$\mathcal{H}_\infty$-CONTROL OF DISCRETE-TIME DESCRIPTOR SYSTEMS

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Abstract. We consider the (sub)optimal $\mathcal{H}_\infty$-control problem for discrete time descriptor systems. Necessary and sufficient optimality conditions are derived in terms of deflating subspaces of palindromic matrix pencils. This approach allows the use of structure preserving matrix techniques which lead to a more robust method compared with currently used algorithms. The approach is suitable for standard systems as well as for index one and higher index systems. We illustrate the results by a numerical example.

1. Introduction. The $\mathcal{H}_\infty$-control problem has been a point of research in many publications [15,16,27,36,38]. For standard state space systems, where the dynamics of the system is modeled by a linear constant coefficient ordinary differential equation, the analysis of this problem is well studied [12] and numerical methods have been developed and integrated in control software packages [2, 5, 17, 28]. The standard discrete-time case is considered in [34].

Most of these methods work well for a wide range of problems in computing suboptimal controllers but the exact computation of the optimal value γ in $\mathcal{H}_\infty$ control is usually difficult [10]. In [3,4] several improvements of the previously known methods were presented to avoid some of the numerical difficulties that arise when approaching the optimum. These methods are based on the solution of structured eigenvalue problems with structured methods.

In this paper we consider a more general situation where the dynamics of the system is constrained, i.e., described by differential-algebraic-equations or descriptor systems. Descriptor systems arise in various applications such as mechanical systems [1, 18, 30, 31, 33] and electric circuit simulation [13]. The $\mathcal{H}_\infty$-control problem for continous-time descriptor systems has been studied in [22] and a numerical robust method for the γ-iteration has been proposed. In this paper we focus on the $\mathcal{H}_\infty$-control problem for discrete-time descriptor systems and provide a numerically robust method for the γ-iteration. We see that there are some major differences in the contrast to the continous-time case. These differences are discussed in full detail in Section 5.

We consider systems of the form

$$\begin{aligned} Ex_{k+1} &= \ \ Ax_k + B_1 w_k \ + B_2 u_k, \qquad x_0 = x^0, \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k, \\ y_k &= C_2 x_k + D_{21} w_k + D_{22} u_k, \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i, j = 1, 2$. (Here, by $\mathbb{R}^{k,l}$ we denote the set of real $k \times l$ matrices.) In this system, $\{x_k\} \in \mathbb{R}^n$ is the sequence of state vectors, $\{u_k\} \in \mathbb{R}^{m_2}$ is the sequence of control input vectors, and $\{w_k\} \in \mathbb{R}^{m_1}$ is the sequence of exogenous inputs that may include noise, linearization errors and un-modelled dynamics. The sequence of vectors $\{y_k\} \in \mathbb{R}^{p_2}$ contains measured outputs, while $\{z_k\} \in \mathbb{R}^{p_1}$ describes regulated outputs or estimation errors. To simplify the notation, throughout the paper we will frequently use x_k, u_k, w_k, y_k and z_k instead.

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We define the $\mathcal{H}_\infty$-norm by

$$\|F\|_\infty = \sup_{\theta \in (-\pi,\pi]} \sigma_{\max}(F(e^{j\theta})),$$

where $\sigma_{\max}(F(e^{j\theta}))$ denotes the maximal singular value of the matrix $F(e^{j\theta})$.
In robust control, $\|F\|_\infty$ is used as a measure of the worst case influence of the disturbances w on the output z, where in this case F is the transfer function mapping noise or disturbance inputs to error signals [38].

The optimal $\mathcal{H}_\infty$ control problem is the task of designing a dynamic controller that minimizes (or at least approximately minimizes) this measure.

Definition 1.1 (The Optimal $\mathcal{H}_\infty$ control problem.). *For the descriptor system (1.1), determine a controller (dynamic compensator)*

$$\mathbf{K}: \quad \begin{aligned} E\hat{x}_{k+1} &= \hat{A}\hat{x}_k + \hat{B}y_k, \\ u_k &= \hat{C}\hat{x}_k + \hat{D}y_k, \end{aligned} \tag{1.2}$$

with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$ *and transfer function* $K(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ *such that the closed-loop system resulting from the combination of (1.1) and (1.2), that is given by*

$$\begin{aligned} Ex_{k+1} &= (A + B_2\hat{D}Z_1C_2)x_k + (B_2Z_2\hat{C})\hat{x}_k + (B_1 + B_2\hat{D}Z_1D_{21})w_k, \\ \hat{E}\hat{x}_k &= \hat{B}Z_1C_2x_k + (\hat{A} + \hat{B}Z_1D_{22}\hat{C})\hat{x}_k + \hat{B}Z_1D_{21}w_k, \\ z_k &= (C_1 + D_{12}Z_2\hat{D}C_2)x_k + D_{12}Z_2\hat{C}\hat{x}_k + (D_{11} + D_{12}\hat{D}Z_1D_{21})w_k, \end{aligned} \tag{1.3}$$

with $Z_1 = (I - D_{22}\hat{D})^{-1}$ *and* $Z_2 = (I - \hat{D}D_{22})^{-1}$, *has the following properties.*

1. *System (1.3) is* internally stable, *i.e., the solution* $\begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}$ *of the system with* $w \equiv 0$ *is* asymptotically stable, *i.e.* $\lim_{k\to\infty} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} = 0$.
2. *The closed-loop transfer function* $T_{zw}(s)$ *from* w *to* z *is minimized in the* $\mathcal{H}_\infty$*-norm.*

Since it is in general very difficult to minimize over the complicated set of stabilizing controllers we study two closely related problems, the *modified optimal* $\mathcal{H}_\infty$ *control problem* and the *suboptimal* $\mathcal{H}_\infty$ *control problem.*

Definition 1.2 (The Modified optimal $\mathcal{H}_\infty$ control problem). *For the descriptor system (1.1) let* Γ *be the set of positive real numbers* γ *for which there exists an internally stabilizing dynamic controller of the form (1.2) so that the transfer function* $T_{zw}(s)$ *of the closed loop system (1.3) satisfies* $\|T_{zw}\|_\infty < \gamma$.

In the modified optimal $\mathcal{H}_\infty$ *control problem we want to determine* $\gamma_{mo} = \inf \Gamma$ *and a corresponding controller (1.2) such that* $\|T_{zw}\|_\infty = \gamma_{mo}$.

Since it is in general possible that there does not exist an internally stabilizing dynamic controller with the property that $\|T_{zw}\|_\infty = \gamma_{mo}$, (in this case $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$) one studies the suboptimal $\mathcal{H}_\infty$ control problem.

Definition 1.3 (The Suboptimal $\mathcal{H}_\infty$ control problem). *For the descriptor system (1.1) and* $\gamma \in \Gamma$ *with* $\gamma > \gamma_{mo}$ *determine an internally stabilizing dynamic controller of the form (1.2) such that the closed loop transfer function satisfies* $\|T_{zw}\|_\infty < \gamma$. *We call such a controller* γ-suboptimal controller *or simply* suboptimal controller.

The outline of the paper is as follows:
In the next section we present the notation and several definitions that are used throughout the paper. Then we discuss the arising difficulties of current methods in discrete-time $\mathcal{H}_\infty$ control theory and motivate the use of (structured) matrix pencils in this context. In Section 3 we develop the theoretical results for discrete-time $\mathcal{H}_\infty$ control for descriptor systems using so called BVD-pencils [9] that extend the idea of the extended symplectic pencil (ESP) that is used for the standard case in [19, 25]. We will see that the discrete-time $\mathcal{H}_\infty$ control can be solved by calculating appropriate deflating subspaces of certain matrix pencils similar to the continuous-time case that is treated in [22]. In order to further improve the numerical treatment of this problem and to make use of structure preserving algorithms we reformulate the results in terms of palindromic matrix pencils, since these allow structure preserving numerical calculations. We show different ways how this reformulation can be carried out without losing necessary information on the eigenvalues and deflating subspaces. In the last section we verify our results by a numerical example.

2. Preliminaries. In this section we introduce some notation and definitions. For symmetric matrices A and B, by $A \geq B$ and $A > B$ we denote that $A - B$ is positive semidefinite and positive definite, respectively. The spectral radius of a matrix $A \in \mathbb{R}^{n,n}$ is denoted by $\rho(A)$. The set of complex numbers with positive real part is denoted by $\mathbb{C}^+$ and the set of positive real numbers by $\mathbb{R}^+$.

Let $\lambda E - A$ be a matrix pencil with $E, A \in \mathbb{R}^{n,n}$. Then $\lambda E - A$ is called *regular* if $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. If $\lambda E - A$ is not regular, then it is said to be *singular*. A pencil $P(\lambda) = \lambda E - A$ is called *symplectic* if $EJ_nE^T = AJ_nA^T$ with $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A pencil of the form

$$P(\lambda) = \lambda \begin{bmatrix} 0 & E & 0 \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ E^T & Q & Y \\ 0 & Y^T & R \end{bmatrix}$$

is called BVD-pencil [9]. For regular pencils, *generalized eigenvalues* are the pairs $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for which $\det(\alpha E - \beta A) = 0$. If $\beta \neq 0$, then the pair represents the finite eigenvalue $\lambda = \alpha/\beta$. If $\beta = 0$, then (α, β) represent the eigenvalue infinity.

The solution and many properties of the *free descriptor system* (with $u_k, w_k = 0$, for all k) can be characterized in terms of the *Weierstraß canonical form (WCF).*

THEOREM 2.1. [14] If $\lambda E - A$ is a regular pencil, then there exist nonsingular matrices $W = \begin{bmatrix} W_f & W_\infty \end{bmatrix} \in \mathbb{R}^{n,n}$ and $V = \begin{bmatrix} V_f & V_\infty \end{bmatrix} \in \mathbb{R}^{n,n}$ so that

$$W^TEV = \begin{bmatrix} W_f^T \\ W_\infty^T \end{bmatrix} E \begin{bmatrix} V_f & V_\infty \end{bmatrix} = \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \tag{2.1a}$$

and

$$W^TAV = \begin{bmatrix} W_f^T \\ W_\infty^T \end{bmatrix} A \begin{bmatrix} V_f & V_\infty \end{bmatrix} = \begin{bmatrix} A_f & 0 \\ 0 & I_{n_\infty} \end{bmatrix}, \tag{2.1b}$$

where A_f is a nonsingular matrix in real Jordan canonical form, whose eigenvalues are the finite eigenvalues of the pencil and N is a nilpotent matrix, also in Jordan canonical form. (Here n_f, n_∞ denote the number of finite or infinite eigenvalues, respectively.)

The index of nilpotency of the nilpotent matrix N in (2.1a) is called the *index* of the system and if E is nonsingular, then the pencil is said to have *index zero.*

DEFINITION 2.2. *A subspace $\mathcal{L} \subset \mathbb{R}^n$ is called* deflating subspace *for the pencil $\lambda E - A$ if for a matrix $X_{\mathcal{L}} \in \mathbb{R}^{n,k}$ with full column rank and $\operatorname{Im} X_{\mathcal{L}} = \mathcal{L}$ there exists matrices $Y_{\mathcal{L}} \in \mathbb{R}^{n,k}$, $R_{\mathcal{L}} \in \mathbb{R}^{k,k}$, and $U_{\mathcal{L}} \in \mathbb{R}^{k,k}$ such that*

$$EX_{\mathcal{L}} = Y_{\mathcal{L}} R_{\mathcal{L}}, \quad AX_{\mathcal{L}} = Y_{\mathcal{L}} U_{\mathcal{L}}. \tag{2.2}$$

A deflating subspace $\mathcal{L}$ of $\lambda E - A$ is called stable (semi-stable) *if all finite eigenvalues of $\lambda R_{\mathcal{L}} - U_{\mathcal{L}}$ are in the open (closed) unit disc.*

DEFINITION 2.3. *A subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ is called* Lagrangian *if it has dimension n and if $x^T J_n y = 0$ for all $x, y \in \mathcal{L}$. A subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ is called generalized Lagrangian if it is a subspace of a Lagrangian.* In the notation of (2.1a)–(2.1b) with

$$\begin{aligned} B_{i,f} &= W_f^T B_i, \quad B_{i,\infty} = W_\infty^T B_i, \\ C_{i,f} &= C_i V_f, \qquad C_{i,\infty} = C_i V_\infty, \qquad i = 1, 2, \end{aligned} \tag{2.3}$$

classical solutions of (1.2) take the form

$$x_k = V_f x_{k,f} + V_\infty x_{k,\infty}, \quad x_0 = V_f x_{0,f} + V_\infty x_{0,\infty}$$

where $x_{k,f}$ and $x_{k,\infty}$ satisfy

$$x_{k+1,f} = A_f x_{k,f} + B_{1,f}\, w_k + B_{2,f} u_k, \tag{2.4a}$$

$$N x_{k+1,\infty} = \quad x_{k,\infty} + B_{1,\infty} w_k + B_{2,\infty} u_k \tag{2.4b}$$

for all k. If the pencil $\lambda E - A$ has index ν, then this system has the explicit solution sequences [6]

$$x_{k,f} = A_f^k x_{0,f} + \sum_{i=0}^{k-1} A_f^{k-i-1} \left(B_{1,f} w_i + B_{2,\infty} u_i \right), \tag{2.5a}$$

$$x_{k,\infty} = -\sum_{i=k}^{\nu-1} N^i \left(B_{1,\infty} w_i + B_{2,\infty} u_i \right). \tag{2.5b}$$

In contrast to standard state space systems where $E = I$, this shows that the initial condition $x_\infty(t_0)$ is restricted by (2.5b). Discrete-time descriptor systems may possess noncausal behavior, i.e. the solution may depend on future values of the sequences x_k, u_k, w_k. This corresponds to the concept of impulsive behavior in the continuous-time case.

Note further that for the closed loop system (1.3) to be *internally stable*, the controller has to be designed in such a way that both x_f and x_∞ are asymptotically stable. While for the finite part this can be guaranteed if the spectrum of the matrix A_f lies in the open unit circle, for the infinite part this has to be explicitly ensured by the construction of the controller.

As in the case of standard state space systems certain conditions will be needed to guarantee the existence of optimal $\mathcal{H}_\infty$ controls. Thus we need to define stabilizability, detectability and controllability conditions, for discrete-time descriptor systems.

DEFINITION 2.4. *Let $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and $C \in \mathbb{R}^{p,n}$. Further, let T_∞, S_∞ be matrices with $\operatorname{Im} T_\infty = \ker E^T$ and $\operatorname{Im} S_\infty = \ker E$.*

i) The triple (E, A, B) is called finite dynamics stabilizable *if* $\operatorname{rank}[\lambda E - A, B] = n$ *for all* $|\lambda| \geq 1$*;*
ii) (E, A, B) *is* impulse controllable *if* $\operatorname{rank}[E, AS_\infty, B] = n$*;*
iii) (E, A, B) *is* strongly stabilizable *if it is both finite dynamics stabilizable and impulse controllable;*
iv) The triple (E, A, C) *is* finite dynamics detectable *if* $\operatorname{rank}[\lambda E^T - A^T, C^T] = n$ *for all* $|\lambda| \geq 1$*;*
v) (E, A, C) *is* impulse observable *if* $\operatorname{rank}[E^T, A^T_\infty, C^T] = n$*;*
vi) $(\lambda E - A, C)$ *is* strongly detectable *if it is both is both finite dynamics detectable and impulse observable.*

After introducing the notation and preliminary results, we derive the theoretical basis for the optimal $\mathcal{H}_\infty$ control problem for discrete-time descriptor systems in the next section. We proceed in several steps. First we recall the well known results in discrete-time $\mathcal{H}_\infty$ control for standard systems ($E = I$). These results usually make use of the discrete algebraic Riccati equation (DARE) which can be solved by finding deflating subspaces of a symplectic matrix pencils under strong assumptions on the invertibility of certain matrices. It was noticed in [19] that this problem in solving discrete algebraic Riccati equations can be circumvented by using the so called Extended Symplectic Pencil. We will adapt and slightly modify this approach. As a next step we will extend these results to descriptor systems of index one by making use of the Weierstrass canonical form (WCF) and to higher index systems by using a preliminary output feedback. Since palindromic matrix pencils can be well treated by efficient structure preserving algorithms we will show how these can be used to further enhance the numerical treatment of the discrete $\mathcal{H}_\infty$- control problem.

3. $\mathcal{H}_\infty$-control for discrete-time descriptor systems. In this section we discuss the theoretical background for the modified optimal $\mathcal{H}_\infty$ problem. As in the case of standard state space systems, see [15, 16, 27, 38], we need several assumptions on the system matrices of (1.1).

Assumptions:

$A1)$ The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable, see Definition 2.4.

$A2)$ $\operatorname{rank} \begin{bmatrix} A - e^{j\theta}E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\theta \in [0, 2\pi]$.

$A3)$ $\operatorname{rank} \begin{bmatrix} A - e^{j\theta}E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\theta \in [0, 2\pi]$.

$A4)$ For matrices T_∞, S_∞ with $\operatorname{Im} S_\infty = \ker E$ and $\operatorname{Im} T_\infty = \ker E^T$ the rank conditions

$$\operatorname{rank} \begin{bmatrix} T^T_\infty A S_\infty & T^T_\infty B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} = n + m_2 - \operatorname{rank} E,$$
$$\operatorname{rank} \begin{bmatrix} T^T_\infty A S_\infty & T^T_\infty B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} = n + p_1 - \operatorname{rank} E$$

hold.

Note that assumption A_1 is in fact stronger than necessary, but it rules out the possibility of having non-causal systems. It is actually sufficient to assume finite dynamics stabilizability and the existence of an equivalent system that is controllable at infinity. But for now we restrict ourselves to systems that satisfy Assumption A_1 which is not a restriction in general, since non-causal systems usually only appear if the modeling of the problem is not performed correctly [7]. In $\mathcal{H}_\infty$- control theory we usually consider the following two subsystems of (1.1), see [34]

$$\begin{aligned} Ex_{k+1} &= Ax_k + B_1 w_k + B_2 u_k, \qquad x_0 = x^0, && (3.1a)\\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k, && (3.1b) \end{aligned}$$

and the dual system

$$\begin{aligned} E^T x_{k+1}^T &= A^T x_k^T + C_1^T w_k + C_2^T u_k, \qquad x_0 = x^0, && (3.2a)\\ y_k^T &= B_1 x_k^T + D_{11}^T w_k^T + D_{21}^T u_k^T. && (3.2b) \end{aligned}$$

First we consider subsystem (3.1). Since we want to minimize the influence of the disturbance w_k on the output z_k by using a control input u_k, we minimize the following objective function. Note that u_k does not appear explicitly in the sequence, but z_k depends on u_k.

$$J(\{x_k\},\{w_k\},\{u_k\}) = \frac{1}{2}\sum_{k=0}^{\infty} \|z_k\|^2 - \gamma^2\|w_k\|^2. \tag{3.3}$$

Then (3.3) can be rewritten as

$$J(\{x_k\},\{w_k\},\{u_k\}) = \frac{1}{2}\sum_{k=0}^{\infty} \begin{bmatrix} x_k^T \\ w_k^T \\ u_k^T \end{bmatrix}^T \begin{bmatrix} C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} \\ D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 & D_{11}^T D_{12} \\ D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix}.$$

We define two matrices to describe the ∞-stage descriptor process for (3.1a).

$$M = \begin{bmatrix} A & B_1 & B_2 & -E & & & & & & & \\ & & & A & B_1 & B_2 & -E & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & & A & B_1 & B_2 & -E \end{bmatrix}, \tag{3.4}$$

$$L = \begin{bmatrix} \hat{L} & & & \\ & \hat{L} & & \\ & & \ddots & \\ & & & \hat{L} \end{bmatrix},$$

with

$$\hat{L} = \begin{bmatrix} C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} \\ D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 & D_{11}^T D_{12} \\ D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \tag{3.5}$$

and the infinite variable ξ by

$$\xi^T = [x_0^T, w_0^T, u_0^T, x_1^T, w_1^T, u_1^T, \dots]^T$$

and rewrite (3.1a) and (3.3) as an optimization problem [23]

$$\text{Minimize } \frac{1}{2}\xi^T L \xi, \text{ subject to } M\xi = f. \tag{3.6}$$

Define the Lagrangian of the optimization problem by

$$\mathcal{L}(\xi,\mu) = \frac{1}{2}\xi^T L\xi + \mu^T(M\xi - f), \tag{3.7}$$

where

$$\mu^T = [\mu_1, \mu_2, \mu_3, \ldots]$$

is the infinite vector of the Lagrangian multipliers. Differentiating the Lagrangian (3.7) leads to the following conditions for the optimal control

$$L\xi + \mu M^T\mu = 0, \tag{3.8}$$

$$M\xi = f. \tag{3.9}$$

We introduce the artificial variable $\mu_0 = 0$. Then (3.8) can be rewritten as

$$\begin{aligned}
0 &= C_1^T C_1 x_k + C_1^T D_{11} w_k + C_1 D_{12} u_k + A^T \mu_{k+1} - E^T \mu_k,\\
0 &= D_{11}^T C_1 x_k + (D_{11}^T D_{11} - \gamma^2) w_k + D_{11} D_{12} u_k + B_1^T \mu_{k+1},\\
0 &= D_{12}^T C_1 x_k + D_{12}^T D_{11} w_k + D_{12}^T D_{12} u_k + B_2^T \mu_{k+1},\\
0 &= E^T \mu_\infty.
\end{aligned}$$

Together with equation (3.1b), we can rewrite this as

$$\begin{bmatrix} 0 & -E & 0 & 0 & 0\\ A^T & 0 & 0 & 0 & 0\\ B_1^T & 0 & 0 & 0 & 0\\ B_2^T & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{k+1}\\ x_{k+1}\\ w_{k+1}\\ u_{k+1}\\ z_{k+1} \end{bmatrix} + \begin{bmatrix} 0 & A & B_1 & B_2 & 0\\ -E^T & C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} & 0\\ 0 & D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 I & D_{11}^T D_{12} & 0\\ 0 & D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} & 0\\ 0 & C_1 & D_{11} & D_{12} & -I \end{bmatrix} \begin{bmatrix} \mu_k\\ x_k\\ w_k\\ u_k\\ z_k \end{bmatrix} = 0. \tag{3.10}$$

Using the last column for eliminations in the second to fourth column yields the equivalent system

$$\left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0\\ A^T & 0 & 0 & 0 & 0\\ \hline B_1^T & 0 & 0 & 0 & 0\\ B_2^T & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \mu_{k+1}\\ x_{k+1}\\ w_{k+1}\\ u_{k+1}\\ z_{k+1} \end{bmatrix} + \left[\begin{array}{cc|ccc} 0 & A & B_1 & B_2 & 0\\ -E^T & 0 & 0 & 0 & C_1^T\\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11}^T\\ 0 & 0 & 0 & 0 & D_{12}^T\\ 0 & C_1 & D_{11} & D_{12} & -I \end{array}\right] \begin{bmatrix} \mu_k\\ x_k\\ w_k\\ u_k\\ z_k \end{bmatrix} = 0,$$

and we introduce the matrix pencil

$$\lambda U_H + V_H(\gamma) := \lambda \left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0\\ A^T & 0 & 0 & 0 & 0\\ \hline B_1^T & 0 & 0 & 0 & 0\\ B_2^T & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & A & B_1 & B_2 & 0\\ -E^T & 0 & 0 & 0 & C_1^T\\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11}^T\\ 0 & 0 & 0 & 0 & D_{12}^T\\ 0 & C_1 & D_{11} & D_{12} & -I \end{array}\right]. \tag{3.11}$$

Similar calculations for the second subsystem (3.2) lead to the pencil

$$\lambda U_J + V_J(\gamma) := \lambda \left[\begin{array}{cc|ccc} 0 & -E^T & 0 & 0 & 0\\ A & 0 & 0 & 0 & 0\\ \hline C_1 & 0 & 0 & 0 & 0\\ C_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & A^T & C_1^T & C_2^T & 0\\ -E & 0 & 0 & 0 & B_1\\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11}\\ 0 & 0 & 0 & 0 & D_{21}\\ 0 & B_1^T & D_{11}^T & D_{21}^T & -I \end{array}\right]. \tag{3.12}$$

Remark: These pencils have the form of the BVD-pencils introduced in [9]. In the next sections we will use these pencils to develop the theory for discrete-time descriptor systems starting with the standard case.

4. Standard Case ($E = I$). First we consider the standard case where $E = I$ and make the following assumptions that are typical in $\mathcal{H}_\infty$-theory, see for example [38].

$A_{st}1)$ (A, B_2) is stabilizable and (A, C_2) is detectable.

$A_{st}2)$ $\operatorname{rank} \begin{bmatrix} A - e^{j\theta}I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\theta \in [0, 2\pi]$,

$A_{st}3)$ $\operatorname{rank} \begin{bmatrix} A - e^{j\theta}I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\theta \in [0, 2\pi]$,

$A_{st}4)$ $\operatorname{rank} D_{12} = m_2$, $\operatorname{rank} D_{21} = p_1$.

Note that $A1) - A4)$ reduce to $A_{st}1) - A_{st}4)$ when setting $E = I$. For the standard system the matrix pencils (3.11) and (3.12) have the forms

$$\lambda U_{H,st} + V_{H,st}(\gamma) :=$$
$$\lambda \left[\begin{array}{cc|ccc} 0 & -I & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 \\ \hline B_1^T & 0 & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & A & B_1 & B_2 & 0 \\ -I & 0 & 0 & 0 & C_1^T \\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11}^T \\ 0 & 0 & 0 & 0 & D_{12}^T \\ 0 & C_1 & D_{11} & D_{12} & -I \end{array}\right] \tag{4.1}$$

and

$$\lambda U_{J,st} + V_{J,st}(\gamma) :=$$
$$\lambda \left[\begin{array}{cc|ccc} 0 & -I & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & A^T & C_1^T & C_2^T & 0 \\ -I & 0 & 0 & 0 & B_1 \\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11} \\ 0 & 0 & 0 & 0 & D_{21} \\ 0 & -B_1^T & D_{11}^T & D_{21}^T & -I \end{array}\right]. \tag{4.2}$$

DEFINITION 4.1. [20] A solution X of the discrete-time algebraic Riccati equation

$$X = A^T XA + Q - (C + B^T XA)^T (R + B^T XB)^{-1}(C + B^T XA)$$

is said to be stabilizing if all eigenvalues of

$$A - B(R + B^T XB)^{-1}(B^T XA + C)$$

are inside the unit circle.

THEOREM 4.2. [34] Consider a system (1.1) and assume that $A_{st}1) - A_{st}4)$ are satisfied. Then the following statements are equivalent:

(i) There exists a dynamic controller of the form (1.2) such that the transfer matrix T_{zw} from w to z of the resulting closed loop system satisfies $\|T_{zw}\|<1$ and such that the resulting closed loop system is internally stable.

(ii) There exist positive semi-definite symmetric matrices P and Q such that

(a) $U > 0$, where

$$U := I - D_{11}^T D_{11} - B_1^T PB_1 + (B_1^T PB_2 + D_{11}^T D_{12})(B_2^T PB_2 + D_{12}^T D_{12})^{-1}(B_2^T PB_1 + D_{12}^T D_{11}).$$

(b) P is a stabilizing solution of the discrete time algebraic Riccati equation

$$P = A^TPA+C_1^TC_1-\begin{bmatrix} B_1^TPA+D_{11}^TC_1 \\ B_2^TPA+D_{12}^TC_1 \end{bmatrix}^T \hat{R}_H^{-1}(P,\gamma)\begin{bmatrix} B_1^TPA+D_{11}^TC_1 \\ B_2^TPA+D_{12}^TC_1 \end{bmatrix}, \tag{4.3}$$

where

$$\hat{R}_H(P,\gamma) = \begin{bmatrix} D_{11}^TD_{11}-\gamma^2 I & D_{11}^TD_{12} \\ D_{12}^TD_{11} & D_{12}^TD_{12} \end{bmatrix} + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} P \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

(c) $V > 0$, where

$$V = I-D_{11}D_{11}^T-C_1QC_1^T+(C_1QC_2^T+D_{11}D_{21}^T)(D_{21}D_{21}^T+C_2QC_2^T)^{-1}(C_2QC_1^T+D_{21}D_{11}^T).$$

(d) Q is a stabilizing solution of the discrete algebraic Riccati equation

$$Q = AQA^T+B_1B_1^T-\begin{bmatrix} C_1QA^T+D_{11}B_1^T \\ C_2QA^T+D_{21}B_1^T \end{bmatrix}^T \hat{R}_J^{-1}(Q,\gamma)\begin{bmatrix} C_1QA^T+D_{11}B_1^T \\ C_2QA^T+D_{21}B_1^T \end{bmatrix}, \tag{4.4}$$

where

$$\hat{R}_J(Q,\gamma) = \begin{bmatrix} D_{11}D_{11}^T-\gamma^2 I & D_{11}D_{21}^T \\ D_{21}D_{11}^T & D_{21}D_{21}^T \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} P \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^T.$$

(e) $\rho(PQ) < \gamma^2$.

Note that in contrast to the continuous-time case $\hat{R}_H$ and $\hat{R}_J$ depend on the solutions P and Q of the discrete-time algebraic Riccati equation. Conditions (a) and (c) in Theroem 4.2 ensure that $\hat{R}_H$ and $\hat{R}_J$ are invertible. To find a stabilizing solution of the discrete algebraic Riccati equation we can make use of the relation to deflating subspaces of symplectic matrix pencils. This means that we can calculate the stabilizing positive semi-definite solution of (4.3) and (4.4) by finding a basis of the stable deflating subspace of the symplectic matrix pencils associated with (4.3) and (4.4), respectively [24, 26, 35]. Then, with

$$R_H = \begin{bmatrix} D_{11}^TD_{11}-\gamma^2 I & D_{11}^TD_{12} \\ D_{12}^TD_{11} & D_{12}^TD_{12} \end{bmatrix}, \quad R_J = \begin{bmatrix} D_{11}D_{11}^T-\gamma^2 I & D_{11}D_{21}^T \\ D_{21}D_{11}^T & D_{21}D_{21}^T \end{bmatrix}, \tag{4.5}$$

we can replace conditions (b) and (d) in Theroem 4.2 by the following two conditions.

(b') There exist matrices $X_{H,1},\ X_{H,2} \in \mathbb{R}^{n,n}$ with $X_{H,1}$ nonsingular such that the columns of $\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix}$ span an n-dimensional semi-stable deflating subspace of the matrix pencil

$$\lambda \begin{bmatrix} 0 & A-\begin{bmatrix} B_1 & B_2 \end{bmatrix} R_H^{-1}\begin{bmatrix} D_{11}^TC_1 \\ D_{12}^TC_1 \end{bmatrix} \\ -I & C_1^TC_1-\begin{bmatrix} C_1^TD_{11} & C_1^TD_{12} \end{bmatrix} R_H^{-1}\begin{bmatrix} D_{11}^TC_1 \\ D_{12}^TC_1 \end{bmatrix} \end{bmatrix} \tag{4.6}$$

$$+\begin{bmatrix} -\begin{bmatrix} B_1 & B_2 \end{bmatrix} R_H^{-1}\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} & -I \\ -A^T+\begin{bmatrix} C_1^TD_{11} & C_1^TD_{12} \end{bmatrix} R_H^{-1}\begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} & 0 \end{bmatrix}.$$

(d') There exist matrices $X_{J,1},\ X_{J,2} \in \mathbb{R}^{n,n}$ with $X_{J,1}$ nonsingular such that the columns of $\begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix}$ span an n-dimensional semi-stable deflating subspace of the matrix pencil

$$\lambda \begin{bmatrix} 0 & A^T - \begin{bmatrix} C_1^T & C_2^T \end{bmatrix} R_J^{-1} \begin{bmatrix} D_{11}B_1^T \\ D_{21}B_1^T \end{bmatrix} \\ -I & B_1B_1^T - \begin{bmatrix} B_1D_{11}^T & B_1D_{21}^T \end{bmatrix} R_J^{-1} \begin{bmatrix} D_{11}B_1^T \\ D_{21}B_1^T \end{bmatrix} \end{bmatrix} + \begin{bmatrix} -\begin{bmatrix} C_1^T & C_2^T \end{bmatrix} R_J^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & -I \\ -A + \begin{bmatrix} B_1D_{11}^T & B_1D_{21}^T \end{bmatrix} R_J^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & 0 \end{bmatrix}. \tag{4.7}$$

Similar to [22] we can now show that conditions (b') and (d') are equivalent to the existence of a semi-stable deflating subspace of (4.1) and (4.2).

LEMMA 4.3. *If the columns of the matrices*

$$Q_H = \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \\ Q_{H,5} \end{bmatrix}, \quad Q_J = \begin{bmatrix} Q_{J,1} \\ Q_{J,2} \\ Q_{J,3} \\ Q_{J,4} \\ Q_{J,5} \end{bmatrix}, \tag{4.8}$$

partitioned conformably with (4.1) and (4.2), span a semi-stable deflating subspace of the pencils (4.1) and (4.2), respectively, then the columns of

$$Q_H = \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \end{bmatrix}, \quad Q_J = \begin{bmatrix} Q_{J,1} \\ Q_{J,2} \end{bmatrix} \tag{4.9}$$

span a semi-stable deflating subspace of (4.6) and (4.7).

Proof. We assume that the columns of Q_H span a semi-stable deflating subspace of the pencil in (3.10), i.e.

$$\begin{bmatrix} 0 & A & B_1 & B_2 & 0 \\ -I & C_1^TC_1 & C_1^TD_{11} & C_1^TD_{12} & 0 \\ 0 & D_{11}^TC_1 & D_{11}^TD_{11} - \gamma^2 I & D_{11}^TD_{12} & 0 \\ 0 & D_{12}^TC_1 & D_{12}^TD_{11} & D_{12}^TD_{12} & 0 \\ 0 & C_1 & D_{11} & D_{12} & -I \end{bmatrix} \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \\ Q_{H,5} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ -A^T & 0 & 0 & 0 & 0 \\ -B_1^T & 0 & 0 & 0 & 0 \\ -B_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \\ Q_{H,5} \end{bmatrix} T_H$$

for T_H with $|\lambda_i(T_H)| \leq 1$, for all i. After some permutations and eliminations we obtain

$$\left[\begin{array}{cc|cc} 0 & A & B_1 & B_2 \\ -I & C_1^TC_1 & C_1^TD_{11} & C_1^TD_{12} \\ \hline 0 & D_{11}^TC_1 & D_{11}^TD_{11} - \gamma^2 I & D_{11}^TD_{12} \\ 0 & D_{12}^TC_1 & D_{12}^TD_{11} & D_{12}^TD_{12} \end{array}\right] \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & I & 0 & 0 \\ -A^T & 0 & 0 & 0 \\ \hline -B_1^T & 0 & 0 & 0 \\ -B_2^T & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \end{bmatrix} T_H.$$

Since R_H is assumed to be invertible, we can use it to eliminate the upper right 2×2 block in $V_H(\gamma)$ and we get

$$\left[\begin{array}{cc|cc} 0 & A - \left[\begin{array}{cc} B_1 & B_2 \end{array}\right] R_H^{-1} \left[\begin{array}{c} D_{11}^T C_1 \\ D_{12}^T C_1 \end{array}\right] & 0 & 0 \\ -I & C_1^T C_1 - \left[\begin{array}{cc} C_1^T D_{11} & C_1^T D_{12} \end{array}\right] R_H^{-1} \left[\begin{array}{c} D_{11}^T C_1 \\ D_{12}^T C_1 \end{array}\right] & 0 & 0 \\ \hline 0 & D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 I & D_{11}^T D_{12} \\ 0 & D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{array}\right] \left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} \left[\begin{array}{cc} B_1 & B_2 \end{array}\right] R_H^{-1} \left[\begin{array}{c} B_1^T \\ B_2^T \end{array}\right] & -I & 0 & 0 \\ -A^T + \left[\begin{array}{cc} C_1^T D_{11} & C_1^T D_{12} \end{array}\right] R_H^{-1} \left[\begin{array}{c} B_1^T \\ B_2^T \end{array}\right] & 0 & 0 & 0 \\ \hline -B_1^T & 0 & 0 & 0 \\ -B_2^T & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \end{array}\right] T_H.$$

Thus $\left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \end{array}\right]$ spans a semi-stable deflating subspace of (4.6). The calculations for (4.7) are equivalent. □

Similarly to [4, 22] we replace condition (e) in Theorem 4.2 by

(e') The matrix

$$\mathcal{Y}(\gamma) = \left[\begin{array}{cc} -\gamma X_{H,2}^T(\gamma) X_{H,1}(\gamma) & X_{H,2}^T(\gamma) X_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) X_{H,2}(\gamma) & -\gamma X_{J,2}(\gamma)^T X_{J,1}(\gamma) \end{array}\right] \tag{4.10}$$

is positive semidefinite and satisfies $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$, where

$$\begin{aligned} \hat{k}_H &= \operatorname{rank} X_{H,2}(\gamma_{H,1}) = \operatorname{rank} X_{H,2}(\gamma_{H,2}), \\ \hat{k}_J &= \operatorname{rank} X_{J,2}(\gamma_{J,1}) \ = \operatorname{rank} X_{J,2}(\gamma_{J,2}). \end{aligned}$$

Using these results we can reformulate the set of conditions in Theorem 4.2 that needs to be checked for the existence of a suitable controller:

$C_{st}1$) There exists a matrix $X_H(\gamma)$ as in (4.8) such that
- $C_{st}1a$) the columns of $X_H(\gamma)$ span a semi-stable deflating subspace of $\lambda U_{H,st} + V_{H,st}(\gamma)$,
- $C_{st}1b$) $\operatorname{rank} X_{H,1}(\gamma) = n$.

$C_{st}2$) There exists a matrix $X_J(\gamma)$ as in (4.8) such that
- $C_{st}2a$) the columns of $X_J(\gamma)$ span a semi-stable deflating subspace of $\lambda U_{J,st} + V_{J,st}(\gamma)$,
- $C_{st}2b$) $\operatorname{rank} X_{J,1}(\gamma) = n$.

$C_{st}3$) Condition (e') is satisfied.

In the modified optimal $\mathcal{H}_\infty$ control problem we want to find the smallest value γ_{mo} such that all these conditions are satisfied.

REMARK 4.4. *Note that condition (a) and (c) in Theorem 4.2 are only needed to ensure that the inverse in the discrete-time algebraic Riccati Equation exists. Since we found an alternative formulation by using matrix pencils, we do not need the existence of the inverse as an extra condition.*

5. Descriptor Systems. In this section we will show that the following generalization of Conditions $C_{st}1$) – $C_{st}3$) is sufficient for the existence of an appropriate controller in the descriptor system case:

$C1)$ There exists a matrix $X_H(\gamma)$ as in (4.8) such that

$C1a)$ the columns of $X_H(\gamma)$ span a semi-stable deflating subspace of $\lambda U_H + V_H(\gamma)$,

$C1b)$ $\operatorname{rank} EX_{H,1}(\gamma) = r$.

$C2)$ There exists a matrix $X_J(\gamma)$ as in (4.8) such that

$C2a)$ the columns of $X_J(\gamma)$ span a semi-stable deflating subspace of $\lambda U_J + V_J(\gamma)$,

$C2b)$ $\operatorname{rank} E^T X_{J,1}(\gamma) = r$.

$C3)$ The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} -\gamma X_{H,2}^T(\gamma) E X_{H,1}(\gamma) & X_{H,2}^T(\gamma) E X_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) E^T X_{H,2}(\gamma) & -\gamma X_{J,2}(\gamma)^T E^T X_{J,1}(\gamma) \end{bmatrix} \tag{5.1}$$

is positive semidefinite and satisfies $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$, where

$$\begin{aligned} \hat{k}_H &= \operatorname{rank} E^T X_{H,2}(\gamma_{H,1}) = \operatorname{rank} E^T X_{H,2}(\gamma_{H,2}), \\ \hat{k}_J &= \operatorname{rank} EX_{J,2}(\gamma_{J,1}) \quad = \operatorname{rank} EX_{J,2}(\gamma_{J,2}). \end{aligned}$$

Remark 5.1. *Note that in the continuous-time case (both for standard and descriptor systems) one has the additional condition that the pencils $\lambda U_H + V_H(\gamma)$ and $\lambda U_J + V_J(\gamma)$ in (3.11) and (3.12) have index one, [22]. This is not necessary here, since the following example shows that even a pencil of higher index can still have the desired number of (semi-)stable eigenvalues.*

Example 5.2. *Let*

$$E = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; S = R = 0.$$

Then

$$\lambda U_H + V_H = \lambda \left[\begin{array}{cc|cc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ \hline 1 & 2 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|cc|c} 0 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 2 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is not of index one, but it has $\operatorname{rank} E = r = 2$ *eigenvalues inside the uni circle, namely* $\lambda = \frac{1}{2}$ *and* $\lambda = 0$.

5.1. The Index 1 Case. To extend the result for the standard case to descriptor systems, where (E, A) is of index one, we can use the WCF to reformulate the descriptor system as a system in standard form and then apply Theorem 4.2. Transforming system (1.1) and using the notation introduced in (2.3), the explicit solution (2.5b) reduces to $x_\infty = -B_{1,\infty} w - B_{2,\infty} u$ and using this, we obtain the standard state space system

$$\begin{aligned} \dot{x}_{k+1,f} = \;& A_f x_{k,f} + B_{1,f} w_k + B_{2,f} u_k, \\ z_k &= C_{1,f} x_{k,f} + (D_{11} - C_{1,\infty} B_{1,\infty}) w_k + (D_{12} - C_{1,\infty} B_{2,\infty}) u_k, \\ y_k &= C_{2,f} x_{k,f} + (D_{21} - C_{2,\infty} B_{1,\infty}) w_k + (D_{22} - C_{2,\infty} B_{2,\infty}) u_k. \end{aligned} \tag{5.2}$$

Lemma 5.3. *Consider system (1.1) and suppose that the index of $\lambda E - A$ is at most one. Then for $i \in \{1, 2, 3, 4\}$, system (1.1) satisfies $Ai)$ if and only if (5.2) satisfies $A_{st}i)$.*

Proof. The proof is completely equivalent to the continuous-time case treated in [22] (Lemma 4.2) and is therefore omitted. □

Next we need to show that the set Γ of all values for γ that satisfy Ci for $i = 1, 2, 3$ is invariant under transformation to WCF.

LEMMA 5.4. *Consider the system (1.1) and assume that the index of $\lambda E - A$ is at most one. Let $\lambda U_H - V_H(\gamma)$ and $\lambda U_J - V_J(\gamma)$ be the pencils as in (4.1) and (4.2) and let $\lambda U_{H,st} - V_{H,st}(\gamma)$, $\lambda U_{J,st} - V_{J,st}(\gamma)$ be the corresponding pencils constructed from the data of the system (5.2).*

Let Γ_H, Γ_J be the set of γ-values that satisfy conditions $C1$) and $C2$) and let $\mathcal{Y}(\gamma)$ be the matrix introduced in (5.1).

Let analogously $\Gamma_{H,st}$, $\Gamma_{J,st}$ and $\mathcal{Y}_{st}(\gamma)$ be correspondingly defined for the standard state space system (5.2). Then,

$$\Gamma_{H,st} = \Gamma_H, \quad \Gamma_{J,st} = \Gamma_J,$$
$$\operatorname{rank} \mathcal{Y}(\gamma) = \operatorname{rank} \mathcal{Y}_{st}(\gamma).$$

Proof. We only prove the statement for the sets associated with $\lambda U_H - V_H(\gamma)$. The proof for the sets associated with $\lambda U_J - V_J(\gamma)$ is analogous.

Introducing the transformation matrix

$$P = \begin{bmatrix} V_f^T & 0 & 0 & 0 & 0 \\ 0 & W_f^T & 0 & 0 & 0 \\ 0 & 0 & I_{m_1} & 0 & 0 \\ 0 & 0 & 0 & I_{m_2} & 0 \\ 0 & -C_{1,\infty} W_\infty^T & 0 & 0 & I_{p_2} \\ V_\infty^T & 0 & 0 & 0 & 0 \\ 0 & W_\infty^T & 0 & 0 & 0 \end{bmatrix}^T \tag{5.3}$$

we obtain that

$$\lambda P^T U_H P - P^T V_H(\gamma) P = \begin{bmatrix} \lambda U_{H,st} - V_{H,st}(\gamma) & 0 & 0 \\ 0 & I_{n_\infty} & 0 \\ 0 & 0 & I_{n_\infty} \end{bmatrix}. \tag{5.4}$$

This directly implies that $\Gamma_{H,st} = \Gamma_H$. Furthermore, we can conclude that the columns of a matrix

$$X_{H,st} = \begin{bmatrix} X_{H,st,1}^T & X_{H,st,2}^T & X_{H,st,3}^T & X_{H,st,4}^T & X_{H,st,5}^T & X_{H,st,6}^T & X_{H,st,7}^T \end{bmatrix}^T$$

partitioned conformably to the block structure of (5.4) span a semi-stable invariant subspace if and only if the columns of

$$\begin{bmatrix} X_{H,1} \\ X_{H,2} \\ X_{H,3} \\ X_{H,4} \\ X_{H,5} \end{bmatrix} = \begin{bmatrix} V_f X_{H,st,1} + V_\infty X_{H,st,6} \\ W_f X_{H,st,2} - W_\infty C_{1,\infty}^T X_{H,st,5} + W_\infty X_{H,st,7} \\ X_{H,st,3} \\ X_{H,st,4} \\ X_{H,st,5} \end{bmatrix}$$

span the semi-stable invariant subspace of $\lambda U_H - V_H(\gamma)$. Using the fact that $EV_\infty = 0$, it follows that $\begin{bmatrix} EX_{H,1}(\gamma)^T & X_{H,2}(\gamma)^T \end{bmatrix}^T$ is a generalized Lagrangian subspace if and only if $\begin{bmatrix} X_{H,st,1}(\gamma)^T & X_{H,st,2}(\gamma)^T \end{bmatrix}^T$ is a Lagrangian subspace. This also implies that $\operatorname{rank} EX_{H,1} = \operatorname{rank} X_{H,st,1}$. Thus, we have $\Gamma_{H,st} = \Gamma_H$.

Since an analogous result holds for the pencils $\lambda U_J - V_J(\gamma)$ and $\lambda U_{J,st} - V_{J,st}(\gamma)$, we conclude that also $\operatorname{rank} \mathcal{Y}(\gamma) = \operatorname{rank} \mathcal{Y}_{st}(\gamma)$. $\square$

With these preparations we have the following result for systems of index one.

PROPOSITION 5.5. *Consider system (1.1) such that the index of the pencil $\lambda E - A$ is at most one, and the pencils $\lambda U_H + V_H(\gamma)$ and $\lambda U_J + V_J(\gamma)$ are as in (3.11) and (3.12), respectively. Suppose that assumptions $A1)$-$A4)$ hold.*

Then there exists an internally stabilizing controller such that the transfer function from w to z satisfies $\|T_{zw}\|_\infty < \gamma$ if and only if γ is such that the conditions $C1)$, $C2)$ and $C3)$ hold.

Proof. The closed-loop transfer function $T_{NW}(s)$ of the system (5.2) with a controller of the form (1.2) is equal to the closed-loop transfer function of the system (1.1) with the same controller.

Since (1.1) is strongly stabilizable (strongly detectable), if and only if system (5.2) is stabilizable (detectable), a controller that internally stabilizes (5.2) also stabilizes the finite dynamics of (1.1).

Therefore the existence of a controller with desired properties for (1.1) is equivalent to the existence of such a controller for (5.2). Since by Lemma 5.3 the validity of assumptions $A1)$-$A4)$ for (5.2) is equivalent to those of (1.1) and, furthermore, also by Lemma 5.4 the corresponding conditions $C1)$-$C3)$ of these two systems are equivalent and thus the statement follows. $\square$

We have seen so far that the the index one case follows from the standard case by some simple transformation. To extend the results to descriptor systems of arbitrary index we study the general case in the following section.

5.2. The General Case. To extend the previous results to general descriptor systems we will use an a-priori feedback that transforms the general system to an index one system such that we can apply the previous results. Using a feedback of the form $u_k = KY_k + \bar{u}_k$ leads to the system

$$\begin{aligned} Ex_{k+1} &= (A + B_2KC_2)x_k + (B_1 + B_2KD_{21})w_k + B_2\bar{u}_k, \qquad x_0 = x^0, \\ z_k &= (C_1 + D_{12}KC_2)x_k + (D_{11} + D_{12}KD_{21})w_k + D_{12}\bar{u}_k, \\ y_k &= C_2x_k + D_{21}w_k. \end{aligned} \tag{5.5}$$

The feedback matrix K will be constructed in a way that system (5.5) has index one. Under the assumption that the system is controllable and observable at ∞, such a feedback always exists [8]. After applying such a feedback we can construct a controller (1.2) for (5.5). A controller for the overall system is then given by

$$\begin{aligned} \hat{E}\hat{x}_{k+1} &= \hat{A}\hat{x}_k + \hat{B}y_k, \\ u_k &= \hat{C}\hat{x}_k + (\hat{D} + K)y_k. \end{aligned} \tag{5.6}$$

We now need to show that this static output feedback does not change the assumptions $A1)$–$A4)$. The proof is similar to the continuous-time case discussed in [22].

LEMMA 5.6. *Consider system (1.1) and let $K \in \mathbb{R}^{m_2,p_2}$ such that the pencil $\lambda E - (A + B_2KC_2)$ is regular. Then for every $i \in \{1, 2, 3, 4\}$ the system (1.1) satisfies $Ai)$ if and only if the system (5.5) satisfies $Ai)$.*

Proof. The invariance of strong stabilizability and strong detectability under output feedback is proved in [11]. The proof for the equivalence of the conditions $A2)$

and $A3$) follows from the identities

$$\begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ KC_2 & I \end{bmatrix} = \begin{bmatrix} A + B_2KC_2 - i\omega E & B_2 \\ C_1 + D_{12}KC_2 & D_{12} \end{bmatrix},$$

$$\begin{bmatrix} I & B_2K \\ 0 & I \end{bmatrix} \begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} A + B_2KC_2 - i\omega E & B_2 \\ C_1 + D_{12}KC_2 & D_{12} \end{bmatrix}.$$

The remaining assertions follows from

$$\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ KC_2S_\infty & I \end{bmatrix} = \begin{bmatrix} T_\infty^T (A + B_2KC_2) S_\infty & T_\infty^T B_2 \\ (C_1 + D_{12}KC_2) T_\infty & D_{12} \end{bmatrix},$$

$$\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} \begin{bmatrix} I & 0 \\ KC_2S_\infty & I \end{bmatrix} = \begin{bmatrix} T_\infty^T (A + B_2KC_2) S_\infty & T_\infty^T (B_1 + B_2KD_{21}) \\ C_2 S_\infty & D_{21} \end{bmatrix}.$$

□

As in the index one case we need to show the set Γ of all suitable γ-parameters is invariant under the preliminary feedback.

LEMMA 5.7. *Consider the system (1.1) and let $K \in \mathbb{R}^{m_2,p_2}$ be such that the pencil $\lambda E - (A + BKC)$ is regular. Let Γ_H, Γ_J be the sets of γ-values such that condition $C1$). and $C2$), respectively, are satisfied. Furthermore, let $\Gamma_{H,K}$, $\Gamma_{J,K}$ be the corresponding quantities for the system (5.5). Then*

$$\Gamma_{H,K} = \Gamma_H, \qquad \Gamma_{J,K} = \Gamma_J, \qquad \operatorname{rank} \mathcal{Y}(\gamma) = \operatorname{rank} \mathcal{Y}_K(\gamma).$$

Proof. We only show the result for the the sets associated with $\lambda U_H - V_H(\gamma)$, the proof for $\lambda U_H - V_H(\gamma)$ is completely analogous.

Let $\lambda U_H - V_H^K$ be the BVD-pencil [9] formed from the data of system (5.5). Then, with the transformation matrices

$$T_{l,K} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & D_{21}^T K^T & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \; T_{r,H} = \begin{bmatrix} I & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & -KC_2 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

we have the identity

$$\lambda T_{l,K} U_H T_{r,H} - T_{l,K} V_H(\gamma) T_{r,H} = \lambda U_H - V_H^K(\gamma).$$

Thus we have that both the index and eigenvalues of $\lambda U_H - V_H(\gamma)$ and $\lambda U_H - V_H^K(\gamma)$ coincide. This directly implies that $\Gamma_{H,K} = \Gamma_H$. The remaining relations follow from the fact that

$$\begin{bmatrix} X_{H,1}^T & X_{H,2}^T & X_{H,3}^T & X_{H,4}^T & X_{H,5}^T \end{bmatrix}^T$$

is a semi-stable invariant subspace of $\lambda U_H - V_H^K(\gamma)$ if and only if

$$\begin{bmatrix} X_{H,1}^T & X_{H,2}^T & X_{H,3}^T & (X_{H,4} - KC_2X_{H,1} + KD_{21}X_{H,3})^T & X_{H,5}^T \end{bmatrix}^T$$

is a semi-stable invariant subspace of $\lambda U_H - V_H(\gamma)$. □ With these auxiliary results, we are now in a position to prove the general Theorem.

THEOREM 5.8. *Consider system (1.1) and the associated pencils $\lambda U_H - V_H(\gamma)$ and $\lambda U_J - V_J(\gamma)$ as in (3.11) and (3.12), respectively. Suppose that assumptions $A1$)-$A4$) hold.*

Then there exists an internally stabilizing controller such that the transfer function from w to z satisfies $\|T_{zw}\|_\infty < \gamma$ if and only if γ is such that the conditions $C1$), $C2$) and $C3$) hold.

Proof. There exists a matrix $K \in \mathbb{R}^{m_2,p_2}$ such that the system (5.5) has index at most one. Lemma 5.6 implies that (5.5) satisfies $A1$)-$A4$) as well. Furthermore, by Lemma 5.7, the validity of the conditions $C1$)–$C3$) for the system (1.1) are equivalent to the respective conditions for system (5.5).

Proposition 5.5 then implies that conditions $C1$)–$C4$) for (5.5) are fulfilled if and only if there exists a desired controller for (5.5).

Since an application of the controller (1.2) to (5.5) results in the same closed loop system as controlling (5.5) with (5.6), the desired result follows immediately. □
Since we want to calculate the optimal value for γ as accurate as possible we should use structured methods for the arising matrix pencils.

6. Palindromic Pencils. In this section we will reformulate the results to make use of structure preserving numerical methods.

DEFINITION 6.1. *A matrix pencil $\lambda A - B$ is called T-palindromic if $B = A^T$.* In the following we will just use the term palindromic pencil for a T-palindromic pencil. We want to find a reformulation of (3.11) and (3.12) that has palindromic structure and preserves the information on eigenvalues and deflating subspaces [9]. In this chapter we introduce two techniques that carry out this transformation and we discuss the advantages and disadvantages of both approaches. The first approach can be found in [32] for the linear-quadratic optimal control problem.

We introduce the new sequences

$$\tilde{x}_k = \sum_{j=0}^{k} x_k,\ \tilde{u}_k = \sum_{j=0}^{k} u_k,\ \tilde{w}_k = \sum_{j=0}^{k} w_k,\ \tilde{z}_k = \sum_{j=0}^{k} z_k,$$

and rewrite subsystem (3.1) as

$$\begin{aligned} E\tilde{x}_{k+1} &= \ \ A\tilde{x}_k + B_1\tilde{w}_k \ \ + B_2\tilde{u}_k, \\ \tilde{z}_k &= C_1\tilde{x}_k + D_{11}\tilde{w}_k + D_{12}\tilde{u}_k, \end{aligned}$$

and with $x_0 = \tilde{x}_0$, $w_0 = \tilde{w}_0$, $u_0 = \tilde{u}$ and $x_i = \tilde{x}_i - \tilde{x}_{i-1}$, $w_i = \tilde{w}_i - \tilde{w}_{i-1}$, $u_i = \tilde{u}_i - \tilde{u}_{i-1}$ for $i > 0$ we can formulate the cost functional as

$$\begin{aligned} J = \frac{1}{4} &\begin{bmatrix} \tilde{x}_0 \\ \tilde{w}_0 \\ \tilde{u}_0 \end{bmatrix}^T \begin{bmatrix} C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} \\ D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 & D_{11}^T D_{12} \\ D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ \tilde{w}_0 \\ \tilde{u}_0 \end{bmatrix} \\ + \sum_{k=0}^{\infty} &\begin{bmatrix} \tilde{x}_k - \tilde{x}_{k-1} \\ \tilde{w}_k - \tilde{x}_{k-1} \\ \tilde{u}_k - \tilde{u}_{k-1} \end{bmatrix}^T \begin{bmatrix} C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} \\ D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 & D_{11}^T D_{12} \\ D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} \tilde{x}_k - \tilde{x}_{k-1} \\ \tilde{w}_k - \tilde{x}_{k-1} \\ \tilde{u}_k - \tilde{x}_{k-1} \end{bmatrix}. \end{aligned}$$

After regrouping we get

$$J = \frac{1}{2}(\sum_{k=0}^{\infty} \begin{bmatrix} \tilde{x}_k \\ \tilde{w}_k \\ \tilde{u}_k \end{bmatrix}^T \begin{bmatrix} C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} \\ D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 & D_{11}^T D_{12} \\ D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ \tilde{w}_k \\ \tilde{u}_k \end{bmatrix}$$
$$- \begin{bmatrix} \tilde{x}_k \\ \tilde{w}_k \\ \tilde{u}_k \end{bmatrix}^T \begin{bmatrix} C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} \\ D_{11}^T C_1 & D_{11}^T D_{11} - \gamma^2 & D_{11}^T D_{12} \\ D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{x}_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix}).$$

Let the matrices M and $\hat{L}$ be defined as in (3.4) and (3.5) respectively and define

$$\tilde{L} = \begin{bmatrix} \hat{L} & -\hat{L} & & \\ & \hat{L} & -\hat{L} & \\ & & \ddots & \ddots \end{bmatrix}.$$

Define a new variable $\tilde{\xi}$ by

$$\tilde{\xi}^T = [\tilde{x}_0^T, \tilde{w}_0^T, \tilde{u}_0^T, \tilde{x}_1^T, \tilde{w}_1^T, \tilde{u}_1^T, \ldots]^T$$

and rewrite the optimization problem (3.6) as

$$\text{Minimize} \quad \frac{1}{2}\tilde{\xi}^T \tilde{L}\tilde{\xi}, \quad \text{subject to} \quad M\tilde{\xi} = 0.$$

Define the Lagrangian of the optimization problem by

$$\mathcal{L}(\tilde{\xi}, \mu) = \frac{1}{2}\tilde{\xi}^T \tilde{L}\tilde{\xi} + \mu^T (M\tilde{\xi}) \tag{6.1}$$

with

$$\mu^T = [\mu_1, \mu_2, \ldots].$$

Differentiating (6.1) leads to the following conditions for the optimal control

$$\begin{aligned} \tilde{L}\tilde{\xi} + \mu M^T &= 0, \\ M\tilde{\xi} &= 0. \end{aligned}$$

We introduce the artificial variable $\mu_0 = 0$. Then the first equation can be rewritten as

$$\begin{aligned} 0 &= C_1^T C_1 \tilde{x}_k + C_1^T D_{11} \tilde{w}_k + C_1 D_{12} \tilde{u}_k - C_1^T C_1 \tilde{x}_{k+1} \\ &\quad - C_1^T D_{11} \tilde{w}_{k+1} - C_1 D_{12} \tilde{u}_{k+1} + A^T \lambda_{k+1} - E^T \lambda_k && (6.2) \\ 0 &= D_{11}^T C_1 \tilde{x}_k + (D_{11}^T D_{11} - \gamma^2) \tilde{w}_k + D_{11} D_{12} \tilde{u}_k \\ &\quad - D_{11}^T C_1 \tilde{x}_{k+1} - (D_{11}^T D_{11} - \gamma^2) \tilde{w}_{k+1} - D_{11} D_{12} \tilde{u}_{k+1} + B_1^T \lambda_{k+1} && (6.3) \\ 0 &= D_{12}^T C_1 \tilde{x}_k + D_{12}^T D_{11} \tilde{w}_k + D_{12} D_{12} \tilde{u}_k \\ &\quad - D_{12}^T C_1 \tilde{x}_{k+1} - D_{12}^T D_{11} \tilde{w}_{k+1} - D_{12} D_{12} \tilde{u}_{k+1} + B_2^T \lambda_{k+1} && (6.4) \end{aligned}$$

and the final condition $\lim_{k\to\infty}(F_1 x_k + F_2 w_k + F_3 u_k) = \lim_{k\to\infty} E^T\lambda_k$ for appropriate matrices $F_1,\ F_2,\ F_3$. If we additionally use the system equations for z_k in terms of the new variables

$$\tilde{z}_k-\tilde{z}_{k-1} = C_1\tilde{z}_k-C_1\tilde{z}_{k-1}+D_{11}\tilde{z}_k-D_{11}\tilde{z}_{k-1}+D_{12}\tilde{z}_k-D_{12}\tilde{z}_{k-1},\ i=1,\ldots,\infty,\ \tilde{z}_0 = z_0, \tag{6.5}$$

we can rewrite (6.2) - (6.5) as

$$\begin{bmatrix} 0 & -E & 0 & 0 & 0 \\ A^T & -C_1^T C_1 & -C_1^T D_{11} & -C_1^T D_{12} & 0 \\ B_1^T & -D_{11}^T C_1 & -(D_{11}^T D_{11}-\gamma^2 I) & -D_{11}^T D_{12} & 0 \\ B_2^T & -D_{12}^T C_1 & -D_{12}^T D_{11} & -D_{12}^T D_{12} & 0 \\ 0 & C_1 & D_{11} & D_{12} & -I \end{bmatrix} \begin{bmatrix} \lambda_{k+1} \\ x_{k+1} \\ w_{k+1} \\ u_{k+1} \\ z_{k+1} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & A & B_1 & B_2 & 0 \\ -E^T & C_1^T C_1 & C_1^T D_{11} & C_1^T D_{12} & 0 \\ 0 & D_{11}^T C_1 & D_{11}^T D_{11}-\gamma^2 I & D_{11}^T D_{12} & 0 \\ 0 & D_{12}^T C_1 & D_{12}^T D_{11} & D_{12}^T D_{12} & 0 \\ 0 & -C_1 & -D_{11} & -D_{12} & I \end{bmatrix} \begin{bmatrix} \lambda_k \\ x_k \\ w_k \\ u_k \\ z_k \end{bmatrix} = 0$$

which is equivalent to

$$\left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0 \\ A^T & 0 & 0 & 0 & -C_1^T \\ \hline B_1^T & 0 & -\gamma^2 I & 0 & -D_{11}^T \\ B_2^T & 0 & 0 & 0 & -D_{12}^T \\ 0 & C_1 & D_{11} & D_{12} & -I \end{array}\right] \begin{bmatrix} \lambda_{k+1} \\ x_{k+1} \\ w_{k+1} \\ u_{k+1} \\ z_{k+1} \end{bmatrix}$$
$$+ \left[\begin{array}{cc|ccc} 0 & A & B_1 & B_2 & 0 \\ -E^T & 0 & 0 & 0 & C_1^T \\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11}^T \\ 0 & 0 & 0 & 0 & D_{12}^T \\ 0 & -C_1 & -D_{11} & -D_{12} & -I \end{array}\right] \begin{bmatrix} \lambda_k \\ x_k \\ w_k \\ u_k \\ z_k \end{bmatrix} = 0.$$

This boundary value problem can be solved by considering the following palindromic matrix pencil and its deflating subspace associated with the eigenvalues inside the unit disk:

$$\lambda U_{H,p} + V_{H,p}(\gamma) =$$
$$\lambda \left[\begin{array}{cc|ccc} 0 & -E & 0 & 0 & 0 \\ A^T & 0 & 0 & 0 & -C_1^T \\ \hline B_1^T & 0 & -\gamma^2 I & 0 & -D_{11}^T \\ B_2^T & 0 & 0 & 0 & -D_{12}^T \\ 0 & C_1 & D_{11} & D_{12} & -I \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & A & B_1 & B_2 & 0 \\ -E^T & 0 & 0 & 0 & C_1^T \\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11}^T \\ 0 & 0 & 0 & 0 & D_{12}^T \\ 0 & -C_1 & -D_{11} & -D_{12} & -I \end{array}\right]. \tag{6.6}$$

Similar calculations for the second subsystem (3.2) lead to

$$\lambda U_{J,p} + V_{J,p}(\gamma) =$$
$$\lambda \left[\begin{array}{cc|ccc} 0 & -E^T & 0 & 0 & 0 \\ A & 0 & 0 & 0 & B_1 \\ \hline C_1 & 0 & -\gamma^2 I & 0 & -D_{11} \\ C_2 & 0 & 0 & 0 & -D_{21} \\ 0 & B_1^T & D_{11}^T & D_{21}^T & -I \end{array}\right] + \left[\begin{array}{cc|ccc} 0 & A^T & C_1^T & C_2^T & 0 \\ -E & 0 & 0 & 0 & B_1 \\ \hline 0 & 0 & -\gamma^2 I & 0 & D_{11} \\ 0 & 0 & 0 & 0 & D_{21} \\ 0 & -B_1^T & -D_{11}^T & -D_{21}^T & -I \end{array}\right]. \tag{6.7}$$

Since we are interested in calculating the semi-stable deflating subspaces, we need the following Lemma to provide a connection of the semi-stable deflating subspaces of (6.6) and (3.11) (and similarly for (6.7) and (3.12)).

LEMMA 6.2. *Let* $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ *be a matrix partitioned conformably with the blocks shown in (6.6). If* $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ *is a deflating subspace of (3.11), i.e.,*

$$U_H X = V_H X T,$$

where T *is a matrix whose spectrum are the semi-stable eigenvalues, then* $\tilde{X} = \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix}$ *is the semi-stable deflating subspace of the palindromic pencil (6.6), i.e.,*

$$U_{H,p}\tilde{X} = V_{H,p}\tilde{X}T.$$

Proof. The pencil $\lambda U_H + V_H(\gamma)$ in (3.11) has the block form

$$\lambda \begin{bmatrix} 0 & F \\ -G^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^T & D \end{bmatrix}$$

with

$$F = \begin{bmatrix} -E & 0 & 0 & 0 \end{bmatrix} \tag{6.8}$$

and

$$G = \begin{bmatrix} A & B_1 & B_2 & 0 \end{bmatrix} \tag{6.9}$$

and we know from [37] that this pencil can be transformed to the following (even) form by Cayley transformation and a "drop-procedure",

$$\lambda U_{H,e} - V_{H,e} = \lambda \begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{F}^T & \tilde{D} \end{bmatrix}. \tag{6.10}$$

If X is a deflating subspace of $\lambda U_H + V_H(\gamma)$ associated with T then (6.10) has the deflating subspace $\tilde{X} = \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix}$ associated with the Cayley transformation $c(T)$ of T [37], i.e.,

$$U_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} c(T) = V_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix}. \tag{6.11}$$

Now we perform an inverse Cayley transformation $(\tilde{U}_H, \tilde{V}_H) = 1/2(U_{H,e} - V_{H,e}, U_{H,e} + V_{H,e})$ and from (6.11) we get

$$\begin{aligned}
&1/2U_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} c(T) = 1/2V_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} \\
\Leftrightarrow &1/2U_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} (T-I)(T+I)^{-1} = 1/2V_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} \\
\Leftrightarrow &1/2U_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} (T-I) = 1/2V_{H,e} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} (T+I) \\
\Leftrightarrow &\tilde{U} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix} T = \tilde{V} \begin{bmatrix} X_1(I+T) \\ 2X_2 \end{bmatrix}.
\end{aligned}$$

Multiplying $(\tilde{U}, \tilde{V})$ by $(\operatorname{diag} I_n, 2I_{n+m_1+m_2})$ we find that $\tilde{X}$ spans the deflating subspace of $\lambda U_{H,p} - V_{H,p}$ associated with T. □
▯

Unfortunately, this approach of get palindromic matrix pencils has some drawbacks which become clear when we lock at the transformations as an algebraic manipulation [9, 32, 37]. We find that this procedure adds $m_1 + m_2 + p_1$ copies of the eigenvalue 1 and -1 which is not desirable in our case. One reason for this is that eigenvalues on the unit circle are usually critical in the numerical treatment. The other drawback is that we are interested in the calculation of deflating subspaces that are associated with the eigenvalues inside the open (closed) unit disk and thus would have to distinguish between the eigenvalues inside the unit disc that are added by the above procedure and the ones that have been there originally. This can be done by considering the kernel of the matrix U_H in (3.11) since it provides the information on the infinite eigenvalues of (3.11) that may have been transformed to 1 by the above procedure. For more details see Section 7.

Another approach that reformulates the pencils (3.11) and (3.12) to a palindromic form is the following adaption of [21] which we show for the pencil (3.11). The calculations for (3.12) are similar. We multiply (3.11) with the matrix

$$\begin{bmatrix} I_n & & & & \\ & \lambda I_n & & & \\ & & \lambda I_{m_1} & & \\ & & & \lambda I_{m_2} & \\ & & & & \lambda I_{p_1} \end{bmatrix}$$

and obtain the quadratic palindromic matrix polynomial

$$\lambda^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ A^T & 0 & 0 & 0 & 0 \\ B_1^T & 0 & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & -E & 0 & 0 & 0 \\ -E^T & 0 & 0 & 0 & C_1^T \\ 0 & 0 & -\gamma^2 I & 0 & D_{11}^T \\ 0 & 0 & 0 & 0 & D_{12}^T \\ 0 & -C_1 & -D_{11} & -D_{12} & -I \end{bmatrix} + \begin{bmatrix} 0 & A & B_1 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (6.12)$$

By performing this multiplication we add $n + m_1 + m_2 + p_1$ eigenvalues at 0 and n eigenvalues at ∞. All other eigenvalues are preserved [9, 21] and we now have a palindromic standard formulation of (3.11) (and by similar calculations also for (3.12)) which contains the desired information concerning the eigenvalues and as well the deflating subspaces, see [9]. Just as in the first method of deriving a palindromic representation we add eigenvalues inside the unit disc. Here we can easily distinguish between the ones that have been there originally and the ones that we added by performing the procedure by simply calculating the kernel of the matrix V_H, see section 7 and [9]. We dis not add any critical eigenvalues, but on the other hand we increased the degree of the matrix polynomial by one and thus it is a necessary next step to reduce the degree back to 1 without changing the palindromic structure or the information on eigenvalues, eigenvectors and deflating subspaces. There are several ways to carry out this linearization. A possible palindromic linearization of a quadratic matrix polynomial $\lambda^2 U^T + \lambda V + U$ is given by

$$\lambda \begin{bmatrix} U^T & V-U \\ U^T & U^T \end{bmatrix} + \begin{bmatrix} U & U \\ V-U^T & U \end{bmatrix}$$

and exists whenever $\lambda^2 U^T + \lambda V + U$ has no eigenvalue at -1. Using this linearization we obtain the palindromic pencil

$$\begin{aligned} &\lambda U_{H,lin} - V_{H,lin} \qquad (6.13)\\ &= \left[\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & -A-E & -B_1 & -B_2 & 0 \\ A^T & 0 & 0 & 0 & 0 & -E^T & 0 & 0 & 0 & C_1^T \\ B_1^T & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I & 0 & D_{11}^T \\ B_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{12}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & -C_1 & -D_{11} & -D_{12} & -I \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A^T & 0 & 0 & 0 & 0 & A^T & 0 & 0 & 0 & 0 \\ B_1^T & 0 & 0 & 0 & 0 & B_1^T & 0 & 0 & 0 & 0 \\ B_2^T & 0 & 0 & 0 & 0 & B_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &- \left[\begin{array}{ccccc|ccccc} 0 & A & B_1 & B_2 & 0 & 0 & A & B_1 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & E & 0 & 0 & 0 & 0 & A & B_1 & B_2 & 0 \\ -E^T-A^T & 0 & 0 & 0 & C_1^T & 0 & 0 & 0 & 0 & 0 \\ -B_1^T & 0 & -\gamma^2 I & 0 & D_{11}^T & 0 & 0 & 0 & 0 & 0 \\ -B_2^T & 0 & 0 & 0 & D_{12}^T & 0 & 0 & 0 & 0 & 0 \\ 0 & -C_1 & -D_{11} & -D_{12} & -I & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

In order to use this palindromic pencil for the verification the conditions C_1), C_2) and C_3) we need to discuss the relationship of the deflating subspaces of (3.11) and (6.13).

LEMMA 6.3. *Let* $[X_1^T, \ldots, X_{10}^T]^T$ *span a stable deflating subspace of the matrix pencil* $\lambda U_{H,lin} - V_{H,lin}$. *Then* $[X_1^T + X_6^T, X_2^T + X_7^T, X_3^T + X_8^T, X_4^T + X_9^T, X_5^T + X_{10}^T]^T$ *spans a stable deflating subspace of* $\lambda U_H - V_H(\gamma)$ *in (3.11).*

Proof. The result follows by some simple matrix calculations. □

7. Numerical Methods. In this section we discuss how we can use the results from the previous sections to calculate the (sub-)optimal value γ_{mo}. We make use of a procedure to verify if for a fixed value γ the desired controller exists and then proceed with a bisection procedure to find γ_{mo}.

If we use the first approach of attaining a palindromic pencil, we need the following steps that we have reached to verify γ_{mo}:

`Procedure 1a` (classification of γ)
`Input:` Data of system (1.1), value $\gamma \geq 0$.
`Output:` Decision whether $\gamma < \gamma_{mo}$ or $\gamma > \gamma_{mo}$.

(1) Formulate palindromic pencil (6.6) and (6.7).
(2) Calculate the eigenvalues of (6.6) and (6.7) and let $s_H := \dim\ker U_H$ and $s_J := \dim\ker U_J$ be the number of stable eigenvalues of the two pencils respectively that are not equal to one and let v_H and v_J be the number of stable eigenvalues of the two pencils respectively including all eigenvalues at one.
IF $v_H < r$ or $v_J < r$, then $\gamma < \gamma_{mo}$, STOP.
ELSEIF $s_H < r$ and $v_H - \min\{\dim\ker(U_H), m_1+m_2+p_1\} < r$, then $\gamma < \gamma_{mo}$ STOP.,
ELSEIF $s_J < r$ and $v_J - \min\{\dim\ker(U_J), m_1+m_2+p_1\} < r$, then $\gamma < \gamma_{mo}$ STOP.
ELSE
(2a) Use a structure preserving method to calculate the deflating subspaces of (6.6) and (6.7) associated with the semi-stable eigenvalues.
IF rank$EX_{H,1} < r$, then $\gamma < \gamma_{mo}$, STOP.
ELSEIF rank$E^T X_{J,1} < r$, then $\gamma < \gamma_{mo}$, STOP.
ELSEIf $\hat{\mathfrak{Y}}$ is not positive semi-definite, then $\gamma < \gamma_{mo}$, STOP.
ELSE $\gamma \geq \gamma_{mo}$.

If we use the second approach in the construction of the palindromic problem, we perform the following steps

`Procedure 1b` (classification of γ)
`Input:` Data of system (1.1), value $\gamma \geq 0$.
`Output:` Decision whether $\gamma < \gamma_{mo}$ or $\gamma > \gamma_{mo}$.

(1) Formulate quadratic palindromic polynomials (6.12) from $\lambda U_H + V_H(\gamma)$ and $\lambda U_J + V_J(\gamma)$.
(2) Use structure preserving linearization to formulate palindromic pencils.
(3) Use a structure preserving method to calculate the eigenvalues and let s_H and s_J be the number of stable eigenvalues of the two pencils respectively that are not equal to zero.
IF $s_H < r$ and $\dim\ker(V_H) < r - s_H$, then $\gamma < \gamma_{mo}$, STOP.
ELSEIF $s_J < r$ and $\dim\ker(V_J) < r - s_J$, then $\gamma < \gamma_{mo}$, STOP.
ELSE
(3a) Use a structure preserving method to calculate the deflating subspaces of the linearized palindromic polynomials associated with the eigenvalues inside the unit disc. IF rank$EX_{H,1} < r$, then $\gamma < \gamma_{mo}$, STOP.
ELSEIF rank$E^T X_{J,1} < r$, then $\gamma < \gamma_{mo}$, STOP.
ELSEIf $\hat{\mathfrak{Y}}$ is not positive semi-definite, then $\gamma < \gamma_{mo}$, STOP.
ELSE $\gamma \geq \gamma_{mo}$.

To actually find γ_{mo} we may use a bisection together with one of the above algorithms:

`Procedure 2` (Bisection)
`Input:` γ_{up} and γ_{lo}
`Output:` γ_{mo}
(1) IF $\gamma_{up} - \gamma_{lo} < d$, d sufficiently small, then $\gamma_{mo} = \gamma_{lo}$.
(2) ELSEIF $\gamma < \gamma_{mo}$, set $\gamma_{lo} = \gamma$ and $\gamma = (\gamma_{mo} + \gamma_{up})/2$ and run Algorithm 1.

ELSE set $\gamma_{up} = \gamma$ and $\gamma = (\gamma_{mo} + \gamma_{up})/2$ and run Algorithm 1a/1b.

8. Example. To illustrate the functionality of our approach, we consider the following example. Let the discrete-time system of the form (1.1) be given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \ C_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \ D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ D_{21} = 1$$

Using the pencils of the form (3.11) and (3.12) and the QZ-algorithm in MATLAB to calculate the eigenvalues and the deflating subspaces associated with the eigenvalues inside the unit disc and Procedure 2 to determine the optimal value for gamma, we computed $\gamma_{opt} = 4.7684$. If we use the palindromic reformulation Procedure 1b and make use of the palindromic structure with the methods from [29] when calculating eigenvalues and deflating subspaces we obtain a smaller value $\gamma_{opt} = 4.4163$. The reason is that the matrices of the eigenvalue problem become more and more ill conditioned when approaching the optimal value for γ and algorithms that do not take the special structure of the pencil into account will not get as closed to the optimal value as methods that make use of the special structure of a matrix pencil.

9. Conclusion. In this paper we developed conditions for the existence of optimal $\mathcal{H}_\infty$-controllers for discrete-time descriptor systems of arbitrary index. The conditions are expressed in terms of matrix pencils. Furthermore we used several approaches to reformulate the results in terms of palindromic matrix pencils that set us in position to apply structured numerical methods which lead to even better results. We illustrated our approach by a numerical example.

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