Lengths of quasi-commutative pairs of matrices

A.E. Guterman[*] O.V. Markova[†] V. Mehrmann[‡]

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Dedicated to Hans Schneider,
driving force of the field of Linear Algebra, role model, advisor, and friend.

Abstract

In this paper we discuss some partial solutions of the length conjecture which describes the length of a generating system for matrix algebras. We consider mainly the algebras generated by two matrices which are quasi-commuting. It is shown that in this case the length function is linearly bounded. We also analyze which particular natural numbers can be realized as the lengths of certain special generating sets and prove that for commuting or product-nilpotent pairs all possible numbers are realizable, however there are non-realizable values between lower and upper bounds for the other quasi-commuting pairs. In conclusion we also present several related open problems.

Keywords: Finite-dimensional algebras; Lengths of sets and algebras; Quasi-commuting matrices

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1 Introduction

In this paper we present partial contributions to an open problem concerning the length function which is important in the study of finite-dimensional algebras. To introduce the problem we first need some notation.

[*]Lomonosov Moscow State University, Fac. Mechanics and Mathematics, 119899 Moscow, Russia, email: alexander.guterman@gmail.com

[†]Lomonosov Moscow State University, Fac. Mechanics and Mathematics, 119899 Moscow, Russia, email: ov_markova@mail.ru

[‡]Inst. f. Mathematik, MA 4-5, TU Berlin, D-10623 Berlin, Germany email: mehrmann@math.tu-berlin.de

1.1 Notation

Let $\mathbb{F}$ be an arbitrary *field* and let $\mathcal{A}$ be a *finite-dimensional associative algebra over* $\mathbb{F}$ with *generating system* $\mathcal{S} = \{a_1, \ldots, a_h\} \subset \mathcal{A}$. Let $\langle S \rangle$ denote the *linear span* (the set of all finite $\mathbb{F}$-linear combinations of elements of $\mathcal{S}$) of the subset S in some vector space over $\mathbb{F}$. For a finite set (alphabet) $B = \{b_1, \ldots, b_m\}$, finite sequences of letters in B are called *words*. Let B^* be the *set of all words over* B, and let F_B be the *free semigroup over* B, i. e., the set B^* equipped with the *concatenation operation*.

The following *length function* plays an important role in the study of finite-dimensional algebras, see e. g., [15].

Definition 1.1. The *length* of a word $b_1 \ldots b_t$, where $b_i \in B$, is equal to t. We adopt the convention that 1 (the empty word) is a *word of length* 0 over B.

By B^t, $t \geq 0$, we denote the set of all words over B of length not greater than t over B and by $B^t \setminus B^{t-1}$ the set of all words of length exactly $t \geq 1$ over B.

Remark 1.2. The products of elements of a generating set $\mathcal{S}$ can be viewed as the images of elements of the free semigroup $F_{\mathcal{S}}$ under the natural homomorphism. We can refer to them as words in the generators and use the natural notation $\mathcal{S}^t$ and $\mathcal{S}^t \setminus \mathcal{S}^{t-1}$.

If $\mathcal{A}$ is an *algebra with unit element* $1_{\mathcal{A}}$ then we set $\mathcal{S}^0 = \{1_{\mathcal{A}}\}$, otherwise $\mathcal{S}^0 = \emptyset$. We denote by $\mathcal{L}_t(\mathcal{S})$ the *linear span of the words* in $\mathcal{S}^t$. Note that $\mathcal{L}_0(\mathcal{S}) = \langle 1_{\mathcal{A}} \rangle = \mathbb{F}$ for *unitary algebras*, and $\mathcal{L}_0(\mathcal{S}) = 0$, otherwise. Let also $\mathcal{L}(\mathcal{S}) = \bigcup\limits_{i=0}^{\infty} \mathcal{L}_i(\mathcal{S})$ be the linear span of all words in the alphabet $\{a_1, \ldots, a_h\}$.

Definition 1.3. The *length of a generating system* $\mathcal{S}$ for the algebra $\mathcal{A}$ is the minimum nonnegative integer k such that $\mathcal{L}_k(\mathcal{S}) = \mathcal{A}$. The length of $\mathcal{S}$ will be denoted by $l(\mathcal{S})$. The *length of an algebra* $\mathcal{A}$ is defined as $l(\mathcal{A}) = \max\limits_{\mathcal{S}} l(\mathcal{S})$, where the maximum is taken over all generating systems of the algebra.

A word $v \in \mathcal{S}^t \setminus \mathcal{S}^{t-1}$ is said to be *reducible over* $\mathcal{S}$, if there exists an index $i < t$ such that $v \in \mathcal{L}_i(\mathcal{S})$, i. e., v can be represented as a linear combination of words of smaller lengths, otherwise it is called *irreducible over* $\mathcal{S}$.

By I, I_n we denote the identity matrix and by O, O_n the zero matrix in $M_n(\mathbb{F})$, the set of all $n \times n$ matrices over $\mathbb{F}$. Let $E_{i,j}$ be the (i,j)-th canonical basis element in $M_n(\mathbb{F})$, i. e., the matrix with one in the (i,j)-th position and zeros otherwise. The spectrum of a square matrix, i. e., the set of its eigenvalues in $\mathbb{F}$ is denoted by $\sigma(A)$, a Jordan block of size k associated with an eigenvalue λ is denoted by $J_k(\lambda)$, and the degree of the minimal polynomial of A is denoted by $\deg(A)$.

1.2 The length conjecture

The problem of the *length evaluation* was first discussed in [17], [18] for the algebra of 3×3 matrices in the context of the mechanics of isotropic continua. The problem of computing the length of the full matrix algebra $M_n(\mathbb{F})$ as a function of the matrix size n was stated in the work [16] and is still an open problem formulated in the following conjecture.

Conjecture 1.4 ([16]). *Let $\mathbb{F}$ be an arbitrary field. Then $l(M_n(\mathbb{F})) = 2n-2$.*

It is known that the conjecture is valid for $n = 2, 3, 4$, see [16], however, the known upper bounds for the length of the matrix algebra are in general nonlinear in n.

Theorem 1.5 ([16, Theorem 1, Remark 1]). *Let $\mathbb{F}$ be an arbitrary field. Then*

$$l(M_n(\mathbb{F})) \leq \left\lceil \frac{n^2+2}{3} \right\rceil,$$

where $\lceil . \rceil$ denotes the least integer function.

An (asymptotic) improvement of this bound was obtained in [15].

Theorem 1.6 ([15, Corollary 3.2]). *If $\mathbb{F}$ is an arbitrary field, then*

$$l(M_n(\mathbb{F})) < n\sqrt{\frac{2n^2}{n-1}+\frac{1}{4}}+\frac{n}{2}-2.$$

While the problem is open in general, there exist linear upper bounds for the lengths of matrix sets satisfying some additional conditions. In [1] the bound $2n-2$ was proved for sets satisfying a modified Poincarré–Birkhoff–Witt property (further, PBW-property), and in [9] a bound was provided for sets containing a rank one matrix. In [5, 14] the linear upper bound $n-1$ was proved for commutative subalgebras in $M_n(\mathbb{F})$ and it was also shown that any integer from $[0, n-1]$ can be realized as the length of a commutative matrix subalgebra. The lengths of commutative sets and algebras were studied further in [13], where an upper bound $l(\mathcal{S}) \leq (m-1)[\log_m d] + [m^{\{\log_m d\}}] - 1$ for the length of a commutative set was obtained, with $d = \dim \mathcal{L}(\mathcal{S})$, $m = \max\{\deg(A) | A \in \mathcal{S}\}$, $[x]$ and $\{x\}$ denote the integer and fractional parts of a real number x.

A summary of linear upper bounds for the lengths of $\mathcal{S}$ is given in the following list.

Remark 1.7.

1. Let $\mathcal{S}$ satisfy a modified PBW-property. Then $l(\mathcal{S}) \leq 2n-2$ if $\mathcal{L}(\mathcal{S}) = M_n(\mathbb{F})$, see [1, Theorem 2.7].

2. Let $\mathcal{S}$ satisfy a modified PBW-property. Then $l(\mathcal{S}) \leq 2n-3$, if $\mathcal{S}$ does not generate the whole algebra $M_n(\mathbb{F})$, see [1, Theorem 3.2].

3. If $\mathcal{S}$ is a commutative set, then $l(\mathcal{S}) \leq n-1$, see [5, Theorem 6.1].

4. If all elements of $\mathcal{S}$ are simultaneously triangularizable over $\mathbb{F}$, then $l(\mathcal{S}) \leq n-1$ by [11, Lemma 4.2].

The existing results, in particular, the bounds given in Remark 1.7, allow us to formulate the following weaker version of Conjecture 1.4:

Conjecture 1.8. *Let $\mathbb{F}$ be an arbitrary field. Then $l(M_n(\mathbb{F}))$ is a linear function in n.*

We believe that Conjecture 1.4 is true, however since it is not proved for many years we formulate the above conjecture. A possible approach to solve it can be to prove that $l(M_n(\mathbb{F})) \leq C \cdot n^{1+\delta}$, where $\delta \to 0$. At the moment, the bound obtained in [15] is the best and provides $\delta = \frac{1}{2}$.

The third item in Remark 1.7 shows that in the case of commuting matrices, i. e., when the algebra is generated by two elements and the products AB and BA coincide, there are good linear upper bounds for the length function. The next natural generalization is to consider the case that the products AB and BA are linearly dependent, namely the matrices A and B commute up to a scalar factor, i. e., there exists $\varepsilon \in \mathbb{F}$ such that $AB = \varepsilon BA$. In this case it is usually said that the matrices A and B are quasi-commuting. This class contains commutative pairs and is contained in the class of matrices with modified PBW-property considered in [1].

Definition 1.9. If A, B in $M_n(\mathbb{F})$ are such that AB and BA are linearly dependent, then we say that A and B *quasi-commute.* If the given factor $\omega \in \mathbb{F}$ in the quasi-commutativity relation $AB = \omega BA$ is important, then we say that A, B *commute up to a factor* ω (or ω*-commute*).

The structure of quasi-commuting pairs and possible values of the commutativity factor have been studied extensively, see [6] for a survey and [8] for some recent characterizations. An important tool in the study of quasi-commuting matrices is the following pre-normal form.

Theorem 1.10 ([3, Theorems 1–2]). *Let $\mathbb{F}$ be an algebraically closed field and let $n \in \mathbb{N}$, $n \geq 2$. If matrices $A, B \in M_n(\mathbb{F})$ satisfy the the identity $AB = \varepsilon BA$ for some $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$ and the matrix AB is not nilpotent, then there exists an integer $0 \leq r \leq n-2$ and an invertible matrix $P \in M_n(\mathbb{F})$ such that*

$$P^{-1}AP = \begin{bmatrix} S & X \\ O & A_r \end{bmatrix}, \ P^{-1}BP = \begin{bmatrix} T & Y \\ O & B_r \end{bmatrix}, \tag{1}$$

where ε is necessarily a primitive root of unity of order $k > 1$ *dividing $n-r$, S and T are triangular $r \times r$ matrices, ST and TS are both nilpotent, and*

$$A_r = \begin{bmatrix} C & O & \dots & O \\ O & \varepsilon C & \dots & O \\ & & \ddots & \\ O & O & \dots & \varepsilon^{k-1}C \end{bmatrix}, \ B_r = \begin{bmatrix} O & O & \dots & O & D_1 \\ D_2 & O & \dots & O & O \\ & & \ddots & & \\ O & O & \dots & D_k & O \end{bmatrix}, \tag{2}$$

where $C \in M_{(n-r)/k}(\mathbb{F})$ is nonsingular, and $D_1, \dots, D_k \in M_{(n-r)/k}(\mathbb{F})$ are arbitrary nonsingular matrices satisfying the relations $D_iC = CD_i$, $i = 1, 2, \dots, k$.

We remark that quasi-commutativity is an important relation in quantum physics, for the details we refer to the classical monographs [7, 10] and the references therein, and in the representation theory of affine Hecke algebras, see [2].

The list from Remark 1.7 provides known linear upper bounds for the values of the length function for several subsets $\mathcal{S} \subseteq M_n(\mathbb{F})$. In this paper we revisit upper bounds and provide new lower bounds for the lengths of quasi-commuting pairs of matrices. We investigate whether all integers satisfying these bounds can be realized as length for a certain generating pair $\mathcal{S} = \{A_1, A_2\} \subseteq M_n(\mathbb{F})$ such that A_1 and A_2 are quasi-commuting? We prove that for commuting or product-nilpotent pairs all possible numbers are realizable and show that there are non-realizable values between lower and upper bounds for the other quasi-commuting pairs.

Our paper is organized as follows. In Section 2 we discuss length bounds for generating sets of two elements. In Section 3 we discuss the special case of ε-commutative matrices, where ε is a primitive root of unity. Section 4 is devoted to the improvement of the bounds provided by Theorem 3.7 under some additional conditions, in particular, the case that at least one of the matrices is non-derogatory or singular. Section 5 contains special cases and concrete computations for small n. We conclude with a summary and some open problems in Section 6.

2 Commuting pairs and quasi-commutative pairs with nilpotent product

Among all quasi-commuting pairs of matrices there are two special classes that can be singled out and where the analysis is easier than for general pairs. These are commuting pairs and quasi-commutative, non-commuting pairs with zero or nilpotent product. In these cases the length of algebras generated by such pairs is smaller than for general quasi-commutative matrix pairs. Below we present these classes explicitly.

Let $\mathcal{S} = \{A, B\}$, where $A, B \in M_n(\mathbb{F})$ satisfy one of the following conditions.

$$\begin{array}{lrcl} 1. & AB & = & BA, \\ 2. & (AB)^n & = & (BA)^n = 0 \text{ and } AB = \alpha BA \text{ for some } \alpha \in \mathbb{F} \setminus \{0\}, \\ 3. & AB & = & 0 \text{ or } BA = 0. \end{array} \tag{3}$$

We will show that in each of these cases $l(\mathcal{S}) \leq n-1$ and, moreover, for any $k = 1, \ldots, n-1$, there exist pairs of matrices $\mathcal{S}_k^\nu = \{A_k^\nu, B_k^\nu\}$ satisfying $l(\mathcal{S}_k^\nu) = k$.

2.1 The commutative case

Let $\mathcal{S} = \{A, B\}$, where $A, B \in M_n(\mathbb{F})$ and suppose that $AB = BA$. Consider the following lemma.

Lemma 2.1. *Let $k, n \in \mathbb{N}$, $n \geq 2$, $\frac{n}{2} + 1 \leq k \leq n-1$, and let $\mathbb{F}$ be an arbitrary field. Consider matrices $A, B \in M_n(\mathbb{F})$ such that A is a Jordan block of size n associated with the eigenvalue 0, i. e., $A = J_n(0)$, and that $B = A^k$. If $\mathcal{S} = \{A, B\}$, then $l(\mathcal{S}) = k - 1$.*

Proof. We have $\mathcal{L}(\{A, B\}) = \mathcal{L}(\{A\})$. Clearly, the matrices A and B satisfy the relations $B^2 = 0$, $BA^{k-1} = 0$, and $A^{k+t} = BA^t$ for all $t \geq 0$. Therefore all words in A and B of length k are reducible, while the words $I, A, \ldots, A^{k-1}$ are irreducible, since $A = J_n(0)$ and $\deg A = n > k$, thus the assertion follows. □

Using Lemma 2.1 we can prove the following theorem for the commutative case.

Theorem 2.2. *Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be an arbitrary field. Then*

1. *for any $k \in \mathbb{N} \cup \{0\}$, $k \leq n-1$, there exists a commutative pair of matrices $\mathcal{S}_k = \{A_k, B_k\} \subset M_n(\mathbb{F})$ such that $l(\mathcal{S}_k) = k$;*

2. for a commutative set $\mathcal{S} = \{A, B\} \subset M_n(\mathbb{F})$ we have:

a. $l(\mathcal{S}) = n-1$ if and only if one of the matrices in $\mathcal{S}$, say A, is non-derogatory and $B = \alpha A + \beta I$ for some $\alpha, \beta \in \mathbb{F}$;

b. $l(\mathcal{S}) = 0$ if and only if both matrices in $\mathcal{S}$ are scalar matrices.

Proof. Let $\mathcal{A}$ denote the algebra generated by $\mathcal{S}$.

1. For $k = 0$ we take $\mathcal{S}_0 = \{I, 2I\}$. When $k = n-1$ we take $\mathcal{S}_{n-1} = \{J_n(1), J_n(0)\}$. Then $l(\mathcal{S}_{n-1}) = l(\{J_n(1)\}) = n-1$ since the matrix $J_n(1)$ is non-derogatory.

   For a fixed $1 \le k \le n-2$ consider $A_k = J_{k+2}(0) \oplus O_{n-k-2}$, $B_k = A_k^{k+1}$ and $\mathcal{S}_k = \{A_k, B_k\}$. Then we have $A_k B_k = B_k A_k = O_n$ and $B_k^2 = O_n$. So by Lemma 2.1, $\mathcal{A}$ as a vector space is spanned by $I, A_k, B_k, A_k^2, \ldots, A_k^k$ and $l(\mathcal{S}_k) = k$.

2. Consider $\mathcal{S} = \{A, B\} \subset M_n(\mathbb{F})$, such that $AB = BA$.

   a. In the case $l(\mathcal{S}) = n-1$, we obtain from [12, Theorem 3] that $\mathcal{A}$ is a subalgebra generated by a non-derogatory matrix, consequently $\dim \mathcal{A} = n$, and by [12, Lemma 2] $\dim \mathcal{L}_1(\mathcal{S}) = 2$. Hence the identity matrix I and one of the matrices in $\mathcal{S}$, say A, form a basis of $\mathcal{L}_1(\mathcal{S})$, thus $B \in \mathcal{S} \subset \mathcal{L}_1(\mathcal{S})$ is equal to a linear combination of the basis elements I and A. If $B = \alpha A + \beta I$, then we also have $l(\mathcal{S}) = l(\{A\}) = \deg A - 1$, thus $l(\mathcal{S}) = n-1$ if and only if the matrix A is non-derogatory.

   b. For $l(\mathcal{S}) = 0$ the assertion is obvious, since the length of any set containing non-scalar matrices is at least 1.

$\square$

2.2 The nilpotent product case

In this subsection we consider the second case of the conditions (3) where the product of two matrices is nilpotent. We need the following lemma.

Lemma 2.3. *Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be an arbitrary field. If for the two matrices in $\mathcal{S} = \{A, B\} \subset M_n(\mathbb{F})$ $(AB)^n = (BA)^n = 0$, and either i) $AB = aBA$ for $a \in \mathbb{F} \setminus \{0\}$, or ii) at least one of the products AB, BA is zero, then $l(\mathcal{S}) \le n-1$.*

Proof. At the beginning we consider the case that the field $\mathbb{F}$ is algebraically closed. We first prove that A and B are simultaneously triangularizable.

i) In the case that $a \neq 0$, this follows from Theorem 1.10. To show this we proceed by induction. The case $n = 1$ is trivial, and suppose that the assertion holds for $n-1$. Assume that $AB = aBA$ for $a \neq 0$. Since $\det((AB)^n) = 0$, we have that at least one of the matrices from $\mathcal{S}$ is singular. Without loss of generality we may assume that $\det B = 0$. Then there exists a vector $v \in \mathbb{F}^n \setminus \{0\}$ such that $Bv = 0$. Thus $B(Av) = a^{-1}A(Bv) = a^{-1}A \cdot 0 = 0$, and hence $BA^iv = 0$ for all $i \in \mathbb{N} \cup \{0\}$. Therefore, the space $W = \langle v, Av, A^2v, \ldots \rangle$ is a common invariant subspace for both A and B, and there exists a vector $w \in W$ such that $Bw = 0$ and $Aw = \lambda w$. Then the matrices A and B can be simultaneously transformed by a similarity transformation to the block form

$$A' = \begin{bmatrix} \lambda & * \\ O_{n-1,1} & A_{n-1} \end{bmatrix}, B' = \begin{bmatrix} 0 & * \\ O_{n-1,1} & B_{n-1} \end{bmatrix},$$

where the matrices A_{n-1} and B_{n-1} are of order $n-1$ and satisfy the same assumptions as A and B, hence by the inductive assumption A_{n-1} and B_{n-1} are simultaneously triangularizable.

ii) Now, let $a = 0$. If $AB = 0$ then either $B = 0$ and A can be transformed to its Jordan form, or $A = 0$, B can be transformed to its Jordan form, or both matrices A and B are singular. In the latter case, there exists a vector $v \in \mathbb{F}^n$ such that $Av = 0$. Then $ABv = O \cdot v = 0$, and hence $AB^iv = 0$ for all $i \in \mathbb{N} \cup \{0\}$. Therefore, the space $W = \{v, Bv, B^2v, \ldots\}$ is a common invariant subspace for both A and B, and there exists a vector $w \in W$ such that $Aw = 0, Bw = \beta w$ and we obtain the assertion via induction on n as in i).

Thus, there exists an invertible matrix $C \in M_n(\mathbb{F})$ such that the algebra $C^{-1}\mathcal{L}(\{A, B\})C$ is a subalgebra in the algebra of upper triangular matrices in $M_n(\mathbb{F})$, hence $l(\mathcal{S}) \leq n-1$, and moreover $l(\mathcal{L}(\{A, B\})) \leq n-1$.

If the field $\mathbb{F}$ is not algebraically closed, then from [14, Proposition 3.19] and the bound in i), we obtain that

$$l(\mathcal{S}) \leq l(\mathcal{L}(\{A, B\})_{\mathbb{F}}) \leq l(\mathcal{L}(\{A, B\})_{\bar{\mathbb{F}}}) \leq n-1,$$

where $\bar{\mathbb{F}}$ denotes the extension field of $\mathbb{F}$. $\square$

Using Lemma 2.3 we obtain the following theorem.

Theorem 2.4. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbb{F}$ be an arbitrary field satisfying $|\mathbb{F}| > 2$, and let $a \in \mathbb{F} \setminus \{0, 1\}$ be arbitrary. Then for any $k \in \mathbb{N}, k \leq n-1$, there exists a pair $\{A_k, B_k\} \subset M_n(\mathbb{F})$ such that $(A_kB_k)^n = 0$, $A_kB_k = aB_kA_k$ and this pair satisfies $l(\{A_k, B_k\}) = k$.*

Proof. Consider an n-dimensional Jordan block with the eigenvalue 0, $J_n = J_n(0)$, and set $D_n = \operatorname{diag}(1, a, \dots, a^{n-1})$. Then

$$J_n D_n = \begin{bmatrix} 0 & a & 0 & \dots & 0 \\ 0 & 0 & a^2 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 0 & a^{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} = a D_n J_n.$$

The word $J_n^{n-1} = E_{1,n}$ is irreducible, since for all words $J_n^k D_n^t$, $k \le n-2$ the element in the position $(1,n)$ is zero. Thus, $l(\{J_n, D_n\}) = n-1$.

For a given $k = 1, \dots, n-1$, using the same notation, consider a pair $A_k = J_{k+1} \oplus O_{n-k-1}$, $B_k = D_{k+1} \oplus I_{n-k-1}$. We then have $\deg A_k = \deg J_{k+1}$, $\deg B_k = \deg D_{k+1}$, and $l(\{A_k, B_k\}) = l(\{J_{k+1}, D_{k+1}\}) = k$. □

The next statement extends Theorem 2.4 to the case $a = 0$. We observe that the case $a = 1$ is considered in Section 2.1.

Theorem 2.5. *Let $n \in \mathbb{N}$, $n \ge 2$ and let $\mathbb{F}$ be an arbitrary field. For any $k \in \mathbb{N}, k \le n-1$ there exist a pair $\{A_k, B_k\} \subset M_n(\mathbb{F})$ with $B_k A_k = 0$, $A_k B_k \ne 0$, and $l(\{A_k, B_k\}) = k$.*

Proof. Considering the pair of matrices $J_n = J_n(0)$, $E_{n,n}$, then the following identities hold.

$$E_{n,n} J_n = 0,\ J_n^r E_{n,n} = E_{n-r,n},\ r = 1 \dots, n-1,\ J_n^r = \sum_{k=1}^{n-r} E_{k,r+k}.$$

Therefore, for any word $V \in \{J_n, E_{n,n}\}^{n-2}$ we have $(V)_{1,n} = 0$. However, $(J_n^{n-1})_{1,n} = 1$, and thus, J_n^{n-1} is an irreducible word and $l(\{J_n, E_{n,n}\}) = n-1$.

For a given $k = 1, \dots, n-1$ consider a pair

$$A_k = \begin{bmatrix} J_{k+1}(0) & O \\ O & O \end{bmatrix},\ B_k = \begin{bmatrix} E_{k+1,k+1} & O \\ O & O \end{bmatrix} \in M_n(\mathbb{F}).$$

Then we have $\deg A_k = \deg J_{k+1}$, $\deg B_k = \deg E_{k+1,k+1}$ and

$$l(\{A_k, B_k\}) = l(\{J_{k+1}, E_{k+1,k+1}\}) = k,$$

here $\deg X$ denotes the degree of the minimal polynomial of X. □

In this section we have discussed two special classes of matrices and investigated the lengths for these classes. We have shown that the case of non-commuting product-nilpotent matrices is similar to the commutative case. In the next section we look at quasi-commutative matrices, where the scalar factor is a root of unity.

3 Pairs of ε-commutative matrices

In this section we discuss the special class of pairs of quasi-commutative matrices for which the factor is a primitive k-th root of unity and we will show that in this case the situation is very different from the commutative and nilpotent case. In the following we will show that in this case we obtain a bound

$$2k - 2 \le l(\mathcal{S}) \le 2n - 2.$$

We begin with a lower bound for the length of a pair of ε-commutative matrices.

Lemma 3.1. *Let $k, n \in \mathbb{N}$, $1 < k \le n$, and $k|n$. Let $\mathbb{F}$ be an algebraically closed field containing a primitive root of unity ε_k of order k. Consider a pair of matrices $\{A, B\} \subset M_n(\mathbb{F})$ such that the matrix AB is invertible and $AB = \varepsilon_k BA$. Then*

$$l(\{A, B\}) \ge 2k - 2.$$

Proof. To prove the bound, we show that the word $A^{k-1}B^{k-1}$ is irreducible over $\{A, B\}$.

Since AB is invertible, by Theorem 1.10 the matrices A and B can be assumed to be in block form (2), where C is a nonsingular square matrix of order n/k, and $D_1, \ldots, D_k$ are arbitrary nonsingular matrices of order n/k which satisfy the relations $D_iC = CD_i$, $i = 1, 2, \ldots, k$.

Since the field is algebraically closed, the matrix C has at least one non-zero eigenvalue λ, and thus A has at least k distinct eigenvalues λ, $\varepsilon\lambda$, $\ldots$, $\varepsilon^{k-1}\lambda$, which implies that $I, A, \ldots, A^{k-1}$ are linearly independent.

Consider a decomposition of the space $M_n(\mathbb{F})$ into a direct sum of subspaces $V_1 \oplus \ldots \oplus V_k$, such that

$$V_i = \left\{ F \otimes Z | F = (\sum_{j=1}^{k-i} f_j E_{j+i,j} + \sum_{q=1}^{i} g_q E_{q,q+k-i}) \in M_k(\mathbb{F}), Z \in M_{\frac{n}{k}}(\mathbb{F}) \right\},$$

where $\otimes$ denotes the Kronecker product of matrices.

We have $\dim V_i = \frac{n^2}{k}$, which means that every matrix in the space V_i has the same block structure as the matrix B^i.

For all $i = 1, \ldots, k$, we have $B^i \in V_i$, and moreover, $A^hB^i \in V_i$ for all $h \in \mathbb{N}$. Furthermore, $B^t \in V_i$, $t \in \mathbb{N}$ if and only if $t \equiv i(\mod k)$. In particular, the word B^{k-1} is irreducible.

The space $\mathcal{L}_{2k-3}(\{A, B\})$ is generated by the words A^tB^m, $t, m \in \mathbb{N} \cup \{0\}, t + m \le 2k - 3$. Since for these words $m \le 2k - 3$, then $m \equiv k - 1(\mod k)$ if and only if $m = k - 1$.

Suppose that $A^{k-1}B^{k-1}$ were reducible, i. e.,

$$A^{k-1}B^{k-1} = \sum_{\substack{t,m\in\mathbb{N}\cup\{0\}\\ t+m\leq 2k-3}} \alpha_{t,m}A^tB^m.$$

We have $A^{k-1}B^{k-1} \in V_{k-1}$, thus the summands from V_j, for all $j \neq k-1$, in the right hand side are zero, i.e.

$$\sum_{\substack{t,m\in\mathbb{N}\cup\{0\}\\ m\neq k-1 \pmod k\\ t+m\leq 2k-3}} \alpha_{t,m}A^tB^m = 0.$$

Then it follows from the above analysis that

$$A^{k-1}B^{k-1} = \sum_{t=0}^{k-2} \alpha_{t,k-1}A^tB^{k-1},$$

and hence, we obtain that

$$(A^{k-1} - \sum_{t=0}^{k-2} \alpha_{t,k-1}A^t)B^{k-1} = 0.$$

The invertibility of B then implies that $A^{k-1} - \sum\limits_{t=0}^{k-2} \alpha_{t,k-1}A^t = 0$, which is a contradiction. □

Making use of Lemma 3.1, we have the following lower bound for the length also in the non-invertible case.

Theorem 3.2. *Let $k, n \in \mathbb{N}$, $1 < k \leq n$. Let $\mathbb{F}$ be an algebraically closed field containing a primitive root of unity ε_k of degree k. Consider matrices $A, B \in M_n(\mathbb{F})$ such that $AB = \varepsilon_k BA$ and $(AB)^n \neq 0$. Then*

$$l(\{A, B\}) \geq 2k - 2.$$

Proof. By Theorem 1.10, we may assume that the matrices A and B have the block form (1), where S and T are upper triangular $r \times r$ matrices, ST and TS are both nilpotent, and A_r and B_r are invertible matrices in $M_{n-r}(\mathbb{F})$ satisfying $A_rB_r = \varepsilon_k B_rA_r$.

From the theorem on the length of block-triangular algebras [11, Corollary 5.4], it follows that

$$l(\{A, B\}) \geq \max\{l(\{S, T\}), l(\{A_r, B_r\})\} \geq l(\{A_r, B_r\}),$$

and thus the lower bound $l(\{A, B\}) \geq l(\{A_r, B_r\}) \geq 2k - 2$ follows from Lemma 3.1. □

To prove an upper bound, we need another lemma.

Lemma 3.3. *Let $n, m \in \mathbb{N}, n \geq 2, m \leq n$. Let $\mathbb{F}$ be a field with $|\mathbb{F}| \geq n$, containing a primitive root of unity ε_m or order m. Consider $A, B \in M_n(\mathbb{F})$ such that $AB = \varepsilon_m BA$. If $m \nmid n$, then the words BA^{n-1} and $B^{n-1}A$ are reducible. Moreover, all words in A, B of lengths $\geq n$ containing at least $n-1$ copies of one letter are reducible.*

Proof. We denote by $H_{s,n-s}(A, B)$ the sum of all words in A, B of length n with s letters A and $n-s$ letters B, $1 \leq s \leq n-1$. With $\mathcal{S} = \{A, B\}$, for any scalar $t \in \mathbb{F}$ we have that $(A+tB)^n \in \mathcal{L}_{n-1}(\{A+tB\}) \subseteq \mathcal{L}_{n-1}(\mathcal{S})$ by the Cayley–Hamilton Theorem [4]. For a set of pairwise distinct scalars $\alpha_1, \ldots, \alpha_{n-1}$ the Vandermonde matrix $V(\alpha_1, \ldots, \alpha_{n-1})$ is invertible. Hence, we obtain that each sum $H_{s,n-s}(A, B)$ belongs to $\mathcal{L}_{n-1}(\mathcal{S})$. In particular,

$$\begin{aligned} H_{n-1,1}(A,B) &= BA^{n-1} + ABA^{n-2} + \ldots + A^{n-1}B \\ &= 1 \cdot BA^{n-1} + \varepsilon_m \cdot BA^{n-1} + \ldots + \varepsilon_m^{n-1} BA^{n-1} \\ &= (\sum_{i=0}^{n-1} \varepsilon_m^i) BA^{n-1}. \end{aligned}$$

Since $\varepsilon_m \neq 1$ and ε_m is not an n-th root of unity, then $\sum\limits_{i=0}^{n-1} \varepsilon_m^i = \frac{\varepsilon_m^n - 1}{\varepsilon_m - 1} \neq 0$, and hence, $BA^{n-1} \in \mathcal{L}_{n-1}(\mathcal{S})$. Considering $H_{1,n-1}(A, B)$ we analogously obtain that $B^{n-1}A \in \mathcal{L}_{n-1}(\mathcal{S})$.

To see the last part of the assertion, we notice that any word in A, B is a scalar multiple of a word of type $B^k A^l$, $k, l \in \mathbb{N} \cup \{0\}$. $\square$

Using Lemma 3.3, we obtain the following upper bound.

Theorem 3.4. *Let $\mathbb{F}$ be an algebraically closed field, $n \in \mathbb{N}$, $n \geq 2$. Consider $A, B \in M_n(\mathbb{F})$ such that the matrix AB is not nilpotent and $AB = \varepsilon BA$ for some primitive root of unity $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$. Then $l(\{A, B\}) \leq 2n-2$ and equality $l(\{A, B\}) = 2n-2$ holds if and only if ε is a primitive root of unity of order n.*

Proof. The upper bound $l(\{A, B\}) \leq 2n-2$ follows from [1, Theorem 2.7]. So it remains to study the case when the equality $l(\{A, B\}) = 2n-2$ is attained. If this is the case, then the word $B^{n-1}A^{n-1}$ must be irreducible over $\{A, B\}$. In Lemma 3.3 we have shown that all words are reducible if ε is a primitive root of unity of order k not dividing n, and hence we just need to consider the case $k|n$.

It follows from Theorem 1.10, that A and B can be taken into the block form (1). We will show then, that for the maximal length $2n-2$ the triangular block cannot occur. Indeed, suppose $r \geq 1$. For the triangular part we have a bound $l(\{S,T\}) \leq r-1$ (see Remark 1.7), while for the invertible part the general bound $l(\{A_r, B_r\}) \leq 2(n-r)-2$ from [1, Theorem 2.7] holds. Then by the theorem on the length of a block-triangular algebra [11, Corollary 5.4] it holds that

$$\begin{aligned} l(\{A,B\}) &\leq l(\{S,T\}) + l(\{A_r,B_r\}) + 1 \leq r-1+2(n-r)-2+1 \\ &= 2n-r-2 < 2n-2. \end{aligned}$$

Therefore, AB is invertible and we have that A, B have the form (2) where ε is a primitive root of unity of order k, C is a nonsingular square matrix of order n/k, $D_1, \dots, D_k$ are arbitrary nonsingular matrices of order n/k satisfying the relations $D_i C = C D_i$, $i = 1, 2, \dots, k$.

If $k = n$, then the pair A, B generates the full matrix algebra $M_n(\mathbb{F})$ and $l(\{A,B\}) = 2n-2$ (analogously to the pair C_n, D_n from Lemma 3.5). If $k|n$ and $1 < k < n$, which is possible only for $n \geq 4$, then we will show that $l(\{A,B\}) < 2n-2$. To show this, we consider several cases for the matrix C.

a) If the matrix C is derogatory, i. e., $\deg C \leq \frac{n}{k} - 1$, then $\deg A \leq k \cdot \deg C \leq k(\frac{n}{k} - 1) = n - k \leq n-2$ and thus, the word A^{n-2} is reducible, and so are both of the words A^{n-1} and $B^{n-1}A^{n-1}$. Hence, $l(\{A,B\}) < 2n-2$.

b) If the matrix C has a pair of eigenvalues α, β, such that $\alpha = \varepsilon\beta$, then also $\deg A \leq n-2$ and $l(\{A,B\}) < 2n-2$.

c) Suppose that C is a non-derogatory matrix such that $\sigma(C) \cap \sigma(\varepsilon C) = \emptyset$, then the matrix A is also non-derogatory.

From the relation $D_i C = C D_i$, we obtain that $D_i = p_i(C)$ for some polynomial p_i of degree not larger than $\frac{n}{k} - 1$, $i = 1, \dots, k$.

Since D_i is invertible, it follows that $\mathcal{L}(\{C\}) = \mathcal{L}(\{C\}) D_i$, $i = 1, \dots, k$, and for each $m = 1, \dots, k$, and all $j = 0, \dots, \frac{n}{k} - 1$ the matrices

$$\begin{bmatrix} O_{\frac{n(m-1)}{k}} & O & O \\ O & C^j & O \\ O & O & O_{n-\frac{nm}{k}} \end{bmatrix}$$

are polynomials in A of degree not larger than $n-1$. Thus for any $t = 0, \dots, k-1$, the matrices $B^t, AB^t, \dots, A^{n-1}B^t$ generate all non-zero blocks of B^t. Finally, $l(\{A,B\}) \leq n-1+k-1 < 2n-3$, since k is a proper divisor of a number $n > 3$, i. e., $k \leq \frac{n}{2} \leq n-2$. $\square$

Finally, we prove the sharpness of the bounds in Theorems 3.2 and 3.4.

Lemma 3.5. *Let $k, n \in \mathbb{N}$, $k, n \geq 2$, and let $\mathbb{F}$ be a field containing primitive roots of unity ε_i of all orders $i = 2, \ldots, n$. Let us introduce the matrices*

$$
\begin{aligned}
C_k &:= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}, \ D_k := \operatorname{diag}(1, \varepsilon_k, \varepsilon_k^2, \ldots, \varepsilon_k^{k-1}) \in M_k(\mathbb{F}), \\
A_k &:= \begin{bmatrix} C_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{bmatrix}, \ B_k := \begin{bmatrix} D_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{bmatrix} \in M_n(\mathbb{F}).
\end{aligned}
$$

Then $A_k B_k = \varepsilon_k B_k A_k$, and $l(\{A_k, B_k\}) = l(\{C_k, D_k\}) = 2k - 2$.

Proof. The proof that $l(\{C_k, D_k\}) = 2k - 2$ is given in [1, Section 3] and the identity $A_k B_k = \varepsilon_k B_k A_k$ holds by construction. We also notice that $\deg A_k = \deg B_k = k+1$, hence the algebra $\mathcal{L}(\{A_k, B_k\})$ is spanned by words $B_k^r A_k^s$, $0 \leq r, s \leq k$. We do not need words $B_k^r A_k^k$ and $B_k^k A_k^s$, since

$$
B_k^k = A_k^k = \begin{bmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & O_{n-k} \end{bmatrix}
$$

and thus $B_k^r A_k^k = B_k^r$ and $B_k^k A_k^s = A_k^s$. □

Our next result shows that the length $2n - 3$ cannot occur.

Theorem 3.6. *Let $\mathbb{F}$ be an algebraically closed field, and let $n \in \mathbb{N}$, $n \geq 2$. If the product AB is not nilpotent, then there does not exist a pair of matrices $\{A, B\} \subset M_n(\mathbb{F})$ such that $AB = \varepsilon BA$ for some root of unity $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$ and $l(\{A, B\}) = 2n - 3$.*

Proof. The proof is similar to the proof of Theorem 3.4. If $l(\{A, B\}) = 2n - 3$ holds, then at least one of the words $B^{n-1}A^{n-2}$ or $B^{n-2}A^{n-1}$ must be irreducible over $\{A, B\}$. In Lemma 3.3 we have shown that all of these words are reducible if ε is a root of unity of order k not dividing n. So we are left to consider the case $k|n$.

It follows from Theorem 1.10, that A and B can be assumed to be in block form (1). We show first that for the length $2n-3$ the triangular block cannot occur. Indeed, if $r \geq 1$ then by the theorem on the length of a block-triangular algebra [11, Corollary 5.4], it follows that

$$
\begin{aligned}
l(\{A, B\}) &\leq l(\{S, T\}) + l(\{A_r, B_r\}) + 1 \leq r - 1 + 2(n - r) - 2 + 1 \\
&= 2n - r - 2 \leq 2n - 3.
\end{aligned}
$$

Since $2n-r-2 < 2n-3$ for $r > 2$, and for $r = 1$ we have $k|(n-1)$, and thus $k \nmid n$, it follows that $l(\{A, B\}) < 2n-3$.

Therefore AB is invertible and we have that A, B are as in (2), where ε is a primitive root of unity of order k, C is a nonsingular square matrix of order n/k, and $D_1, \ldots, D_k$ are arbitrary nonsingular matrices of order n/k satisfying the relations $D_iC = CD_i$, $i = 1, 2, \ldots, k$.

If $k = n$, then $l(\{A, B\}) = 2n-2$ as was shown in Theorem 3.4, so let $k|n$ and $1 < k < n$, which is possible only for $n \geq 4$. We will then show that $l(\{A, B\}) < 2n-3$ and consider again different cases for C.

a) If the matrix C is derogatory, i. e., $\deg C \leq \frac{n}{k} - 1$, then $\deg A \leq k \cdot \deg C \leq k(\frac{n}{k} - 1) = n - k \leq n-2$. Hence, the word A^{n-2} is reducible, and so are both of the words $B^{n-2}A^{n-1}$ and $B^{n-1}A^{n-2}$. Thus $l(\{A, B\}) < 2n-3$.

b) If the matrix C has a pair of eigenvalues α, β, such that $\alpha = \varepsilon\beta$, then also $\deg A \leq n-2$ and $l(\{A, B\}) < 2n-3$.

c) If C is a non-derogatory matrix such that $\sigma(C) \cap \sigma(\varepsilon C) = \emptyset$, then the bound $l(\{A, B\}) < 2n-3$ follows as in the analogous case of Theorem 3.4. □

We summarize the results of this section in the following theorem.

Theorem 3.7. *Let $\mathcal{S} = \{A, B\}$, where $A, B \in M_n(\mathbb{F})$ satisfy $AB = \varepsilon_k BA$ for a degree k primitive root of unity $\varepsilon_k \in \mathbb{F} \setminus \{0\}$, and do not satisfy any of the conditions in (3).*

1. Then

$$2k - 2 \leq l(\mathcal{S}) \leq 2n - 2.$$

and, moreover, both these bounds are sharp.

2. There is no matrix pair $A_0, B_0 \in M_n(\mathbb{F})$ such that $A_0B_0 = \varepsilon B_0A_0$ for some $\varepsilon \in \mathbb{F}$ satisfying $l(\{A_0, B_0\}) = 2n-3$.

As a corollary we have a smaller bound for the case that the value $2n-2$ cannot be attained.

Corollary 3.8. *Let $\mathbb{F}$ be an algebraically closed field, $n \in \mathbb{N}$, $n \geq 2$. Consider $A, B \in M_n(\mathbb{F})$ such that the matrix AB is not nilpotent and $AB = \varepsilon BA$ for some root of unity $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$, of degree $k < n$. Then $l(\{A, B\}) \leq 2n-4$.*

So far we have always assumed that the considered field is algebraically closed. In the proof of [14, Proposition 3.19] it has been shown that the length of a subset $\mathcal{S} \subseteq M_n(\mathbb{F})$ is the same as that considered as a subset of $M_n(\mathbb{K})$ for any extension field $\mathbb{K} \supseteq \mathbb{F}$. Thus, Theorem 3.7, and the bounds from Theorems 3.2 and 3.4 hold for the length of a quasi-commuting pair of matrices over arbitrary fields and the length $l(\{A, B\}) = 2n-3$ is not realizable if $AB = \varepsilon BA$ for some root of unity $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$ if AB is not nilpotent.

Furthermore, we have the following corollary.

Corollary 3.9. *Let $\mathbb{F}$ be an algebraically closed field. Then all even numbers between 2 and $2n-2$ are realizable as lengths of quasi-commuting matrix pairs.*

The following tables display the realizability of different natural numbers that can arise for subalgebras generated by two quasi-commutative matrices of small size. We put a "+" for the realizable values and "-" for the non-realizable values.

For $n = 3$ we have the following realizability pattern.

$l(\{A,B\})$	0	1	2	3	4
$AB = BA$	+	+	+	-	-
$AB = aBA,\ a \neq 1,\ (AB)^3 = 0$	-	+	+	-	-
$(AB)^3 \neq 0,\ AB = -BA$	-	-	+	-	-
$(AB)^3 \neq 0,\ AB = \varepsilon_3 BA$	-	-	-	-	+

For $n = 4$ the realizability is as follows.

$l(\{A,B\})$	0	1	2	3	4	5	6
$AB = BA$	+	+	+	+	-	-	-
$AB = aBA,\ a \neq 1,\ (AB)^4 = 0$	-	+	+	+	-	-	-
$(AB)^4 \neq 0,\ AB = -BA$	-	-	+	+	+	-	-
$(AB)^4 \neq 0,\ AB = \varepsilon_3 BA$	-	-	-	-	+	-	-
$(AB)^4 \neq 0,\ AB = \varepsilon_4 BA$	-	-	-	-	-	-	+

4 Upper bounds depending on k and the multiplicity of the eigenvalue 0

In this section we provide upper bounds, as functions of n, k and r, where r is the (algebraic) multiplicity of zero as an eigenvalue of AB, for the lengths of pairs $A, B \in M_n(\mathbb{F})$, such that AB is not nilpotent and $AB = \varepsilon_k BA$, where ε_k is a primitive root of unity of degree k.

We first discuss invertible ε-commuting pairs.

Remark 4.1. If $aAB = BA$ for some $a \in \mathbb{F}$, then the longest word in A and B that is not trivially reducible is the word $A^{\deg A - 1} B^{\deg B - 1}$ of length $\deg A + \deg B - 2$, and hence $l(\{A,B\}) \leq \deg A + \deg B - 2$.

In the non-derogatory case we have the following bound.

Lemma 4.2. *Let $k, n \in \mathbb{N}$, $1 < k \leq n$, $k|n$. Let $\mathbb{F}$ be a field containing a primitive root of unity ε_k of degree k. Consider matrices $A, B \in M_n(\mathbb{F})$ such that the matrix A is non-derogatory, the matrix AB is invertible, and $AB = \varepsilon_k BA$. Then*

$$l(\{A, B\}) \leq n + k - 2.$$

Proof. Since the length of a set in the extension field is that of the field, we may assume that the field $\mathbb{F}$ is algebraically closed, and since AB is invertible, by Theorem 1.10, the matrices A and B can be assumed to be in the form (2), where C is a nonsingular square matrix of order n/k, and $D_1, \ldots, D_k$ are arbitrary nonsingular matrices of order n/k satisfying relations $D_iC = CD_i$, $i = 1, 2, \ldots, k$.

Since A is non-derogatory, C is also non-derogatory and $\sigma(C) \cap \sigma(\varepsilon_k C) = \emptyset$. Then from $D_iC = CD_i$ we obtain that $D_i = p_i(C)$ for some polynomial p_i of degree not larger than $\frac{n}{k} - 1$, $i = 1, \ldots, k$, and, since D_i is invertible, it follows that $\mathcal{L}(\{C\}) = \mathcal{L}(\{C\})D_i$, $i = 1, \ldots, k$.

For each $m = 1, \ldots, k$, and all $j = 0, \ldots, \frac{n}{k} - 1$ the matrices

$$\begin{bmatrix} O_{\frac{n(m-1)}{k}} & O & O \\ O & C^j & O \\ O & O & O_{n-\frac{nm}{k}} \end{bmatrix}$$

are polynomials in A of degree not larger than $n - 1$. Thus, for any $t = 0, \ldots, k-1$ the matrices $B^t, AB^t, \ldots, A^{n-1}B^t$ generate all non-zero blocks of B^t, and finally, we get $l(\{A, B\}) \leq n - 1 + k - 1 = n + k - 2$. $\square$

The bound of Lemma 4.2 is sharp as the following corollary shows.

Corollary 4.3. *Let $k, n \in \mathbb{N}$, $n \geq 2$, $1 < k \leq n$ and $k|n$. Let $\mathbb{F}$ be a field containing a primitive root of unity ε_k of degree k, then the pair $\{A, B\} \subset M_n(\mathbb{F})$,*

$$A = \begin{bmatrix} J_{\frac{n}{k}}(1) & O & \ldots & O \\ O & \varepsilon_k J_{\frac{n}{k}}(1) & \ldots & O \\ & & \ddots & \\ O & O & \ldots & \varepsilon^{k-1} J_{\frac{n}{k}}(1) \end{bmatrix}, \ B = \begin{bmatrix} O & O & \ldots & O & I_{\frac{n}{k}} \\ I_{\frac{n}{k}} & O & \ldots & O & O \\ & & \ddots & & \\ O & O & \ldots & I_{\frac{n}{k}} & O \end{bmatrix},$$

satisfies $AB = \varepsilon_k BA$, and $l(\{A, B\}) = n + k - 2$.

Proof. As was shown in the proof of Lemma 4.2, the set

$$\mathcal{S} = \{B^t, AB^t, \ldots, A^{n-1}B^t | t = 0, \ldots, k-1\} \subset \{A, B\}^{n+k-2}$$

is a basis for $\mathcal{L}(\{A, B\})$. If we show that the word $A^{n-1}B^{k-1}$ is irreducible over $\{A, B\}$, then the assertion follows.

For this, note that the matrix B satisfies $B^k = I$. Hence, if $m = qk+r \geq k$ then $B^m = B^r$, i. e., is reducible. In this case any word $A^sB^m = A^sB^r$, $s \in \mathbb{N} \cup \{0\}$ is also reducible to a word, containing a power of B bounded by $k-1$. Therefore, if $A^{n-1}B^{k-1}$ is a reducible word, then it can be expressed as a linear combination of words from $\mathcal{S}$, which is a contradiction to the linear independence of the set $\mathcal{S}$. $\square$

In the derogatory case we obtain another bound:

Lemma 4.4. *Let $k, n \in \mathbb{N}$, $1 < k < n, k|n$ and let $\mathbb{F}$ be a field containing a primitive root of unity ε_k of degree k. Consider matrices $A, B \in M_n(\mathbb{F})$ such that both matrices A and B are derogatory, the matrix AB is invertible, and $AB = \varepsilon_k BA$. Then*

$$l(\{A, B\}) \leq 2(n-k) - 2.$$

Proof. Since the length does not change when going to an extension field, without loss of generality we may assume that the field $\mathbb{F}$ is algebraically closed.

Since AB is invertible, again using Theorem 1.10, the matrices A and B can be assumed to be of the form (2) where $D_1, \ldots, D_k$ are arbitrary nonsingular matrices of order n/k subject to the relations $D_iC = CD_i$, $i = 1, 2, \ldots, k$.

Since the matrix A is derogatory we have two possibilities:

a) The matrix C is derogatory, i. e., $\deg C \leq \frac{n}{k} - 1$, then

$$\deg A \leq k \cdot \deg C \leq k(\frac{n}{k} - 1) = n - k.$$

b) The matrix C is non-derogatory and has at least one pair of eigenvalues α, β, such that $\alpha = \varepsilon\beta$. Then k numbers $\alpha, \varepsilon_k\alpha, \ldots, \varepsilon^{k-1}$ are eigenvalues of the matrix A, such that there are at least two Jordan blocks associated with each of them. Then also $\deg A \leq n - k$.

Since ε_k^{-1} is also a primitive root of unity of order k, then $BA = \varepsilon_k^{-1}AB$, so we can replace A by B, and the Jordan forms of A and B have same structure, see, e.g. [6, Theorem 5]. Hence $\deg B \leq n - k$.

Combining these bounds, by Remark 4.1 we obtain $l(\{A, B\}) \leq 2(n - k) - 2$. $\square$

Note that the bound from Lemma 4.4 provides better result for the length function than the bound from Lemma 4.2 if and only if $n = 2k$.

As a direct corollary, we obtain the following result.

Corollary 4.5. *Let $k, n, q \in \mathbb{N}$, $n \geq 4$, $1 < k < n$, $k|n$, $1 \leq q \leq \frac{n}{k} - 1$ and let $\mathbb{F}$ be a field containing a primitive root of unity ε_k of order k. Then the pair*

$$A_q = \begin{bmatrix} C_q & O & \dots & O \\ O & \varepsilon_k C_q & \dots & O \\ & & \ddots & \\ O & O & \dots & \varepsilon^{k-1} C_q \end{bmatrix}, \ B = \begin{bmatrix} O & O & \dots & O & I_{\frac{n}{k}} \\ I_{\frac{n}{k}} & O & \dots & O & O \\ & \ddots & & & \\ O & O & \dots & I_{\frac{n}{k}} & O \end{bmatrix} \in M_n(\mathbb{F})$$

with $C_q = J_{\frac{n}{k}-q}(1) \oplus I_q$, satisfies $A_q B = \varepsilon_k B A_q$ and $l(\{A_q, B\}) = n + k(1 - q) - 2$.

Proof. Noting that $\deg A_q = k(\frac{n}{k} - q) = n - kq$, the proof is similar to that of Corollary 4.3. $\square$

This corollary provides an example of the sharpness in Lemma 4.4, namely we get the following:

Example 4.6. *Let A_q, B, C_q be defined in the Corollary 4.5, where we choose $n = 2k$ and $q = 1$. Then $l(\{A_q, B\}) = n+k(1-q)-2 = 2k-2 = 2(n-k)-2$, which provides the exact value of the upper bound of the length in Lemma 4.4.*

Finally we have the following upper bound.

Theorem 4.7. *Let $k, n \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, $n \geq 3$, $0 \leq r \leq n-2$, $1 < k \leq n-r$, $k|(n-r)$, and let $\mathbb{F}$ be a field containing a primitive root of unity ε_k of order k. Consider matrices $A, B \in M_n(\mathbb{F})$ satisfying $AB = \varepsilon_k BA$ and suppose that AB has an eigenvalue 0 of multiplicity r. Then*

$$l(\{A, B\}) \leq \max\{2(n-k) - r, n + k\} - 2.$$

Proof. Again we may assume without loss of generality that the field $\mathbb{F}$ is algebraically closed, otherwise we could just apply the result in the extension field. By Theorem 1.10 we may assume that A and B are already in the form (1), where ST and TS are both nilpotent, and A_r and B_r are invertible matrices in $M_{n-r}(\mathbb{F})$ satisfying $AB = \varepsilon_k BA$.

The upper bound on the length of the pair A, B then follows from the theorem on the length of a block-triangular algebra [11, Corollary 5.4], which gives

$$\begin{aligned} l(\{A_r, B_r\}) &\leq \max\{l(\{S, T\}), l(\{A_r, B_r\})\} \leq l(\{A, B\}) \\ &\leq l(\{S, T\}) + l(\{A_r, B_r\}) + 1. \end{aligned}$$

Since S, T are triangular, then $l(\{S, T\}) \leq r - 1$ by [11, Lemma 4.2]. If at least one of the matrices A_r and B_r is non-derogatory, then $l(\{A_r, B_r\}) \leq n - r + k - 2$ by Lemma 4.2, and if both A_r and B_r are derogatory, then $l(\{A_r, B_r\}) \leq 2(n - r - k) - 2$ by Lemma 4.4. $\square$

5 Special cases

In this section we discuss several special cases which in particular exclude certain possibilities.

Proposition 5.1. *Let $\mathbb{F}$ be an algebraically closed field, $n \in \mathbb{N}$, $n \geq 2$. There does not exist a pair of matrices $\{A, B\} \subset M_n(\mathbb{F})$ that satisfy $(AB)^n \neq 0$, $AB = \varepsilon BA$ for $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 0, 1$ and for the length we have $l(\{A, B\}) = 1$.*

Proof. The assertion follows from Theorem 3.2, since $2k-2 \geq 2$ for $k \geq 2$. $\square$

In the next result we show that for the order k root of unity and $n \times n$ matrices we can get a length $(m+1)k - 1$ for any $m \in \mathbb{N}$ and $n \geq (m+1)k$.

Theorem 5.2. *Let $k, m \in \mathbb{N}$, $k \geq 2$ and let $\mathbb{F}$ be a field containing a primitive root of unity ε_k of order k. For any $n \geq (m+1)k$ there exist matrices $A_n, B_n \in M_n(\mathbb{F})$ such that $(A_n B_n)^n \neq 0$, $A_n B_n = \varepsilon_k B_n A_n$ and $l(\{A_n, B_n\}) = (m+1)k - 1$.*

Proof. Let $n = (m+1)k$. Consider the matrices

$$A_n = \begin{bmatrix} O_k & L & L & \dots & L \\ O_{m,k} & J_m(1) & O_m & \dots & O_m \\ O_{m,k} & O_m & \varepsilon_k J_m(1) & \dots & O_m \\ & & & \ddots & \\ O_{m,k} & O_m & O_m & \dots & \varepsilon_k^{k-1} J_m(1) \end{bmatrix},$$

$$B_n = \begin{bmatrix} J_k(0) & \varepsilon_k^{k-1} U & \dots & \varepsilon_k U & U \\ O_{m,k} & O_m & \dots & O_m & J_m(1) \\ O_{m,k} & J_m(1) & \dots & O_m & O_m \\ & & \ddots & & \\ O_{m,k} & O_m & \dots & J_m(1) & O_m \end{bmatrix},$$

where L is an arbitrary nonzero $k \times m$ matrix. We determine the matrix U from the relation $A_n B_n = \varepsilon_k B_n A_n$:

$$\varepsilon_k (J_k(0) L + \varepsilon_k^{k-1} U J_m(1)) = L J_m(1),$$

which gives
$$U = L - \varepsilon_k J_k(0) L (J_m(1))^{-1}.$$
Furthermore, we have
$$A_n^k = \begin{bmatrix} O_k & LJ_m(1)^{k-1} & \varepsilon_k^{k-1} LJ_m(1)^{k-1} & \dots & (\varepsilon_k^{k-1})^{k-1} LJ_m(1)^{k-1} \\ O_{m,k} & J_m(1)^k & O_m & \dots & O_m \\ O_{m,k} & O_m & J_m(1)^k & \dots & O_m \\ & & & \ddots & \\ O_{m,k} & O_m & O_m & \dots & J_m(1)^k \end{bmatrix}.$$
Next we calculate B_n^k to show that $B_n^k = A_n^k$ independent of the choice of L. If we denote $B_n = \begin{bmatrix} J_k(0) & Y_1 \\ O & B \end{bmatrix}$ and $B_n^k = \begin{bmatrix} J_k(0)^k & Y_k \\ O & B^k \end{bmatrix}$, then $J_k(0)^k = 0$, $B^k = J_m(1)^k \oplus \dots \oplus J_m(1)^k$, and
$$Y_k = \sum_{i=0}^{k-1} J_k(0)^i Y_1 B^{k-i-1} = \sum_{i=0}^{k-1} J_k(0)^i \begin{bmatrix} \varepsilon_k^{k-1} U & \dots & \varepsilon_k U & U \end{bmatrix} B^{k-i-1}.$$
For $s = 1, \dots, k$ we obtain
$$\begin{aligned} (Y_k)_{k-s+1} &= \sum_{i=0}^{k-1} \varepsilon_k^{i+s} J_k(0)^i U J_m(1)^{k-i-1} \\ &= \sum_{i=0}^{k-1} \varepsilon_k^{i+s} J_k(0)^i (L - \varepsilon_k J_k(0) L (J_m(1))^{-1}) J_m(1)^{k-i-1} \\ &= \sum_{i=0}^{k-1} \varepsilon_k^{i+s} J_k(0)^i L J_m(1)^{k-i-1} - \sum_{j=1}^{k} \varepsilon_k^{j+s} J_k(0)^j L J_m(1)^{k-j-1} \\ &= \varepsilon_k^s L J_m(1)^{k-1} - \varepsilon_k^{k+s} J_k(0)^k L J_m(1)^{-1} = \varepsilon_k^s L J_m(1)^{k-1}. \end{aligned}$$
Since $(k-1)(k-s) \equiv s (\mod k)$, then the assertion $B_n^k = A_n^k$ follows. Thus, we have the relations
$$\begin{aligned} A_n B_n &= \varepsilon_k B_n A_n, \\ B_n^k &= A_n^k, \\ A_n (A_n^k - I)^m &= 0. \end{aligned}$$
Therefore, any word in A_n, B_n is a scalar multiple of a word of the same length containing at most $k-1$ letters B_n. Moreover, all words, containing at least $km+1 = n-k+1$ letters A_n are reducible, while the matrices $I, A_n, \dots, A_n^{km}$ are linearly independent. Thus, $l(\{A_n, B_n\}) < km+1+k-1 = k(m+1) = n$.

Suppose that the word $A_n^{km}B_n^{k-1}$ were reducible. Applying the arguments from Lemma 3.1, we obtain that

$$A_n^{km}B_n^{k-1} = \sum_{j=0}^{km-1} \alpha_j A_n^j B_n^{k-1}. \tag{4}$$

Then $\alpha_0 = 0$, since the $(1,k)$ elements satisfy

$$(B_n^{k-1})_{1,k} = 1, \ (A_n^p B_n^{k-1})_{1,k} = 0 \text{ for } p \geq 1.$$

Denote by A the $(n-k)\times(n-k)$ submatrix of A_n located in the last $n-k$ rows and columns.

Then the identity (4) implies that $A^{km}B^{k-1} = \sum\limits_{j=1}^{km-1} \alpha_j A^j B^{k-1}$, and since the matrices A and B are invertible, it follows that $A^{km-1} = \sum\limits_{j=1}^{km-1} \alpha_j A^{j-1}$, which is a contradiction, and hence, $l(\{A_n, B_n\}) = k(m+1)-1 = n-1$.

For $n \geq (m+1)k+1$ just consider $A_n = O_{n-(m+1)k} \oplus A_{(m+1)k}$, $B_n = O_{n-(m+1)k} \oplus B_{(m+1)k}$. Then the matrices A_n, B_n satisfy the same relations as $A_{(m+1)k}, B_{(m+1)k}$, and thus $l(\{A_n, B_n\}) = (m+1)k-1$. $\square$

Our next special case deals with the case of quasi-commutative pairs associated with the root of unity -1.

Proposition 5.3. *Let $m,n \in \mathbb{N}$, $n \geq 3$, and $2m \leq n$. If $\mathbb{F}$ is a field of characteristics different from 2, then the pair*

$$A = \begin{bmatrix} J_m(1) & O & O \\ O & -J_m(1) & O \\ O & O & O_{n-2m} \end{bmatrix}, \ B = \begin{bmatrix} O & I_m & O \\ I_m & O & O \\ O & O & O_{n-2m} \end{bmatrix} \in M_n(\mathbb{F}),$$

satisfies $AB = -BA$ and $l(\{A,B\}) = 2m$.

Proof. The proof is similar to the proof of Corollary 4.3. $\square$

Corollary 4.3 shows that for even numbers $n \geq 6$, n not being a power of 2, there exist odd numbers k such that $k|n$, for which $n+k-2$ is an odd number that is realizable as a length of an ε-commuting pair. For example, in the smallest case $n = 6$, we have that $k = 3$ and $n+k-2 = 7$ is realizable.

Proposition 5.4. *Let $n \in \mathbb{N}$, $n \geq 3$, and let $\mathbb{F}$ be a field containing primitive roots of unity ε_{n-1} and ε_n of orders $n-1$ and n, respectively. Consider matrices $A, B \in M_n(\mathbb{F})$, such that AB is not nilpotent.*

i) If $AB = \varepsilon_n BA$, then $l(\{A, B\}) = 2n - 2$.

ii) If $AB = \varepsilon_{n-1} BA$, then $l(\{A, B\}) = 2n - 4$.

Proof. The assertion follows from Theorem 3.7 and Corollary 3.8, since if $AB = \varepsilon_n BA$, then

$$2n - 2 = 2\,\mathrm{ord}\,(\varepsilon_n) - 2 \leq l(\{A, B\}) \leq 2n - 2,$$

and if $AB = \varepsilon_{n-1} BA$, then

$$2n - 4 = 2\,\mathrm{ord}\,(\varepsilon_{n-1}) - 2 \leq l(\{A, B\}) \leq 2n - 4,$$

here $\mathrm{ord}\,(\alpha)$ denotes the order of the element α in a group. $\square$

Proposition 5.5. *For given $n, r \in \mathbb{N}$ and any two matrices $A, B \in M_n(\mathbb{F})$ having the block form (1), the words $A^r B^{n-r}$ and $B^r A^{n-r}$ are reducible.*

Proof. The Cayley-Hamilton theorem [4] implies that for matrices $S, T \in M_r(\mathbb{F})$, there exist polynomials $f(x)$ and $g(x)$ from $\mathbb{F}[x]$ of degrees not greater than $r - 1$ such that $S^r = f(S)$ and $T^r = g(T)$, and similarly there exist polynomials $u(x)$ and $v(x)$ from $\mathbb{F}[x]$ of degrees not greater than $n - r - 1$ such that $A_r^{n-r} = u(A_r)$ and $B_r^{n-r} = v(B_r)$. Therefore,

$$(A^r - f(A))(B^{n-r} - v(B)) = \begin{bmatrix} O & * \\ O & * \end{bmatrix} \begin{bmatrix} * & * \\ O & O \end{bmatrix} = O,$$

$$(B^r - g(B))(A^{n-r} - u(A)) = \begin{bmatrix} O & * \\ O & * \end{bmatrix} \begin{bmatrix} * & * \\ O & O \end{bmatrix} = O.$$

Expanding these equations, we obtain that the words $A^r B^{n-r}$ and $B^r A^{n-r}$ are reducible. $\square$

We have the following special cases for $n = 5$.

Theorem 5.6. *Let $\mathbb{F}$ be an arbitrary field, and consider matrices $A, B \in M_5(\mathbb{F})$, such that $AB = \omega BA$ for some $\omega \in \mathbb{F}$ and denote $\mathcal{S} = \{A, B\}$. Then*

1. *$l(\mathcal{S}) = 8$ if and only if ω is a primitive root of unity of order 5;*
2. *$l(\mathcal{S}) = 6$ if and only if ω is a primitive root of unity of order 4;*
3. *$l(\mathcal{S}) \leq 4$ for all other values of ω.*

Proof. 1. If $\omega = 1$, i. e., the matrices A and B commute, or if ω is not a root of unity, which implies that AB is a nilpotent matrix, then $l(\mathcal{S}) \leq 4$ by Lemma 2.3.

2. If ω is a primitive root of unity of order $k = 4$ or 5, then $l(\mathcal{S}) = 2k - 2$ by Theorems 3.4, 3.6, and 3.7.

3. If ω is a primitive root of unity of order $k = 2$ or 3, then again without loss of generality we may assume that the field $\mathbb{F}$ is algebraically closed. Since k and 5 are co-prime, by Theorem 1.10, the matrices A and B can be assumed to be in block form (1) with the triangular block of size $1 \leq r \leq 3$.

Consider words in A and B of length 5, then the words A^5 and B^5 are reducible by Cayley-Hamilton theorem. The condition $k \nmid 5$ also implies that the words A^4B and AB^4 are reducible by Lemma 3.3. Thus to prove the bound $l(\mathcal{S}) \leq 4$ it remains to show the reducibility of the words A^3B^2 and A^2B^3.

a) If $r = 2$ or $r = 3$, then the words A^rB^{5-r} and B^rA^{5-r} are reducible by Proposition 5.5.

b) If $r = 1$, then $k|4$ and $k \leq 3$, and thus $k = 2$ and $\omega = -1$. By (2) we have

$$A = \begin{bmatrix} s & X_1 & X_2 \\ O & C & O \\ O & O & -C \end{bmatrix}, B = \begin{bmatrix} t & Y_1 & Y_2 \\ O & O & D_1 \\ O & D_2 & O \end{bmatrix},$$

where $st = 0$, $C, D_1, D_2 \in M_2(\mathbb{F})$.

If $C = \gamma I_2$ for some $\gamma \in \mathbb{F}$, then $(B - tI_5)(A^2 - \gamma^2 I_5) = 0$ similarly to a). Hence a word BA^2 is reducible, which implies that both words $B^2A^3 = A^3B^2$ and $B^3A^2 = A^2B^3$ containing it as a subword are also reducible.

Suppose that the matrix C is not a scalar multiple of the identity. A 2×2 matrix which is not a multiple of the identity is always non-derogatory, thus the matrices D_1 and D_2 commuting with C are linear polynomials in C, and $D_1D_2 = D_2D_1$. Thus,

$$A^2 = \begin{bmatrix} s^2 & X_1' & X_2' \\ O & C^2 & O \\ O & O & C^2 \end{bmatrix}, B^2 = \begin{bmatrix} t^2 & Y_1' & Y_2' \\ O & D_1D_2 & O \\ O & O & D_2D_1 \end{bmatrix}.$$

The matrices C^2 and D_1D_2 can also be expressed as linear polynomials in C, i. e., $C^2 = \gamma_1 C + \gamma_0 I_2$, $D_1D_2 = \delta_1 C + \delta_0 I_2$.

Suppose that $\lambda_1, \lambda_2 \in \mathbb{F}$ are the eigenvalues of C. Then

$$(C^2 - (\lambda_1\gamma_1 + \gamma_0)I_2)(D_1D_2 - (\lambda_2\delta_1 + \delta_0)I_2) = \gamma_1\delta_1\chi_C(C) = 0,$$

where $\chi_C(t)$ denotes the characteristic polynomial of C, and

$$(A^2 - (\lambda_1\gamma_1 + \gamma_0)I_5)(B^2 - (\lambda_2\delta_1 + \delta_0)I_5) = \begin{bmatrix} u & Z_1 & Z_2 \\ O & O & O \\ O & O & O \end{bmatrix}.$$

Multiplying this identity by the matrix $A - sI_5$ or $B - tI_5$ from the left we obtain the zero matrix on the right-hand side of the equation. Therefore, the words A^3B^2 and $BA^2B^2 = A^2B^3$ are reducible. $\square$

As a summary, for $n = 5$ the realizability is depicted in the following table.

$l(\{A,B\})$	0	1	2	3	4	5	6	7	8
$AB = BA$	+	+	+	+	+	-	-	-	-
$AB = aBA,\ a \neq 1,\ (AB)^4 = 0$	-	+	+	+	+	-	-	-	-
$(AB)^5 \neq 0,\ AB = -BA$	-	-	+	+	+	-	-	-	-
$(AB)^5 \neq 0,\ AB = \varepsilon_3 BA$	-	-	-	-	+	-	-	-	-
$(AB)^5 \neq 0,\ AB = \varepsilon_4 BA$	-	-	-	-	-	-	+	-	-
$(AB)^5 \neq 0,\ AB = \varepsilon_5 BA$	-	-	-	-	-	-	-	-	+

We have shown that for small values of n, the number $2n-5$ is realizable as the length of a ε-commuting pair with non-nilpotent product for $n = 4$ with $\varepsilon = -1$ (Theorem 5.2) and for $n = 6$ with ε being a primitive root of unity of order 3 and not realizable for $n = 3$ and $n = 5$ (Proposition 5.1 and Theorem 5.6).

In the following we apply the bounds from Section 4 to show that the number $2n - 5$ is not realizable as the length of an ε-commuting pair for $n > 6$.

Theorem 5.7. *Let $\mathbb{F}$ be an arbitrary field, and let $n \in \mathbb{N}$, $n \geq 7$. If the product AB is not nilpotent, then there does not exist a pair of matrices $\{A, B\} \subset M_n(\mathbb{F})$ such that $AB = \varepsilon BA$ for some root of unity $\varepsilon \in \mathbb{F}$, $\varepsilon \neq 1$ and $l(\{A, B\}) = 2n - 5$.*

Proof. Again without loss of generality we may assume that the field $\mathbb{F}$ is algebraically closed. Suppose that $AB = \varepsilon_k BA$ for a primitive root of unity of order k. By Theorem 1.10, the matrices A and B can be assumed to be in block form (2), and we have the following different options for the size r of the triangular block, in each case we show that $l(\{A, B\}) < 2n - 5$.

1. Suppose that $r = 0$ or $r = 1$. In this case $k|(n-r)$, but by Theorem 3.2, $k \neq n - r$. Thus, $2 \leq k \leq \frac{n-r}{2} \leq \frac{n}{2} < n - 3$ for $n > 6$. Hence,

$$n + k - 2 < n + (n - 3) - 2 = 2n - 5,$$

and

$$2(n-k)-r-2 \leq 2(n-2)-2 = 2n-6 < 2n-5.$$

Then from Theorem 4.7 we obtain that

$$l(\{A,B\}) \leq \max\{2(n-k)-r, n+k\}-2 = \max\{2(n-k), n+k\}-2 < 2n-5.$$

2. Suppose that $r = 2$ or $r = 3$.

If $k \nmid n$, then the words $A^{n-1}B^{n-4}$ and $A^{n-4}B^{n-1}$ are reducible by Lemma 3.3, and the words A^rB^{n-r} and B^rA^{n-r} are reducible by Proposition 5.5. Since $2 < n-3$ and $3 < n-2$ for $n > 6$, this statement implies that the words $A^{n-3}B^{n-2}$ and $A^{n-2}B^{n-3}$ of length $2n-5$ are also reducible, which means that all words in A and B of length $2n-5$ are reducible, and hence $l(\{A,B\}) < 2n-5$.

If $k|n$, then a combination of the two conditions $k|n$ and $k|(n-r)$ for a prime r, implies that $k = r$. Consequently,

$$n+k-2 = n+r-2 \leq n+1 < 2n-5,$$

and

$$2(n-k)-r-2 = 2(n-r)-r-2 = 2n-3r-2 \leq 2n-8 < 2n-5,$$

and again applying Theorem 4.7, we obtain that $l(\{A,B\}) < 2n-5$.

3. Suppose that $r \geq 4$. Then applying the bound from the proof of Theorem 3.4, we have $l(\{A,B\}) \leq 2n-r-2 \leq 2n-6 < 2n-5$. $\square$

Corollary 5.8. *Let $n \in \mathbb{N}$, $n > 6$, and let $\mathbb{F}$ an arbitrary field. Consider matrices $A, B \in M_n(\mathbb{F})$, such that AB is not nilpotent and $AB = \varepsilon BA$. Then $l(\{A,B\}) = 2n-4$ if and only if ε is a primitive root of unity of order $n-1$.*

Proof. The sufficiency follows from Proposition 5.4.

On the other hand, the proof of Theorem 5.7 shows that if ε is a root of unity of order not greater than $n-2$, then $l(\{A,B\}) < 2n-5 < 2n-4$, while if ε is a primitive root of unity of order n, then $l(\{A,B\}) = 2n-2$ by Proposition 5.4. $\square$

6 Open problems and Conclusions

We have made some progress on the characterization of lengths of sets of matrices for the case of matrix pairs, in particular, for the quasi-commutative case, i. e., $A, B \in M_n(\mathbb{F})$ with $(AB)^n \neq 0$, and $AB = \varepsilon BA$.

We can summarize our results and open questions on the realizability of different natural numbers as lengths of $\{A, B\}$ for a given coefficient ε for $n \geq 6$ in the following table (where "+" stands for the realizable values and "-" for the non-realizable ones, "?" for unknown situations, and "??" if the answer is unknown in the odd case.)

$l(\{A,B\})$ / ord (ε)	1	2	3	4	5	$\dots$	n-1	n	n+1	$\dots$	2n-6	2n-5	2n-4	2n-3	2n-2
1	+	+	+	+	+	$\dots$	+	-	-	$\dots$	-	-	-	-	-
2	-	+	+	+	+	$\dots$	+	??	?	$\dots$	?	-	-	-	-
3	-	-	-	+	+	$\dots$	+?	??	??	$\dots$	?	-	-	-	-
$\vdots$						$\dots$			$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	-	-	-	-	-	$\dots$	-	-	-	$\dots$	+	-	-	-	-
$n-1$	-	-	-	-	-	$\dots$	-	-	-	$\dots$	-	-	+	-	-
n	-	-	-	-	-	$\dots$	-	-	-	$\dots$	-	-	-	-	+

Here "+?" sign for ε_3 means that $n-1$ is realizable if $n \neq 1(\mod 3)$, but for $n \equiv 1(\mod 3)$ the situation with $n-1$ is unknown as well as with any number of the form $3p$.

On top of the general open problem, and the question how to extend the results for pairs of matrices to sets with more than two elements, the following special case also remains open.

Problem 6.1. Let $\mathbb{F}$ be an algebraically closed field, $\mathcal{S} = \{A, B\}$, where $A, B \in M_n(\mathbb{F})$ satisfy $AB = \varepsilon BA$, $\varepsilon \in \mathbb{F} \setminus \{0\}$, and does not satisfy any of the conditions (3). What odd integers m greater or equal to the size of matrices are realizable as lengths of $\mathcal{S}$?

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