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Abstract We discuss the direct discretization of the input/output map of linear time-invariant systems with distributed inputs and outputs. At first, the input and output signals are discretized in space and time, resulting in a matrix representation of an approximated input/output map. Then the system dynamics is approximated, in order to calculate the matrix representation numerically. The discretization framework, corresponding error estimates, a SVD-based system reduction method and a numerical application in optimal flow control are presented.

1 Introduction

The control of complex physical systems is a big challenge in many engineering applications as well as in mathematical research. Typically, these control systems are modeled by infinite-dimensional state space systems on the basis of (instationary and nonlinear) partial differential equations (PDEs). The difficulty is that on the one hand, space-discretizations resolving most of the state information typically lead to very large semi-discrete systems, on the other hand, popular design techniques for real-time controllers like robust control require linear models of very moderate size.

Numerous approaches to bridge this gap are proposed in the literature, see e.g. [1; 4]. In many applications it is sufficient to approximate the high order model by a low-order model that captures the essential state dynamics. To determine such low-order models one can use physical insight [17; 21] and/or mathematical methods like proper orthogonal decomposition [5] or balanced truncation [1; 20]. In this paper we focus on the situation where for the design of appropriate controllers it is sufficient

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to approximate the *input/output (i/o) map* of the system, schematically illustrated in Figure 1.

For such configurations, empirical or simulation-based black-box system identification [3; 14], and mathematical model reduction techniques like balanced truncation [12], moment matching [10] or recent variants of proper orthogonal decomposition [23] are common tools to extract appropriate low order models. Typically the bottleneck in these methods is the computational effort to compute the reduced order model form the semi-discretized model which often is of very high order.

In contrast to this, we investigate a new and integral approach to derive directly low-order models with error estimates for the i/o behavior but instead of semi-discretizing the system in space and then reducing this large model, we focus directly on the i/o map of the *original* infinite-dimensional system, in the following sections denoted by

$$\mathbb{G} : \mathscr{U} \to \mathscr{Y}, \qquad u = u(t,\theta) \mapsto y = y(t,\xi)$$

and we suggest a framework for its direct discretization for a general class of *linear time-invariant* systems (introduced in Section 2.1). Here u and y are input and output signals from Hilbert spaces $\mathscr{U}$ and $\mathscr{Y}$, respectively, which may vary in time t and space $\theta \in \Theta$ and $\xi \in \Xi$, with appropriate spatial domains Θ and Ξ. The framework consists of two steps.

1. *Approximation of signals (cf. Section 3).* We choose finite-dimensional subspaces $\bar{\mathscr{U}} \subset \mathscr{U}$ and $\bar{\mathscr{Y}} \subset \mathscr{Y}$ with bases $\{u_1,\dots,u_{\bar{p}}\} \subset \bar{\mathscr{U}}$ and $\{y_1,\dots,y_{\bar{q}}\} \subset \bar{\mathscr{Y}}$, and denote the corresponding orthogonal projections by $\mathbb{P}_{\bar{\mathscr{U}}}$ and $\mathbb{P}_{\bar{\mathscr{Y}}}$, respectively. Then, the approximation
$$\mathbb{G}_S = \mathbb{P}_{\bar{\mathscr{Y}}} \mathbb{G} \mathbb{P}_{\bar{\mathscr{U}}}$$
has a matrix representation $\mathbf{G} \in \mathbb{R}^{\bar{q}\times\bar{p}}$.
2. *Approximation of system dynamics (cf. Section 4).* Frequently, $\mathbb{G}$ arises from a linear PDE state space model. Then the components $\mathbf{G}_{ij} = (y_i, \mathbb{G}u_j)_{\mathscr{Y}}$ can be approximated by *numerically simulating* the state space model successively for inputs u_j, $j = 1,\dots,\bar{p}$ and by testing the resulting outputs against all $y_1,\dots,y_{\bar{q}}$.

We discuss several features of this framework.

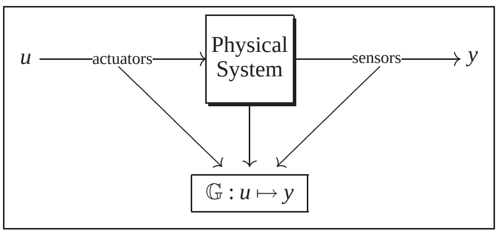


Fig. 1 Schematic illustration of an input/output map, corresponding to a physical system, given e.g. by a set of equations or a numerical solver (black-box approach).

Error estimation (cf. Section 5). The total error ε_{DS} can be estimated by the *signal* approximation error ε_S and the *dynamical* approximation error ε_D, i.e.,

$$\underbrace{\|\mathbb{G}-\mathbb{G}_{DS}\|}_{=:\varepsilon_{DS}} \leq \underbrace{\|\mathbb{G}-\mathbb{G}_S\|}_{=:\varepsilon_S} + \underbrace{\|\mathbb{G}_S-\mathbb{G}_{DS}\|}_{=:\varepsilon_D}, \tag{1}$$

where the norms still have to be specified. Here $\mathbb{G}_{DS}$ denotes the numerically estimated approximation of $\mathbb{G}_S$. Theorem 3 shows how to choose $\bar{\mathscr{U}}$ and $\bar{\mathscr{Y}}$ and the accuracy tolerances for the numerical solutions of the underlying PDEs such that ε_S and ε_D balance and that $\varepsilon_S + \varepsilon_D < \texttt{tol}$ for a given tolerance $\texttt{tol}$.

Progressive reduction of the signal error. Choosing hierarchical bases in $\bar{\mathscr{U}}$ and $\bar{\mathscr{Y}}$, the error ε_S can be progressively reduced by adding further basis functions $u_{\bar{p}+1}, u_{\bar{p}+2}, \ldots$ and $y_{\bar{q}+1}, y_{\bar{q}+2}, \ldots$ resulting in additional columns and rows of the matrix representation.

Control Design (cf. Section 6.2). The matrix representation $\mathbf{G} = [\mathbf{G}_{ij}]$ may directly be used in control design, or a state realization of the i/o model $\mathbb{G}_{DS}$ can be used as basis for many classical control design algorithms.

2 I/o maps of linear time-invariant systems

For $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, $L^2(\Omega)$ denotes the usual Lebesgue space of square-integrable functions, and $H^\alpha(\Omega)$, $\alpha \in \mathbb{N}_0$ denotes the corresponding Sobolev spaces of α-times weakly differentiable functions. We interpret functions v, which vary in space and time, optionally as classical functions $v : [0,T] \times \Omega \to \mathbb{R}$ with values $v(t;x) \in \mathbb{R}$, or as *abstract* functions $v : [0,T] \to X$ with values in a function space X such as $X = H^\alpha(\Omega)$. Correspondingly, $H^\alpha(0,T;H^\beta(\Omega))$, with $\alpha, \beta \in \mathbb{N}_0$, denotes the space of equivalence classes of functions $v : [0,T] \to H^\beta(\Omega)$ with $t \mapsto \|v\|_{H^\beta(\Omega)}$ being α-times weakly differentiable, for details see e.g. [7]. We introduce Hilbert spaces

$$H^{\alpha,\beta}((0,T)\times\Omega) := H^\alpha(0,T;L^2(\Omega)) \cap L^2(0,T;H^\beta(\Omega)), \tag{2}$$

$$\|v\|_{H^{\alpha,\beta}((0,T)\times\Omega)} := \|v\|_{H^\alpha(0,T;L^2(\Omega))} + \|v\|_{L^2(0,T;H^\beta(\Omega))}, \tag{3}$$

see e.g. [18]. By $C([0,T];X)$ and $C^\alpha([0,T];X)$ we denote the space of functions $v : [0,T] \to X$ which are continuous respectively α-times continuously differentiable.

For two normed spaces X and Y, $\mathscr{L}(X,Y)$ denotes the set of bounded linear operators $X \to Y$, and we abbreviate $\mathscr{L}(X) := \mathscr{L}(X,X)$. For $\alpha \in \mathbb{N}$, $L^\alpha(0,T;\mathscr{L}(X,Y))$ denotes the space of operator-valued functions $K : [0,T] \to \mathscr{L}(X,Y)$ with $t \mapsto \|K(t)\|_{\mathscr{L}(X,Y)} = \sup_{x\neq 0} \|K(t)x\|_Y / \|x\|_X$ lying in $L^\alpha(0,T)$. Vectors, often representing a discretization of a function v, are written in corresponding small bold letters $\mathbf{v}$, whereas matrices, often representing a discrete version of an operator like $\mathbb{G}$ or G, are written in bold capital letters $\mathbf{G}$. $\mathbb{R}^{\alpha\times\beta}$ stands for the set of real $\alpha \times \beta$ matrices, and $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker tensor product of two matrices $\mathbf{A}$ and $\mathbf{B}$.

2.1 I/o maps of ∞-dimensional state space systems

We consider infinite-dimensional linear time-invariant systems of first order

$$\partial_t z(t) = Az(t) + Bu(t), \quad t \in (0,T], \tag{4a}$$
$$z(0) = z^0, \tag{4b}$$
$$y(t) = Cz(t), \quad t \in [0,T]. \tag{4c}$$

Here for every time $t \in [0,T]$, the state $z(t)$ is supposed to belong to a Hilbert space Z like $Z = L^2(\Omega)$, where Ω is a subset of $\mathbb{R}^{d_\Omega}$ with $d_\Omega \in \mathbb{N}$. A is a densely defined unbounded operator $A : Z \supset D(A) \to Z$, generating a C^0-semigroup $(S(t))_{t\geq 0}$ on Z. The control operator B belongs to $\mathscr{L}(U,Z)$ and the observation operator C to $\mathscr{L}(Z,Y)$, where $U = L^2(\Theta)$ and $Y = L^2(\Xi)$ with subsets $\Theta \subset \mathbb{R}^{d_1}$ and $\Xi \subset \mathbb{R}^{d_2}$, $d_1, d_2 \in \mathbb{N}$.

We recall how a linear bounded i/o-map $\mathbb{G} \in \mathscr{L}(\mathscr{U},\mathscr{Y})$ with

$$\mathscr{U} = L^2(0,T;U) \quad \text{and} \quad \mathscr{Y} = L^2(0,T;Y)$$

can be associated to (4), for details see e.g. [22, Ch. 4]. It is well-known that for initial values $z_0 \in D(A)$ and controls $u \in C^1([0,T];Z)$, a unique *classical solution* $z \in C([0,T];Z) \cap C^1((0,T);Z)$ of (4) exists. For $z_0 \in Z$ and $u \in \mathscr{U}$, one has

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)\,ds, \quad t \in [0,T], \tag{5}$$

the so called *mild solution* of (4). Hence, the output signal $y(t) = Cz(t)$ is well-defined and belongs to $\mathscr{Y} \cap C([0,T];Y)$. In particular, the output signals $y(u) \in \mathscr{Y}$ arising from input signals $u \in \mathscr{U}$ and zero initial conditions $z_0 \equiv 0$ allow to define the linear i/o-map $\mathbb{G} : \mathscr{U} \to \mathscr{Y}$ of the system (4) by $u \mapsto y(u)$. It is possible to represent $\mathbb{G}$ as a convolution with the kernel function $K \in L^2(-T,T;\mathscr{L}(U,Y))$,

$$K(t) = \begin{cases} CS(t)B, & t \geq 0 \\ 0, & t < 0 \end{cases}. \tag{6}$$

Then the i/o-map $\mathbb{G}$ of (4) has the representation

$$(\mathbb{G}u)(t) = \int_0^T K(t-s)u(s)\,ds, \quad t \in [0,T], \tag{7}$$

and belongs to $\mathscr{L}(\mathscr{U},\mathscr{Y}) \cap \mathscr{L}(\mathscr{U},C([0,T],\mathscr{Y}))$, c.f. [24].

To obtain error estimates we will assume additional smoothness of the input and output signals, i.e., according to definition (2)

$$\mathbb{G}_{|\mathscr{U}_s} \in \mathscr{L}(\mathscr{U}_s,\mathscr{Y}_s), \quad \text{with } \mathscr{U}_s = H^{\alpha_1,\beta_1}((0,T)\times\Theta), \quad \mathscr{Y}_s = H^{\alpha_2,\beta_2}((0,T)\times\Xi), \tag{8}$$

for some $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{N}$ different from zero.

Remark 1. Assumption (8) is fulfilled by many linear time-invariant systems with distributed controls and observations, like the heat equation with homogeneous Neumann boundary conditions and more general parabolic equations, see [18] and [19]. For Stokes systems, results similar to (7) and (8) are obtained by working with appropriate subspaces of divergence-free functions, see [25]. Wave equations with second order time derivatives can be represented in form of (4) and (7) by means of an order reduction. Hyperbolic systems do not have the smoothing property of parabolic systems, such that results like (8) demand input signals of higher regularity in time, see [18, p. 95]. However, systems with boundary control or pointwise observation do not fit directly into the setting (4).

2.2 I/o map of linearized Navier-Stokes systems

In order to apply our concepts to flow control we consider the linear differential-algebraic equation (DAE) system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} A & J \\ -J^T & Q \end{bmatrix} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) + \mathbf{Bu}(t) \\ 0 \end{bmatrix} \quad \text{in } (0,T] \tag{9}$$

$$\mathbf{v}(0) = \mathbf{v}_0 \tag{10}$$

arising from the spatial discretization of a linearized (around a steady-state solution) Navier-Stokes equation. Here $\mathbf{v}(t) \in \mathbb{R}^{n_v}$ and $\mathbf{p}(t) \in \mathbb{R}^{n_p}$ represent the spatially discretized velocity and pressure, $\mathbf{f}(t) \in \mathbb{R}^{n_v}$ contains the volume forces and the boundary conditions and $\mathbf{Bu}(t) \in \mathbb{R}^{n_v}$ stands for the discretized input function $u(t)$ mapped into the field of volume forces via an operator B. The system is defined by constant coefficient matrices M, A, J, Q of appropriate size.

Assume that $\mathscr{A} := \begin{bmatrix} A & J \\ -J^T & Q \end{bmatrix}$ is invertible which is equivalent to the unique solvability of the stationary problem. In addition we assume that A^{-1} exists, which is the case for the Stokes and for reasonably formulated Oseen linearizations. As a conclusion we have that $S := Q + J^T A^{-1} J$ is invertible and

$$\mathscr{A}^{-1} = \begin{bmatrix} [I - A^{-1} J S^{-1} J^T] A^{-1} & -A^{-1} J S^{-1} \\ S^{-1} J^T A^{-1} & S^{-1} \end{bmatrix}.$$

Premultiplying (9) by $\mathscr{A}^{-1}$ and setting $E_{11} = [I - A^{-1} J S^{-1} J^T] A^{-1} M$ and $E_{21} = S^{-1} J^T A^{-1} M$ we get the equivalent system

$$\begin{bmatrix} E_{11} \dot{\mathbf{v}}(t) \\ E_{21} \dot{\mathbf{v}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} E_{11} M^{-1} [\mathbf{f}(t) + \mathbf{Bu}(t)] \\ E_{21} M^{-1} [\mathbf{f}(t) + \mathbf{Bu}(t)] \end{bmatrix}, \quad \text{in } (0,T],$$

$$\mathbf{v}(0) = \mathbf{v}_0.$$

To express the solution of this system explicitly we introduce some terms from DAE calculus, see e.g., [16].

Consider a linear DAE initial value problem defined by its matrix pair $(\mathscr{E},\mathscr{A})$:

$$\begin{aligned} \mathscr{E}\dot{x}+\mathscr{A}x &= f(t), \quad \text{for } t\in[0,T], \\ x(0) &= x_0. \end{aligned} \tag{11}$$

Definition 1. A matrix pair $(\mathscr{E},\mathscr{A})$ with $\mathscr{E},\mathscr{A}\in\mathbb{R}^{n\times n}$ is regular, if $\det(\lambda\mathscr{E}+\mathscr{A})$ does not vanish identically for all $\lambda\in\mathbb{C}$.

For a regular matrix pair one has the Weierstraß canonical form.

Theorem 1 (Weierstraß canonical form, [11]). *Let $(\mathscr{E},\mathscr{A})$ be regular matrix pair. Then, there exist nonsingular matrices $P_1,P_2\in\mathbb{R}^{n\times n}$ such that*

$$(P_1\mathscr{E}P_2,P_1\mathscr{A}P_2)=\left(\begin{bmatrix} I_d & 0\\ 0 & N\end{bmatrix},\begin{bmatrix} T & 0\\ 0 & I_a\end{bmatrix}\right),$$

where T is a matrix in real Jordan canonical form and N is a nilpotent matrix also in Jordan canonical form. Moreover, it is allowed that one or the other block is not present.

If the index of nilpotency of N in Theorem 1 is ν, then one defines $\operatorname{ind}(\mathscr{E},\mathscr{A}):=\nu$, saying $(\mathscr{E},\mathscr{A})$ or the corresponding DAE has (differentiation) index ν.

Setting $\operatorname{ind}(\mathscr{E}):=\operatorname{ind}(\mathscr{E},I)$ it follows by the definition that $\nu=\operatorname{ind}(\mathscr{E},I)$ is the smallest integer for which $\operatorname{rank}\mathscr{E}^{\nu+1}=\operatorname{rank}\mathscr{E}^{\nu}$ holds. In this case the differentiation index of $\mathscr{E}$ is equivalent to the matrix index, used in the following definition of a generalized inverse:

Definition 2. Let $\mathscr{E}\in\mathbb{R}^{n,n}$ have $\operatorname{ind}(\mathscr{E})=k$. A matrix $X\in\mathbb{R}^{n,n}$ satisfying

$$\begin{aligned} &\text{(D1)} \quad \mathscr{E}X=X\mathscr{E} \\ &\text{(D2)} \quad X\mathscr{E}X=X \\ &\text{(D3)} \quad X\mathscr{E}^{k+1}=\mathscr{E}^{k} \end{aligned}$$

is called a Drazin inverse $\mathscr{E}^D$ of $\mathscr{E}$.

The Drazin inverse $\mathscr{E}^D$ is well defined and unique. By means of the introduced concepts one obtains the following solution formula:

Theorem 2. *Let $\mathscr{E},\mathscr{A}\in\mathbb{R}^{n,n}$ be a commuting regular matrix pair. Furthermore, let $f\in C^{\nu}(0,T;\mathbb{R}^n)$ with $\nu=\operatorname{ind}(\mathscr{E},\mathscr{A})$. Then every solution $x\in C^1(0,T;\mathbb{R}^n)$ of $\mathscr{E}\dot{x}+\mathscr{A}x=f(t)$ has the form*

$$\begin{aligned} x(t)=e^{-\mathscr{E}^D\mathscr{A}t}\mathscr{E}^D\mathscr{E}q+\int_0^t e^{-\mathscr{E}^D\mathscr{A}(t-s)}\mathscr{E}^D f(s)ds &+ \\ (I-\mathscr{E}^D\mathscr{E})\sum_{i=0}^{\nu-1}(-\mathscr{E}\mathscr{A}^D)^i\mathscr{A}^D f^{(i)}(t)& \end{aligned}$$

for some $q \in \mathbb{C}^n$.

Thus, if q exists such, that

$$x_0 = \mathscr{E}^D \mathscr{E} q + (I - \mathscr{E}^D \mathscr{E}) \sum_{i=0}^{\nu-1} (-\mathscr{E}\mathscr{A}^D)^i \mathscr{A}^D f^{(i)}(0),$$

then (11) possesses a unique solution, provided that $\mathscr{E}$ and $\mathscr{A}$ form a regular commuting matrix pair and f is sufficiently smooth. Note that for regular matrix pairs the commutativity requirement is not a restriction, see e.g. [16].

In the present case with $\mathscr{A} = I$ and $\mathscr{E} = E := \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix}$ this formula simplifies. One can show that $\operatorname{ind}(E) = 2$ and $\operatorname{ind}(E_{11}) \leq 2$, and thus we get the expressions

$$E^D = \begin{bmatrix} E_{11}^D & 0 \\ E_{21} E_{11}^{D^2} & 0 \end{bmatrix} \quad , \quad E^D E = \begin{bmatrix} E_{11}^D E_{11} & 0 \\ E_{21} E_{11}^D & 0 \end{bmatrix}$$
$$\exp\left(-E^D t\right) = \begin{bmatrix} \exp(-E_{11}^D t) & 0 \\ E_{21} E_{11}^D \exp(-E_{11}^D t) & 0 \end{bmatrix}.$$

In addition we assume that $\operatorname{ind}(E_{11}) = 1$ which can be shown for the Stokes case and should be demanded of a reasonable discretization for the more general Oseen case, c.f. [8]. Thus, the explicit solution of (9) reads

$$\begin{aligned} \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{p}(t) \end{bmatrix} &= \begin{bmatrix} \exp(-E_{11}^D t) E_{11}^D E_{11} q_v \\ E_{21} \exp(-E_{11}^D t) E_{11}^D q_v \end{bmatrix} \\ &\quad + \int_0^t \begin{bmatrix} \exp(-E_{11}^D (t-\tau)) E_{11} E_{11}^D M^{-1} [\mathbf{f}(\tau) + \mathbf{Bu}(\tau)] \\ E_{21} \exp(-E_{11}^D (t-\tau)) (E_{11}^D)^2 E_{11} M^{-1} [\mathbf{f}(\tau) + \mathbf{Bu}(\tau)] \end{bmatrix} \mathrm{d}\tau \\ &\quad + \begin{bmatrix} 0 \\ E_{21} [I - E_{11}^D E_{11}] M^{-1} [\mathbf{f}(t) + \mathbf{Bu}(t)] \end{bmatrix}, \end{aligned} \tag{12}$$

if $\mathbf{f}(t) + \mathbf{Bu}(t)$ is differentiable and $\mathbf{v}_0$ is a consistent initial value. Note that for the solution of the velocity component one only needs $\mathbf{f}(t) + \mathbf{Bu}(t)$ to be integrable and q_v such that $E_{11} E_{11}^D q_v = \mathbf{v}_0$.

Applying an output operator C^T to the first component in (12) one gets an explicit formula for the output $\tilde{y}(t) = C^T \mathbf{v}(t) = C^T \mathbf{v}(t;u)$, depending on the input u. We will consider only the part which depends on u and define the i/o map $\mathbb{G} : \mathscr{U} \to \mathscr{Y}$ via the formula

$$y(t) = (\mathbb{G}u)(t) = C^T \int_0^t \exp(-E_{11}^D (t-\tau)) E_{11} E_{11}^D M^{-1} \mathbf{Bu}(\tau) \mathrm{d}\tau. \tag{13}$$

Due to the linear character of the equations the output $\tilde{y}$ can be recovered by adding the response of the uncontrolled system to y. Note that $\mathbb{G}$ is of type (7).

3 Discretization of signals

In order to discretize the input signals $u \in \mathscr{U}$ and $y \in \mathscr{Y}$ in space and time, we choose four families $\{U_{h_1}\}_{h_1>0}$, $\{Y_{h_2}\}_{h_2>0}$, $\{\mathscr{R}_{\tau_1}\}_{\tau_1>0}$ and $\{\mathscr{S}_{\tau_2}\}_{\tau_2>0}$ of subspaces $U_{h_1} \subset U$, $Y_{h_2} \subset Y$, $\mathscr{R}_{\tau_1} \subset L^2(0,T)$ and $\mathscr{S}_{\tau_2} \subset L^2(0,T)$ of finite dimensions $p(h_1) = dim(U_{h_1})$, $q(h_2) = dim(Y_{h_2})$, $r(\tau_1) = dim(\mathscr{R}_{\tau_1})$ and $s(\tau_2) = dim(\mathscr{S}_{\tau_2})$. We then define

$$\mathscr{U}_{h_1,\tau_1} = \{u \in \mathscr{U} : u(t;\cdot) \in U_{h_1}, u(\cdot;\theta) \in \mathscr{R}_{\tau_1} \quad \text{for almost every } t \in [0,T], \theta \in \Theta\},$$
$$\mathscr{Y}_{h_2,\tau_2} = \{y \in \mathscr{Y} : y(t;\cdot) \in Y_{h_2}, y(\cdot;\xi) \in \mathscr{S}_{\tau_2} \quad \text{for almost every } t \in [0,T], \xi \in \Xi\}.$$

We denote the orthogonal projections onto these subspaces by $P_{\mathscr{S},\tau_2} \in \mathscr{L}(L^2(0,T))$ $\mathbb{P}_{\mathscr{U},h_1,\tau_1} \in \mathscr{L}(\mathscr{U})$ and $\mathbb{P}_{\mathscr{Y},h_2,\tau_2} \in \mathscr{L}(\mathscr{Y})$. As first step of the approximation of $\mathbb{G}$, we define

$$\mathbb{G}_S = \mathbb{G}_S(h_1,\tau_1,h_2,\tau_2) = \mathbb{P}_{\mathscr{Y},h_2,\tau_2}\mathbb{G}\mathbb{P}_{\mathscr{U},h_1,\tau_1} \in \mathscr{L}(\mathscr{U},\mathscr{Y}). \tag{14}$$

In order to obtain a matrix representation of $\mathbb{G}_S$, we introduce families of bases $\{\mu_1,\dots,\mu_p\}$ of U_{h_1}, $\{\nu_1,\dots,\nu_q\}$ of Y_{h_2}, $\{\phi_1,\dots,\phi_r\}$ of $\mathscr{R}_{\tau_1}$ and $\{\psi_1,\dots,\psi_s\}$ of $\mathscr{S}_{\tau_2}$ and corresponding mass matrices $\mathbf{M}_{U,h_1} \in \mathbb{R}^{p\times p}$, $\mathbf{M}_{Y,h_2} \in \mathbb{R}^{q\times q}$, $\mathbf{M}_{\mathscr{R},\tau_1} \in \mathbb{R}^{r\times r}$ and $\mathbf{M}_{\mathscr{S},\tau_2} \in \mathbb{R}^{s\times s}$, for instance via

$$[\mathbf{M}_{U,h_1}]_{ij} = (\mu_j,\mu_i)_U, \qquad i,j = 1,\dots,p.$$

These mass matrices induce, for instance via

$$(\mathbf{v},\mathbf{w})_{p;w} = \mathbf{v}^T\mathbf{M}_{U,h_1}\mathbf{w} \quad \text{for all } \mathbf{v},\mathbf{w} \in \mathbb{R}^p,$$

weighted scalar products and corresponding norms in the respective spaces, which we indicate by a subscript w, like $\mathbb{R}^p_w$ with $(\cdot,\cdot)_{p;w}$ and $\|\cdot\|_{p;w}$, in contrast to the canonical spaces like $\mathbb{R}^p$ with $(\cdot,\cdot)_p$ and $\|\cdot\|_p$. We represent signals $u \in \mathscr{U}_{h_1,\tau_1}$ and $y \in \mathscr{Y}_{h_2,\tau_2}$ as

$$u(t;\theta) = \sum_{k=1}^{p}\sum_{i=1}^{r}\mathbf{u}_i^k\phi_i(t)\mu_k(\theta), \qquad y(t;\xi) = \sum_{l=1}^{q}\sum_{j=1}^{s}\mathbf{y}_j^l\psi_j(t)\nu_k(\xi), \tag{15}$$

where $\mathbf{u}_i^k$ are the elements of a block-structured vector $\mathbf{u} \in \mathbb{R}^{pr}$ with p blocks $\mathbf{u}^k \in \mathbb{R}^r$, and the vector $\mathbf{y} \in \mathbb{R}^{qs}$ is defined similarly. Then

$$\|u\|_{\mathscr{U}} = \|\mathbf{u}\|_{pr;w}, \quad \text{and} \quad \|y\|_{\mathscr{Y}} = \|\mathbf{y}\|_{qs;w},$$

where $\|\cdot\|_{pr;w}$ and $\|\cdot\|_{qs;w}$ denote the weighted norms with respect to the mass matrices

$$\mathbf{M}_{\mathscr{U},h_1,\tau_1} = \mathbf{M}_{U,h_1} \otimes \mathbf{M}_{\mathscr{R},\tau_1} \in \mathbb{R}^{pr\times pr}, \quad \mathbf{M}_{\mathscr{Y},h_2,\tau_2} = \mathbf{M}_{Y,h_2} \otimes \mathbf{M}_{\mathscr{S},\tau_2} \in \mathbb{R}^{qs\times qs},$$

i.e., the corresponding coordinate isomorphisms $\kappa_{\mathscr{U},h_1,\tau_1} \in \mathscr{L}(\mathscr{U}_{h_1,\tau_1}, \mathbb{R}^{pr}_w)$ and $\kappa_{\mathscr{Y},h_2,\tau_2} \in \mathscr{L}(\mathscr{Y}_{h_2,\tau_2}, \mathbb{R}^{qs}_w)$ are unitary.

Finally, we obtain a matrix representation $\mathbf{G}$ of $\mathbb{G}_S$ by setting

$$\mathbf{G} = \mathbf{G}(h_1,\tau_1,h_2,\tau_2) = \kappa_{\mathscr{Y}} \mathbb{P}_{\mathscr{Y}} \mathbb{G} \mathbb{P}_{\mathscr{U}} \kappa_{\mathscr{U}}^{-1} \in \mathbb{R}^{qs\times pr}, \tag{16}$$

where the dependencies on h_1,τ_1,h_2,τ_2 have been partially omitted. Considering

$$\mathbf{H} = \mathbf{H}(h_1,\tau_1,h_2,\tau_2) := \mathbf{M}_{\mathscr{Y},h_2,\tau_2}\mathbf{G} \in \mathbb{R}^{qs\times pr}$$

as a block-structured matrix with $q \times p$ blocks $\mathbf{H}^{kl} \in \mathbb{R}^{s\times r}$ and block elements $\mathbf{H}^{kl}_{ij} \in \mathbb{R}$, we obtain the representation

$$\mathbf{H}^{kl}_{ij} = [\mathbf{M}_{\mathscr{Y}} \kappa_{\mathscr{Y}} \mathbb{P}_{\mathscr{Y}} \mathbb{G}(\mu_l \phi_j)]^k_i = (\nu_k \psi_i, \mathbb{G}(\mu_l \phi_j))_{\mathscr{Y}}. \tag{17}$$

To have a discrete analogon of the $\mathscr{L}(\mathscr{U},\mathscr{Y})$-norm, we introduce for given $\mathscr{U}_{h_1,\tau_1}$ and $\mathscr{Y}_{h_2,\tau_2}$ the weighted matrix norm

$$\|\mathbf{G}(h_1,\tau_1,h_2,\tau_2)\|_{qs\times pr;w} := \sup_{\mathbf{u}\in\mathbb{R}^{pr}} \frac{\|\mathbf{G}\mathbf{u}\|_{qs;w}}{\|\mathbf{u}\|_{pr;w}} = \|\mathbf{M}^{1/2}_{\mathscr{Y},h_2,\tau_2} \mathbf{G} \mathbf{M}^{-1/2}_{\mathscr{U},h_1,\tau_1}\|_{qs\times pr}, \tag{18}$$

and we write $(h_1',\tau_1',h_2',\tau_2') \le (h_1,\tau_1,h_2,\tau_2)$ if the inequality holds component-wise.

Lemma 1 ([24, p. 44]). *For all* $(h_1,\tau_1,h_2,\tau_2) \in \mathbb{R}^4_+$, *we have*

$$\|\mathbf{G}(h_1,\tau_1,h_2,\tau_2)\|_{qs\times pr;w} = \|\mathbb{G}_S(h_1,\tau_1,h_2,\tau_2)\|_{\mathscr{L}(\mathscr{U},\mathscr{Y})} \le \|\mathbb{G}\|_{\mathscr{L}(\mathscr{U},\mathscr{Y})}. \tag{19}$$

If the subspaces $\{\mathscr{U}_{h_1,\tau_1}\}_{h_1,\tau_1>0}$ *and* $\{\mathscr{Y}_{h_2,\tau_2}\}_{h_2,\tau_2>0}$ *are nested in the sense that*

$$\mathscr{U}_{h_1,\tau_1} \subset \mathscr{U}_{h_1',\tau_1'}, \quad \mathscr{Y}_{h_2,\tau_2} \subset \mathscr{Y}_{h_2',\tau_2'} \quad \text{for } (h_1',\tau_1',h_2',\tau_2') \le (h_1,\tau_1,h_2,\tau_2), \tag{20}$$

then $\|\mathbf{G}(h_1,\tau_1,h_2,\tau_2)\|_{qs\times pr;w}$ *monotonically grows and* $\|\mathbf{G}(h_1,\tau_1,h_2,\tau_2)\|_{qs\times pr;w}$ *is convergent for* $(h_1,\tau_1,h_2,\tau_2) \searrow 0$.

4 Approximation of system dynamics

We discuss the efficient approximation of $\mathbb{G}_S$ respectively of its matrix representation $\mathbf{G} = \mathbf{M}^{-1}_{\mathscr{Y}}\mathbf{H}$. For time-invariant systems with distributed control and observation, this task reduces to the approximation of the convolution kernel $K \in L^2(0,T;\mathscr{L}(U,Y))$.

4.1 Kernel function approximation

We recall the notation of the general linear time-invariant system (4) for a state $z(t) \in Z$ for $t \in [0,T]$,

$$\partial_t z(t) = Az(t) + Bu(t), \quad t \in (0,T], \tag{21a}$$
$$z(0) = z^0, \tag{21b}$$
$$y(t) = Cz(t), \quad t \in [0,T], \tag{21c}$$

that can be associated with a bounded i/o map

$$\mathbb{G} : \mathscr{U} \to \mathscr{Y} : (\mathbb{G}u)(t) = \int_0^T K(t-s)u(s)\,ds, \quad t \in [0,T], \tag{22}$$

where $\mathscr{U}$ and $\mathscr{Y}$ denote the Hilbert spaces of the input and output signals.

Inserting (22) in (17), by a change of variables we obtain

$$\mathbf{H}_{ij}^{kl} = \int_0^T \int_0^T \psi_i(t)\phi_j(s)(\nu_k, K(t-s)\mu_l)_Y \,ds\,dt = \int_0^T \mathbf{W}_{ij}(t)\mathbf{K}_{kl}(t)\,dt,$$

with matrix-valued functions $\mathbf{W} : [0,T] \to \mathbb{R}^{s\times r}$ and $\mathbf{K} : [0,T] \to \mathbb{R}^{q\times p}$,

$$\mathbf{W}_{ij}(t) = \int_0^{T-t} \psi_i(t+s)\phi_j(s)\,ds, \qquad \mathbf{K}_{kl}(t) = (\nu_k, K(t)\mu_l)_Y,$$

and thus

$$\mathbf{H} = \mathbf{M}_{\mathscr{Y}}\mathbf{G} = \int_0^T \mathbf{K}(t) \otimes \mathbf{W}(t)\,dt. \tag{24}$$

For systems of the form (21), the matrix-valued function $\mathbf{K}$ is given by

$$\mathbf{K}_{kl}(t) = (\nu_k, CS(t)B\mu_l)_Y = (c_k^*, S(t)b_l)_Z,$$

where $c_k^* = C^*\nu_k \in Z$ and $b_l = B\mu_l$ for $k = 1,\dots,q$ and $l = 1,\dots,p$. Hence, $\mathbf{K}$ can be calculated by solving p homogeneous systems

$$\dot{z}_l(t) = Az_l(t), \qquad t \in (0,T], \tag{25a}$$
$$z_l(0) = b_l, \tag{25b}$$

since (25) has the mild solution $z_l(t) = S(t)b_l \in C([0,T]; L^2(\Omega))$. We obtain an approximation $\tilde{\mathbf{H}}$ of $\mathbf{H}$ by replacing $z_l(t)$ by numerical approximations $z_{l,\mathtt{tol}}(t)$, i.e.,

$$\tilde{\mathbf{H}} = \int_0^T \tilde{\mathbf{K}}(t) \otimes \mathbf{W}(t)\,dt, \tag{26}$$

with $\tilde{\mathbf{K}}_{kl}(t) = (\nu_k, Cz_{l,\mathtt{tol}}(t))_Y = (c_k^*, z_{l,\mathtt{tol}}(t))_Z$. Here the subscript `tol` indicates that the error $z_l - z_{l,\mathtt{tol}}$ is assumed to satisfy some tolerance criterion which will be specified later. The corresponding approximation $\mathbb{G}_{DS}$ of $\mathbb{G}_S$ is given by

$$\mathbb{G}_{DS} = \kappa_{\mathscr{Y}}^{-1} \tilde{\mathbf{G}} \kappa_{\mathscr{U}} \mathbb{P}_{\mathscr{U}}, \quad \text{with } \tilde{\mathbf{G}} = \mathbf{M}_{\mathscr{Y}}^{-1} \tilde{\mathbf{H}} \tag{27}$$

and depends on h_1, h_2, τ_1, τ_2 and `tol`.

4.2 Dynamics approximation error

The following proposition relates the system dynamics error ε_D to the errors made in solving the PDE (25) for $l = 1, \dots, p$.

Proposition 1 ([24, p. 51]). *The system dynamics error* $\varepsilon_D := \|\mathbb{G}_S - \mathbb{G}_{DS}\|_{\mathscr{L}(\mathscr{U},\mathscr{Y})}$ *satisfies*

$$\varepsilon_D \leq \sqrt{T} \|\mathbf{K} - \tilde{\mathbf{K}}\|_{L^2(0,T;\mathbb{R}_w^{q\times p})} \leq p\sqrt{T} \sqrt{\frac{\lambda_{\max}(\mathbf{M}_{Y,h_2})}{\lambda_{\min}(\mathbf{M}_{U,h_1})}} \max_{1\leq l\leq p} \|\mathbf{K}_{:,l} - \tilde{\mathbf{K}}_{:,l}\|_{L^2(0,T;\mathbb{R}^q)}. \tag{28}$$

Here $\mathbf{K}_{:,l}$ *and* $\tilde{\mathbf{K}}_{:,l}$ *denote the lth column of* $\mathbf{K}(t)$ *and* $\tilde{\mathbf{K}}(t)$*, respectively,* $\lambda_{\max}(\mathbf{M}_{Y,h_2})$ *is the largest eigenvalue of* $\mathbf{M}_{Y,h_2}$ *and* $\lambda_{\min}(\mathbf{M}_{U,h_1})$ *the smallest eigenvalue of* $\mathbf{M}_{U,h_1}$. $\mathbb{R}_w^{q\times p}$ *denotes the space of real* $q \times p$*-matrices equipped with the weighted matrix norm* $\|\mathbf{M}\|_{q\times p;w} = \sup_{\mathbf{u}\neq 0} \|\mathbf{M}\mathbf{u}\|_{q;w}/\|\mathbf{u}\|_{p;w}$.

4.3 Error estimation for the homogeneous PDE

In order to approximate the system dynamics, the homogeneous PDE (25) has to be solved via a fully-discrete numerical scheme for p different initial values. A *first* goal is to choose the time and space grids (and possibly other discretization parameters) such that

$$\|\mathbf{K}_{:,l} - \tilde{\mathbf{K}}_{:,l}\|_{L^2(0,T;\mathbb{R}^q)} < \mathtt{tol} \quad \text{resp.} \quad \|\mathbf{K}_{:,l}^w - \tilde{\mathbf{K}}_{:,l}^w\|_{L^2(0,T;\mathbb{R}^q)} < \mathtt{tol} \tag{29}$$

is *guaranteed* for a given `tol` > 0 by means of reliable error estimators. A *second* goal is to achieve this accuracy in a *cost-economic* way.

Discontinuous Galerkin time discretizations combined with standard Galerkin space discretizations provide an appropriate framework to derive corresponding error estimates, also for the case of adaptively refined grids which are in general no longer quasi-uniform, see e.g. [15]. We distinguish two types of error estimates.

Global state error estimates measure the error $(z_l - z_{l,\mathtt{tol}})$ in some global norm. For parabolic problems, a priori and a posteriori estimates for the error in $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;L^\infty(\Omega))$ can be found in [9]. Such results permit to guarantee (29) in view of

$$\|\mathbf{K}_{:,l} - \tilde{\mathbf{K}}_{:,l}\|_{L^2(0,T;\mathbb{R}^q)} \leq \|C\|_{\mathscr{L}(Z,Y)} \left(\sum_{i=1}^{q} \|v_i\|_Y^2 \right)^{1/2} \|z - z_{\mathtt{tol}}^{(l)}\|_{L^2(0,T;Z)}. \tag{30}$$

Goal-oriented error estimates are used to measure the error $\|\mathbf{K}_{:,l} - \tilde{\mathbf{K}}_{:,l}\|_{L^2(0,T;\mathbb{R}^q)}$ directly. This may be advantageous, since (30) may be very conservative. A general introduction to goal-oriented error estimation and strategies for mesh adaption is given in the monograph by Bangerth and Rannacher [2].

Assumption 1 Given a tolerance $\mathtt{tol} > 0$, we can ensure (by using appropriate error estimators and mesh refinements) that the solutions z_l of (25) and the solutions $z_{l,\mathtt{tol}}$ calculated by means of an appropriate fully-discrete numerical scheme satisfy

$$\|\mathbf{K}_{:,l} - \tilde{\mathbf{K}}_{:,l}\|_{L^2(0,T;\mathbb{R}^q)} < \mathtt{tol}, \qquad l = 1, \ldots, p. \tag{31}$$

5 Total error estimates

We present estimates for the total error in the approximation of $\mathbb{G}$. Using general-purpose ansatz spaces $\mathscr{U}_{h_1,\tau_1}$ and $\mathscr{Y}_{h_2,\tau_2}$ for the signal approximation, we only obtain error results in a weaker $\mathscr{L}(\mathscr{U}_s, \mathscr{Y})$-norm.

Theorem 3 ([24, p. 55]). *Consider the i/o map* $\mathbb{G} \in \mathscr{L}(\mathscr{U}, \mathscr{Y})$ *of the infinite-dimensional linear time-invariant system* (7) *and assume that*

(i) $\mathbb{G}_{|\mathscr{U}_s} \in \mathscr{L}(\mathscr{U}_s, \mathscr{Y}_s)$ *with spaces of higher regularity in space and time*

$$\mathscr{U}_s = H^{\alpha_1,\beta_1}((0,T) \times \Theta), \quad \mathscr{Y}_s = H^{\alpha_2,\beta_2}((0,T) \times \Xi), \quad \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{N}.$$

(ii) The families of subspaces $\{\mathscr{U}_{h_1,\tau_1}\}_{h_1,\tau_1}$ *and* $\{\mathscr{Y}_{h_2,\tau_2}\}_{h_2,\tau_2}$ *satisfy*

$$\|u - \mathbb{P}_{\mathscr{U},h_1,\tau_1} u\|_{\mathscr{U}} \leq (c_{\mathscr{R}} \tau_1^{\alpha_1} + c_U h_1^{\beta_1}) \|u\|_{\mathscr{U}_s}, \qquad u \in \mathscr{U}_s,$$
$$\|y - \mathbb{P}_{\mathscr{Y},h_2,\tau_2} y\|_{\mathscr{Y}} \leq (c_{\mathscr{S}} \tau_2^{\alpha_2} + c_Y h_2^{\beta_2}) \|y\|_{\mathscr{Y}_s}, \qquad y \in \mathscr{Y}_s,$$

with positive constants $c_{\mathscr{R}}$, $c_{\mathscr{S}}$, c_U *and* c_Y.

(iii) The error in the solution for the state dynamics can be made arbitrarily small, i.e. for a given tolerance equation (31) *holds.*

Let $\delta > 0$ *be given. Then one can choose subspaces* $\mathscr{U}_{h_1^*,\tau_1^*}$ *and* $\mathscr{Y}_{h_2^*,\tau_2^*}$ *such that*

$$\|\mathbb{G} - \mathbb{G}_{DS}\|_{\mathscr{L}(\mathscr{U}_s,\mathscr{Y})} < \delta.$$

Moreover, the signal error $\varepsilon_S' := \|\mathbb{G} - \mathbb{G}_S\|_{\mathscr{L}(\mathscr{U}_s,\mathscr{Y})}$ *and the system dynamics error* $\varepsilon_D := \|\mathbb{G}_S - \mathbb{G}_{DS}\|_{\mathscr{L}(\mathscr{U},\mathscr{Y})}$ *are balanced in the sense that* $\varepsilon_S', \varepsilon_D < \delta/2$.

6 Application to Flow Control

As a test case we present a 2D driven cavity problem modelled by the Oseen equations for the velocity v and the pressure p in $\Omega \times I := (-1,1)^2 \times (0,0.1]$. Let $\Omega_c = (-0.2,0.2) \times (-0.7,-0.5)$ and and $\Omega_m = (-0.1,0.1) \times (0,0.6)$ be rectangular subsets of Ω where the control is active and the observation takes place, respectively, c.f. Figure 2.

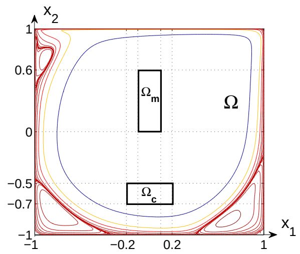


Fig. 2 Schematic illustration of a 2D driven cavity flow and the domains of control and observation, Ω_c and Ω_m, respectively.

Setting $Y = [L^2(0,1)]^2$ and $U = [L^2(0,1)]^2$, we define $C \in \mathscr{L}([L^2(\Omega)]^2, Y)$ and $B \in \mathscr{L}(U, [L^2(\Omega)]^2)$ by

$$(Cv)(\xi) = \int_{-0.1}^{0.1} \frac{v(x_1, \theta_m \xi)}{0.2}\, dx_1, \quad (Bu)(x_1,x_2) = \begin{cases} u(\theta_c x_1), & (x_1,x_2) \in \Omega_c, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta_m : [0,1] \to [0,0.6]$ and $\theta_c : [-0.2,0.2] \to [0,1]$ are affine linear mappings, that adjust the spatial extensions of the signal spaces to the respective domains Ω_c and Ω_m. By definition, B maps the two input signal components into the control domain such that they are homogeneous in x_2-direction. The output is extracted as the average in x_1-direction of the velocity within the observation domain.

Thus the considered system reads

$$\begin{aligned} v_t + (v_\infty \cdot \nabla)v + (v \cdot \nabla)v_\infty + \nabla p - \frac{1}{Re}\triangle v &= (v_\infty \cdot \nabla)v_\infty + Bu, \\ \nabla \cdot v &= 0, \\ y &= C^T v, \end{aligned}$$

with initial and boundary conditions $v|_{t=0} = v_\infty$ and $v|_{\partial\Omega} = g$.

Here g defines the boundary data for the driven cavity with moving upper lid and v_∞ denotes the steady state solution.

A stabilized $Q1-P0$ finite element discretization of the state space converts the above system into a DAE of type (9), see e.g. [6] for technical issues. To distinguish the spatially discretized quantities from the continuous we use bold letters, e.g. $\mathbf{v}$

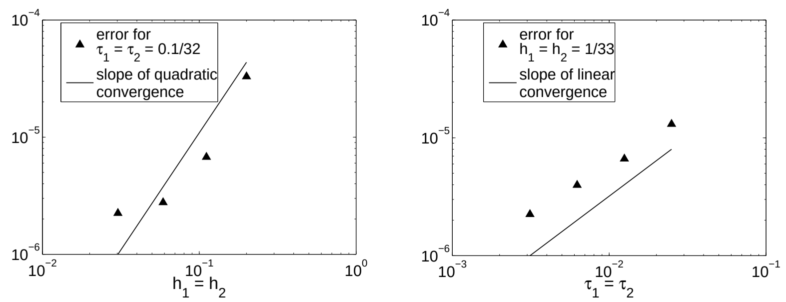
denotes the spatially discretized velocity v. For an appropriate v_∞, c.f. [13], the obtained system meets the assumptions necessary to establish the corresponding i/o map $\mathbb{G}$ via (13).

The considered system was investigated on a uniform rectangular 128×128 grid with the Reynolds number $Re = 3333$. For the numerical estimation of $\mathbb{G}_{DS}$ a modified projection algorithm [13] with 128 timesteps was used.

6.1 Tests of convergence in signal approximation

The following numerical convergence tests have all been carried out with approximations $\mathbb{G}_{DS}(h_1, \tau_1, h_2, \tau_2, \texttt{tol})$ of the i/o-map $\mathbb{G}$ corresponding to the spatially discretized system (6). Hierarchical linear finite elements in U_{h_1} and Y_{h_2} and Haar wavelets in $\mathscr{R}_{\tau_1}$ and $\mathscr{S}_{\tau_2}$ have been chosen.

To check the convergence in the signal approximation numerically, we chose the testsignal $\hat{u}(t;\theta) = [\sin(10\pi t)\sin(10\pi\theta)\ 0]^T$ with its numerically computed system response $y = \mathbb{G}\hat{u}$. As a measure for the error the relative deviation $\|y - \tilde{y}\|_{\mathscr{Y}} \|\hat{u}\|_{\mathscr{U}}$ with $\tilde{y} = \mathbb{G}_{DS}\hat{u}$ for varying discretization parameters h_1, τ_1, h_2, τ_2 was taken. Figure 3 (a) shows the evolution of the signal approximation error for a fixed time discretization $\tau_1 = \tau_2 = 0.1 \cdot 2^{-5}$ and varying space resolution of the signals. In Figure 3 (b) the roles of the space and time discretization are changed.



(a) (b)

Fig. 3 Relative output errors $\|y - \tilde{y}\|_{\mathscr{Y}} \|\hat{u}\|_{\mathscr{U}}$ with $\tilde{y} = \mathbb{G}_{DS}\hat{u}$ errors for (a) varying $h_1 = h_2$ and fixed $\tau_1 = \tau_2 = 0.1/32$ and (b) for varying $\tau_1 = \tau_2$ and fixed $h_1 = h_2 = 1/33$.

The convergence is in the region where it is assumed for approximations using piecewise constant or piecewise linear finite elements. Note that it is necessary to balance space and time resolution properly, as indicated by the breakdown of the quadratic convergence of the space discretization on the lowest level of the time approximation error as shown in Figure 3 (a).

6.2 Application to Optimal Flow Control

We investigate the use of the i/o-map approximation in optimization problems

$$\min J(u,y) \qquad \text{subject to } y = \mathbb{G}u, \quad u \in \mathscr{U}. \tag{32}$$

Here, $J : \mathscr{U} \times \mathscr{Y} \to \mathbb{R}$ is the cost functional $J(u,y) = \frac{1}{2}\|y - y_D\|^2_{\mathscr{Y}} + \alpha\|u\|^2_{\mathscr{U}}$, $y_D \in \mathscr{Y}$ is an aspired system's output signal, and $\alpha > 0$ is a regularization parameter. We define the discrete cost functional

$$\bar{J}_{\mathbf{h}} : \mathbb{R}^{pr} \times \mathbb{R}^{qs} \to \mathbb{R}, \qquad \bar{J}_{\mathbf{h}}(\mathbf{u},\mathbf{y}) = \frac{1}{2}\|\mathbf{y} - \mathbf{y}_D\|^2_{qs;w} + \alpha\|\mathbf{u}\|^2_{pr;w}, \tag{33}$$

with $\mathbf{y}_D = \kappa_{\mathscr{Y},h_2,\tau_2}\mathbb{P}_{\mathscr{Y},h_2,\tau_2}y_D$, and instead of (32) we solve

$$\min \bar{J}_{\mathbf{h}}(\mathbf{u},\mathbf{y}) \qquad \text{subject to } \mathbf{y} = \tilde{\mathbf{G}}\mathbf{u}, \quad \mathbf{u} \in \bar{U} \tag{34}$$

with the solution $\bar{\mathbf{u}}$ of (34) characterized by

$$(\tilde{\mathbf{G}}^T\mathbf{M}_{\mathscr{Y}}\tilde{\mathbf{G}} + \alpha\mathbf{M}_{\mathscr{U}})\bar{\mathbf{u}} = \tilde{\mathbf{G}}^T\mathbf{M}_{\mathscr{Y}}\mathbf{y}_D. \tag{35}$$

As the target y_D we chose the output corresponding to the input $u_0 \equiv [1\ 1]^T$ and solved (35) with the finite dimensional i/o map $\tilde{\mathbf{G}} \in \mathbb{R}^{34\cdot16\times34\cdot16}$ of the driven cavity case and $\alpha = 10^{-7}$.

Solving the linear equation system took 0.0537 seconds and yielded a $\bar{\mathbf{u}}$ which reduced the energy by 21.3% while causing a relative deviation in the output of 2.26%, c.f. Figure 4.

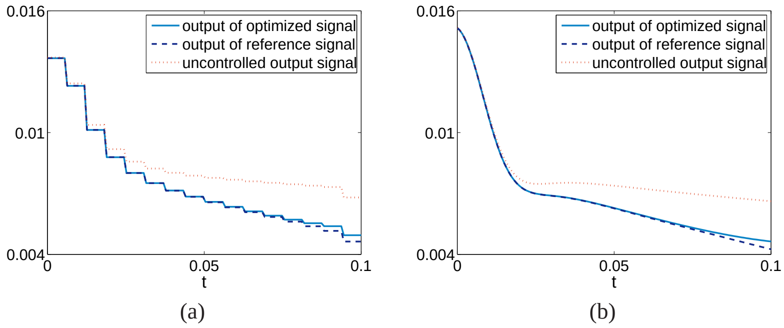


Fig. 4 Illustration of the x_1-component of (a) the output signals $\tilde{\mathbf{G}}\mathbf{u}_0$ and $\tilde{\mathbf{G}}\bar{\mathbf{u}}$ and (b) $\mathbb{G}u_0$ and $\mathbb{G}\bar{u}$ for varying t and $\xi = 0.5$. Here $\mathbf{u}_0$ and $\bar{u}$ represent u_0 and $\bar{\mathbf{u}}$ in the discrete and continuous input space, respectively.

7 Conclusion and Acknowledgement

The presented method provides a completely algebraic representation of the input/output behavior of a linear time-invariant system, like the Oseen problem. The developed error estimates guarantee a desired quality of the approximation. As illustrated by the numerical example this i/o map also leads to useful solutions of the inverse problem of determining controls with very short calculation times.

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