necessary, and if we want a strangeness-free system as a behavior and as a free system, then the preceding procedure is followed except the reinterpretation of variables and the third block is omitted. Otherwise the procedure is identical.

## 4 Conclusions

We have described detailed regularization procedures for general linear and nonlinear descriptor systems and shown that under some general hypothesis every system can be reformulated as a system that is strangeness-free as a behavior system as well as free system. Regularization procedures for inconsistent systems and initial conditions have been described as well as procedures for the construction of regularizing state feedbacks and for the removal of feedthrough terms. In all cases it is possible to achieve a reformulated systems which allows the application of standard simulation, control and optimization techniques.

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