A note on Potter's theorem for quasi-commutative matrices

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Dedicated to Thomas Laffey on the occasion of his 65th birthday

Abstract

We discuss the converse of a theorem of Potter stating that if the matrix equation $AB = \omega BA$ is satisfied with ω a primitive qth root of unity, then $A^q + B^q = (A+B)^q$. We show that both conditions have to be modified to get a converse statement and we present a characterization when the converse holds for these modified conditions and $q = 3$ and a conjecture for the general case. We also present some further partial results and conjectures.

Key words. quasi-commutative matrices, roots of unity, Potter's theorem
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1 Introduction

In [3], H. S. A. Potter published a note on the matrix equation

$$AB = \omega BA, \tag{1}$$

with A, B square matrices and $\omega \in \mathbb{C}$. He called a pair of complex $n \times n$ matrices A and B satisfying (1) *quasi-commutative* but we prefer the name *ω-commutative* to indicate the dependence on the scalar ω. ω-commutative matrices are of importance in quantum mechanics, see [4]. Potter showed the following result.

Theorem 1 *[3] Let $A, B \in \mathbb{C}^n$ satisfy (1), where ω is a primitive qth root of unity. Then*

$$A^q + B^q = (A+B)^q. \tag{2}$$

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In a recent historical note [2], some unpublished work of Helmut Wielandt was discussed, in which a simple proof of Theorem 1 was given as well as some extensions. It was shown by a counterexample that the converse of Potter's theorem does not hold in general.

In this paper we, therefore, study the question under which conditions the converse statement in Theorem 1 holds. In view of the counterexample in [2] it is clear that we need further conditions that go beyond (2). To see which conditions are candidates, for a fixed integer q, we make the following observations.

Observations: Let $A, B \in \mathbb{C}^{n,n}$, let ω be a primitive q-th root of unity and let k, ℓ be integers such that k and q, as well as ℓ and q are relative prime. If A, B are ω-commutative, then

1. sA, tB are ω-commutative for all $s, t \in \mathbb{C}$.
2. A, A^iB are ω-commutative for all $i \in \mathbb{N}$, because
$$A(A^iB) = A^i(AB) = \omega(A^iB)A.$$
3. A^kB and B are ω^k-commutative, since
$$(A^kB)B = A^{k-1}(AB)B = \omega A^{k-1}BAB = \omega^k B(A^kB).$$
4. A^k and B are ω^k-commutative, since
$$A^kB = A^{k-1}AB = \omega A^{k-1}BA = \omega^k BA^k.$$
5. A and B^ℓ are ω^ℓ-commutative.

In view of these observations it seems natural to require that for appropriate integers j, ℓ and for all $s, t \in \mathbb{C}$ instead of (2) the stronger condition

$$\left((sA)^j + (tB)^\ell\right)^q = (sA)^{jq} + (tB)^{\ell q} \tag{3}$$

holds. Note that the ω-commutativity of A, B implies (3) for all $s, t \in \mathbb{C}$ with $j = \ell = 1$, as was already observed in [2].

On the other hand we prove in the special case that $q = 3$ and that AB has distinct eigenvalues, that A, B satisfy (3) for $j = 1, \ell = 1$, $j = 2, \ell = 1$ and for all $s, t \in \mathbb{C}$, if and only if the identities

$$0 = (AB - \omega BA)(AB - \omega^2 BA),\ 0 = (AB - \omega^2 BA)(AB - \omega BA) \tag{4}$$

hold.

It is clear that (1) implies (4) but we show via counterexamples that the converse does not hold for general primes q. Thus, it looks reasonable that we should try to relate the conditions (3) and extended product formulas such as

$$\Pi_{i=1}^{q-1}(AB - \omega^{\sigma(i)}BA) = 0, \tag{5}$$

for all permutations σ of $(1, 2, \ldots, q-1)$.

We present this relationship in the case $q = 3$ and also some further partial results.

In order to simplify the presentation, in the following we say that $A, B \in \mathbb{C}^{n,n}$ satisfy $(s, t; j, \ell, q)$ if (3) holds for positive integers j, ℓ, q, *for all* $s, t \in \mathbb{C}$.

2 The case $q = 3$.

In this section we discuss the case $q = 3$. We begin our analysis with an observation relating (3) to sums of products in A, B.

Proposition 2 *Let* $A, B \in \mathbb{C}^{n,n}$.

i) A, B *satisfy* $(s, t; 1, 1, 3)$ $\forall s, t \in \mathbb{C}$ *iff*

$$\begin{aligned} A^2B + ABA + BA^2 &= 0, && (6) \\ AB^2 + BAB + B^2A &= 0. && (7) \end{aligned}$$

ii) $A, B \in \mathbb{C}^{n,n}$ *satisfy* $(s, t; 2, 1, 3)$ $\forall s, t \in \mathbb{C}$ *iff*

$$\begin{aligned} A^4B + A^2BA^2 + BA^4 &= 0, && (8) \\ A^2B^2 + BA^2B + B^2A^2 &= 0. && (9) \end{aligned}$$

iii) $A, B \in \mathbb{C}^{n,n}$ *satisfy* $(s, t; 1, 2, 3)$ $\forall s, t \in \mathbb{C}$ *iff*

$$\begin{aligned} A^2B^2 + AB^2A + B^2A^2 &= 0, && (10) \\ AB^4 + B^2AB^2 + B^4A &= 0. && (11) \end{aligned}$$

Proof. The proof is straightforward by comparing coefficients in (3). $\square$

Let us carry out some manipulations involving (6)–(11).

First of all, if (9) and (10) hold, then taking their difference we obtain that

$$AB^2A = (AB)(BA) = BA^2B = (BA)(AB), \tag{12}$$

i.e. AB and BA commute. If we have this condition then some of the discussed questions significantly simplify, however for general $q > 3$, the conditions in (3) will not imply that AB and BA commute.

Question: What are the minimum requirements in terms of conditions of the form (3) that guarantee that AB and BA commute.

If (6)-(9) hold then we obtain the following identities

$$\begin{aligned} ABAB &= B^2A^2, && (13) \\ BABA &= A^2B^2, && (14) \\ A^3(BA) &= (BA)A^3, && (15) \\ (AB)A^3 &= A^3(AB). && (16) \end{aligned}$$

Combining these observations we have the following Proposition.

Proposition 3 *Suppose that* $A, B \in \mathbb{C}^{n,n}$ *satisfy* $(s, t; 1, 1, 3)$, $(s, t; 2, 1, 3)$, *and* $(s, t; 1, 2, 3)$ *for all* $s, t \in \mathbb{C}$. *Then (5) holds with* $q = 3$ *and* ω *being a primitive* 3*rd root of unity.*

Proof. By assumption (6)–(11) hold, implying that also (12)–(16) hold. Then we have

$$\begin{aligned}(AB-\omega BA)(AB-\omega^2 BA) &= ABAB-\omega^2 ABBA-\omega BAAB+BABA\\ &= ABAB-(\omega^2+\omega)AB^2A+BABA\\ &= ABAB+AB^2A+BABA\\ &= B^2A^2+AB^2A+A^2B^2=0,\end{aligned}$$

where we have used (12)–(14) and (10). The proof of the identity $(AB-\omega^2 BA)(AB-\omega BA)=0$ is similar. □

We see from Proposition 3 that if we require all three conditions $(s,t;1,1,3)$, $(s,t;2,1,3)$ and $(s,t;1,2,3)$ for all $s,t\in\mathbb{C}$, then we already have the product formulas (5). An immediate question then is whether we can weaken the assumptions in Proposition 3 and still get the product formulas. Under a generic assumption the answer is positive as the next result shows.

Theorem 4 *Suppose that $A,B\in\mathbb{C}^{n,n}$ are such that AB has pairwise distinct eigenvalues and that A,B satisfy $(s,t;1,1,3)$ for all $s,t\in\mathbb{C}$.*

If A,B also satisfy either $(s,t;2,1,3)$ or $(s,t;1,2,3)$ for all $s,t\in\mathbb{C}$, then (5) holds with $q=3$ and ω being a primitive 3rd root of unity.

Proof. It suffices to give a proof for the case that A,B satisfy $(s,t;2,1,3)$, the other case follows by exchanging the roles of A and B. Our assumptions imply that (6)–(9) hold and thus also (16).

We can replace A,B by $S^{-1}AS, S^{-1}BS$, where $S\in\mathbb{C}^{n,n}$ is invertible. Hence we may assume w.l.o.g. that A is in Jordan canonical form, i.e.

$$A=\begin{bmatrix} G & 0\\ 0 & N\end{bmatrix},\quad B=\begin{bmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix},$$

where G is invertible and N is nilpotent.

Partition AB conformably with A as

$$AB=\begin{bmatrix} X_{11} & X_{12}\\ X_{21} & X_{22}\end{bmatrix}.$$

Using (16) it follows that $X_{1,2}, X_{2,1}$ satisfy the Sylvester equations $G^3X_{1,2}-X_{1,2}N^3=0$ and $X_{2,1}G^3-N^3X_{2,1}=0$, respectively. Since G^3 and N^3 have no common eigenvalues, it follows from the usual theory of the Sylvester equation [1] that $X_{1,2}=0$ and $X_{2,1}=0$, i.e.

$$AB=\begin{bmatrix} X_{11} & 0\\ 0 & X_{22}\end{bmatrix}. \tag{17}$$

Then by (17) we have

$$AB=\begin{bmatrix} X_{11} & 0\\ 0 & X_{22}\end{bmatrix}=\begin{bmatrix} GB_{11} & GB_{12}\\ NB_{21} & NB_{22}\end{bmatrix},$$

and so, since G is nonsingular, it follows from (17) that $B_{12} = 0$. Using (15) we obtain that $B_{21}G^4 = N^4 B_{21} G$ and hence, by factoring out G and using the Sylvester equation, it follows also that $B_{21} = 0$. Moreover, by (16) and the assumption that all eigenvalues of AB are pairwise distinct, we have that A^3 is diagonalizable and hence, it follows that G and N^3 are diagonal, i.e. the size of each Jordan block in N is at most 3.

If A is singular then, furthermore, we have that N is a single Jordan block. This follows, since $\operatorname{rank} A \leq n-2$ would imply $\operatorname{rank} AB \leq n-2$, which would contradict the assumption that AB has pairwise distinct eigenvalues. We actually claim that $N = 0$. To see this, we show that N cannot be of size $k = 2$ or $k = 3$.

Suppose that $k = 2$, i.e.

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B_{22} = [b_{i,j}].$$

Then from (6) it follows that

$$0 = N^2 B_{22} + N B_{22} N + B_{22} N^2 = N B_{22} N,$$

which implies that $b_{2,1} = 0$. But then AB has a double eigenvalue 0 which is a contradiction.

Suppose that $k = 3$, i.e.

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; B_{22} = [b_{i,j}],$$

then from (6) we obtain that $b_{3,1} = 0$, $b_{2,1} + b_{3,2} = 0$ and $b_{1,1} + b_{2,2} + b_{3,3} = 0$. Inserting these identities we obtain

$$N B_{22} = \begin{bmatrix} b_{2,1} & b_{2,2} & b_{2,3} \\ 0 & -b_{2,1} & -b_{1,1} - b_{2,2} \\ 0 & 0 & 0 \end{bmatrix}$$

Making use of (13) and the fact that

$$B_{22}^2 N^2 = \begin{bmatrix} 0 & 0 & b_{1,1}^2 + b_{1,2} b_{2,1} \\ 0 & 0 & b_{2,1} b_{1,1} + b_{2,2} b_{2,1} \\ 0 & 0 & b_{3,2} b_{2,1} \end{bmatrix},$$

it follows that $b_{3,2} b_{2,1} = -b_{2,1}^2 = 0$ and hence $N B_{22}$ has a multiple eigenvalue at 0, which is a contradiction.

In summary, we have that either A is nonsingular or $N = 0 \in \mathbb{C}^{1,1}$.

Consider now the blocks $G = \operatorname{diag}(g_1, \ldots, g_\ell), B_{11}$ and set

$$B_{11} = \begin{bmatrix} \beta_{1,1} & \ldots & \beta_{1,\ell} \\ \vdots & \ddots & \vdots \\ \beta_{\ell,1} & \ldots & \beta_{\ell,\ell} \end{bmatrix}.$$

From the $(1,1)$ block of (6) we obtain that

$$(g_i^2 + g_i g_j + g_j^2)\beta_{i,j} = 0,\ i,j = 1,2,\dots \ell$$

so $\beta_{i,j} = 0$ unless g_i/g_j is a primitive 3rd root of unity, i.e. $e^{2\pi\imath/3}$ or $e^{4\pi\imath/3}$.

Suppose that $g_i \neq g_j$ and that $g_i, \omega g_i, \omega^2 g_i, g_j, \omega g_j, \omega^2 g_j$ are pairwise distinct numbers with multiplicities $m_1, m_2, \dots, m_6$, respectively, where we may assume that $m_1, m_4 > 0$, and let $\tilde{m} = \sum_{r=1}^{6} m_r$.

If follows that there exists an $(\tilde{m}, \tilde{m})$ principal submatrix of B_{11} that has the block structured form

$$\left[\begin{array}{ccc|ccc} 0 & * & * & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * & 0 \end{array}\right]$$

with diagonal blocks of sizes $m_i \times m_i$ and blocks denoted by $*$ to be discussed later. This implies that B_{11} is block diagonal, and hence it suffices to consider the diagonal blocks separately. Thus, we may just consider an $m \times m$ principal submatrix with eigenvalues $g_i, \omega g_i, \omega^2 g_i$, multiplicities m_1, m_2, m_3 and $m_1 + m_2 + m_3 = m$.

Since we can scale the matrix equations, we may assume w.l.o.g. that $g_i = 1$ and we consider the associated $m \times m$ principal submatrices in G, B_{11} which then have the form

$$\hat{A} = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & \omega I_{m_2} & 0 \\ 0 & 0 & \omega^2 I_{m_3} \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 & \hat{B}_{1,2} & \hat{B}_{1,3} \\ \hat{B}_{2,1} & 0 & \hat{B}_{2,3} \\ \hat{B}_{3,1} & \hat{B}_{3,2} & 0 \end{bmatrix},$$

partitioned accordingly.

Consider the relevant blocks in (7) and (9) which are

$$\begin{aligned}
\hat{B}^2 &= \begin{bmatrix} \hat{B}_{1,2}\hat{B}_{2,1} + \hat{B}_{1,3}\hat{B}_{3,1} & \hat{B}_{1,3}\hat{B}_{3,2} & \hat{B}_{1,2}\hat{B}_{2,3} \\ \hat{B}_{2,3}\hat{B}_{3,1} & \hat{B}_{2,1}\hat{B}_{1,2} + \hat{B}_{2,3}\hat{B}_{3,2} & \hat{B}_{2,1}\hat{B}_{1,3} \\ \hat{B}_{3,2}\hat{B}_{2,1} & \hat{B}_{3,1}\hat{B}_{1,2} & \hat{B}_{3,1}\hat{B}_{1,3} + \hat{B}_{3,2}\hat{B}_{2,3} \end{bmatrix}, \\
\hat{B}\hat{A} &= \begin{bmatrix} 0 & \omega\hat{B}_{1,2} & \omega^2\hat{B}_{1,3} \\ \hat{B}_{2,1} & 0 & \omega^2\hat{B}_{2,3} \\ \hat{B}_{3,1} & \omega\hat{B}_{3,2} & 0 \end{bmatrix}, \\
\hat{A}\hat{B} &= \begin{bmatrix} 0 & \hat{B}_{1,2} & \hat{B}_{1,3} \\ \omega\hat{B}_{2,1} & 0 & \omega\hat{B}_{2,3} \\ \omega^2\hat{B}_{3,1} & \omega^2\hat{B}_{3,2} & 0 \end{bmatrix}, \\
\hat{B}^2\hat{A} &= \begin{bmatrix} \hat{B}_{1,2}\hat{B}_{2,1} + \hat{B}_{1,3}\hat{B}_{3,1} & \omega\hat{B}_{1,3}\hat{B}_{3,2} & \omega^2\hat{B}_{1,2}\hat{B}_{2,3} \\ \hat{B}_{2,3}\hat{B}_{3,1} & \omega(\hat{B}_{2,1}\hat{B}_{1,2} + \hat{B}_{2,3}\hat{B}_{3,2}) & \omega^2\hat{B}_{2,1}\hat{B}_{1,3} \\ \hat{B}_{3,2}\hat{B}_{2,1} & \omega\hat{B}_{3,1}\hat{B}_{1,2} & \omega^2(\hat{B}_{3,1}\hat{B}_{1,3} + \hat{B}_{3,2}\hat{B}_{2,3}) \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
\hat{A}\hat{B}^2 &= \begin{bmatrix} \hat{B}_{1,2}\hat{B}_{2,1}+\hat{B}_{1,3}\hat{B}_{3,1} & \hat{B}_{1,3}\hat{B}_{3,2} & \hat{B}_{1,2}\hat{B}_{2,3} \\ \omega\hat{B}_{2,3}\hat{B}_{3,1} & \omega(\hat{B}_{2,1}\hat{B}_{1,2}+\hat{B}_{2,3}\hat{B}_{3,2}) & \omega\hat{B}_{2,1}\hat{B}_{1,3} \\ \omega^2\hat{B}_{3,2}\hat{B}_{2,1} & \omega^2\hat{B}_{3,1}\hat{B}_{1,2} & \omega^2(\hat{B}_{3,1}\hat{B}_{1,3}+\hat{B}_{3,2}\hat{B}_{2,3}) \end{bmatrix}, \\
\hat{B}\hat{A}\hat{B} &= \begin{bmatrix} \omega\hat{B}_{1,2}\hat{B}_{2,1}+\omega^2\hat{B}_{1,3}\hat{B}_{3,1} & \omega^2\hat{B}_{1,3}\hat{B}_{3,2} & \omega\hat{B}_{1,2}\hat{B}_{2,3} \\ \omega^2\hat{B}_{2,3}\hat{B}_{3,1} & \hat{B}_{2,1}\hat{B}_{1,2}+\omega^2\hat{B}_{2,3}\hat{B}_{3,2} & \hat{B}_{2,1}\hat{B}_{1,3} \\ \omega\hat{B}_{3,2}\hat{B}_{2,1} & \hat{B}_{3,1}\hat{B}_{1,2} & \hat{B}_{3,1}\hat{B}_{1,3}+\omega\hat{B}_{3,2}\hat{B}_{2,3} \end{bmatrix}.
\end{aligned}
$$

Comparing blocks and using that $1+\omega+\omega^2=0$, the equations associated with off-diagonal blocks in (7) are automatically satisfied and from the diagonal blocks we obtain

$$
\begin{aligned}
(2+\omega)\hat{B}_{1,2}\hat{B}_{2,1}+(2+\omega^2)\hat{B}_{1,3}\hat{B}_{3,1} &= 0, \\
(1+2\omega)\hat{B}_{2,1}\hat{B}_{1,2}+\omega(2+\omega)\hat{B}_{2,3}\hat{B}_{3,2} &= 0, \\
(1+2\omega^2)\hat{B}_{3,1}\hat{B}_{1,3}+\omega(1+2\omega)\hat{B}_{3,2}\hat{B}_{2,3} &= 0.
\end{aligned} \tag{18}
$$

Considering (9) instead of (7) means just to replace $\hat{A}$ by $\hat{A}^2$ or ω by ω^2 in (18) which gives the three extra conditions

$$
\begin{aligned}
(2+\omega^2)\hat{B}_{1,2}\hat{B}_{2,1}+(2+\omega)\hat{B}_{1,3}\hat{B}_{3,1} &= 0, \\
(1+2\omega^2)\hat{B}_{2,1}\hat{B}_{1,2}+\omega^2(2+\omega^2)\hat{B}_{2,3}\hat{B}_{3,2} &= 0, \\
(1+2\omega)\hat{B}_{3,1}\hat{B}_{1,3}+\omega^2(1+2\omega^2)\hat{B}_{3,2}\hat{B}_{2,3} &= 0.
\end{aligned} \tag{19}
$$

We will now show that (18) and (19) together imply that

$$
\begin{aligned}
\hat{B}_{1,2}\hat{B}_{2,1} &= 0,\ \hat{B}_{2,1}\hat{B}_{1,2}=0, \\
\hat{B}_{1,3}\hat{B}_{3,1} &= 0,\ \hat{B}_{3,1}\hat{B}_{1,3}=0, \\
\hat{B}_{2,3}\hat{B}_{3,2} &= 0,\ \hat{B}_{3,2}\hat{B}_{2,3}=0.
\end{aligned} \tag{20}
$$

This follows from

$$
\det\begin{bmatrix} 2+\omega & 2+\omega^2 \\ 2+\omega^2 & 2+\omega \end{bmatrix} = 3(\omega-\omega^2)\neq 0
$$

and

$$
\det\begin{bmatrix} 1+2\omega & 2\omega+\omega^2 \\ 1+2\omega^2 & \omega+2\omega^2 \end{bmatrix} = 3\omega(\omega-1)\neq 0.
$$

By (20) we have that

$$
\begin{aligned}
&(\hat{A}\hat{B}-\omega\hat{B}\hat{A})(\hat{A}\hat{B}-\omega^2\hat{B}\hat{A}) = \\
&\begin{bmatrix} (1-\omega^2)(\omega-\omega^2)\hat{B}_{1,2}\hat{B}_{2,1} & 0 & 0 \\ 0 & (\omega-1)(\omega^2-1)\hat{B}_{2,3}\hat{B}_{3,2} & 0 \\ 0 & 0 & (\omega^2-\omega)(1-\omega)\hat{B}_{3,1}\hat{B}_{1,3} \end{bmatrix} = 0,
\end{aligned}
$$

and similarly

$$(\hat{A}\hat{B} - \omega^2\hat{B}\hat{A})(\hat{A}\hat{B} - \omega\hat{B}\hat{A}) =$$
$$\begin{bmatrix} (1-\omega)(\omega^2-\omega)\hat{B}_{1,3}\hat{B}_{3,1} & 0 & 0 \\ 0 & (\omega-\omega^2)(1-\omega^2)\hat{B}_{2,1}\hat{B}_{1,2} & 0 \\ 0 & 0 & (\omega^2-1)(\omega-1)\hat{B}_{3,2}\hat{B}_{2,3} \end{bmatrix} = 0.$$

□

Question: Can we drop the generic assumption that AB has pairwise distinct eigenvalues in the assumptions of Theorem 4.

In the proof of Theorem 4 we have seen that for $q = 3$ from conditions (6)–(9) and the fact that AB has distinct eigenvalues, it follows that the algebraic and geometric multiplicity of 0 as an eigenvalue of A is at most 1. We make the following conjecture:

Conjecture: *Suppose that $A, B \in \mathbb{C}^{n,n}$ are such that AB has pairwise distinct eigenvalues, and such that for a fixed prime number q, $q \geq 2$, A, B satisfy $(s, t; j, \ell, q)$ for some (appropriate) integers $j, \ell \in \mathbb{N}$, and for all $s, t \in \mathbb{C}$. Then the algebraic multiplicity of 0 as an eigenvalue of A or B is at most 1.*

The proof of Theorem 4 suggests the following question on the converse of Theorem 4.

Question: Let q be a prime and suppose that $\omega \in \mathbb{C}$ is a primitive qth root of unity, and that $A, B \in \mathbb{C}^{n,n}$ are such that AB has distinct eigenvalues and satisfies (5) for all permutations. Is it true that if we require certain trace conditions then $(s, t; 1, 1, q)$ holds for all $s, t \in \mathbb{C}$.

We present an answer to this question again in the case $q = 3$.

Theorem 5 *Let A and B be complex square matrices satisfying (5), where ω is a primitive 3rd root of unity. Suppose that the eigenvalues of AB are pairwise distinct and that $\operatorname{tr}(A^jB^j) = 0$ for $j = 1, 2, \ldots$, with $3 \not| j$. Then A, B satisfy $(s, t; 1, 1, 3)$ for all $s, t \in \mathbb{C}$.*

Proof. From the two identities in (5) we get

$$\begin{aligned} 0 &= ABAB - \omega^2 AB^2A - \omega BA^2B + BABA, \\ 0 &= ABAB - \omega AB^2A - \omega^2 BA^2B + BABA. \end{aligned} \tag{21}$$

Subtracting these equations we get

$$(\omega - \omega^2)AB^2A - (\omega - \omega^2)BA^2B = 0$$

and thus, since ω is a primitive 3rd root of unity, we see that $AB^2A = BA^2B$ and hence AB and BA commute.

We may assume w.l.o.g. that $AB =: D = \operatorname{diag}(d_1, \ldots, d_n)$. Since AB commutes with BA and since the d_i are pairwise distinct, it follows that also $BA =: F = \operatorname{diag}(f_1, \ldots, f_n)$ is diagonal and $f_i = d_{\sigma(i)}$, $i = 1, 2, \ldots, n$ for some permutation σ. It then follows from (5) that

$$(d_i - \omega d_{\sigma(i)})(d_i - \omega^2 d_{\sigma(i)}) = 0,\ i = 1, 2, \ldots, n$$

or for convenience

$$(d_{\phi(i)} - \omega d_i)(d_{\phi(i)} - \omega^2 d_i) = 0,\ i = 1, 2, \dots, n, \tag{22}$$

where $\phi = \sigma^{-1}$.

Since the d_i are pairwise distinct we have at most one i such that $d_i = 0$.

Case 1: If AB is nonsingular, then (22) implies that ϕ has no fixed points. We may write ϕ as a product of disjoint cycles and want to show that the length l of each cycle is $l \le 3$. For this, suppose that there exists a cycle of length $l \ge 3$. W.l.o.g. we may assume that the cycle is $(1, 2, 3, \dots, l)$ and then we have

$$\begin{aligned}
(d_2 - \omega d_1)(d_2 - \omega^2 d_1) &= 0,\\
(d_3 - \omega d_2)(d_3 - \omega^2 d_2) &= 0,\\
&\vdots\\
(d_l - \omega d_{l-1})(d_l - \omega^2 d_{l-1}) &= 0,\\
(d_1 - \omega d_l)(d_1 - \omega^2 d_l) &= 0.
\end{aligned}$$

Suppose first that $d_2 = \omega d_1$. Then, we claim that from the second equation we have $d_3 = \omega d_2$. If this were not so, then $d_3 = \omega^2 d_2 = \omega^3 d_1 = d_1$ which is a contradiction. Thus, $d_2 = \omega d_1$ implies $d_3 = \omega d_2$. If $l \ge 4$, then the next equation gives a contradiction, because then $d_4 = \omega d_3 = \omega^2 d_2 = d_1$ or $d_4 = \omega^2 d_3 = d_2$. Hence $l = 3$ and $d_3 = \omega d_2 = \omega^2 d_1$.

Suppose now that $d_2 = \omega^2 d_1$. Then an analogous argument shows that $l = 3$ and $d_3 = \omega^2 d_2 = \omega d_1$.

For $l = 2$ and the cycle $(1, 2)$ we have either $d_2 = \omega d_1$ or $d_2 = \omega^2 d_1$. So for the diagonal matrices $D = AB$ and $F = BA$, there are four types of pairs of principal submatrices.

$$\begin{array}{ll}
(i) & (d_1, \omega d_1), (d_2, \omega d_2), \dots, (d_r, \omega d_r)\\
 & (\omega d_1, d_1), (\omega d_2, d_2), \dots, (\omega d_r, d_r) \text{ for some } r \ge 0,\\
(ii) & (d_{r+1}, \omega^2 d_{r+1}), (d_{r+2}, \omega^2 d_{r+2}), \dots, (d_{r+v}, \omega^2 d_{r+v})\\
 & (\omega^2 d_{r+1}, d_{r+1}), (\omega^2 d_{r+2}, d_{r+2}), \dots, (\omega^2 d_{r+v}, d_{r+v}) \text{ for some } v \ge 0,\\
(iii) & (\tilde{d}_1, \omega \tilde{d}_1, \omega^2 \tilde{d}_1), (\tilde{d}_2, \omega \tilde{d}_2, \omega^2 \tilde{d}_2), \dots, (\tilde{d}_w, \omega \tilde{d}_w, \omega^2 \tilde{d}_w)\\
 & (\omega^2 \tilde{d}_1, \tilde{d}_1, \omega \tilde{d}_1), (\omega^2 \tilde{d}_2, \tilde{d}_2, \omega \tilde{d}_2), \dots (\omega^2 \tilde{d}_w, \tilde{d}_w, \omega \tilde{d}_w) \text{ for some } w \ge 0,\\
(iv) & (\hat{d}_1, \omega^2 \hat{d}_1, \omega \hat{d}_1), (\hat{d}_2, \omega^2 \hat{d}_2, \omega \hat{d}_2), \dots, (\hat{d}_u, \omega^2 \hat{d}_u, \omega \hat{d}_u)\\
 & (\omega \hat{d}_1, \hat{d}_1, \omega^2 \hat{d}_1), (\omega \hat{d}_2, \hat{d}_2, \omega^2 \hat{d}_2), \dots, (\omega \hat{d}_u, \hat{d}_u, \omega^2 \hat{d}_u) \text{ for some } u \ge 0.
\end{array} \tag{23}$$

Since $DA = ABA = AF$, it follows that

$$a_{i,j}(d_i - f_j) = 0,\ i, j = 1, 2, \dots, n, \tag{24}$$

i.e. $a_{i,j} = 0$, unless $d_i = f_j$. Thus, the structure of D, F, and the assumption that the eigenvalues of AB are pairwise distinct, imply that A, B are block diagonal with a similar block structure as that of D, F given by (23), i.e. A and likewise B is the direct sum of r blocks of size 2×2 (as in (23(i)), v blocks of size 2×2 (as in (23(ii)), w blocks of size 3×3 (as in (23(iii)), and u blocks of size 3×3 (as in (23(iv)).

In order for A, B to satisfy $(s,t;1,1,3)$ for all $s,t \in \mathbb{C}$, we need, in particular, that

$$\begin{aligned} 0 &= (sA+tB)^3 - (sA)^3 - (tB)^3 \\ &= s^2t(A^2B + ABA + BA^2) + st^2(AB^2 + BAB + B^2A) \\ &= s^2t(AD + DA + FA) + st^2(DB + FB + BF). \end{aligned} \tag{25}$$

For blocks of type (iii) we then have that A and B are ω-commutative and for blocks of type (iv) we have that A and B are ω^2-commutative. Hence, that $(s,t,1,1,3)$ holds for all $s,t \in \mathbb{C}$ follows directly from the original theorem of Potter.

Thus it remains to consider types (i) and (ii) in (23). We want to show that they cannot occur under our assumptions and for this we make use of the trace conditions.

Note that blocks of type (iii) and (iv) only contribute trivially to the traces under consideration. To see this suppose that ω is primitive root of 1 and that X, Y are ω-commutative, so $XY = \omega YX$. Assume also that j is a positive integer that is not divisible by 3. Then, clearly, $X^jY^j = \omega^{j^2}Y^jX^j$. Considering traces we see that the trace of X^jY^j is zero. Applying this observation for X, Y, ω to A, B, ω, and A, B, ω^2, respectively, the assertion on the blocks of type (iii) and (iv) follows.

Since (5) has to hold for the two possible orders of products, we may assume w.l.o.g. that $\omega = e^{2\pi\imath/3}$ and we introduce $\eta = e^{2\pi\imath/6}$,

$$z_i = \begin{cases} \eta d_i, \ 1 \le i \le r, \\ \omega d_i, \ r+1 \le i \le r+v, \end{cases}$$

and the set $S = \{z_1, \dots, z_{r+v}\}$.

Let j be any positive integer such that $j = 6\ell + 2$ for some integer $\ell \ge 0$. Consider now blocks of A, B corresponding to typical blocks of type (i) in (23). It follows from (24) that they have the form

$$\hat{A}_i = \begin{bmatrix} 0 & a_{i,i+1} \\ a_{i+1,i} & 0 \end{bmatrix}, \hat{B}_i = \begin{bmatrix} 0 & b_{i,i+1} \\ b_{i+1,i} & 0 \end{bmatrix},$$

respectively. From

$$\hat{A}_i\hat{B}_i = \begin{bmatrix} a_{i,i+1}b_{i+1,i} & 0 \\ 0 & a_{i+1,i}b_{i,i+1} \end{bmatrix} = \begin{bmatrix} d_i & 0 \\ 0 & \omega d_i \end{bmatrix},$$

we have

$$b_{i,i+1} = \frac{\omega d_i}{a_{i+1,i}}, \ b_{i+1,i} = \frac{d_i}{a_{i,i+1}}.$$

Hence, for $i = 1, \dots, r$, we have

$$\begin{aligned} \mathrm{tr}(\hat{A}_i^j\hat{B}_i^j) &= \mathrm{tr}(\hat{A}_i^{2(3\ell+1)}\hat{B}_i^{2(3\ell+1)}) \\ &= 2(a_{i,i+1}a_{i+1,i}b_{i,i+1}b_{i+1,i})^{3\ell+1} = 2(\omega d_i^2)^{3\ell+1} \\ &= 2(\eta^2 d_i^2)^{3\ell+1} = 2z_i^{6\ell+2} = 2z_i^j. \end{aligned}$$

For typical blocks of type (ii) an analogous computation (replacing ω by ω^2) yields for $i = r+1, \dots, r+v$ the traces

$$\operatorname{tr}(\hat{A}_i^j \hat{B}_i^j) = 2(\omega^2 d_i^2)^{3\ell+1} = 2z_i^{6\ell+2} = 2z_i^j.$$

For $j = 2, 8, 14, 20, \dots,$ (actually we only need a finite number of these) then the trace conditions are

$$\sum_{i=1}^{r+v} z_i^j = 0,$$

i.e. we have a system

$$M\hat{z} := \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ z_1^6 & z_2^6 & \dots & z_{r+v-1}^6 & z_{r+v}^6 \\ z_1^{12} & z_2^{12} & \dots & z_{r+v-1}^{12} & z_{r+v}^{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} z_1^2 \\ z_2^2 \\ \vdots \\ z_{r+v-1}^2 \\ z_{r+v}^2 \end{bmatrix} = 0. \tag{26}$$

Note that the matrix M may have equal columns. In order to deal with this situation, we define an equivalence relation $\sim$ on S by setting $z_i \sim z_k$ if and only if $z_i^6 = z_k^6$, which is the case exactly if $d_i^6 = d_k^6$. If we keep only one column for each equivalence class and sufficiently many rows to get a square matrix then this is a Vandermonde matrix. Denoting the equivalence classes in S under this equivalence relation by S_i, we have that for each S_i, $\sum z_k^2 = 0$, where we only sum over the pairwise different elements. We denote this by

$$\sum_{S_i} z_k^2 = 0. \tag{27}$$

Now let $j = 6\ell + 1$ for some integer $\ell \geq 0$. Then for $i = 1, \dots r$ we have

$$\begin{aligned} \operatorname{tr}(\hat{A}_i^j \hat{B}_i^j) &= \operatorname{tr}(\hat{A}_i \hat{A}_i^{6\ell} \hat{B}_i^{6\ell} \hat{B}_i) \\ &= (a_{i,i+1} a_{i+1,i} b_{i,i+1} b_{i+1,i})^{3\ell} \operatorname{tr}(\operatorname{diag}(d_i, \omega d_i)) = (\omega d_i^2)^{3\ell}(1+\omega) d_i \\ &= -\omega^2 (\eta^2 d_i^2)^{3\ell} d_i = -\eta^4 (z_i^{6\ell} d_i) = z_i^j. \end{aligned}$$

For typical blocks of type (ii), i.e. $i = r+1, \dots, r+v$, an analogous computation yields that

$$\operatorname{tr}(\hat{A}_i^j \hat{B}_i^j) = (\omega^2 d_i^2)^{3\ell}(1+\omega^2) d_i = -\omega z_i^{6\ell} d_i = -z_i^j.$$

For $j = 1, 7, 13, 19, \dots$ the trace conditions are then

$$\sum_{i=1}^{r} z_i^j - \sum_{i=r+1}^{r+v} z_i^j = 0, \tag{28}$$

which gives a system $M\tilde{z} = 0$, with M as in (26) and

$$\tilde{z} = \left[\ z_1, z_2, \dots, z_r, -z_{r+1}, -z_{r+2}, \dots, -z_{r+v}\ \right]^T.$$

Proceeding analogously as before and introducing for each i subclasses $S_{i,1} = \{z_k : z_k \sim z_i,\ 1 \leq k \leq r\}$, $S_{i,2} = \{z_k : z_k \sim z_i,\ r+1 \leq k \leq r+v\}$, we obtain that for each S_i

$$\sum_{S_{i,1}} z_k = \sum_{S_{i,2}} z_k. \tag{29}$$

We will now show that for $r + v > 0$ we obtain a contradiction. W.l.o.g. we may assume that $r > 0$. Recall that for any $1 \leq i \leq r$, together with d_i also ωd_i is an eigenvalue of D and for any $r+1 \leq i \leq r+v$ together with d_i also $\omega^2 d_i$ is an eigenvalue of D. Note further that the numbers $d_1, \ldots, d_{r+v}$ are pairwise distinct and nonzero, but this does not have to be the case for $z_1, z_2, \ldots, z_{r+v}$. For the equivalence class associated with z_1, we have

$$S_1 = \{\eta^\ell z_1, 0 \leq \ell \leq 5\} \cap S,$$

and we see that $S_{1,1}$ (which is nonempty, since $z_1 \in S_{1,1}$) can contain at most 2 elements, one of the set $\{z_1, \omega z_1, \omega^2 z_1\}$ and one of $\{\eta z_1, -z_1, \eta^5 z_1\}$. The same holds for $S_{1,2}$.

If $S_{1,1}$ contains only one element, then this has to be z_1. By (29) it then follows that $S_{1,2} = \{z_1\}$ but then (27) is violated. If $S_{1,1}$ contains two elements, then we have 3 cases. If $S_{1,1} = \{z_1, -z_1\}$, then $S_{1,2} = \{\omega^\ell z_1, -\omega^\ell z_1\}$ for some $0 \leq \ell \leq 2$. But then

$$\sum_{S_1} z_k^2 = 2(1 + \omega^{2\ell}) z_1^2 \neq 0$$

gives a contradiction to (27). If $S_{1,1} = \{z_1, \eta z_1\}$, then (29) implies that $S_{1,2} = S_{1,1}$, but then (27) is not satisfied. The contradiction for the case that $S_{1,1} = \{z_1, \eta^5 z_1\}$ is obtained analogously.

Case 2: If AB is singular, then there can be only one d_i that is 0, w.l.o.g let this be d_n. Then (22) implies that $\phi(n) = n$ or $f_n = d_n = 0$. From (24) we obtain $a_{j,n} = a_{n,j} = 0$ for all $1 \leq j \leq n-1$ and by symmetry of the roles of A, B, also $b_{j,n} = b_{n,j} = 0$ for all $1 \leq j \leq n-1$. Then A and B are (after an appropriate permutation) direct sums of an $n-1 \times n-1$ matrix and a 1×1 zero matrix. For the $n-1 \times n-1$ matrix Case 1 can be applied. □

Remark 6 It should be noted that in the proof of Theorem 5 only a finite number of trace conditions have been used. It is, however, not clear what the minimal number of trace conditions is.

One could now ask whether if A, B satisfy (5) then already A, B are ω-commutative. The following example shows that this is not the case even for $q = 3$.

Example 7 Let

$$\hat{A} = \begin{bmatrix} 0 & 0 & \hat{a}_{1,3} \\ \hat{a}_{2,1} & 0 & 0 \\ 0 & \hat{a}_{3,2} & 0 \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 & \frac{\omega}{\hat{a}_{2,1}} & 0 \\ 0 & 0 & \frac{\omega^2}{\hat{a}_{3,2}} \\ \frac{1}{\hat{a}_{1,3}} & 0 & 0 \end{bmatrix},$$

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \tilde{a}_{1,3} \\ \tilde{a}_{2,1} & 0 & 0 \\ 0 & \tilde{a}_{3,2} & 0 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0 & \frac{\omega^2}{\tilde{a}_{2,1}} & 0 \\ 0 & 0 & \frac{\omega}{\tilde{a}_{3,2}} \\ \frac{1}{\tilde{a}_{1,3}} & 0 & 0 \end{bmatrix},$$

and set

$$A = \mathrm{diag}(\hat{A}, \tilde{A}), \;\; B = \mathrm{diag}(\hat{B}, \tilde{B}).$$

Then due to (5) and the fact that AB is diagonal, $(AB - \omega BA)(AB - \omega^2 BA) = (AB - \omega^2 BA)(AB - \omega BA) = 0$, while $AB - \omega BA \neq 0$ and $AB - \omega^2 BA \neq 0$. An analogous example can be constructed for every prime q.

3 The case $q > 3$.

For $q > 3$ the situation becomes rather complicated and technical and we do not have proofs to analogous theorems such as Theorem 4 or Theorem 5.

The first obvious question is the following.

Question: Let $q > 3$ be a prime and suppose that $A, B \in \mathbb{C}^{n,n}$ satisfy (5). Does this imply (3) for $j, \ell = 1$ and for all $s, t \in \mathbb{C}$? Furthermore, is it then true that for all $j, \ell \in \mathbb{N}$ with j and q as well as ℓ and q relatively prime, the formula (5) holds with A^j replacing A and B^ℓ replacing B.

The answer to both parts of the question is negative as the following example shows.

Example 8 Consider the following matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{15} \\ a_{21} & 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix}, \;\; B = \begin{bmatrix} 0 & b_{12} & 0 & 0 & 0 \\ 0 & 0 & b_{23} & 0 & 0 \\ 0 & 0 & 0 & b_{34} & 0 \\ 0 & 0 & 0 & 0 & b_{45} \\ b_{51} & 0 & 0 & 0 & 0 \end{bmatrix}$$

and set as before $D = \mathrm{diag}(d_1, \ldots, d_5) = AB$ and $F = \mathrm{diag}(f_1, \ldots, f_5) = BA$.

Let now $d_2 = \omega d_1$, $d_3 = \omega^3 d_1$, $d_4 = \omega^2 d_1$, and $d_5 = \omega^4 d_1$, where ω is a primitive 5th root of unity, and choose for simplicity $d_1 = 1$. Then $D = AB = \mathrm{diag}(1, \omega, \omega^3, \omega^2, \omega^4)$ and a simple calculation yields that $F = BA = \mathrm{diag}(\omega, \omega^3, \omega^2, \omega^4, 1)$ which is associated with the permutation $\sigma = (2, 3, 4, 5, 1)$. A simple calculation shows that (5) holds for all permutations. Actually it is enough to check one permutation, since D and F commute.

Considering now (3) for $q = 5$, we first consider the coefficient of $s^4 t$ which is

$$A^4 B + A^3 BA + A^2 BA^2 + ABA^3 + BA^4,$$

and we show that this term is equal to 0.

Due to the special structure of A, B, since AB is diagonal, it follows that the graph structure of each of the summands is the same and equal to that of A^3, so only the elements in positions $(1,3), (2,4), (3,5), (4,1), (5,2)$ do not vanish automatically. We will show that these vanish as well.

The element in position $(1,3)$ is

$$\begin{aligned} & a_{15}a_{54}a_{43}a_{32}b_{23} + a_{15}a_{54}a_{43}b_{34}a_{43} + a_{15}a_{54}b_{45}a_{54}a_{43} \\ & + a_{15}b_{51}a_{15}a_{54}a_{43} + b_{12}a_{21}a_{15}a_{54}a_{43} \\ & = a_{15}a_{54}a_{43}(d_1 + d_2 + d_3 + d_4 + d_5) = 0, \end{aligned}$$

since $d_1 + d_2 + d_3 + d_4 + d_5 = 0$. The proof for the other positions is analogous.

Now we consider the coefficient of s^3t^2, which is

$$\begin{aligned} &A^3B^2 + A^2BAB + A^2B^2A + ABA^2B + ABABA + AB^2A^2 \\ &\quad + BA^3B + BA^2BA + BABA^2 + B^2A^3 \end{aligned}$$

and here the element in position $(1,5)$ is

$$\begin{aligned} &a_{15}a_{54}a_{43}b_{34}b_{45} + a_{15}a_{54}b_{45}a_{54}b_{45} + a_{15}a_{54}b_{45}b_{51}a_{15} + a_{15}b_{51}a_{15}a_{54}b_{45} \\ &+a_{15}b_{51}a_{15}b_{51}a_{15} + a_{15}b_{51}b_{12}a_{21}a_{15} + b_{12}a_{21}a_{15}a_{54}b_{45} + b_{12}a_{21}a_{15}b_{51}a_{15} \\ &+b_{12}a_{21}b_{12}a_{21}a_{15} + b_{12}b_{23}a_{32}a_{21}a_{15} \\ &= a_{15}[d_4d_5 + d_5^2 + 2d_1d_5 + d_1^2 + 2d_1d_2 + d_2d_5 + d_2^2 + d_2d_3] \\ &= a_{15}[\omega + \omega^3 + 2\omega^4 + 1 + 2\omega + 1 + \omega^2 + \omega^4] \\ &= a_{15}[2 + 3\omega + \omega^2 + \omega^3 + 3\omega^4] \neq 0. \end{aligned}$$

It thus follows that (3) does not hold for $j, \ell = 1$ and for all $s, t \in \mathbb{C}$, i.e. we have given an example where A, B are not ω-commutative, (5) holds for all permutations, while (3) does not hold for all $s, t \in \mathbb{C}$.

This example also gives a counterexample to the other question, since if we replace (A, B) by (A^2, B) in the product formula (5), then we have

$$\begin{aligned} G &= A^2B = AD = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{15}\omega^4 \\ a_{21} & 0 & 0 & 0 & 0 \\ 0 & a_{32}\omega & 0 & 0 & 0 \\ 0 & 0 & a_{43}\omega^3 & 0 & 0 \\ 0 & 0 & 0 & a_{54}\omega^2 & 0 \end{bmatrix}, \\ H &= BA^2 = FA = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{15}\omega \\ a_{21}\omega^3 & 0 & 0 & 0 & 0 \\ 0 & a_{32}\omega^2 & 0 & 0 & 0 \\ 0 & 0 & a_{43}\omega^4 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix}. \end{aligned}$$

In (5) we then obtain

$$\begin{aligned} &(G - \omega H)(G - \omega^2 H)(G - \omega^3 H)(G - \omega^4 H) \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & a_{15}(\omega^4 - \omega^2) \\ a_{21}(1 - \omega^4) & 0 & 0 & 0 & 0 \\ 0 & a_{32}(\omega - \omega^3) & 0 & 0 & 0 \\ 0 & 0 & a_{43}(\omega^3 - 1) & 0 & 0 \\ 0 & 0 & 0 & a_{54}(\omega^2 - \omega) & 0 \end{bmatrix} \times \\ &\begin{bmatrix} 0 & 0 & 0 & 0 & a_{15}(\omega^4 - \omega^3) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32}(\omega - \omega^4) & 0 & 0 & 0 \\ 0 & 0 & a_{43}(\omega^3 - \omega) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \end{aligned}$$

$$\times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{21}(1-\omega) & 0 & 0 & 0 & 0 \\ 0 & a_{32}(\omega-1) & 0 & 0 & 0 \\ 0 & 0 & a_{43}(\omega^3-\omega^2) & 0 & 0 \\ 0 & 0 & 0 & a_{54}(\omega^2-\omega^3) & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & a_{15}(\omega^4-1) \\ a_{21}(1-\omega^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54}(\omega^2-\omega^4) & 0 \end{bmatrix}.$$

The $(5,1)$ element of the product is

$$a_{54}a_{43}a_{32}a_{21}(\omega-1)(1-\omega^2)(\omega^2-\omega)(\omega^3-\omega)$$

and hence nonzero.

This example shows that the case that $q > 3$ is very difficult and we pose it as an open question to characterize the relationship between the product form (5) of the ω-commutativity and the formula (3).

4 Conclusion

We have discussed the relationship between the product version $\Pi_{i=1}^{q-1}(AB - \omega^{\sigma(i)}BA) = 0$, of the ω-commutativity condition and the condition $\left((sA)^j + (tB)^\ell\right)^q = (sA)^{jq} + (tB)^{\ell q}$ for some (appropriate) integers j, ℓ and for all $s, t \in \mathbb{C}$, where ω is a primitive qth root of unity. We have (except for some generic condition) characterized the case $q = 3$ and indicated by examples that the case of a prime $q > 3$ presents a real challenge.

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