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Volker Mehrmann Manideepa Saha

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The Frobenius-Jordan form of nonnegative matrices

Volker Mehrmann[†] Manideepa Saha[*]

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Abstract

In this paper we use preferred and quasi-preferred bases of generalized eigenspaces associated with the spectral radius of nonnegative matrices to analyze the existence and uniqueness of a variant of the Jordan canonical form which we call *Frobenius-Jordan form*. It is a combination of the classical Jordan canonical form in the part associated with the eigenvalues that are different from the spectral radius, while it is like the Frobenius normal form in the part associated with the spectral radius. Based on the Frobenius-Jordan form, spectral and combinatorial properties of nonnegative matrices are discussed. In particular, we analyze the existence of nonnegative graph representations of the generalized eigenspace associated with the spectral radius.

Keywords: Preferred basis, quasi-preferred basis, Frobenius-Jordan form
AMS Subject Classification: 15A48, 15A21, 15A18, 05C50

In memoriam Michael Neumann and Uriel Rothblum

1 Introduction

In this paper we discuss several open questions concerning the relation between the spectral and combinatorial properties of nonnegative matrices. It is well-known that for the invariant subspace associated with the spectral radius of a nonnegative matrix there exist several types

[*]Dept. of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam, India, 781039. saha@math.tu-berlin.de, manideepa@iitg.ernet.in . This author was supported by Einstein Foundation Berlin, IMU Berlin Einstein Foundation Program.

[†]Institut für Mathematik, MA 4-5, TU Berlin, Strasse de 17. Juni 136, D-10623 Berlin, Germany. mehrmann@math.tu-berlin.de .

of nonnegative bases which have a nice combinatorial structure, see [2, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17], called *preferred basis and quasi-preferred basis.* In this paper we use these bases to analyze the existence and uniqueness of a variant of the Jordan canonical form named *Frobenius-Jordan form* which is a combination of the classical Jordan canonical form [5] in the part associated with the eigenvalues that are different from the spectral radius, while it is like the Frobenius normal form [4] in the part associated with the spectral radius.

The paper proceeds as follows. Section 2 contains some notation and preliminary results mostly introduced in [8]. In Section 3 we introduce the *Frobenius-Jordan form* of a matrix and show the existence and uniqueness (up to similarity transformation) of such a form for nonnegative matrices. Furthermore, we investigate some graph theoretical properties of nonnegative matrices with the help of the Frobenius-Jordan form. In Section 4 we consider the special so called graph-representations of nonnegative bases for nonnegative matrices. We derive necessary conditions for the existence of such graph bases and show that they need not always exist. We conclude with a summary and some further open questions.

2 Notation and Preliminaries

This section contains the basic notation that is used in this paper and some preliminary results, mostly from [8]. We denote the set $\{1, 2, \ldots, n\}$ by $< n >$. For a real $n \times m$ matrix $A = [a_{i,j}]$ and an n-vector $x = [x_i]$, we use the following terminology and notation.

- $A \geqslant 0$ (A is nonnegative) if $a_{i,j} \geqslant 0$, for all $i \in < n >$, $j \in < m >$.
- $A > 0$ (A is semipositive) if $A \geqslant 0$ and $A \neq 0$.
- $A \gg 0$ (A is strictly positive) if $a_{i,j} > 0$, for all $i \in < n >$, $j \in < m >$.

For a real $n \times n$ matrix $A = [a_{i,j}]$ we denote

- by $\sigma(A)$ the spectrum of A;
- by $\rho(A) = \max_{\lambda \in \sigma(A)}\{|\lambda|\}$, the spectral radius of A;
- by $N(A)$ the nullspace of A, and by $n(A)$ the nullity of A, i. e., its dimension;
- by $\mathrm{ind}_\lambda(A)$ the size of the largest Jordan block associated with the eigenvalue λ;
- by $E_\lambda(A)$, the generalized eigenspace of A corresponding to the eigenvalue λ, i. e., $N((\lambda I - A)^n)$.

Definition 2.1 *An $n \times n$ matrix A is said to be* reducible *if there exists a permutation matrix Π such that*

$$\Pi A \Pi^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{2.1}$$

where B and D are square matrices or in the case that $n = 1$ and $A = 0$. Otherwise A is called irreducible.

If A is a reducible and in the form (2.1), and if a diagonal block is reducible, then this block can be reduced further via permutation similarity. If this process is continued, then finally there exists a suitable permutation matrix Π such that $\Pi A \Pi^T$ is in block triangular from

$$\Pi A \Pi^T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ 0 & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{pp} \end{bmatrix}, \tag{2.2}$$

where each block $A_{i,i}$ is square and is either irreducible or a 1×1 null matrix. This block triangular form is called the *Frobenius normal form* of A. An irreducible matrix consists of one block in the Frobenius normal form.

If $A = [A_{i,j}]$ is an $n \times n$ nonnegative matrix in Frobenius normal form with p block rows and columns, and when discussing matrix-vector multiplication with A or the structure of eigenvectors of A, we partition vectors b analogously in p vector components b_i conformably with A, and we define the *(block) support* of b via $\text{blocksupp}(b) = \{i \in < p >: b_i \neq 0\}$.

To the Frobenius normal form (2.2) of A we associate the directed *reduced graph* $R(A)$ of A with p vertices, and a directed edge from i to j if and only if $A_{i,j} \neq 0$. Note that due to the block triangular structure of A, $R(A)$ may contain loops but no other cycles.

Definition 2.2 *Let A be an $n \times n$ matrix in Frobenius normal form (2.2). For any two vertices i and j in $R(A)$ we say that j has access to i if $j = i$ or if there is a path in the reduced graph from j to i. In this case we write $j \to i$. Otherwise, we write $j \nrightarrow i$.*

For be a set W of vertices in the vertex set $V(A)$ of $R(A)$ we introduce the following sets.

$$\begin{aligned}
\text{below}(W) &= \{i \in V(A) : \text{ there exists } j \in W \text{ such that } i \to j\}; \\
\text{above}(W) &= \{i \in V(A) : \text{ there exists } j \in W \text{ such that } j \to i\}; \\
\text{top}(W) &= \{i \in W : \text{ there exists } j \in W, \text{ such that } i \to j \text{ implies } i = j\}; \\
\text{bottom}(W) &= \{i \in W : \text{ there exists } j \in W, \text{ such that } j \to i \text{ implies } i = j\}.
\end{aligned}$$

Important objects that we will use in this paper to combine the spectral and combinatorial structure of nonnegative matrices are the level and height characteristics.

Definition 2.3 *Let A be an $n \times n$ matrix in Frobenius normal form (2.2).*

(i) *A vertex i in $R(A)$ is called a* singular vertex *(or a basis vertex of $\rho(A)I - A$) if $A_{i,i}$ is singular. We denote the set of all singular vertices of $R(A)$ by $H(A)$. We define the* singular graph *$S(A)$ associated with $R(A)$ as the graph with vertex set $H(A)$ and (i, j) is an edge if and only if $i = j$ or there is a path from i to j.*

(ii) *The* level of a singular vertex *i in $R(A)$, denoted by* level(i), *is the maximal number of singular vertices on a path in $R(A)$ that terminates at i.*

(iii) *Let x be a block-vector with q blocks, partitioned according to the Frobenius normal form of A. The* level of *x, denoted by* level(x), *is defined to be* $\max\{\text{level}(i) : i \in \text{blocksupp}(x)\}$.

(iv) *For a vector $x \neq 0$ in the eigenspace $E_0(A)$, we define the* height of *x, denoted by* height(x), *to be the minimal nonnegative integer k such that $A^k x = 0$.*

Let m be the maximal level of a singular vertex in $R(A)$. The level characteristic *$\lambda(A)$ of A is defined to be the tuple $(\lambda_1, \dots, \lambda_m)$, where λ_k is the number of singular vertices of $R(A)$ of level k.*

If A is singular, then let t be the maximal positive integer such that $n(A^t) > n(A^{t-1})$. We define the height characteristic *$\eta(A)$ of A to be the tuple $(\eta_1, \dots, \eta_t)$, where $\eta_k = n(A^k) - n(A^{k-1})$.*

The other essential objects in our analysis are appropriately chosen sets of basis vectors for the eigenspace associated with the spectral radius.

Definition 2.4 *Let A be a square matrix in Frobenius normal form (2.2), and let $H(A) = \{\alpha_1, \dots, \alpha_q\}$, with $\alpha_1 < \dots < \alpha_q$ be the set of singular vertices in $R(A)$.*

A set of vectors $x^1 = [x^1_i], \dots, x^q = [x^q_i] \geqslant 0$ is called a quasi-preferred set *for A if*

$$x^i_j \gg 0 \text{ if } j \to \alpha_i, \text{ and } x^i_j = 0 \text{ if } j \not\to \alpha_i$$

for all $i = 1, \dots, q$ and $j = 1, \dots, p$.

If in addition we have

$$-Ax^i = \sum_{k=1}^{q} c_{i,k} x^k, \ i = 1, \dots, q,$$

where the $c_{i,k}$ satisfy

$$c_{i,k} > 0 \text{ if } \alpha_i \to \alpha_k, i \neq k; \text{ and } c_{i,k} = 0 \text{ if } \alpha_k \not\to \alpha_i \text{ or } i = k$$

for all $i, k = 1, \dots, q$, then the set of vectors $x^1, \dots, x^q$ is said to be a preferred set *for A.*

A (quasi-)preferred set that forms a basis for $E_0(A)$ is called (quasi-)preferred basis *for A.*

In the following we often make use of the interplay of a nonnegative matrix A and the M-matrix $M = \rho(A)I - A$.

Definition 2.5 *An $n \times n$ matrix M is called an* M-matrix *if it can be written as $M = sI - A$, where $A \geqslant 0$ and $s \geqslant \rho(A)$.*

The following results are well-known.

Theorem 2.6 *[8] Let M be an M-matrix. If x is a nonnegative vector in $E_0(M)$, then* $\mathrm{height}(x) = \mathrm{level}(x)$.

Theorem 2.7 *[8] (Preferred Basis Theorem) If M is an M-matrix, then there exists a nonnegative preferred basis for the generalized eigenspace $E_0(M)$ of M.*

After having introduced the basic concepts, in the next section we introduce the Frobenius-Jordan form of a nonnegative matrix.

3 The Frobenius Jordan Form of a Nonnegative matrix

In this section we prove the existence of a Frobenius-Jordan form for a nonnegative matrix and discuss the combinatorial properties.

Theorem 3.1 *Let A be an $n \times n$ nonnegative matrix with the spectral radius ρ. Then there exists a nonsingular matrix $T = [T_1\ T_2]$ such that the columns of T_1 form a quasi-preferred basis of $E_\rho(A)$ and such that*

$$T^{-1}AT = \begin{bmatrix} Z_F & Z_{FJ} \\ 0 & Z_J \end{bmatrix} = Z, \tag{3.1}$$

where $AT_1 = T_1 Z_F$, Z_F is nonnegative, in block upper-triangular form

$$Z_F = \begin{bmatrix} \rho I_{n_1} & Z_{1,2} & \dots & Z_{1,t} \\ 0 & \rho I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & Z_{t-1,t} \\ 0 & \dots & \dots & \rho I_{n_t} \end{bmatrix}, \tag{3.2}$$

$\sigma(Z_F) = \{\rho\}$, $\rho \notin \sigma(Z_J)$, and Z_J is in Jordan canonical form. If, furthermore, for $j = 1, \dots, t-1$, none of the blocks $Z_{j,j+1}$ has a zero column, then the block-sizes $n_1, \dots, n_t$ are invariants.

Proof. Consider the M-matrix $M = \rho I - A$. Without loss of generality, we may assume that M is in Frobenius normal form (2.2), and let $\alpha_1 < \alpha_2 < \ldots < \alpha_q$ be the singular vertices of M. Since M is an M-matrix, by Theorem 2.7 it follows that M has a preferred basis $\{x^1, x^2, \ldots, x^q\}$ for the generalized eigenspace $E_0(M)$, with $Mx^i = -\sum_{k=1}^{q} \hat{c}_{k,i} x^k$, so that

$$Ax^i = \rho x^i + \sum_{\substack{k=1 \\ k \neq i}}^{q} \hat{c}_{k,i} x^k, \tag{3.3}$$

where the $\hat{c}_{k,i}$ satisfy $\hat{c}_{k,i} > 0$ if $\alpha_k \to \alpha_i$, and $\hat{c}_{k,i} = 0$ if $\alpha_k \nrightarrow \alpha_i$ for $i, k \in \{1, \ldots, q\}$, $i \neq k$. If we set $\hat{T}_1 = [x^1, \ldots, x^q]$, then equation (3.3) implies that $A\hat{T}_1 = \hat{T}_1 \hat{C}$ with

$$\hat{C} = \begin{bmatrix} \rho & \hat{c}_{12} & \ldots & \hat{c}_{1q} \\ 0 & \rho & \ldots & \hat{c}_{2q} \\ \vdots & \vdots & \ddots & \hat{c}_{q-1,q} \\ 0 & 0 & \ldots & \rho \end{bmatrix},$$

nonnegative and we can determine a permutation matrix Π_1 such that $AT_1 = T_1 Z_F$, with $T_1 = \hat{T}_1 \Pi_1$ and $Z_F = \Pi_1^T \hat{C} \Pi$ is as in (3.2). Since the columns of T_1 are linearly independent, we can extend them to a basis of the space $T^{'} = [T_1 \ T_1^{'}]$ and, thus, we obtain

$$AT^{'} = T^{'} \begin{bmatrix} Z_F & Z_{12} \\ 0 & Z_2 \end{bmatrix}$$

for some matrices Z_{12} and Z_2. Let V_2 be a nonsingular matrix such that $V_2^{-1} Z_2 V_2 = Z_J$ is in Jordan canonical form and consider the matrix

$$T = T^{'} \mathrm{diag}(I, V_2) = [T_1 \ T_2].$$

Then $T^{-1}AT$ is as in (3.1) and the fact that $\rho \notin \sigma(Z_J)$ is clear by construction.

It remains to show that the block-sizes $n_1, \ldots, n_t$ of Z_F are invariant if none of the blocks $Z_{i,i+1}$ in (3.2) has a zero column. Set $m_0 = 0, m_i = n_1 + \ldots + n_i$ and $X^i = [x^{m_{i-1}+1}, \ldots, x^{m_i}]$, for $i = 1, 2, \ldots, t$. Let $(\lambda_1, \ldots, \lambda_\ell)$ be the level characteristics of A, with ℓ being the length of the longest chain in A.

We first prove by induction on i that for $i \in \{1, \ldots, t\}$ we have $\mathrm{height}(x^j) = i$, for all $j \in \{m_{i-1} + 1, \ldots, m_i\}$. For $j \in < n_1 >$, we have $Mx^j = 0$, due to (3.1) and the fact that the columns $x^1, \ldots, x^q$ of T_1 form a quasi-preferred basis for $E_0(M)$. This shows that $\mathrm{height}(x^j) = 1$, for $j \in < n_1 >$.

Now assume that for any i with $i < k \leqslant \ell$, we have $\mathrm{height}(x^j) = i$, for all $j \in \{m_{i-1} + 1, \ldots, m_i\}$. Thus, we have $M^i X^i = 0$ and the columns of $M^{i-1} X^i$ are nonzero.

But then $AT_1 = T_1 Z_F$ implies that $-MX^k = X^1 Z_{1,k} + \ldots + X^{k-1} Z_{k-1,k}$. Multiplying with M^{k-1} and M^{k-2} respectively from the left, we obtain $M^k X^k = 0$ and $M^{k-1} X^k = \left(M^{k-2} X^{k-1}\right) Z_{k-1,k} \neq 0$, see Lemma 19 in [3], as each column of both the nonnegative matrices $Z_{k-1,k}$ and $\pm M^{k-2} X^{k-1}$ is nonzero, since either $M^{k-2} X^{k-1}$ or $-M^{k-2} X^{k-1}$ must be nonnegative. This shows that $\text{height}(x^j) = k$, for all $j \in \{m_{k-1}+1, \ldots, m_k\}$. As a consequence of this and Theorem 2.6, we conclude that for $i \in \{1, \ldots, t\}$, $\text{level}(x^j) = i$ for all $j \in \{m_{i-1}+1, \ldots, m_i\}$. Thus, we have $n_i = \lambda_i$ and $t = \ell$. □

We call a matrix Z as defined in Theorem 3.1 a *Frobenius-Jordan form of* A and Z_F the *leading diagonal block* of this form.

The following example shows that, without further requirements, the Frobenius-Jordan form may not be unique.

Example 3.2 Let

$$A = \left[\begin{array}{cc|cc|cc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right]$$

Then $\rho(A) = 4$ and $E_4(M) = N((4I - A)^3) = \{x = [x_i] \in \mathbb{R}^{6,1} : x_1 = x_2,\ x_3 = x_4,\ x_5 = x_6\}$ with $M = 4I - A$. Consider the quasi-preferred bases spanned by the columns of $X = [x^1\ x^2\ x^3]$ and $Y = [y^1\ y^2\ y^3]$, with $x^1 = [1\ 1\ 0\ 0\ 0\ 0]^T$, $x^2 = [1\ 1\ 1\ 1\ 0\ 0]^T$, $x^3 = [1\ 1\ 1\ 1\ 1\ 1]^T$, and $y^1 = x^1$; $y^2 = 2x^2$; $y^3 = 4x^3$. Then we have

$$AX = X \begin{bmatrix} 4 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} = XZ_{F_1},\ AY = Y \begin{bmatrix} 4 & 4 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix} = YZ_{F_2}.$$

Thus, the leading diagonal block of a Frobenius-Jordan form (and hence the Frobenius-Jordan form) of a nonnegative matrix is not unique.

Our next theorem shows that any two Frobenius-Jordan forms of a matrix A are related by a similarity transformation via a block upper triangular matrix.

Theorem 3.3 *Let*

$$Z_F = \begin{bmatrix} \rho I_{n_1} & Z_{1,2} & \ldots & Z_{1,t} \\ 0 & \rho I_{n_2} & \ldots & Z_{2,t} \\ \vdots & \vdots & \ddots & Z_{t-1,t} \\ 0 & 0 & \ldots & \rho I_{n_t} \end{bmatrix}$$

and

$$\tilde{Z}_F = \begin{bmatrix} \rho I_{n_1} & \tilde{Z}_{1,2} & \dots & \tilde{Z}_{1,t} \\ 0 & \rho I_{n_2} & \dots & \tilde{Z}_{2,t} \\ \vdots & \vdots & \ddots & \tilde{Z}_{t-1,t} \\ 0 & 0 & \dots & \rho I_{n_t} \end{bmatrix}$$

be the leading diagonal blocks corresponding to two Frobenius-Jordan forms of a nonnegative matrix A. Then there exist a block upper triangular matrix $F = [F_{i,j}]$ with the same block structure and the blocks $F_{1,1}, \dots, F_{t,t}$ are diagonal with positive diagonal entries, such that

$$\tilde{Z}_F = F^{-1} Z_F F,$$

where

$$F = \begin{bmatrix} F_{1,1} & F_{1,2} & \dots & F_{1,t} \\ 0 & F_{2,2} & \dots & F_{2,t} \\ \vdots & \vdots & \ddots & F_{t-1,t} \\ 0 & 0 & \dots & F_{t,t} \end{bmatrix} \tag{3.4}$$

In particular, we have $\tilde{Z}_{i,i+1} = F_{i,i}^{-1} Z_{i,i+1} F_{i+1,i+1}$.

Proof. By possibly permuting rows and the corresponding columns of the matrix A we may assume that A has the form

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,t} \\ 0 & A_{2,2} & \dots & A_{2,t} \\ \vdots & \vdots & \ddots & A_{t-1,t} \\ 0 & 0 & \dots & A_{t,t} \end{bmatrix},$$

where for each $i = 1, \dots, t$, $A_{i,i}$ is the submatrix of A associated with the classes of level i.

Let $H(M) = \{\alpha_1, \dots, \alpha_q\}$, with $\alpha_1 < \alpha_2 < \dots < \alpha_q$ be the set of singular vertices in $R(M)$ with $M = \rho I - A$. Let Z_F and $\tilde{Z}_F$ be the leading diagonal blocks of two Frobenius-Jordan forms of A corresponding to the nonsingular matrices $T = [T_1\ T_2]$ and $\tilde{T} = [\tilde{T}_1\ \tilde{T}_2]$, so that the columns of $T_1 = [x^1 \dots x^q]$ and $\tilde{T}_1 = [\tilde{x}^1 \dots \tilde{x}^q]$ both form quasi-preferred bases, respectively, with $AT_1 = T_1 Z_F$ and $A\tilde{T}_1 = \tilde{T}_1 \tilde{Z}_F$. Since both T_1 and $\tilde{T}_1$ are bases for the generalized eigenspace of $M = \rho I - A$, there exists a nonsingular matrix $F \in \mathbb{R}^{q,q}$ such that $\tilde{T}_1 = T_1 F$ with $F = [f_{i,j}]$, where q is the algebraic multiplicity of the eigenvalue 0. Thus, for any $i \in < q >$ we have

$$\tilde{x}^i = f_{1,i} x^1 + f_{2,i} x^2 + \dots + f_{k,i} x^k. \tag{3.5}$$

Let $i \in < q >$ and consider the set $V = \{\alpha_j \in H(M) : f_{j,i} \neq 0\}$. We now show that $V \subseteq \text{below}(\alpha_i)$.

Suppose first that $\alpha_j \in \mathrm{top}(V)$ but $\alpha_j \notin \mathrm{below}(\alpha_i)$. Then $x^i_{\alpha_j} = 0 = \tilde{x}^i_{\alpha_j}$, but $\alpha_j \in \mathrm{top}(V)$ implies that $f_{j,i} \neq 0$, and if $f_{r,i} \neq 0$ and $\alpha_j \to \alpha_r$ (in which case $x^r_{\alpha_j} \gg 0$), then $r = j$.

Thus from equation (3.5) we obtain $\tilde{x}^i_{\alpha_j} = f_{j,i} x^j_{\alpha_j}$ which implies that $f_{j,i} = 0$, which is a contradiction. Hence, we have $\mathrm{top}(V) \subseteq \mathrm{below}(\alpha_i)$.

Suppose next that $\alpha_j \in V \backslash \mathrm{top}(V)$. Then there exists $\alpha_r \in \mathrm{top}(V)$ such that $\alpha_j \to \alpha_r$ and $j \neq r$, which implies that $\alpha_j \in \mathrm{below}(\alpha_i)$, because $\mathrm{top}(V) \subseteq \mathrm{below}(\alpha_i)$. This shows that $V \subseteq \mathrm{below}(\alpha_i)$, i. e., $f_{j,i} = 0$ if $\alpha_j \nrightarrow \alpha_i$.

Since A is in Frobenius normal form with irreducible diagonal blocks, it follows that F can be partitioned into the form (3.4), where each $F_{i,i}$ is corresponding to level i. Since $A\tilde{T}_1 = \tilde{T}_1 \tilde{Z}_F$ and $\tilde{T}_1 = T_1 F$, it follows that $T_1 Z_F F = T_1 F \tilde{Z}_F$, which implies that $Z_F F = F \tilde{Z}_F$. □

One may raise the question whether every possible Jordan form as in (3.1) with a nonnegative basis T_1 stems from a quasi-preferred basis. This is not the case as the following example shows.

Example 3.4 The matrix

$$A = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

has $\rho(A) = 2$. Consider the nonnegative basis of $E_2(A)$ spanned by the columns of $T = [x_1\ x_2]$, with $x_1 = [1\ 1\ 1]^T, x_2 = [2\ 3\ 3]^T$. Then $A[x_1\ x_2] = [x_1\ x_2] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = TZ$, where $Z = 2I$ is the leading block of Frobenius-Jordan form of A, but the columns of T do not form a quasi-preferred basis for A. Note that in this example $\mathrm{ind}_2(A) = 1$.

Since not every nonnegative basis with columns that satisfy condition (3.1) in Theorem 3.1 is a quasi-preferred basis one may ask whether there is a weaker relation.

Example 3.5 The matrix

$$A = \left[\begin{array}{c|cc|c} 2 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

has $\rho(A) = 2$ and $E_2(A) = \{x = [x_i] \in \mathbb{R}^4 : x_2 = x_3\}$.

Consider the nonnegative basis of $E_2(A)$ spanned by the columns of $T = [x^1\ x^2\ x^3]$ with $x^1 = [1\ 0\ 0\ 0]^T$, $x^2 = [1\ 1\ 1\ 0]^T$, $x^3 = [0\ 1\ 1\ 1]^T$. Then,

$$AT = T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} =: TZ_F,$$

where Z_F is the leading diagonal block of a Frobenius-Jordan form of A. Here we have $x_1^3 = 0$ but $1 \to 3$, $x_1^2 > 0$ but $1 \nrightarrow 2$.

In Theorem 3.3 we have shown that Frobenius-Jordan forms of a nonnegative matrix may not be unique, but any two Frobenius-Jordan forms are related via a block-upper triangular similarity transformation with diagonal blocks. In our next theorem we will show that such matrices can be connected by a continuous path.

Let $\mathcal{C}^k([s_0, s_1], \mathbb{R}^{n,n})$ denote the set of k-times continuously differentiable functions from the real interval $[s_0, s_1]$ to $\mathbb{R}^{n,n}$. Then for $k \geqslant 0$ and $s_0 < s_1 < \infty$, two matrix functions $A, B \in \mathcal{C}^k([s_0, s_1], \mathbb{R}^{n,n})$ are called *k-smoothly similar*, if there exists a pointwise nonsingular matrix $T \in \mathcal{C}^k([s_0, s_1], \mathbb{R}^{n,n})$ such that $T^{-1}(s)B(s)T(s) = A(s)$ for all $s \in [s_0, s_1]$. This property is characterized by the following Theorem.

Theorem 3.6 *[18] Let $A \in \mathcal{C}^k([s_0, s_1], \mathbb{R}^{n,n})$ with $k \geqslant 0$ and $s_0 < s_1 < \infty$ and let $\Delta(\lambda, s) = \det(\lambda I - A(s))$ be the characteristic polynomial of the matrix $A(s)$. If the multiplicity m of the distinct roots of $\Delta(\lambda, s)$ remains constant for all $s \in [s_0, s_1]$, then there exists an enumeration $\lambda_1(s), \ldots, \lambda_m(s)$ of the roots of $\Delta(\lambda, s)$ such that $\lambda_j \in \mathcal{C}^k([s_0, s_1], \mathbb{R})$, for $j = 1, \ldots, m$.*

Furthermore, if $A(s)$ is similar to a matrix $J(s)$, in which each eigenvalue $\lambda_j(s)$, $j = 1, \ldots, m$ is in a constant number of Jordan blocks of $J(s)$ with a dimension that is constant in s, then $A(s)$ is k-smoothly similar to $J(s)$.

Theorem 3.7 *All possible Frobenius-Jordan forms of a nonnegative matrix can be connected by a convex combination.*

Proof. Let A be a nonnegative matrix and without loss of generality we assume that A is in Frobenius normal form having p diagonal blocks and algebraic multiplicity q. Let Z_1 and Z_2 be the two Frobenius-Jordan forms of A and let Z_{F_1} and Z_{F_2} be the respective leading diagonal blocks. Then by Theorem 3.3, there is a block upper-triangular invertible matrix F such that $Z_{F_2} = F^{-1} Z_{F1} F$.

Consider now the linear matrix valued function $Z : [0, 1] \to \mathbb{R}^{n,n}$ defined by

$$Z(s) = sZ_{F_1} + (1 - s)Z_{F_2}$$

Observe that $Z(s)$ is similar to Z_{F_1} for all $s \in [0, 1]$. So, if J is the Jordan matrix of Z_{F_1}, i. e., $V^{-1} Z_{F_1} V = J$, then for all $s \in [0, 1]$, $Z(s)$ is similar to the Jordan matrix J, which is independent of s. Thus, by Theorem 3.6, there exists a smooth matrix valued function $U(s)$ satisfying $U^{-1}(s)Z(s)U(s) = J$.

The Sylvester equation $Z_{F_1} F(s) = F(s)Z(s)$ has the general solution $F(s) = VXU^{-1}(s)$, where X is the general solution of the Sylvester equation $JX = XJ$. Then F has the same smoothness as U^{-1} and for all $s \in [0,1]$, with $T(s) = T_1F(s)$, we have $AT(s) = T(s)Z(s)$. □

In order to characterize different Frobenius-Jordan forms of the same matrix A, we study a Frobenius-Jordan where the leading block has the maximal number of nonzeros. We denote this leading block by $Z_{F,\max}$.

Remark 3.8 Since $Z_{F,\max}$ contains the maximal number of nonzero entries, it follows that if we replace Z_{F_1} by $Z_{F,\max}$ in the proof of Theorem 3.7, then we can perform a convex combination between any leading diagonal block and $Z_{F,\max}$ and between any other leading blocks with fewer nonzeros, we have a way to get all possible zeros in the leading diagonals of a Frobenius-Jordan matrices of the nonnegative matrix A.

Corollary 3.9 *Let A be a nonnegative matrix and let T be invertible such that $Z = T^{-1}AT$, as in (3), has a leading block $Z_{F,\max}$ that has a maximal number of nonzeros. Then there is no other Frobenius-Jordan form with leading block Z_F and maximal number of nonzeros but with different positions for the zero elements.*

Proof. Suppose there would be two leading blocks Z_{F_1}, Z_{F_2} of a Frobenius-Jordan form with the same maximal number of nonzeros, but with different zero/nonzero patterns. Connecting Z_{F_1}, Z_{F_2} via a convex combination, implies that at least one zero has to become nonzero, and one nonzero has to become zero, which is not possible for a convex combination of two nonnegative matrices. □

Corollary 3.10 *Let A be a nonnegative matrix and let T be invertible such that $Z = T^{-1}AT$, as in (3), has a leading block $Z_{F,\max}$ that has a maximal number of nonzeros. Then each column of $Z_{F,\max}$ also contains a maximal number of nonzeros.*

Proof. Suppose that there would be a leading block Z_F of a Frobenius-Jordan form which has a column that contains more nonzeros than that of $Z_{F,\max}$. Connecting Z_F and $Z_{F,\max}$ via a convex combination, implies that the corresponding column of the resulting leading block contains more nonzeros than that of $Z_{F,\max}$, whereas other nonzeros of the resulting leading block will be in the same positions as that of in $Z_{F,\max}$. But this contradicts the maximality of $Z_{F,\max}$. □

Example 3.11 In Example 3.2, Z_{F_1} and Z_{F_2} both contain a maximal number of nonzeros and they are not permutationally similar, whereas they are diagonally similar.

Example 3.12 Let

$$A = \left[\begin{array}{cc|ccc|cc|cc} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \end{array}\right].$$

It is easy to check that $\rho(A) = 4$ and for $M = 4I - A$ we have $\mathrm{ind}(M) = 3$, and

$$\begin{aligned} E_4(A) &= N(M^3) \\ &= \{x = [x_i] \in \mathbb{R}^{9,1} \mid x_1 = x_2;\ x_3 = x_5 = x_6;\ 28x_6 - 21x_7 = x_1; \\ &\qquad 4(x_8 - x_9) = 9x_7 - 10x_6\}. \end{aligned}$$

Consider the two preferred bases spanned by the columns of $X = [x^1\ x^2\ x^3\ x^4]$ and $Y = [y^1\ y^2\ y^3\ y^4]$, where

$$\begin{array}{rclrcl} x^1 &=& [7\ 7\ 1\ 1\ 1\ 1\ 1\ 1\ \tfrac{5}{4}]^T, & y^1 &=& [1\ 1\ 1\ 1\ 1\ \tfrac{11}{14}\ 1\ 1\ \tfrac{5}{7}]^T, \\ x^2 &=& [0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0]^T, & y^2 &=& x^2, \\ x^3 &=& [0\ 0\ 0\ 0\ 0\ 21\ 28\ 11\ \tfrac{1}{2}]^T, & y^3 &=& [0\ 0\ 0\ 0\ 0\ \tfrac{3}{4}\ 1\ \tfrac{1}{2}\ \tfrac{1}{8}]^T, \\ x^4 &=& [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1]^T, & y^4 &=& x^4. \end{array}$$

We have $AX = XZ_{F_X}$ and $AY = YZ_{F_Y}$ with

$$Z_{F_X} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 14 & 4 & 0 & 0 \\ \frac{2}{7} & 0 & 4 & 0 \\ \frac{19}{14} & 0 & 77 & 4 \end{bmatrix}, \quad Z_{F_Y} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ \frac{8}{7} & 0 & 4 & 0 \\ \frac{17}{7} & 0 & \frac{11}{4} & 4 \end{bmatrix}.$$

If Z_{F_X} and Z_{F_Y} were diagonally similar and $D = \mathrm{diag}(d_1, d_2, d_3, d_4)$ such that $Z_{F_X} D = D Z_{F_Y}$, then D would have to satisfy the homogeneous linear system $7d_1 = d_2$, $d_1 = 4d_3$, $19d_1 = 34d_4$, $28d_3 = d_4$, which however only has the trivial solution, and, hence, Z_{F_X} and Z_{F_Y} are not diagonally similar.

In our next theorem we show that the subgraph of A corresponding to the leading block of any leading block in a Frobenius-Jordan form is a subgraph of $Z_{F,\max}$. For this we will make use of the following lemma.

Lemma 3.13 *[6] Let $A \in \mathbb{R}^{n,n}$ be in Frobenius normal form (2.2) and let $x \in \mathbb{R}^n$ be partitioned analogously. Then* $\text{blocksupp}(Ax) \subseteq \text{below}(\text{blocksupp}(x))$.

Lemma 3.14 *Let $A \in \mathbb{R}^{n,n}$ be a nonnegative matrix in Frobenius-Jordan form (3.1) with leading block $Z_F = [z_1, \ldots, z_q]$ that corresponds to the quasi-preferred basis spanned by the columns of $T_1 = [x^1, \ \ldots, \ x^q]$. Let $M = \rho I - A$ and let $H(M) = \{\alpha_1, \ldots, \alpha_q\}$, with $\alpha_1 < \ldots < \alpha_q$ be the set of singular vertices in $R(M)$. Then, for any $i \in < q >$,* $\text{blocksupp}(z_i) \subseteq \text{below}(\alpha_i)$.

Proof. Let $Z_F = [z_{i,j}]$. Then $AT_1 = T_1 Z_F$ implies that

$$Ax^i = \rho x^i + \sum_{k=1}^{i-1} z_{k,i} x^k, \ i = 1, \ldots, q. \tag{3.6}$$

We have to show that for every $i \in < q >$ the inclusion $\text{blocksupp}(z_i) \subseteq \text{below}(\alpha_i)$ holds, which is equivalent to $\text{top}(\text{blocksupp}(z_i)) \subseteq \text{below}(\alpha_i)$.

Let $\alpha_k \in \text{top}(\text{blocksupp}(z_i))$. Then (3.6) implies that

$$(Ax^i)_{\alpha_k} = \rho x^i_{\alpha_k} + z_{\alpha_k,i} x^k_{\alpha_k}. \tag{3.7}$$

If $(Ax^i)_{\alpha_k} = \rho x^i_{\alpha_k}$, then $z_{\alpha_k,i} = 0$, which is a contradiction. So we must have that $\alpha_k \in \text{blocksupp}((\rho I - A)x^i)$ and so by Lemma 3.13 we have $\alpha_k \in \text{blocksupp}(x^i)$. Then from the definition of the quasi-preferred basis it follows that $\alpha_k \in \text{below}(\alpha_i)$. □

With this lemma we can now prove the following theorem.

Theorem 3.15 *All possible graphs associated with a leading block of a Frobenius-Jordan form of A are subgraphs of the graph of $Z_{F,\max}$.*

Proof. Consider the M-matrix $M = \rho I - A$. Suppose that $MT_{\max} = T_{\max} Z_{F,\max}$, with $Z_{F,\max} = [\hat{z}_1, \ \ldots, \ \hat{z}_q]$, such that the columns of T_{max} form a quasi-preferred basis. By Theorem 2.7, there exist a preferred basis spanned by the columns of $Y = [y^1, \ \ldots, \ y^q]$ for $E_0(M)$ and let $MY = YZ_F$ with $Z_F = [z_1, \ \ldots, \ z_q]$ be the corresponding part of the Frobenius-Jordan form. Then by definition, for $i \in < q >$ we have $\text{blocksupp}(z_i) = \text{below}(\alpha_i)$. But by Lemma 3.14, $\text{blocksupp}(\hat{z}_i) \subseteq \text{below}(\alpha_i)$ and by Corollary 3.10, we must have $\text{blocksupp}(\hat{z}_i) = \text{below}(\alpha_i)$. □

As a consequence of Theorem 3.15 we have that every leading block with a maximal number of nonzeros is associated with a preferred basis, while all the leading blocks with fewer nonzeros only are related to quasi-preferred bases.

If we perform a convex combination between Z_F and $Z_{F,\max}$ then the sign pattern will become the sign pattern of $Z_{F,\max}$ for any $s > 0$. Since $Z_{F,\max}$ corresponds to a preferred basis, it is likely that for every $s \in (0,1]$ the corresponding basis is a preferred basis. We do not know whether this is true.

In this section we have introduced Frobenius-Jordan forms and analyzed the relationship between different such forms, in the next section we discuss special graph bases for the generalized eigenspaces associated with $\rho(A)$.

4 Nonnegative permuted graph basis for nonnegative matrices

In this section we investigate the existence of a *nonnegative row permuted graph basis* for the generalized eigenspace of a nonnegative matrix. Here we say that T_1 associated with the spectral radius of A is a *permuted graph basis*, [11], if it is of the form $T_1 = \Pi \begin{bmatrix} I \\ Y \end{bmatrix}$ with a permutation matrix Π and nonsingular matrix Y. We call such a basis *nonnegative permuted graph basis* if $Y \geqslant 0$. The following example shows that this does not always exist for every nonnegative matrix.

Example 4.1 Let

$$A = \left[\begin{array}{cc|cc|ccc|cc|cc} 0 & 2 & & & & & & & & & \\ 4 & 0 & & & & & & & & & \\ \hline 0 & 1 & 0 & 4 & & & & & & & \\ 1 & 0 & 4 & 0 & & & & & & & \\ \hline 0 & 0 & 0 & 0 & 2 & 0 & 2 & & & & \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & & & & \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & & & & \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & & \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 2 & 2 \end{array}\right]$$

Then $\rho(A) = 4$ and for $M = 4I - A$ we have $\mathrm{ind}(M) = 2$ so that

$$\begin{aligned} E_4(A) \quad = \quad & \{x = [x_i] \in \mathbb{R}^{11,1} \mid x_1 = x_2 = 0, x_3 = x_4, x_5 = x_6 = x_7, 21x_9 = 28x_8, \\ & 4x_{11} = x_5 + 4x_{10} - 2x_8\} \end{aligned}$$

If x^1, x^2, x^3, x^4 is any quasi-preferred basis for $E_0(M)$, then we have

$$\begin{aligned} x^2 &= [0,\ 0,\ 0,\ 0,\ \xi,\ \xi,\ \xi,\ 0,\ 0\ , \omega,\ \omega + \frac{\xi}{4}]^T, \\ x^3 &= [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \epsilon,\ \frac{21\epsilon}{28},\ \psi + \frac{\epsilon}{2},\ \epsilon]^T, \\ x^4 &= [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \zeta,\ \zeta]^T \end{aligned}$$

with nonnegative $\psi, \epsilon, \zeta, \xi, \omega$. But this implies that $\frac{\zeta}{\epsilon} \geqslant \frac{\zeta}{\psi + \frac{\epsilon}{2}}$ and $\frac{\zeta}{\frac{\xi}{4} + \omega} \leqslant \frac{\zeta}{\omega}$, which are contradicting inequalities. Hence , there does not exist any nonnegative permuted graph basis for A. Note that in this example, the level and height characteristic are different, since $\lambda(M) = (2, 2) \neq \eta(M) = (3, 1)$.

Example 4.2 Consider the matrix

$$A = \left[\begin{array}{c|c|cc} 2 & 0 & 0 & 0 \\ \hline 1 & 2 & 0 & 0 \\ \hline 6 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \end{array}\right],$$

with $\rho(A) = 2$ and for $M = 2I - A$ we have $\mathrm{ind}(M) = 3$, and

$$E_2(A) = \left\{ x = [x_i] \in \mathbb{R}^4 : 3x_1 + x_4 = x_2 + x_3 \right\}.$$

Here again, A does not possess any nonnegative permuted graph basis, whereas level and height characteristic are equal, $\lambda(M) = \eta(M) = (1, 1, 1)$.

To obtain criteria for the existence of nonnegative permuted graph bases we have the following result.

Lemma 4.3 *If a nonnegative matrix A possess a nonnegative permuted graph basis for the generalized eigenspace $E_\rho(A)$, then each block that corresponding to the leading diagonal of a Frobenius-Jordan form of A in the matrix*

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix},$$

partitioned as (2.2) with columns that form a quasi-preferred basis will contribute one row to the identity.

Proof. Without loss of generality we may assume that A is in block lower triangular Frobenius normal form with a spectral radius of algebraic multiplicity q . Since A is a nonnegative matrix, it has a quasi-preferred basis, given by the columns of $X = [x^1, \ldots, x^q]$. Write X as

$$X = \begin{bmatrix} x_1^1 & 0 & \ldots & 0 \\ x_2^1 & x_2^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_p^1 & x_p^2 & \ldots & x_p^q \end{bmatrix}.$$

Suppose that $\beta := \{\beta_1, \ldots, \beta_q\}$ is the set of indices that are associated with the identity, i. e., if $\bar{\beta} =< n > \backslash \beta$, then there exists a permutation $\Pi = [\Pi_\beta^T, \Pi_{\bar{\beta}}^T]^T$ defined by the indices in β and $\bar{\beta}$ such that $X_\beta := \Pi_\beta X$ is invertible, and

$$\Pi X X_\beta^{-1} = \begin{bmatrix} X_\beta \\ X_{\bar{\beta}} \end{bmatrix} X_\beta^{-1} = \begin{bmatrix} I \\ Y \end{bmatrix},$$

with $Y \geqslant 0$.

If the assertion would not hold, then there would exist an index $j \in \langle q \rangle$ such that $j \notin \beta$. Let $\hat{j}$ be the largest such index. Then $\hat{j} \neq q$ because otherwise X_β would have a zero column which is a contradiction. If $\hat{j} < q$, then the $q - \hat{j} \times q - \hat{j}$ submatrix in the lower right corner of X_β is lower triangular with a zero in the first diagonal position and hence again we have a contradiction. As a consequence there does not exist such an index and the proof is complete. □

Lemma 4.3 implies that no block in a Frobenius-Jordan form can contribute more than one row to the identity. Thus the identity cannot be larger than the number of blocks. However, as we have seen, there may not exist a nonnegative permuted graph basis, which means that some blocks do not at all contribute rows to the identity.

However, if each block is to contribute exactly one row to the identity, then we must have the following relation.

Corollary 4.4 *Suppose that A is a nonnegative matrix having a nonnegative permuted graph basis. If it has a quasi-preferred basis $\{x^1, \ldots, x^q\}$ with $x^i = [x_j^i]$ such that there exist unique $k_1, \ldots, k_q$ with $\min_j \frac{(x_i^{i-1})_j}{(x_i^i)_j} = \frac{(x_i^{i-1})_{k_i}}{(x_i^i)_{k_i}}$, then each of $k_1, \ldots, k_q$ will contribute a row to the identity of the nonnegative permuted graph basis.*

Proof. Consider the matrix $X = [x^1 \ \ldots \ x^q]^T$. Thus there exists indices $j_1, \ldots, j_q$ from each block that contribute rows to the identity of the nonnegative permuted graph basis. We now show that for each $i = 1, \ldots, q$

$$\frac{(x_i^{i-1})_{k_i}}{(x_i^i)_{k_i}} \geqslant \frac{(x_i^{i-1})_{j_i}}{(x_i^i)_{j_i}}$$

Then the result will follow from the uniqueness of the k_i. Clearly for each $i = 1, \ldots, q$ both the indices k_i and j_i are from the same block. Since the columns of the matrices X and $\Pi \begin{bmatrix} I \\ Y \end{bmatrix}$ with some nonnegative Y both are bases for the generalized eigenspace, there exists a matrix $B \in \mathbb{R}^{q,q}$ such that $X = \Pi \begin{bmatrix} I \\ Y \end{bmatrix} B$. It can be easily seen that,

$$B = \begin{bmatrix} (x_1^1)_{j_1} & 0 & \ldots & 0 \\ (x_2^1)_{j_2} & (x_2^2)_{j_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(x_q^1\right)_{j_q} & \left(x_q^2\right)_{j_q} & \ldots & \left(x_q^q\right)_{j_q} \end{bmatrix}$$

which implies that for each $i = 1, \ldots, q$ there exist nonnegative scalars (the elements of Y) α_i, β_i such that

$$\begin{aligned} \left(x_i^i\right)_{k_i} &= \alpha_i \left(x_i^i\right)_{j_i} \\ \left(x_i^{i-1}\right)_{k_i} &= \beta_i \left(x_{i-1}^{i-1}\right)_{j_{i-1}} + \alpha_i \left(x_i^{i-1}\right)_{j_i} \end{aligned} \tag{4.1}$$

Since both β_i and $\left(x_{i-1}^{i-1}\right)_{j_{i-1}}$ are nonnegative, the claim follows from the equations in (4.1). □

Corollary 4.4 gives a computational criterion to check the existence of nonnegative permuted graph basis. One computes a preferred bases and checks the inequalities and their uniqueness. If this holds then a nonnegative permuted graph basis exists, if not then it is an open problem to guarantee the existence.

We have seen that not every nonnegative matrix possesses a nonnegative permuted graph basis even though they possess the same level and height characteristic. It is also an open problem to characterize the class of nonnegative matrices that have a nonnegative permuted graph basis.

5 Conclusion

We have presented a variant of the Jordan canonical form for nonnegative matrices and shown the uniqueness of such canonical form up to block triangular similarity transformation. We also studied some graphical properties of nonnegative matrices with the help of this canonical form. We have shown that all such possible canonical forms can be connected by a linear path and that the nonzero pattern of a leading block in the Frobenius-Jordan form is unique. Finally we have presented some necessary conditions for the existence of nonnegative permuted graph basis for nonnegative matrices and we have demonstrated the fact that not every nonnegative matrix has such bases by an example.

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