A NOTE ON THE EIGENVALUES OF SADDLE POINT MATRICES

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Abstract. Results of Benzi and Simoncini (Numer. Math. 103 (2006), pp. 173–196) on spectral properties of block 2×2 matrices are generalized to the case of a symmetric positive semidefinite block at the (2,2) position. More precisely, a sufficient condition is derived when a (nonsymmetric) saddle point matrix of the form $[A \;\; B^T; -B \; C]$ with $A = A^T > 0$, full rank B, and $C = C^T \geq 0$, is diagonalizable and has real and positive eigenvalues.

Key words. saddle point problem, eigenvalues, Stokes problem, normal matrices

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1. Introduction. Many applications in science and engineering require solving large linear algebraic systems in saddle point form; see [1] for an extensive survey. In such problems, the system matrix often is of the form

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}, \tag{1.1}$$

where $A = A^T \in \mathbb{R}^{n \times n}$ is positive definite ($A > 0$), $B \in \mathbb{R}^{m \times n}$ has full rank m, and $C = C^T \in \mathbb{R}^{m \times m}$ is positive semidefinite ($C \geq 0$). The matrix in (1.1) is congruent to the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}$, where $S = -(C + BA^{-1}B^T)$ with $S = S^T < 0$. Hence the matrix in (1.1) is indefinite with n positive and m negative eigenvalues, which represents a significant challenge for linear solvers such as Krylov subspace methods.

It has been noted by several authors (see [1, p. 23] for references), that the matrix

$$\mathcal{A} \equiv \begin{bmatrix} A & B^T \\ -B & C \end{bmatrix}, \tag{1.2}$$

which is obtained from (1.1) by multiplying the second block row by (-1) is *positive stable*, i.e. has only eigenvalues with positive real parts; see, e.g., [1, Theorem 3.6] for a proof of this statement. What is even more appealing is that, under certain conditions, the matrix $\mathcal{A}$ is diagonalizable with all its eigenvalues real and positive. This may be advantageous when solving a linear system with $\mathcal{A}$ using a Krylov subspace method, and in addition this gives rise to a three-term recurrence conjugate gradient type method based on a positive definite inner product. The first instance of this fact has been observed by Fischer et al. [4], who considered $\mathcal{A}$ with $A = \eta I > 0$, and $C = 0$. Recently, the results of [4] have been extended by Benzi and Simoncini [2] to matrices $\mathcal{A}$ with $A = A^T > 0$ and $C = 0$. The purpose of this note is to generalize these results to $\mathcal{A}$ with a symmetric positive semidefinite (2,2) block C. This is of interest in stabilized discretizations of Stokes and generalized Stokes problems; see, e.g. [3, Chapters 5–6] and [2, Section 4] for examples.

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2. Main result. Consider a matrix $\mathcal{A}$ as in (1.2) with $A = A^T > 0$, B of full rank, and $C = C^T \geq 0$, and define the symmetric matrix

$$\mathcal{M}_C(\gamma) \equiv \begin{bmatrix} A - \gamma I & B^T \\ B & \gamma I - C \end{bmatrix}, \tag{2.1}$$

where γ is a yet to be specified real scalar. Note that the matrix $\mathcal{M}_0(\gamma)$ (i.e. $\mathcal{M}_C(\gamma)$ with $C = 0$) is equal to the matrix G defined in [2, p. 182]. This relation and the results for $\mathcal{M}_0(\gamma)$ in [2] are key ingredients in our derivation below. An elementary computation shows that

$$\mathcal{M}_C(\gamma)\mathcal{A} = \mathcal{A}^T \mathcal{M}_C(\gamma). \tag{2.2}$$

We will now derive conditions on the blocks A, B, and C of $\mathcal{A}$ and on γ so that $\mathcal{M}_C(\gamma)$ is positive definite. If these conditions are satisfied, then

$$\mathcal{A} = \mathcal{M}_C(\gamma)^{-1} \mathcal{A}^T \mathcal{M}_C(\gamma), \tag{2.3}$$

i.e., $\mathcal{A}$ is similar to its transpose by a symmetric positive definite similarity transformation. From a classical result of Taussky [8, Section 3] it then follows that $\mathcal{A}$ is similar to a real symmetric matrix. Since $\mathcal{A}$ is known to be positive real, we see that a positive definite $\mathcal{M}_C(\gamma)$ is a sufficient condition for $\mathcal{A}$ to be diagonalizable with all its eigenvalues real and positive.

First note that $\mathcal{M}_C(\gamma)$ is congruent to the block diagonal matrix

$$\begin{bmatrix} A - \gamma I & 0 \\ 0 & S \end{bmatrix}, \quad \text{where} \quad S = (\gamma I - C) - B(A - \gamma I)^{-1} B^T.$$

Therefore a *necessary* (but not sufficient) condition in order to make $\mathcal{M}_C(\gamma)$ positive definite is that

$$\lambda_{\min}(A) > \gamma > \lambda_{\max}(C). \tag{2.4}$$

In the following we will restrict our attention to γ satisfying (2.4). In case A and C are such that $\lambda_{\max}(C) \geq \lambda_{\min}(A)$, which particularly includes the case of singular A, the approach presented here does not work, and we are unaware of any conditions that guarantee $\mathcal{A}$ being diagonalizable with positive real eigenvalues. However, the case $\lambda_{\min}(A) > \lambda_{\max}(C)$ is of practical interest, particularly in the context of stabilized discretizations of Stokes or generalized Stokes problems. For example, the stabilized Stokes coefficient matrix in [3, p. 240] is of the form (1.1) with the (2,2) block given by $-C = -\beta h^2 D$, where β is a nonnegative stabilization parameter and h is the mesh size (here a uniform mesh is assumed for simplicity). The matrix D is symmetric positive semidefinite and has norm 4, giving $\lambda_{\max}(C) = 4\beta h^2$, which is is a very small number unless the stabilization parameter β is chosen very large. In particular, for any symmetric positive definite A, $\lambda_{\min}(A) > \lambda_{\max}(C)$ holds for all $\beta < \frac{1}{4} h^{-2} \lambda_{\min}(A)$.

Next, using a standard result on the eigenvalues of symmetric matrices (cf. e.g. [5, Theorem 8.1.5]),

$$\begin{aligned} \lambda_{\min}(\mathcal{M}_C(\gamma)) &\geq \lambda_{\min}\left(\begin{bmatrix} A - \gamma I & B^T \\ B & \gamma I \end{bmatrix} \right) + \lambda_{\min}\left(\begin{bmatrix} 0 & 0 \\ 0 & -C \end{bmatrix} \right) \\ &= \lambda_{\min}\left(\mathcal{M}_0(\gamma)\right) - \lambda_{\max}(C). \end{aligned} \tag{2.5}$$

Hence a *sufficient* condition so that $\mathcal{M}_C(\gamma)$ is positive definite is

$$\lambda_{\min}\left(\mathcal{M}_0(\gamma)\right) > \lambda_{\max}(C)\,. \tag{2.6}$$

To derive properties on A, B, C, and γ so that (2.6) holds, we consider the eigenvalue problem $\mathcal{M}_0(\gamma)\,[x^T; y^T]^T = \theta\,[x^T; y^T]^T$, or

$$(i)\;\; (A-\gamma I)x + B^T y = \theta x\,, \quad \text{and} \quad (ii)\;\; Bx + \gamma y = \theta y\,.$$

If there exists an eigenvalue θ with $\theta = \gamma$, then $\theta = \gamma > \lambda_{\max}(C)$ since we have restricted our attention to γ satisfying (2.4). If $\theta \neq \gamma$ we can transform equation (ii) into its equivalent form $y = (\theta-\gamma)^{-1}Bx$, which, inserted into (i) yields

$$(A-\gamma I)x + (\theta-\gamma)^{-1}B^T Bx = \theta x\,.$$

Note that we must have $x \neq 0$ for if otherwise equation (ii) would yield $y = 0$, a contradiction to the fact that $[x^T, y^T]^T$ is an eigenvector. After multiplying from the left with x^T and some algebraic manipulations we obtain the equation

$$\theta + \gamma^2 \frac{x^T x}{x^T A x} = \theta^2 \frac{x^T x}{x^T A x} + \gamma - \frac{x^T B^T B x}{x^T A x}\,. \tag{2.7}$$

As in the proof of [2, Corollary 3.2], we can bound the left hand side of (2.7) from above by $\theta + \gamma^2/\lambda_{\min}(A)$, and the right hand side from below by

$$\gamma - \frac{x^T B^T B x}{x^T A x} \geq \gamma - \lambda_{\max}(BA^{-1}B^T)\,,$$

which yields the following lower bound on θ,

$$\theta \geq \gamma - \frac{\gamma^2}{\lambda_{\min}(A)} - \lambda_{\max}(BA^{-1}B^T)\,. \tag{2.8}$$

To maximize the lower bound on θ we set $\gamma = \gamma^* \equiv \frac{1}{2}\lambda_{\min}(A)$. This value of γ is also used in [2], and it is there determined by a slightly different argument in the proof of Proposition 3.1. With $\gamma = \gamma^*$, (2.8) becomes

$$\theta \geq \frac{1}{4}\lambda_{\min}(A) - \lambda_{\max}(BA^{-1}B^T)\,. \tag{2.9}$$

Combining this with (2.6) shows that $\mathcal{M}_C(\gamma^*)$ is positive definite when

$$\lambda_{\min}(A) > 4\left(\,\lambda_{\max}(C) + \lambda_{\max}(BA^{-1}B^T)\,\right). \tag{2.10}$$

Note that if (2.10) holds, and $\gamma = \gamma^*$, then the necessary condition (2.4) on γ is satisfied. We summarize our discussion in the following theorem.

PROPOSITION 2.1. *Consider the matrix $\mathcal{A}$ as in (1.2) with symmetric positive definite $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ of full rank m, and symmetric positive semidefinite $C \in \mathbb{R}^{m\times m}$, and let $\gamma^* \equiv \frac{1}{2}\lambda_{\min}(A)$. If (2.10) holds, then the matrix $\mathcal{M}_C(\gamma^*)$ in (2.1) is positive definite, and $\mathcal{A}$ is diagonalizable with all its eigenvalues real and positive.*

This proposition is a generalization of results previously obtained in [4, 2]:

Fischer et al. [4] consider $\mathcal{A}$ with $A = \eta I > 0$ and $C = 0$. The condition (2.10) then reads $\eta > 2\sigma_{\max}(B)$, where $\sigma_{\max}(B)$ denotes the largest singular value of B.

This is precisely the condition derived in [4, pp. 531–532], and the matrix $\mathcal{M}_0(\eta/2)$ in (2.1) is equal to the matrix in [4, Equation (2.3)] multiplied by $\eta/2$.

Benzi and Simoncini [2, Section 3] consider $\mathcal{A}$ with $A = A^T > 0$ and $C = 0$. Their matrix G in [2, p. 182] is equal to $\mathcal{M}_0(\gamma)$ in (2.1), and [2, Proposition 3.1] is equivalent with Proposition 2.1 above. For the case $C = \beta I \geq 0$, [2, Corollary 2.6] shows that if $\lambda_{\min}(A) \geq 3\beta + 4\lambda_{\max}(BA^{-1}B^T)$, then $\mathcal{A}$ has real eigenvalues. The condition on $\beta = \lambda_{\max}(C)$ in this special case is a bit weaker than (2.10). Note however that (2.10) not only implies real eigenvalues but also diagonalizability of $\mathcal{A}$.

In the terminiology of [6] and under the condition (2.10), the matrix $\mathcal{A}$ is normal of degree one with respect to the symmetric positive definite matrix $\mathcal{M}_C(\gamma^*)$. According to [6, Theorem 3.1], $\mathcal{A}$ must be diagonalizable. If we write the eigendecomposition as $\mathcal{A} = W\Lambda W^{-1}$, where the eigenvalues and eigenvectors of $\mathcal{A}$ are ordered so that the same eigenvalues form a single block on the diagonal of Λ, then $\mathcal{M}_C(\gamma^*)$ must be of the form $\mathcal{M}_C(\gamma^*) = (WDW^T)^{-1}$, where D is a symmetric positive definite block diagonal matrix with block sizes corresponding to those of Λ, cf. [6, Theorem 3.1]. With $\hat{W} = WD^{-1/2}$, $\mathcal{M}_C(\gamma^*) = (\hat{W}\hat{W}^T)^{-1}$, and thus

$$\kappa(\mathcal{M}_C(\gamma^*)) = \|\mathcal{M}_C(\gamma^*)\| \, \|\mathcal{M}_C(\gamma^*)^{-1}\| = \kappa(\hat{W})^2$$

(cf. [2, pp. 184–185], where a similar result is derived in a different way, and subsequently used to bound the residual norm of a Krylov subspace method applied to the matrix $\mathcal{A}$). An estimate for these quantities can be found as follows: First, by [5, Theorem 8.1.5] and [2, Corollary 3.2],

$$\lambda_{\max}(\mathcal{M}_C(\gamma^*)) \leq \lambda_{\max}\left(\mathcal{M}_0(\gamma)\right) \approx \lambda_{\max}(A)\,,$$

and second, by (2.5) and (2.9),

$$\lambda_{\min}(\mathcal{M}_C(\gamma^*)) \geq \frac{1}{2}\gamma^* - (\lambda_{\max}(C) + \lambda_{\max}(BA^{-1}B^T))\,.$$

Combining these two inequalities yields

$$\kappa(\mathcal{M}_C(\gamma^*)) = \frac{\lambda_{\max}(\mathcal{M}_C(\gamma^*))}{\lambda_{\min}(\mathcal{M}_C(\gamma^*))} \approx \frac{\lambda_{\max}(A)}{\frac{1}{2}\gamma^* - (\lambda_{\max}(C) + \lambda_{\max}(BA^{-1}B^T))}\,.$$

For $C = 0$ this result corresponds to the one given in [2, Corollary 3.2].

Since $\mathcal{A}$ is normal of degree one with respect to $\mathcal{M}_C(\gamma^*)$, $\mathcal{A}$ admits an optimal three-term recurrence for computing Krylov subspace bases that are orthogonal with respect to the inner product generated by $\mathcal{M}_C(\gamma^*)$, $\langle x, y\rangle \equiv y^T\mathcal{M}_C(\gamma^*)x$; see [6] for details. Therefore, a three-term recurrence conjugate gradient type method based on this inner product can be constructed. For a practical application of such method a preconditioner that is symmetric positive definite with respect to this inner product should be available, and the inner product matrix $\mathcal{M}_C(\gamma^*)$ should be well conditioned. While the condition number of $\mathcal{M}_C(\gamma^*)$ depends on the conditioning of the eigenvectors of $\mathcal{A}$ and can be estimated as shown above, the construction of such preconditioners is an open problem.

Finally, as a simple example we consider the matrix

$$\mathcal{A} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & b & 0 \\ 0 & 2 & 0 & 0 & b \\ 0 & 0 & 3 & 0 & 0 \\ \hline -b & 0 & 0 & 2c & -c \\ 0 & -b & 0 & -c & 2c \end{array}\right], \quad b \neq 0\,, \quad c \geq 0\,.$$

Elementary computations show that

$$\lambda_{\min}(A) = 1\,, \quad \lambda_{\max}(BA^{-1}B^T) = b^2\,, \quad \lambda_{\max}(C) = 3c\,,$$

and hence the sufficient condition (2.10) becomes

$$1 > 12c + 4b^2\,.$$

If we choose $b = 1/2$, then this condition is not satisfied for any $c \geq 0$, and indeed a MATLAB [7] computation reveals that the matrix $\mathcal{A}$ is not diagonalizable for $c = 0$, and has eigenvalues with nonzero imaginary parts for $c > 0$. On the other hand, if we choose $c = 1/12$, then a MATLAB computation shows that $\mathcal{A}$ has five distinct real and positive eigenvalues whenever $|b| \leq 0.4056855$.

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