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FREQUENCY DOMAIN METHODS AND DECOUPLING OF LINEAR CONSTANT COEFFICIENT INFINITE DIMENSIONAL DIFFERENTIAL ALGEBRAIC SYSTEMS*

TIMO REIS† AND CAREN TISCHENDORF‡

Abstract. We discuss the analysis of constant coefficient linear differential algebraic equations $E\dot{x}(t) = Ax(t) + q(t)$ on infinite dimensional Hilbert spaces. We give solvability criteria of these systems which are mainly based on Laplace transformation. Furthermore, we investigate decoupling of these systems, motivated by the decoupling of finite dimensional differential algebraic systems by the Kronecker normal form. Applications are given by the analysis of mixed systems of ordinary differential, partial differential and differential algebraic equations.

Key words. partial differential-algebraic equations, index, infinite dimensional linear system theory

AMS subject classifications. 34A09, 34A30, 93A10, 34G10, 35E20

1. Introduction. In today's engineering applications, there an increasing interest in partial differential algebraic equations (PDAE's), which are mainly coupled systems of partial differential equations (PDE's) and differential-algebraic equations (DAE's). More concrete, they appear e.g. in modelling and simulation of electrical circuits with further effects which are modelled by PDE's. These effects can be parasitic like transmission lines or heat conduction [2, 14, 23] as well as they could be the result of a more reliable modelling of complex components like semiconductor devices [3, 29, 33]. Moreover, PDAE's are the outcome of mathematical models of several mechanical systems like elastic multibody systems [10] or biomechanical systems like blood flow networks. In order to study these problems in a mathematically systematic way, we are led to differential algebraic systems

$$F\left(\dot{x}(t), x(t), t\right) = 0 \tag{1.1}$$

in an abstract setting, the so called abstract DAE's (ADAE's). The unknown function $x(\cdot)$ is now a path in an appropriate (mostly infinite dimensional) Hilbert space, and the Frechét derivative $\frac{\mathrm{d}}{\mathrm{d}\dot{x}}F(\dot{x}, x, t)$. has a nontrivial nullspace, in general. In this work, we focus on the linear constant coefficient case

$$E\dot{x}(t) = Ax(t) + q(t) \tag{1.2}$$

and making use of that gaining additional structure. $E : X \to Z$ is now a bounded linear operator and X, Z are some Hilbert spaces. In many practical cases, A is often acting on some product spaces and it is a block operator containing differential and evaluation operators, for example. Hence, it is natural to assume that it is unbounded in general and is defined on some proper subspace $D(A) \subset X$.
The aim of this work is a step-by-step generalization of the known theory for the finite dimensional version of (1.2), in which case we have square matrices $E, A \in \mathbb{R}^{n\times n}$.
The finite dimensional systems are well-studied and subject of various textbooks like e.g. [4], [5] and [15]. The matrix pair (E, A) is said to be *regular*, if $\det(sE - A)$ does

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not vanish identically, i.e. $\det(sE - A) \not\equiv 0$. For regular matrix pairs, it is known that there exist invertible matrices $T, W \in \mathbb{R}^{n\times n}$, such that

$$(WET, WAT) = \left(\begin{pmatrix} N & \\ & I \end{pmatrix}, \begin{pmatrix} I & \\ & \bar{A} \end{pmatrix} \right), \tag{1.3}$$

where $N \in \mathbb{R}^{n_\infty \times n_\infty}$ is nilpotent and $\bar{A} \in \mathbb{R}^{(n-n_\infty)\times(n-n_\infty)}$ is an arbitrary square matrix. The representation (1.3) is called *Kronecker normal form* of (E, A). Further, the nilpotency index $\nu \in \mathbb{N}$ of N, i.e. the number with $N^{\nu-1} \neq 0$, $N^\nu = 0$, is well-defined by (E, A) and called the *Kronecker index*. Multiplying a finite dimensional DAE of the form (1.2) from the left side with W and insert the identity $I = TT^{-1}$, we get

$$WET(T^{-1}\dot{x}(t)) = WET(T^{-1}x(t)) + Wq. \tag{1.4}$$

If we introduce $(x_1\ x_2) = T^{-1}x$ and $(x_1\ x_2) = Wq$, the equivalent DAE in *Kronecker normal form* is obtained, namely the following decoupled differential equations

$$N\dot{x}_1(t) = x_1(t) + q_1(t) \tag{1.5a}$$
$$\dot{x}_2(t) = \bar{A}x_2(t) + q_2(t). \tag{1.5b}$$

(1.5a) contains algebraic equations and some further hidden relations, being algebraic, when (1.5a) is differentiated, and thus it is called the *(hidden) algebraic constraints*. The second expression (1.5b) is nothing but an ordinary differential equation extracted from the DAE (1.2) and is therefore called the *inherent ODE*. Altogether, solutions of these equations are given by

$$x_1(t) = -\sum_{k=0}^{\nu-1} N^k q_1^{(k)}(t), \quad x_2(t) = e^{\bar{A}t}x_2(0) + \int_0^t e^{\bar{A}(t-\tau)}q_2(\tau)d\tau. \tag{1.6}$$

In [4] and [13], for example, algorithms for the computations of T and W are presented. Due to (1.6), it can be seen that the Kronecker index ν of (E, A) is the minimal integer which satisfies an inequality

$$\|x(T)\| \leq c_T \cdot \sum_{k=0}^{\nu} \|q^{(k)}(\cdot)\|_{L_2([0,T],\mathbb{R}^n)} \tag{1.7}$$

for some positive constant c. $L_2([0, T], \mathbb{R}^n)$ denotes the Lebesque space of square integrable functions with values in $\mathbb{R}^n$.

In this work, we will perform an analysis as in (1.7) as well as we generalize the decoupling framework to infinite dimensional descriptor systems. We will obtain a form as follows

$$\begin{pmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} I & K \\ 0 & \mathfrak{U} \\ 0 & R \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}. \tag{1.8}$$

The second two lines, i.e. the system

$$\begin{array}{rcl} \dot{x}_2(t) & = & \mathfrak{U}x_2 + q_2(t) \\ 0 & = & x_2(t) + q_3(t), \end{array} \tag{1.9}$$

play the role of the inherent ODE for the finite dimensional case. This type of equations is called an *abstract boundary control system*, since, in an abstract setting, boundary controlled systems can be written in this way. After solving (1.9) for x_2, we obtain for the first component of the state vector

$$x_1(t) = -\sum_{k=0}^{\nu-1} N^k(q_1^{(k)} + Kx_2(t)).$$

An extraordinary role is taken by the coupling term K. Due to the existence of the Kronecker normal form, in the finite dimensional case there can be always found a representation with $K = 0$. However, this is not true for infinite dimensional DAE's. It will turn out that K has to satisfy a certain boundedness condition in order to guarantee that it can be eliminated. The proof of the existence of the form (1.8) is constructive and requires some projector chain to be existent and stagnant. There, we lean against the results of [17]. Besides that E is bounded, we will make the assumption that the *generalized resolvent* $(sE - A)^{-1}$ is bounded and analytic for s in some complex half-plane and has there, it has at most polynomial growth in s. With the Laplace transformation of the ADAE, we will shift the problem into some frequency domain spaces, namely the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$. From that, we will derive some criteria for the solvability of ADAE's. Many examples of practical relevance, especially coupled systems of DAE's and PDE's, fulfill the requirements, we make on E and A.

We briefly resume the actual state of affairs concerning ADAE's. [12] considers systems (1.2), where he assumes that E is indeed injective but not boundedly invertible. Although these assumptions are almost disjoint to those, we make, the mathematical methods for the analysis of the solvability are based on Laplace transform and hence, they are somehow similar to our guesses. The application of that work mainly focuses on PDE's with spacial singularities. In the papers of *Thaller et. al.* (e.g. [31, 32]), besides the boundedness $E : X \to Z$, they additionally assume that

$$\{x \in D(A) : \; Ax \in \operatorname{im} E\} \cap \ker E = \{0\}.$$

It will turn out that this assumption is equivalent to the index of the ADAE being less than 2, in our formulation. Moreover, both assume that the spaces, we denoted by Z and X, coincide. As we will see with the given examples, this is not reasonable for the consideration of coupled systems. More related to this work is [17]. There, abstract differential algebraic systems of the form

$$A(t)\frac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t)) + B(t)x(t) = q(t) \tag{1.10}$$

are considered and an extraction of the algebraic relations is performed. Especially, this approach is applied and in [3, 29, 33] in modelling and simulation of analog circuits. Indeed, this approach is more general than ours since time-varying operators are considered. On the other hand, the presented theory is close to the given practical examples and for the applicability of those results, some inspired homogenizations have to be performed. Moreover, there is no uniform theory for the solvability of this type of equations. Since we assume *time-invariance*, i.e. our operators E and A do not depend on time, a much bigger mathematical framework is disclosed, of which we can make use of. As an example, we can apply methods based on Laplace transformation. Possible applications of this paper are given by the structure analysis of coupled systems. Especially, by the transformation into (1.8), we get inside into the system

behavior. A theory which could benefit from this work is that of *consistent initialization* of PDAE's (see [25]). In our abstract formalism, this denotes that for a given inhomogeneity $q(\cdot)$ we determine the x_0, for which the abstract equation (1.2) possesses a solution with $x(0) = x_0$. It is clear that not every initial condition is allowed since even in the finite dimensional case, it has to fulfill the algebraic constraints and the hidden ones. In this work, we will always assume homogeneous initial condition and hence, we will claim some extra conditions on $q(\cdot)$. Another application is given by the perturbation analysis for these systems, i.e. the sensitivity of the solution with respect to the inhomogeneity. Outgoing from the decoupling theory of systems (1.2), another possible application is the control and observation of PDAE systems [24]. A generalization of the system theoretic concepts of reachability and observability to the infinite dimensional differential algebraic case can be performed with the help of the presented theory. For finite dimensional DAE systems, these concepts are well-known and subject of various works like e.g. [4, 5, 9, 16]. As well, there is an advanced theory about the control and observation of infinite dimensional ODE systems available. With the decoupling of ADAE's, we are able to link these two theories.
This work is organized as follows: In the first section, the functional analytic framework is presented. The needed spaces are introduced. Thereafter, the solvability of abstract differential algebraic systems is analyzed in Section 2. Furthermore, the concept of perturbation index and its interpretation in both time and frequency domain is treated. In section 3, we briefly repeat the decoupling theory for finite dimensional constant coefficient differential algebraic systems under an alternative point of view and, outgoing from that, we consider the infinite dimensional case. We introduce the ADAE index, a generalization of the Kronecker index and deal with its comparison to the perturbation index. Before this work is concluded, an example from analytic circuit theory is given in the fourth section in order to illustrate the applicability of the presented theory.

2. Preliminaries. In this section, we collect some necessary fundamentals. Let X, Z be complex Hilbert spaces throughout this work. The space of bounded linear operators from X to Z is denoted by $L_b(X, Z)$ and associated with the usual operator norm $\|\cdot\|_{L_b(X,Z)}$. A linear operator Q with $Q^2 = Q$ is called *projector onto* $\operatorname{im} Q$ *and along* $\ker Q$. The complementary projector $P = I - Q$ satisfies $\operatorname{im} P = \ker Q$ and $\ker P = \operatorname{im} Q$. It is known that for all subspaces $X_1, X_2 \subset X$ with $X_1 \oplus X_2 = X$, there exists a projector onto X_1 and along X_2. Moreover, that projector is bounded if and only if both X_1 and X_2 are closed (see [27]).
Further, we introduce several function spaces needed throughout this work. More details about the definitions can be found in [1, 8, 11, 21] and [27]. Let I be an interval containing zero during this section. The space of distributions on I with values in X is denoted by $\mathcal{D}'(I, X)$ and the Lebesque space consisting of measurable and square integrable functions mapping from I to X is denoted by $L^2(I, X)$. Outgoing from that, we introduce the weighted Lebesque spaces

$$L^2_\omega(I, X) = \{f(\cdot) \in \mathcal{D}'(I, X) : \; e^{-\omega\cdot} f(\cdot) \in L^2(I, X)\}$$

associated with the norm $\|f\|^2_{L^2_\omega(I,X)} = \|e^{-\omega\cdot} f(\cdot)\|_{L^2(I,X)}$.
For $k \in N$, we denote $f^{(k)}$ to be the k-th distributional derivative of f and define the *Sobolev space with homogeneous initial values*

$$H^k_{0,\omega}(I, X) = \{f \in \mathcal{D}'(I, X) : \; f^{(k)} \in L^2_\omega(I, X) \text{ and } f^{(i)}(0) = 0 \text{ for } i = 0, \ldots, k-1\}$$

and $H_0^k(I,X) = H_{0,0}^k(I,X)$. The norms are given by $\|f\|_{H_{0,\omega}^k(I,X)} = \|f^{(k)}\|_{L_\omega^2(I,X)}$.
It can be seen that the distributional derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} : H_0^k(I,X) \to H_0^{k-1}(I,X)$$

is an isometry. The inverse mapping is given by integration $(\int f)(t) := \int_0^t f(\tau)\tau$, and therefore, it holds $\| \int f(\cdot)\|_{H_0^k(I,X)} = \|f(\cdot)\|_{H_0^{k-1}(I,X)}$.
This leads us to define the Sobolev spaces $H_{0,\omega}^{-k}(I,X)$ with negative exponents, namely

$$H_{0,\omega}^{-k}(I,X) = \{f \in \mathcal{D}'(I,X) : (\textstyle\int)^k f \in L_\omega^2(I,X)\}.$$

The norm in that Hilbert space is given by $\|f\|_{H_{0,\omega}^{-k}(I,X)} := \| (\int)^k f\|_{H_{0,\omega}^{-k+1}(I,X)}$. For $k \in \mathbb{Z}$, $T > 0$ and $f \in H_{0,\omega}^k([0,\infty),X)$, it can be seen that $f|_{[0,T]}$ being the restriction of f to the interval $[0,T]$, we have $f|_{[0,T]} \in H_0^k([0,T],X)$, where . For convenience, we shortly write $\|f\|_{H_0^k([0,T],X)} := \| f|_{[0,T]} \|_{H_0^k([0,T],X)}$.
The complex half-plane consisting of numbers whose real part exceeds $\omega \in \mathbb{R}$ is denoted by $\mathbb{C}_\omega^+$. The *Hardy space* $\mathcal{H}_2(\mathbb{C}_\omega^+,X)$ consists of all holomorphic functions $f : \mathbb{C}_\omega^+ \to X$ with the property

$$\|f\|_{\mathcal{H}_2(\mathbb{C}_\omega^+,X)} := \sup_{\gamma>\omega} \|f(\gamma + i\cdot)\|_{L^2(\mathbb{R},X)} < \infty.$$

For $k \in \mathbb{Z}$, we define the *monomial weighted* $\mathcal{H}_2$ *space* $s^k \cdot \mathcal{H}_2(\mathbb{C}_\omega^+,X)$ associated with the norm

$$\|f\|_{s^k\cdot\mathcal{H}_2(\mathbb{C}_\omega^+,X)} := \sup_{\gamma>\omega} \left\| \frac{1}{(\cdot)^k} f(\cdot) \right\|_{\mathcal{H}_2(\mathbb{C}_\omega^+,X)}$$

Furthermore, it can be seen that the multiplication with s defines an isometry from $s^k \cdot \mathcal{H}_2(\mathbb{C}_\omega^+,X)$ to $s^{k+1} \cdot \mathcal{H}_2(\mathbb{C}_\omega^+,X)$.
In the following, we address the relation between the introduced Hardy and Sobolev spaces. The Laplace transform $\mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}dt$ is known to be an isometry from $L_\omega^2([0,\infty),X)$ to $\mathcal{H}_2(\mathbb{C}_\omega^+,X)$ [34, 37]. For $f \in V := H_\omega^k([0,\infty),X) \cap L_\omega^2([0,\infty),X)$, the differentiation rule of Laplace transform [11] reads

$$\mathcal{L}(f^{(k)})(s) = s^k \mathcal{L}(f)(s).$$

Since V is dense in $H_\omega^k([0,\infty),X)$, the Laplace transform extends to an isometry $\mathcal{L}_k$ mapping from $H_{0,\omega}^k([0,\infty),X)$ to the space $s^{-k}\cdot\mathcal{H}_2(\mathbb{C}_\omega^+,X)$ via $\mathcal{L}_k(f) := s^k\mathcal{L}\left((\int)^k f\right)$.
In terms of better overview, we skip the index and write $\mathcal{L}$ instead of $\mathcal{L}_k$ if it can be seen from the context.
In addition, the Hardy space $\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)$ is the space of functions with values in $L_b(X,Y)$ being holomorphic and bounded on $\mathbb{C}_\omega^+$ and it is a Banach space associated with the norm

$$\|F\|_{\mathcal{H}_\infty(C_\omega^+,X,Y)} := \sup_{s\in\mathbb{C}_\omega^+} \|F(s)\|_{L_b(X,Y)}.$$

Further, we define the monomial weighted $\mathcal{H}_\infty$ spaces $s^k \cdot \mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)$ and, similar to $s^k\mathcal{H}_2$, the norm in that space reads

$$\|F\|_{s^k\cdot\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)} := \left\| \frac{1}{(\cdot)^k} F(\cdot) \right\|_{\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)}.$$

An operator valued holomorphic function $G \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$ defines a bounded linear operator $\mathcal{H}_2(\mathbb{C}_\omega^+, X) \to \mathcal{H}_2(\mathbb{C}_\omega^+, Y)$ via pointwise multiplication $(Gf)(s) := G(s)f(s)$ and in [37] it is shown that

$$\|G\|_{L_b(\mathcal{H}_2(\mathbb{C}_\omega^+,X),\mathcal{H}_2(\mathbb{C}_\omega^+,Y))} = \|G\|_{\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)}. \tag{2.1}$$

Further, G defines a bounded map $\mathcal{F}$ from $L^2_\omega([0,\infty), X)$ to $L^2_\omega([0,\infty), Y)$ by

$$x \mapsto \mathcal{L}^{-1}\left(G(\cdot)\hat{x}(\cdot)\right). \tag{2.2}$$

The norms these operator norms equal, i.e.

$$\|\mathcal{F}\|_{L^2_\omega([0,\infty),X),\, L^2_\omega([0,\infty),Y)} = \|G\|_{\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)}.. \tag{2.3}$$

We repeat the following results from [7] and [36].

Theorem 2.1. *Let $G \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$ and the corresponding $\mathcal{F}$ be defined with (2.2). Then, for all $T > 0$, $y := \mathcal{F}x$, the restriction $y|_{[0,T]} \in L^2([0,T], Y)$, only depends on $x|_{[0,T]} \in L^2([0,T], X)$. Furthermore, for some constant $c_T > 0$, we have an estimate*

$$\|y\|_{L_2([0,T],Y)} \le c_T \|x\|_{L_2([0,T],X)}. \tag{2.4}$$

Conversely, if $\mathcal{F} \in L_b(L^2_\omega([0,\infty), X), L^2_\omega([0,\infty), Y))$ satisfies that properties, there exists a $G \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$, such that $\mathcal{F}x = \mathcal{L}^{-1}\left(G(\cdot)\hat{x}(\cdot)\right)$.

We can conclude that x does not depend on future values of the inhomogeneity i.e. $q(t_2)$ has no influence on $x(t_1)$ for $t_1 < t_2$. This property is called *causality*.
Subsequently, we formulate a generalization of the previous statements to the monomial weighted case. It can be seen that $G \in s^k\mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$ defines a bounded linear operator $s^l\mathcal{H}_2(\mathbb{C}_\omega^+, X) \to s^{l+k}\mathcal{H}_2(\mathbb{C}_\omega^+, Y)$ by pointwise multiplication. Moreover, we have

$$\|G\|_{L_b(s^l\mathcal{H}_2(\mathbb{C}_\omega^+,X),s^{l+k}\mathcal{H}_2(\mathbb{C}_\omega^+,Y))} = \|G\|_{s^k\mathcal{H}_\infty(\mathbb{C}_\omega^+,X,Y)} \text{ for all } l,k \in \mathbb{Z}. \tag{2.5}$$

Further, G defines map from $\mathcal{F} \in L_b(H^l_{0,\omega}([0,\infty), X), H^{l+k}_{0,\omega}([0,\infty), Y))$ by

$$x \mapsto \mathcal{F}x = \mathcal{L}^{-1}\left(G(\cdot)\hat{x}(\cdot)\right) \tag{2.6}$$

and the corresponding operator norms coincide. As a conclusion of Theorem 2.1, we formulate the following.

Corollary 2.2. *Let $k, l \in \mathbb{Z}$, $G \in s^k\mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$ and, furthermore, the operator $\mathcal{F}$ is defined as $y := \mathcal{F}x = \mathcal{L}^{-1}\left(G(\cdot)\hat{x}(\cdot)\right)$. Then for all $T > 0$, the restriction $y|_{[0,T]} \in H_0^{l+k}([0,T], Y)$ only depends on $x|_{[0,T]} \in H_0^l([0,T], X)$ and the following inequality holds for some constant $c_T > 0$*

$$\|y\|_{H_0^{l+k}([0,T],Y)} \le c_T \|x\|_{H_0^l([0,T],X)}. \tag{2.7}$$

Conversely, for $\mathcal{F} \in L_b(H^l_{0,\omega}([0,\infty), X), H^{l+k}_{0,\omega}([0,\infty), Y))$ with the properties above, there exists a $G \in s^k\mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Y)$, such that

$$\mathcal{F}x = \mathcal{L}^{-1}\left(G(\cdot)\hat{x}(\cdot)\right) \text{ for all } x \in H^l_{0,\omega}([0,\infty), X)$$

Overall, by the Laplace transform, we discovered a duality between the defined Sobolev and Hardy spaces. Since the Laplace transform is closely related to Fourier transform, motivated by practical examples, one often speaks of *frequency domain* if Hardy spaces are meant and, correspondingly, the *time domain* is identified with Sobolev (or Lebesque) spaces in literature like [37], as an example.
Further relations between the defined spaces are that, by restriction of analytic functions on $\mathbb{C}^+_{\omega_2}$ to some smaller half-plane, we have the following dense inclusions

$$\begin{array}{ll} s^k\mathcal{H}_2(\mathbb{C}^+_{\omega_1}, X) \subset s^{k_2}\mathcal{H}_2(\mathbb{C}^+_{\omega_2}, X) & \text{for } \omega_1 \leq \omega_2 \\ s^{k_1}\mathcal{H}_2(\mathbb{C}^+_{\omega_1}, X) \subset s^{k}\mathcal{H}_2(\mathbb{C}^+_{\omega_2}, X) & \text{for } k_1 \leq k_2 \text{ and } 0 > \omega_1 \leq \omega_2. \end{array} \tag{2.8}$$

By a Laplace transform, one can obtain that analogous relations hold for the Sobolev spaces, namely

$$\begin{array}{ll} H^k_{0,\omega_1}([0,\infty), X) \subset H^k_{0,\omega_2}([0,\infty), X) & \text{for } \omega_1 \leq \omega_2 \\ H^{k_1}_{0,\omega_1}([0,\infty), X) \subset H^{k_2}_{0,\omega_2}([0,\infty), X) & \text{for } k_1 \leq k_2 \text{ and } 0 > \omega_1 \leq \omega_2. \end{array} \tag{2.9}$$

For the $\mathcal{H}_\infty$ spaces, there hold inclusions being analogous to (2.8), but, however, these inclusions are not dense.

3. Solvability of Abstract Differential Algebraic Systems. The aim of this section is to derive conditions for infinite dimensional systems

$$E\dot{x}(t) = Ax(t) + q(t) \tag{3.1}$$

to be solvable for $x(\cdot)$ in some distributional function space with values in X. As in the introduction, we generally assume that X, Z are some Hilbert spaces and for the operators, we assume $E \in L_b(X, Z)$ and $A : D(A) \subset X \to Z$ is assumed to be densely defined. The inhomogeneity $q(\cdot)$ is a Z-valued of sufficient smoothness. This will be precised throughout this section. At first, we analyze the solvability of (3.1). In analogy to the finite dimensional case, we call the operator pair (E, A) is said to be *regular*, if the *resolvent set* defined as

$$\rho(E, A) := \{\lambda \in \mathbb{C} :\ (\lambda E - A)^{-1} \in L_b(Y, X) \text{ exists}\}$$

is non-empty. Moreover, it can be shown that the A with domain $D(A)$ is closed if (E, A) is regular. In contrast to the finite dimensional case, the regularity of (E, A) alone does not suffice to guarantee the solvability of (3.1). This even holds if $X = Z$ and E is just the identity, where (3.1) stands for an abstract ordinary differential equation. There, the question of solvability is intimately connected with the property of A being the generator of a semigroup, a generalization of the matrix exponential to the infinite dimensional case (see [21]).
Subsequently, we give a frequency-domain criterion for the solvability of abstract differential algebraic systems and the smoothness of the solution $x(\cdot)$.

THEOREM 3.1. *Let $\bar{\nu} \in \mathbb{N}, \omega \in \mathbb{R}$ and $(sE - A)^{-1} \in s^{\bar{\nu}} \cdot \mathcal{H}_\infty(\mathbb{C}^+_\omega, Z, X)$ and let $q \in H^{\nu}_{0,\omega}([0,\infty), Z)$. Then, there exists a unique $x(\cdot) \in H^{\nu-\bar{\nu}}_{0,\omega}(X)$ satisfying the abstract differential algebraic equation (3.1). Moreover, there exist constants $c, c_T > 0$, such that for all $T > 0$ holds*

$$\|x\|_{H^{\nu-\bar{\nu}}_{0,\omega}([0,\infty),X)} \leq c\, \|q\|_{H^{\nu-\bar{\nu}}_{0,\omega}([0,\infty),Z)} \tag{3.2a}$$

$$\|x\|_{H^{\nu-\bar{\nu}}_{0}([0,T],X)} \leq c_T \|q\|_{H^{\nu-\bar{\nu}}_{0}([0,T],Z)}. \tag{3.2b}$$

Proof. We show that

$$\bar{x}(\cdot) = \mathcal{L}^{-1}\left((sE-A)^{-1}\mathcal{L}(q)(s)\right)(\cdot), \tag{3.3}$$

is the unique solution of (3.1). Then, the inequalities (3.2a) and (3.2b) are satisfied by Corollary 2.2.
At first, we show that $\bar{x}$ is satisfies (3.1). Due to the assumption $q \in H^{\nu}_{0,\gamma}([0,\infty), Z)$, we have $\mathcal{L}(q)(s) \in s^{-\nu} \cdot \mathcal{H}_2(\mathbb{C}^+_\omega, Z)$ and therefore

$$\mathcal{L}(\bar{x})(s) = (sE-A)^{-1}\mathcal{L}(q)(s) \in s^{\bar{\nu}}\cdot\mathcal{H}_\infty(\mathbb{C}^+_\omega, Z, X)s^{-\nu}\cdot\mathcal{H}_2(\mathbb{C}^+_\omega, Z) \subset s^{\bar{\nu}-\nu}\cdot\mathcal{H}_2(\mathbb{C}^+_\omega, X).$$

Then, we can deduce $\bar{x}(\cdot) \in H^{\nu-\bar{\nu}}_{0,\omega}([0,\infty), X)$ and we calculate

$$\begin{aligned}
E\dot{\bar{x}} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{-1}\left(E(sE-A)^{-1}\mathcal{L}(q)(s)\right)\\
&= \mathcal{L}^{-1}\left(sE(sE-A)^{-1}\mathcal{L}(q)(s)\right)\\
&= \mathcal{L}^{-1}\left(A(sE-A)^{-1}\mathcal{L}(q)(s) + \mathcal{L}(q)(s)\right)\\
&= A\mathcal{L}^{-1}\left((sE-A)^{-1}\mathcal{L}(q)(s)\right) + \mathcal{L}^{-1}\left(\mathcal{L}(q)(s)\right)\\
&= A\bar{x} + q.
\end{aligned}$$

The second last equality holds due to the facts that A with domain $D(A)$ is closed and $(sE-A)^{-1}$ maps into $D(A)$.
Now we show that the $\bar{x}(\cdot)$ chosen above is the unique solution. Let $x(\cdot) \in H^{\nu-\bar{\nu}}_{0,\omega}(X)$ satisfying (3.1). By a Laplace transform of (3.1), we obtain

$$sE\mathcal{L}(x)(s) = A\mathcal{L}(x)(s) + \mathcal{L}(q)(s)$$

and hence $\mathcal{L}(x)(s) = (sE-A)^{-1}\mathcal{L}(q)(s) = \mathcal{L}(\bar{x})(s)$. Inverse Laplace transform of this equation yields the desired result. ∎
The previous theorem states uniqueness of the solution $x(\cdot) \in H^{\nu-\bar{\nu}}_{0,\omega}([0,\infty), X)$, i.e. we require a zero initial condition $x(0) = 0$ for the solution trajectory. Moreover, due to Corollary 2.2, it can be seen that $(sE-A)^{-1} \in s^{\bar{\nu}-1}\mathcal{H}_\infty(\mathbb{C}^+_\omega, Z, X)$ if and only if the solution mapping of (3.1), namely

$$\begin{array}{rcl} H^{k-\bar{\nu}}_{0,\omega}([0,T], Z) & \longrightarrow & H^{k+1}_{0,\omega}([0,T], X) \\ q(\cdot) & \mapsto & x(\cdot), \end{array} \tag{3.4}$$

is continuous for all $T > 0$ and some (and hence any) $k \in \mathbb{Z}$. Hence, due to Corollary 2.2 and the fact that point evaluation is a bounded mapping from $H^1_0([0,T], Z)$ to $\mathbb{R}$, the $\nu_p \in \mathbb{N}$ defined as

$$\nu_p = \min\{l \in \mathbb{Z} \;:\; (sE-A)^{-1} \in s^{l-1}\mathcal{H}_\infty(\mathbb{C}^+_\omega, Z, X) \text{ for some } \omega \in \mathbb{R}\}$$

is the minimal number for which an estimate

$$\|x(T)\| \leq c_T \cdot \sum_{k=0}^{\nu_p} \|q^{(k)}(\cdot)\|_{L_2([0,T],\mathbb{R}^n)} \tag{3.5}$$

holds for some positive constant c_T. As in the introduction, the constant ν_p is said to be the *perturbation index*. This concept is often used when dealing with DAE's and is therefore subject of various textbooks (see e.g. [4, 5, 35]). We have to notice that these references define the perturbation index slightly different since they take the L_∞-norm instead of L_2-norms, in principle. In [30], the perturbation index is introduced in terms of L_p-spaces. We treated the special case $p = 2$.

4. Decoupling of Differential Algebraic Equations.

4.1. Finite Dimensional DAE's. We briefly present an alternative approach for the decoupling of (finite dimensional) DAE's by using the results of [18]. Later, that approach is taken over to the infinite dimensional case. We define the subsequent matrix chain from which we will derive the equivalence transforms for decoupling the DAE (1.2).

$$
\begin{aligned}
&E_0 := E, \\
&A_0 := A \\
&Q_i \in \mathbb{R}^{n\times n},\ Q_i^2 = Q_i,\ \operatorname{im} Q_i = \ker E_i,\ \sum_{j=0}^{i-1} \ker E_i \subset \ker Q_i = \operatorname{im} P_i \\
E_{i+1} &= E_i - A_i Q_i, \\
A_{i+1} &= A_i P_i.
\end{aligned}
$$

It has to be mentioned that our notation is slightly different to that of [18]. There, the DAE reads $A\dot{x}(t) + Bx(t) = q(t)$ and is a (A, B) regular matrix pair. Consequently, the matrix chain was built by $A_{i+1} := A_i + B_i Q_i$, $B_{i+1} = B_i P_i$. It is clear that both are totally equivalent. Our notation is mainly used in control theory, like e.g. in [9]. The chain above is guaranteed to stagnate and further, the minimal $\nu \in \mathbb{N}$ such that $Q_\nu = 0$ is called *tractability index.* The concept of tractability index has been subject of various publications of MAERZ ET. AL. like e.g. [15] and is also applicable to the case of non-constant-coefficient DAE's of the form (1.10) (see e.g. [19]).
Besides the fact that $\nu < n$, it holds that the space $N_{\Sigma j} := \bigoplus_{i=0}^{j-1} \operatorname{im} Q_i$ does not depend on the particular choice of the projectors Q_i ([19], Theorem 2.3). The following matrix is known to be a projector onto $N_{\Sigma j}$

$$Q_{\Sigma j} := Q_0 + P_0 Q_1 + \ldots + P_0 \cdots P_{j-2} Q_{j-1}.$$

Its complementary projector reads $P_{\Sigma j} = P_0 \cdots P_{j-1}$. The subsequent equivalent transforms can be performed to E and A

$$
\begin{aligned}
E &= E_0 P_0 = (E_0 - A_0 Q_0) P_0 = E_1 P_0 = E_1 P_1 P_0 = (E_1 - A_1 Q_1) P_1 P_0 && \text{(4.1a)}\\
&= E_2 P_1 P_0 = \cdots = E_\nu P_{\nu-1} \cdots P_0 = E_\nu (I - Q_0 - Q_1 - \cdots - Q_{\nu-1}) && \text{(4.1b)}
\end{aligned}
$$

$$
\begin{aligned}
A &= A_0 P_0 + A_0 Q_0 = -(E_0 - A_0 Q_0) Q_0 + A_0 P_0 = -E_1 Q_0 + A_1 && \text{(4.1c)}\\
&= -(E_1 - A_1 Q_1)(P_1 Q_0 + Q_1) + A_1 P_1 = -E_2 (P_1 Q_0 + Q_1) + A_2 && \text{(4.1d)}\\
&= \cdots = -E_\nu (Q_0 + \cdots + Q_{\nu-1}) + A_\nu. && \text{(4.1e)}
\end{aligned}
$$

Since $\ker E_\nu = \{0\}$, the inverse E_ν^{-1} exists. Premultiplying E and A with the matrices $-Q_0 P_1 \cdots P_{\nu-1} E_\nu^{-1}$, ..., $-Q_{\nu-2} P_{\nu-1} E_\nu^{-1}$ and $-Q_{\nu-1} E_\nu^{-1}$ respectively, we obtain the following for $i = 1, \ldots, \nu - 1$

$$
\begin{aligned}
&-Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^{-1} E \\
&= -Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^{-1} E_\nu (I - Q_0 - \cdots - Q_{\nu-1}) \\
&= Q_i - Q_i P_{i+1} \cdots P_{\nu-1} && \text{(4.2a)}
\end{aligned}
$$

$$
\begin{aligned}
&-Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^{-1} A \\
&= -Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^{-1} (-E_\nu (Q_0 + \cdots + Q_{\nu-1}) + A_\nu) \\
&= Q_i P_{i+1} \cdots P_{\nu-1} (Q_0 + \cdots + Q_{\nu-1}) - Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^{-1} A_\nu \\
&= Q_i - Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^{-1} A_\nu. && \text{(4.2b)}
\end{aligned}
$$

Further, we compute

$$\begin{aligned}
&P_0\cdots P_{\nu-1}E_\nu^{-1}E\\
&=P_0\cdots P_{\nu-1}E_\nu^{-1}E_\nu(I-Q_0-\cdots-Q_{\nu-1})\\
&=P_0\cdots P_{\nu-1}(I-Q_0-\cdots-Q_{\nu-1})\\
&=P_0\cdots P_{\nu-1}
\end{aligned}\tag{4.2c}$$

$$\begin{aligned}
&P_0\cdots P_{\nu-1}E_\nu^{-1}A\\
&=P_0\cdots P_{\nu-1}E_\nu^{-1}(-E_\nu(Q_0+\cdots+Q_{\nu-1})+A_\nu)\\
&=P_0\cdots P_{\nu-1}A_\nu.
\end{aligned}\tag{4.2d}$$

We define the vector spaces $X_1 := \operatorname{im} Q_0\times\cdots\times\operatorname{im} Q_{\nu-1}$, $X_2=\operatorname{im} P_{\Sigma\nu}$ and it can be seen that the linear transformations $W:\mathbb{R}^n\to X_1\times X_2$ and $T: X_1\times X_2\to\mathbb{R}^n$ defined below are invertible. Let $z\in Z$, $x_i\in\operatorname{im} Q_i$ for $i=0,\ldots,\nu-1$ and $x_P\in\operatorname{im} P_{\Sigma_\nu}$. Then we declare

$$Wz := \begin{pmatrix}\begin{pmatrix}-Q_0P_1\cdots P_{\nu-1}E_\nu^{-1}z\\ -Q_1P_2\cdots P_{\nu-1}E_\nu^{-1}z\\ \vdots\\ -Q_{\nu-1}E_\nu^{-1}z\end{pmatrix}\\ \\ P_0\cdots P_{\nu-1}E_\nu^{-1}z\end{pmatrix},\qquad T\cdot\begin{pmatrix}\begin{pmatrix}x_0\\ x_1\\ \vdots\\ x_{\nu-1}\end{pmatrix}\\ \\ x_P\end{pmatrix} = x_P+\sum_{i=0}^{\nu-1}P_0\cdots P_{i-1}Q_ix_i.$$

Then, by the matrix calculations (4.2a) and (4.2c), we yield

$$WET=\begin{pmatrix}N_Q & 0\\ \\ 0 & P_{\Sigma\nu}\end{pmatrix},\tag{4.3}$$

and $N_Q: X_1\to X_1$ reads

$$N_Q=\begin{pmatrix}0 & Q_0Q_1 & Q_0P_1Q_2 & Q_0P_1P_2Q_3 & \cdots & Q_0P_1\cdots P_{\nu-2}Q_{\nu-1}\\
 & \ddots & Q_1Q_2 & Q_1P_2Q_3 & & \vdots\\
 & & \ddots & \ddots & \ddots & \vdots\\
 & & & \ddots & \ddots & Q_{\nu-3}P_{\nu-2}Q_{\nu-1}\\
 & & & & \ddots & Q_{\nu-2}Q_{\nu-1}\\
 & & & & & 0\end{pmatrix}.$$

Moreover, we obtain by (4.2b) and (4.2d)

$$WAT=\begin{pmatrix}\begin{pmatrix}Q_0 & & \\ & \ddots & \\ & & Q_{\nu-1}\end{pmatrix} & -\begin{pmatrix}Q_0P_1\cdots P_{\nu-1}E_\nu^{-1}A_\nu\\ Q_1P_2\cdots P_{\nu-1}E_\nu^{-1}A_\nu\\ \vdots\\ Q_{\nu-1}E_\nu^{-1}A_\nu\end{pmatrix}\\ \\ 0 & P_0\cdots P_{\nu-1}E_\nu^{-1}A_\nu\end{pmatrix}.\tag{4.4}$$

It can be seen that N_Q is nilpotent and its nilpotency index does not exceed ν. The fact that it even equals ν can be made clear as follows. We argument that $N_Q^{\nu-1}$ is not the zero matrix. We calculate

$$N_Q^{\nu-1} = \begin{pmatrix} 0 & \cdots & 0 & Q_0 \cdots Q_{\nu-1} \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix} \neq 0.$$

Now assume that $Q_0 \cdots Q_{\nu-1} x = 0$. Hence, it holds $Q_1 \cdots Q_{\nu-1} x \in \ker Q_0 \cap \operatorname{im} Q_1$. From that, we calculate

$$0 = E_1 Q_1 \cdots Q_{\nu-1} x = (E_0 - A_0 Q_0) Q_1 \cdots Q_{\nu-1} x = E_0 Q_1 \cdots Q_{\nu-1} x.$$

This implies $Q_1 \cdots Q_{\nu-1} x \in \ker E_0 = \operatorname{im} Q_0$. Together with the assumption that $Q_1 \cdots Q_{\nu-1} x$ is in $\ker Q_0$, we get $Q_1 \cdots Q_{\nu-1} x = 0$. Iterating this argumentation, we get $Q_{\nu-1} x = 0$ and thus $\ker Q_0 \cdots Q_{\nu-1} = \ker Q_{\nu-1} \neq \{0\}$ and hence, both $Q_0 \cdots Q_{\nu-1}$ and N_Q are non-zero matrices.
By a construction of bijective transformations T_1, T_2 with $T_1 : X_1 \to \mathbb{R}^{n_\infty}$ and $T_2 : X_2 \to \mathbb{R}^{n-n_\infty}$, we get

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} WET \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}^{-1} = \begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix} \tag{4.5a}$$

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} WAT \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}^{-1} = \begin{pmatrix} I & K \\ 0 & \tilde{A} \end{pmatrix}. \tag{4.5b}$$

We observe that the system not yet in Kronecker normal form since a possibly non-vanishing coupling term $K \in \mathbb{R}^{n_\infty \times (n-n_\infty)}$ appears. We will call such a representation a *decoupling form* during this work. A solution of a differential algebraic equation in that form, i.e.

$$\begin{pmatrix} N & \\ & I \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} I & K \\ & \bar{A} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

can be obtained by backsubstitution, namely

$$x_1(t) = -\sum_{k=0}^{\nu-1} N^k \left(K x_2^{(k)}(t) + q_1^{(k)}(t) \right), \text{ where } x_2(t) = \int_0^t e^{\bar{A}(t-s)} q_2(s) ds. \tag{4.6}$$

In order to get a Kronecker normal form with the technique based of matrix chains, the projectors Q_i have to be chosen in a way such that

$$Q_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^{-1} A_j$$

for $j = 0, \ldots, \nu - 1$. Then the terms $Q_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^{-} A_\nu$ vanish and we get a decoupling form with $K = 0$. Then, a transformation of the kind (4.5a), (4.5b) yields a Kronecker normal form. These projectors are called *canonical* and the work [18] deals with their computation. We will deal with canonical projectors in Section 4.3.

4.2. Decoupling in Infinite Dimensions. Here, we develop decoupling forms (4.5a), (4.5b) for the infinite dimensional case. Since projectors are available in infinite dimensional spaces as well, the approach of the previous subsection is useful for that. The following theorem represents the main result concerning decoupling of abstract differential algebraic systems.

THEOREM 4.1. *Let X, Z be Hilbert spaces and let (E, A) be a regular operator pair with $E : X \to Z$ be bounded and $A : D(A) \subset X \to Z$ be densely defined. Moreover, let the operator chain*

$$\begin{aligned}
E_0 &:= E,\\
A_0 &:= A\\
Q_i &\in L_b(X) \cap L_b(D(A)),\ Q_i^2 = Q_i,\ \operatorname{im} Q_i = N_i,\ \sum_{j=0}^{i-1} \ker E_i \subset \ker Q_i = \operatorname{im} P_i\\
N_i &= \ker E_i\\
W_i &\in L_b(Z),\ W_i^2 = W_i,\ \ker W_i = \operatorname{im} E_i,\\
E_{i+1} &= E_i - A_i Q_i,\\
A_{i+1} &= A_i P_i,
\end{aligned}$$

be existent and stagnant, i.e. there exists a ν, such that $N_\nu = \{0\}$. Further, let

$$\operatorname{im} E + A\left(\sum_{k=0}^{\nu-1} N_k\right) \tag{4.7}$$

be closed. Then, there exist Hilbert spaces X_1, X_2, X_3 and mappings $W \in L_b(Z, X_1 \times X_2 \times X_3)$, $T \in L_b(X_1 \times X_2, X)$, where T is bijective and W is injective and has dense range, i.e. $\overline{\operatorname{im} W} = X_1 \times X_2 \times X_3$, such that

$$WET = \begin{pmatrix} N & 0\\ 0 & I\\ 0 & 0\end{pmatrix} : X_1 \times X_2 \longrightarrow X_1 \times X_2 \times X_3, \tag{4.8a}$$

$$WAT = \begin{pmatrix} I & K\\ 0 & \mathfrak{U}\\ 0 & R\end{pmatrix} : X_1 \times D(K) \cap D(\mathfrak{U}) \cap D(R) \longrightarrow X_1 \times X_2 \times X_3, \tag{4.8b}$$

and, in particular, we have that $N \in L_b(X_1)$ is a nilpotent operator whose nilpotency order is ν.

The number ν is called *ADAE index*. This concept was first published in [17] as a generalization of the tractability index to infinite dimensions. Further, that theory is applied and further refined in [29], [3] and [33].

We show the result of Theorem 4.1 constructively and similar to the computations (4.3) and (4.4) for matrix pairs. Here, the main difference to that case is that the injectivity of E_ν only implies the existence of a left inverse E_ν^-, i.e. $E_\nu^- E_\nu = I$ due to the infinite dimensionality of X and Z. The fact that this left inverse can be chosen such that it is bounded is shown in Appendix A. For that, we need the technical condition that the space (4.7) is closed. Although there is a freedom in the choice of the Q_i, it holds that $\sum_{k=0}^{\nu-1} N_k = \operatorname{im} Q_{\Sigma\nu}$ is an invariant of the pair (E, A) (see [19]), and hence, we have no redundancies in our requirements. In the known cases of practical relevance, the existence of the projector chain as the closedness of (4.7) is guaranteed. It remains to be an open question if the regularity of (E, A) and the closedness of the range of E suffices to imply $E_\nu^- \in L_b(Z, X)$.

Proof. [Theorem 4.1] Let $E_\nu^- \in L_b(Z, X)$ be chosen in a way that $E_\nu^- E_\nu = I$ and $E_\nu E_\nu^- = I - W_\nu$. By means of the projectors, we define the Hilbert spaces X_1, X_2 and X_3 by $X_1 := \operatorname{im} Q_0 \times \cdots \times \operatorname{im} Q_{\nu-1}$, $X_2 := \operatorname{im} P_0 \cdots P_{\nu-1}$ and $X_3 := \operatorname{im} W_\nu$ and the transformations $W \in L_b(X_1 \times X_2 \times X_3, Z)$, $T \in L_b(X_1 \times X_2, X)$ by

$$Wz = \begin{pmatrix} \begin{pmatrix} -Q_0 P_1 \cdots P_{\nu-1} E_\nu^- z \\ -Q_1 P_2 \cdots P_{\nu-1} E_\nu^- z \\ \vdots \\ -Q_{\nu-1} E_\nu^- z \end{pmatrix} \\ P_0 \cdots P_{\nu-1} E_\nu^- z \\ W_\nu z \end{pmatrix}, \qquad T \cdot \begin{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\nu-1} \end{pmatrix} \\ x_P \end{pmatrix} = x_P + \sum_{i=0}^{\nu-1} P_0 \cdots P_{i-1} Q_i x_i.$$

It can be seen that T is bijective and W is injective. The relation $\overline{\operatorname{im} W} = X_1 \times X_2 \times X_3$ is implied by the fact $\overline{\operatorname{im} E_\nu^-} = X$.
Now we determine the products WET and WAT. It can be seen that the following expressions read as in (4.3) and (4.4)

$$\begin{pmatrix} \begin{pmatrix} -Q_0 P_1 \cdots P_{\nu-1} E_\nu^- \\ -Q_1 P_2 \cdots P_{\nu-1} E_\nu^- \\ \vdots \\ -Q_{\nu-1} E_\nu^- \end{pmatrix} \\ P_0 \cdots P_{\nu-1} E_\nu^- \end{pmatrix} E\,T, \qquad \begin{pmatrix} \begin{pmatrix} -Q_0 P_1 \cdots P_{\nu-1} E_\nu^- \\ -Q_1 P_2 \cdots P_{\nu-1} E_\nu^- \\ \vdots \\ -Q_{\nu-1} E_\nu^- \end{pmatrix} \\ P_0 \cdots P_{\nu-1} E_\nu^- \end{pmatrix} A\,T.$$

For the remaining part, by using the relations (4.1b) and (4.1e), which hold here as well, we observe

$$\begin{aligned} W_\nu E &= W_\nu E_\nu P_{\nu-1} \cdots P_0 = 0 \\ W_\nu A &= W_\nu(-E_\nu(P_{\nu-1} \cdots P_1 Q_0 + \cdots + P_{\nu-1} Q_{\nu-2} + Q_{\nu-1}) + A_\nu) = W_\nu A_\nu \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} W_\nu ET &= \begin{pmatrix} 0 & \cdots & 0 & 0 \end{pmatrix}, \\ W_\nu AT &= \begin{pmatrix} 0 & \cdots & 0 & W_\nu A_\nu \end{pmatrix}. \end{aligned}$$

Altogether, we get the form (4.8a), (4.8b), and in particular

$$N = \begin{pmatrix} 0 & Q_0 Q_1 & Q_0 P_1 Q_2 & Q_0 P_1 P_2 Q_3 & \cdots & Q_0 P_1 \cdots P_{\nu-2} Q_{\nu-1} \\ & \ddots & Q_1 Q_2 & Q_1 P_2 Q_3 & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & Q_{\nu-3} P_{\nu-2} Q_{\nu-1} \\ & & & & \ddots & Q_{\nu-2} Q_{\nu-1} \\ & & & & & 0 \end{pmatrix}, \quad (4.9a)$$

$$K = -\begin{pmatrix} Q_0 P_1 \cdots P_{\nu-1} E_\nu^- A_\nu \\ Q_1 P_2 \cdots P_{\nu-1} E_\nu^- A_\nu \\ \vdots \\ Q_{\nu-1} E_\nu^- A_\nu \end{pmatrix}, \tag{4.9b}$$

$$\mathfrak{U} = P_0 \cdots P_{\nu-1} E_\nu^- A_\nu \tag{4.9c}$$

$$R = W_\nu A_\nu. \tag{4.9d}$$

We can show analogously to the argumentation in the finite dimensional case that ν coincides with the nilpotency index of N. □

The result of Theorem 4.1 leads to the analysis of systems of the type (1.8). In the projector computations above, we see that the additional relation $0 = Rx_2 + q_3$ on the space X_3 is a consequence of the range defect of E_ν. Due to the dimension formula, X_3 is trivial in the finite dimensional case. Hence we get the abstract boundary control system

$$\begin{aligned} \dot{x}_2(t) &= \mathfrak{U}x_2(t) + q_2(t) \\ 0 &= Rx_2(t) + q_3(t). \end{aligned} \tag{4.10}$$

These systems are well-studied in a system theoretic framework (see e.g. [28, 8]). In [28] it is shown that (4.10) can be transformed into a system

$$\dot{x}_2(t) = \bar{A}x_2(t) + q_2(t) + \bar{B}q_3(t), \tag{4.11}$$

where $\bar{A}$ is a restriction of $\mathfrak{U}$ to the space $D(\bar{A}) = \ker R \cap D(\mathfrak{U})$ and $\bar{B}$ is an operator having its range on some larger space containing X_3, namely the dual of $D(A^*)$, where A^* is the adjoint of A.

Hence the dynamics of the system are mainly determined by the properties of $\bar{A}$. For the solvability of (4.10) and (4.11), $\bar{A}$ has to generate a semigroup $T_{\bar{A}}$. By the Hille-Yosida Theorem [21], $\bar{A}$ is the generator of a strongly continuous semigroup if there exist constants $M > 0, \gamma \in \mathbb{R}$ such that

$$\|(\lambda I - \bar{A})^{-k}\| \leq \frac{M}{(\lambda - \gamma)^k} \qquad \text{for all } k \in \mathbb{N}, \lambda > \gamma.$$

If this is fulfilled, a solution of (1.8) can be found by first solving for x_2 by using the variation of constants formula (see [21]) and then, by inserting this solution, the first component x_1 can be obtained. Analogously to (4.6), we get

$$x_1(t) = -\sum_{k=0}^{\nu-1} N^k \left(Kx_2^{(k)}(t) + q_1^{(k)}(t)\right), \tag{4.12a}$$

$$\text{where} \quad x_2(t) = \int_0^t T_{\bar{A}}(t-s)(q_2(s) + \bar{B}q_3(s)). \tag{4.12b}$$

In order to analyze the property of $\bar{A}$ being a generator of a strongly continuous semigroup, we have to take a closer look at the generalized resolvent. For $\lambda \in \rho(E, A)$, we have

$$\begin{pmatrix} \lambda N - I & -K \\ 0 & \lambda I - \mathfrak{U} \\ 0 & -R \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} (\lambda N - I)^{-1}N & -(\lambda N - I)^{-1}K\,(\lambda I - \mathfrak{U})^{-1}\big|_{\ker R} & (\lambda N - I)^{-1}K\,R^{-1}\big|_{\ker(\lambda I - \mathfrak{U})} \\ 0 & (\lambda I - \mathfrak{U})^{-1}\big|_{\ker R} & -R^{-1}\big|_{\ker(\lambda I - \mathfrak{U})} \end{pmatrix},$$

whereas $z_1 = (\lambda I - \mathfrak{U})^{-1}\big|_{\ker R} x_1$ and $z_2 = R^{-1}\big|_{\lambda I - \mathfrak{U}} x_2$ are defined to be the solutions of the equations

$$\begin{aligned} (\lambda I - \mathfrak{U})z_1 &= x \qquad & (\lambda I - \mathfrak{U})z_2 &= 0 \\ Rz_1 &= 0 & Rz_2 &= x_2. \end{aligned}$$

It can be seen that the unique solvability of these equations is a result of the initial requirement that the operator pair (E, A) is regular. Alternatively, we can express according to (4.12b)

$$\begin{aligned} (\lambda I - \mathfrak{U})^{-1}\big|_{\ker R} &= (\lambda I - \bar{A})^{-1} \\ R^{-1}\big|_{\ker(\lambda I - \mathfrak{U})} &= (\lambda I - \bar{A})^{-1}\bar{B}, \end{aligned}$$

and hence, we have $\rho(E, A) = \rho(\bar{A})$. The property $(sE - A)^{-1} \in s^{\nu_p} \cdot \mathcal{H}_\infty$ is necessary for $\bar{A}$ generating a strongly continuous semigroup. The sufficiency is not guaranteed, it can only be shown that $\bar{A}$ fulfills $\|(\lambda I - \bar{A})^{-1}\| \le M(\gamma - \lambda)^{\tilde{\nu}}$ for some $\tilde{\nu} < \nu_p$, $M, \gamma > 0$ and all $\lambda > \gamma$. Operators having this property generate a so-called *integrated semigroup*, or also called *distributive semigroup*. These semigroups are not continuous functions but distributions of highest order $\tilde{\nu}$ with values in $T_{\bar{A}}$. Integrated semigroups are treated in [20], as an example. They are not subject of this paper.
Since $(sN - I)^{-1}$ is a polynomial with operators as coefficients and has the degree $\nu - 1$, we have

$$\nu \le \min\{l \in \mathbb{Z} \;:\; (sE - A)^{-1} \in s^{l-1}\mathcal{H}_\infty(\mathbb{C}^+_\omega, Z, X) \text{ for some } \omega \in \mathbb{R}\}$$

Hence $\nu \le \nu_p$, i.e. the ADAE index does not exceed the perturbation index. In contrast to the finite dimensional case, this inequality is strict in general.
Trivially, as for finite dimensional DAE's, the decoupling form is not unique. In the following we expose, how two such forms differ. Further, the question arises, if the property of $\bar{A}$ generating a strongly continuous semigroup depends on the particular choice of the decoupling form. Indeed, it can be shown that this property is an invariant of the pair (E, A). This will be a conclusion of the subsequent result. From that, we will obtain some useful consequences for the inherent abstract ODE

THEOREM 4.2. *Let* $(W_1ET_1, W_1AT_1) = (\tilde{E}_1, \tilde{A}_1)$ *and* $(W_2ET_2, W_2AT_2) = (\tilde{E}_2, \tilde{A}_2)$ *be two decoupling forms of the pair* (E, A)*, where* $W_1 \in L_b(Z, X_{11} \times X_{12} \times X_{13})$*,* $W_2 \in L_b(Z, X_{21} \times X_{22} \times X_{23})$*,* $T_1 \in L_b(X_{11} \times X_{12}, X)$ *and* $T_2 \in L_b(X_{21} \times X_{22}, X)$ *are some transformations. In particular, let*

$$\tilde{E}_1 := \begin{pmatrix} N_1 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad \tilde{A}_1 := \begin{pmatrix} I & K_1 \\ 0 & \mathfrak{U}_1 \\ 0 & R_1 \end{pmatrix}, \quad \tilde{E}_2 := \begin{pmatrix} N_2 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad \tilde{A}_2 := \begin{pmatrix} I & K_2 \\ 0 & \mathfrak{U}_2 \\ 0 & R_2 \end{pmatrix}.$$

Then, N_1 *and* N_2 *are similar, i.e. there exists a bounded and boundedly invertible* $T_N \in L_b(X_{11}, X_{21})$ *with* $T_N^{-1} \in L_b(X_{21}, X_{11})$ *such that* $N_1 = T_N^{-1}N_2T_N$*. Additionally, the operators* $\bar{A}_1$ *and* $\bar{A}_2$ *to be the restrictions of* $\mathfrak{U}_1$ *to the space* $\ker R_1$ *and* $\mathfrak{U}_2$ *to* $\ker R_2$ *are similar.*

Proof. Let $\tilde{T} := T_1^{-1}T_2$ and $\tilde{W} := W_2W_1^{-1}$ be partitioned as

$$\tilde{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}.$$

For $\lambda \in \rho(E, A)$, consider

$$(\lambda \tilde{E}_1 - \tilde{A}_1)^{-1}\tilde{E}_1 = \begin{pmatrix} (\lambda N_1 - I)^{-1} N_1 & -(\lambda N_1 - I)^{-1} K_1 (\lambda I - \bar{A}_1)^{-1} \\ 0 & (\lambda I - \bar{A}_1)^{-1} \end{pmatrix},$$

and, analogously $(\lambda \tilde{E}_2 - \tilde{A}_2)^{-1}\tilde{E}_2$. Since $\tilde{T}^{-1}(\lambda \tilde{E}_1 - \tilde{A}_1)^{-1}\tilde{E}_1 \tilde{T} = (\lambda \tilde{E}_2 - \tilde{A}_2)^{-1}\tilde{E}_2$, we have the identity

$$\begin{aligned} \begin{pmatrix} 0 & T_{11}K_{2*} + T_{12}(\lambda I - \bar{A}_2)^{-\nu} \\ 0 & T_{21}K_{2*} + T_{22}(\lambda I - \bar{A}_2)^{-\nu} \end{pmatrix} &= \tilde{T}\left((\lambda \tilde{E}_2 - \tilde{A}_2)^{-1}\tilde{E}_2\right)^{\nu} \\ &= \left((\lambda \tilde{E}_1 - \tilde{A}_1)^{-1}\tilde{E}_1\right)^{\nu} \tilde{T} \\ &= \begin{pmatrix} K_{1*}T_{21} & K_{1*}T_{22} \\ (\lambda I - \bar{A}_1)^{-\nu} T_{21} & (\lambda I - \bar{A}_1)^{-\nu} T_{22} \end{pmatrix} \end{aligned}$$

for some operators K_{1*}, K_{2*}. Hence, we can conclude $T_{21} = 0$. By an analogous argumentation, it is clear that $\bar{T}^{-1}$ has the same block structure as $\bar{T}$ and thus, both T_{11} and T_{22} are boundedly invertible. Moreover, the equality $A_2 = \bar{W} A_1 \bar{T}$ implies

$$\begin{pmatrix} I & K_2 \\ 0 & \mathfrak{U}_2 \\ 0 & R_2 \end{pmatrix} = \begin{pmatrix} W_{11}T_{11} & W_{11}(T_{12} + K_2 T_{22}) + W_{12}\mathfrak{U}_2 T_{22} + W_{13} R_2 T_{22} \\ W_{21}T_{11} & W_{21}(T_{12} + K_2 T_{22}) + W_{22}\mathfrak{U}_2 T_{22} + W_{23} R_2 T_{22} \\ W_{31}T_{11} & W_{31}(T_{12} + K_2 T_{22}) + W_{32}\mathfrak{U}_2 T_{22} + W_{33} R_2 T_{22} \end{pmatrix},$$

and therefore $W_{21} = 0$, $W_{31} = 0$ and $W_{11} = T_{11}^{-1}$. By making use of that and the relation $E_2 = \bar{W} E_1 \bar{T}$, we obtain

$$\begin{pmatrix} N_2 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W_{11}N_1T_{11} & W_{11}N_1T_{12} + W_{12}T_{22} \\ 0 & W_{22}T_{22} \\ 0 & W_{32}T_{22} \end{pmatrix},$$

and this implies $W_{32} = 0$, $W_{22} = T_{22}^{-1}$.
Since $N_2 = T_{11}^{-1} N_1 T_{11}$, N_1 and N_2 are similar. In addition, we have

$$\mathfrak{U}_2 = T_{22}^{-1}\mathfrak{U}_1 T_{22} + W_{23} R_1 T_{22}$$

and $R_2 = W_{33} R_1 T_{22}$. Hence, $\bar{A}_2$ is the restriction of $T_{22}^{-1}\mathfrak{U}_1 T_{22} + W_{23} R_1 T_{22}$ to the space $\ker W_{33} R_1 T_{22}$, and therefore, it is the restriction of $T_{22}^{-1}\mathfrak{U}_1 T_{22}$ to the space $\ker R_1 T_{22}$. From that, we get $\bar{A}_2 = T_{22}^{-1}\bar{A}_1 T_{22}$, which completes the proof. □

COROLLARY 4.3. *Let the same preliminaries hold as in Theorem 4.2. Then, if $\bar{A}_1$ generates a strongly continuous semigroup, the same holds for $\bar{A}_2$.*

Proof. This statement is immediately concluded by Theorem 4.2 since $\bar{A}_2 = T_{22}^{-1}\bar{A}_1 T_{22}$ for some bounded and boundedly invertible operator T_{22} and therefore the semigroup $T_{\bar{A}_2}(\cdot)$ generated by $\bar{A}_2$ is given by $T_{22}^{-1} T_{\bar{A}_1}(\cdot) T_{22}$, where $T_{\bar{A}_1}(\cdot)$ is the semigroup generated by $\bar{A}_1$. □

We will call the semigroups $T_{\bar{A}_1}(\cdot)$ and $T_{\bar{A}_2}(\cdot)$ *inherent semigroups* of E and A. From the proof above, we can conclude that all inherent semigroups of E and A are similar. In the following, we give some a priori criteria on E and A possessing a strongly continuous inherent semigroup.

THEOREM 4.4. *Let a system $E\dot{x} = Ax + q$ with ADAE index ν be given. Further, let $Q_i : i = 0, \ldots, \nu - 1$ be the projector chain as in the proof of Theorem 4.1 and $P_{\Sigma_\nu} = P_0 \cdots P_{\nu-1}$. Then, the following three statements are equivalent:*

(i) There exists a decoupling form with a strongly continuous inherent semigroup
(ii) All decoupling forms have a strongly continuous inherent semigroup
(iii) There exist $M, \gamma > 0$, such that for all $k \in \mathbb{N}, \lambda > \gamma$ holds

$$\|P_{\Sigma_\nu}((\lambda E - A)^{-1}E)^k P_{\Sigma_\nu}\| \leq \frac{M}{(\lambda-\gamma)^k}. \tag{4.13}$$

Proof. We only have to prove the equivalence between (i) and (ii) was already shown in Corollary 4.3. Let (E, A) be given and let W, T be transformations, such that (WET, WAT) is in decoupling form (4.8a) and (4.8b). Since $\sum_{i=0}^{\nu-1} \operatorname{im} Q_i = \ker P_{\Sigma\nu}$ does not depend on the particular choice of the projectors, we can conclude that

$$P_{\Sigma\nu} = T\begin{pmatrix} 0 & H \\ 0 & I \end{pmatrix} T^{-1}$$

for some $H \in L_b(X_1, X_2)$. Thus, for $\lambda \in \rho(E, A)$ we compute

$$\begin{aligned} & P_{\Sigma\nu}((\lambda E - A)^{-1}E)^k P_{\Sigma\nu} \\ &= T\begin{pmatrix} 0 & H \\ 0 & I \end{pmatrix} \begin{pmatrix} (\lambda N - I)N & (\lambda N - I)NK(\lambda I - \bar{A})^{-1} \\ 0 & (\lambda I - \bar{A})^{-1} \end{pmatrix}^k \begin{pmatrix} 0 & H \\ 0 & I \end{pmatrix} T^{-1} \\ &= T\begin{pmatrix} 0 & 0 \\ 0 & (\lambda I - \bar{A})^{-k} \end{pmatrix} T^{-1}. \end{aligned}$$

Hence, if $\bar{A}$ generates a strongly continuous semigroup, there exist constants $m > 0$ and $\gamma \in \mathbb{R}$, such that $\|(\lambda I - \bar{A})^{-k}\| \leq \frac{m}{(\gamma-\lambda)^k}$ for all $k \in \mathbb{N}$ and $\lambda > \gamma$. Thus, for $M = m\|T\|\|T^{-1}\|$, the relation (4.13) is valid.
Conversely, if statement (iii) is fulfilled, we have $\|(\lambda I - \bar{A})^{-k}\| \leq \frac{m}{(\gamma-\lambda)^k}$ for the constant $m := M\|T\|\|T^{-1}\|$ and for all $k \in \mathbb{N}, \gamma > \lambda$ and therefore, by the Hille-Yosida Theorem, $\bar{A}$ generates a strongly continuous semigroup. ∎

4.3. Complete Decouplings. For regular matrix pairs (E, A) it is well-known, that there exists transformation matrices W, T, such that (WET, WAT) reads as in (1.3), i.e. we can always find transformations, such that the coupling term K vanishes, which we call now a *complete decoupling.* Now, one may ask if this is also possible in the infinite dimensional case. In fact, there are practically motivated examples of ADAE's, where a complete decoupling is not possible, which we will confirm with an example at the end of this section. The following theorem states whether such a complete decoupling is possible.

THEOREM 4.5. *Let (E, A) be an operator pair. Then, there exist transformations W_1, T_1, such that*

$$(W_1ET_1, W_1AT_1) = \left(\begin{pmatrix} N_1 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & \mathfrak{U}_1 \\ 0 & R_1 \end{pmatrix} \right) \tag{4.14}$$

if and only if transformations W, T exist such that (WET, WAT) has the decoupling form (1.8), and, additionally $K\mathfrak{U}^k \in L_b(X_2, X_1)$ for $k = 0, \ldots, \nu - 1$.

For the proof, we use the projector approach as in the proof of Theorem 4.1 as well. In order to achieve a complete decoupling, especially, we have to choose the kernels of the decoupling projectors Q_i in a particular way, namely canonical. Our approach is inspired by the work [18], where these complete decouplings of matrix pairs are realized for matrix pairs. The presented method was based on an iteration method leading to the canonical projectors in finitely many steps. The problem in generalizing this to the infinite dimensional case is that one to pay attention for possible unboundednesses. This is the reason for the additional assumption $K\mathfrak{U}^k \in L_b(X_2, X_1)$ for $k = 0, \ldots, \nu - 1$. Before we state the proof, the following useful lemma is presented. The proof of that Lemma is extensive and left to Appendix B.

LEMMA 4.6. *Let an operator pair (E, A) be decoupled with the projectors $Q_0, \ldots, Q_{\nu-1}$, with*

$$K = -\begin{pmatrix} Q_0P_1\cdots P_{\nu-1}E_\nu^- A_\nu \\ Q_1P_2\cdots P_{\nu-1}E_\nu^- A_\nu \\ \vdots \\ Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^- A_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathfrak{U} = P_{\Sigma\nu}E_\nu^- A_\nu \tag{4.15}$$

such that $K(P_{\Sigma\nu}E_\nu^- A_\nu)^l \in L_b(X)$ for $l = 0, \ldots, k$.
Then, a decoupling with the projectors $\bar{Q}_j := -Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_j$ leads to decoupling of (E, A), namely

$$\left(\begin{pmatrix} \bar{N} & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & \bar{K} \\ 0 & \bar{\mathfrak{U}} \\ 0 & \bar{R} \end{pmatrix}\right).$$

Especially $\bar{K}$ reads

$$\bar{K} = -\begin{pmatrix} \bar{Q}_0\bar{P}_1\cdots \bar{P}_{\nu-1}\bar{E}_\nu^- \bar{A}_\nu \\ \bar{Q}_1\bar{P}_2\cdots \bar{P}_{\nu-1}\bar{E}_\nu^- \bar{A}_\nu \\ \vdots \\ \bar{Q}_{i-1}\bar{P}_i\cdots \bar{P}_{\nu-1}\bar{E}_\nu^- \bar{A}_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \bar{\mathfrak{U}} = \bar{P}_{\Sigma\nu}\bar{E}_\nu^- \bar{A}_\nu. \tag{4.16}$$

Moreover, we have $\bar{K}\bar{\mathfrak{U}}^l \in L_b(X)$ for $l = 0, \ldots, k-1$.

The constructed $\bar{Q}_i$ are indeed projectors, since

$$\begin{aligned}(-Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_j)^2 &= Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_jQ_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_j \\ &= -Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- E_\nu Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_j \\ &= -Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_j.\end{aligned}$$

Their boundedness of is a consequence of $\bar{Q}_j = Q_j - Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_\nu$, which holds due to the relation (4.2b) and the boundedness of $Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^- A_\nu$. Now we show Theorem 4.5.

Proof. Since the zero operator is obviously bounded, the existence of a form (4.14) clairly implies the second assertion. In opposition to that, the converse implication is more exertive to prove. Let an operator pair

$$(E,A) = \left(\begin{pmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & K \\ 0 & \mathfrak{U} \\ 0 & R \end{pmatrix} \right)$$

be given with E, A mapping from their domains in $X_1 \times X_2 \times X_3$ to $X_1 \times X_2$ and assume that $K\mathfrak{U}^l$ is bounded for $l = 0, \dots, \nu - 1$. The decoupling procedure can be performed with projectors of the form

$$Q_i = \begin{pmatrix} \hat{Q}_i & 0 \\ 0 & 0 \end{pmatrix},$$

Then, we get

$$Q_{\Sigma\nu} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad E_\nu^- = \begin{pmatrix} (N-I)^{-1} & 0 & 0 \\ 0 & I & 0 \end{pmatrix},$$

$$A_\nu = \begin{pmatrix} P_{\Sigma j} & K \\ 0 & \mathfrak{U} \\ 0 & R \end{pmatrix}, \quad P_{\Sigma\nu} E_\nu^- A_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \hat{A} \end{pmatrix},$$

we obtain K as in (4.9b) with

$$Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^- A_\nu = \begin{pmatrix} 0 & \hat{Q}_i \hat{P}_{i+1} \cdots \hat{P}_{\nu-1} (N-I)^{-1} K \\ 0 & 0 \end{pmatrix}$$

and hence $Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^- A_\nu (P_{\Sigma\nu} E_\nu^- A_\nu)^l$ is bounded for $l = 0, \dots, \nu - 1$.
Now, successively using Lemma 4.6, in the $nu - 1$-th iteration, we get a bounded

$$\bar{K} = \begin{pmatrix} \bar{Q}_0 \bar{P}_1 \cdots \bar{P}_{\nu-1} \bar{E}_\nu^- A_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which can be further eliminated by a new construction of the $\bar{Q}_i$ and we finally get a representation, where the coupling term $\bar{K}$ vanishes. □

For operator pairs consisting of bounded E and A which possess a decoupling form, the condition that $K\mathfrak{U}^l$ is bounded for all $l \in \mathbb{N}$ is trivially fulfilled and hence, the complete decoupling is possible in any case. Therefore, Theorem 4.5 goes with the case of regular matrix pairs $(E, A) \in (\mathbb{R}^{n \times n})^2$, where, as a matter of course, both E and A are bounded.
As an example, where the canonical projectors do not exist, we present the following. Let $E : X \to Z$, $E : D(A) \subset X \to Z$ be given by

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & -C_1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & C_0 \end{pmatrix}.$$

C_p denotes the evaluation operator which maps a function f to its value at $h \in [0,1]$, i.e. $C_p f = f(p)$. The spaces Z, X and $D(A)$ are

$$Z = \mathbb{R}^3 \times L_2([0,1],\mathbb{R}) \times \mathbb{R}, \quad X = \mathbb{R}^3 \times L^2([0,1],\mathbb{R}), \quad D(A) = \mathbb{R}^3 \times H^1([0,1],\mathbb{R}),$$

where $H^1[0,1]$ denotes the Sobolev space [1]

$$H^1([0,1],\mathbb{R}) := \left\{ f \in L^2([0,1],\mathbb{R}),\ \frac{\partial}{\partial x} f \in L_2([0,1],\mathbb{R}) \right\}.$$

The operator pair (E, A) is regular, and the generalized inverse reads

$$(sE - A)^{-1} \begin{pmatrix} x_1 \\ x_2 \\ f \\ x_3 \end{pmatrix} = \begin{pmatrix} e^{-s} x_3 + \int_0^1 e^{-s(1-y)} f(y) dy \\ s e^{-s} x_3 + s \int_0^1 e^{-s(1-y)} f(y) dy - x_2 \\ e^{-sx} x_3 + \int_0^x e^{-s(x-y)} f(y) dy \end{pmatrix}$$

for all $s \in \mathbb{C}$. It can be seen that this generalized resolvent is located in the space $\mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X)$ for all $\omega < 0$. Moreover, the system is already in decoupling form, the ADAE index reads $\nu = 2$ and projectors are given by

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we now construct $\bar{Q}_0, \bar{Q}_1$ according to Lemma 4.6, we get

$$\bar{Q}_1 = -Q_1 E_2^- A_1 = \begin{pmatrix} 1 & 0 & C_1 \\ 1 & 0 & C_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\bar{Q}_1$ contains some evaluation operators, it is not a bounded projector anymore. Therefore a complete decoupling of this pair is not possible with the presented method. We can even show that a complete decoupling does not exist. Assume that there exists a complete decoupling, i.e. transformations W, T, such that

$$(WET, WAT) = \left(\begin{pmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & \mathfrak{U} \\ 0 & R \end{pmatrix} \right).$$

The relations between W and T bringing one decoupling form into another have been investigated in the proof of Theorem 4.2, and hence, we get

$$W = \begin{pmatrix} T_{11}^{-1} & W_{12} & W_{13} \\ 0 & T_{22}^{-1} & W_{23} \\ 0 & 0 & W_{33} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T, W are partitioned according to the block structure of (WET, WAT) and (E, A). Therefore, we can derive

$$\begin{aligned} \begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1} &= \begin{pmatrix} sI - T_{22}^{-1} \frac{\partial}{\partial x} T_{22} + W_{23} C_0 T_{22} \\ W_{33} C_0 T_{22} \end{pmatrix}^{-1} \\ &= T_{22}^{-1} \begin{pmatrix} sI - \frac{\partial}{\partial x} \\ C_0 \end{pmatrix}^{-1} \begin{pmatrix} T_{22} & -W_{23} W_{33}^{-1} \\ 0 & W_{33}^{-1} \end{pmatrix}. \end{aligned}$$

From that and the fact

$$\begin{pmatrix} sI - \frac{\partial}{\partial x} \\ C_0 \end{pmatrix}^{-1} \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, L^2([0,1],\mathbb{R}), L^2([0,1],\mathbb{R}) \times \mathbb{R}),$$

it can be conclude that

$$\begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1} \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X_2, X_2 \times X_3).$$

Especially, for

$$F = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

we obtain $F(sE - A)^{-1}H = se^{-s}$.
Now, let

$$WH = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad FT = \begin{pmatrix} F_1 & F_2 \end{pmatrix}$$

be partitioned according to the block structure of WET and WAT. Then, we obtain

$$\begin{aligned} se^{-s} &= F(sE - A)^{-1}H = FT(sWET - WAT)^{-1}WH \\ &= \begin{pmatrix} F_1 & F_2 \end{pmatrix} \begin{pmatrix} sN - I & 0 \\ 0 & sI - \mathfrak{U} \\ 0 & -R \end{pmatrix}^{-1} \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} \\ &= F_1(-sN - I)H_1 + F_2 \begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1} \begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \end{aligned}$$

Thus we can express the function se^{-s} as a sum of the polynomial $F_1(-sN - I)H_1$ and the function

$$F_2 \begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1} \begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \in \mathcal{H}_\infty(\mathbb{R}, \mathbb{R}),$$

but this is a contradiction, since se^{-s} has a pole of infinite order at ∞. This argumentation yields that a complete decoupling is not possible in that case.

5. Example: An Electrical Circuit with a Transmission Line. We present a simple practical example to demonstrate the reliability of the discussed decoupling theory. Consider an electrical circuit containing a transmission line as below. The voltage and current courses $V(x,t), I(x,t)$ along the transmission line, which is assumed to have length one, satisfy the telegraph equations (see [26])

$$\begin{aligned} C_T \frac{\partial}{\partial t} V(x,t) &= -G_T I(x,t) - \frac{\partial}{\partial x} V(x,t) \\ L_T \frac{\partial}{\partial t} I(x,t) &= -\frac{\partial}{\partial x} I(x,t) - R_T V(x,t), \end{aligned}$$

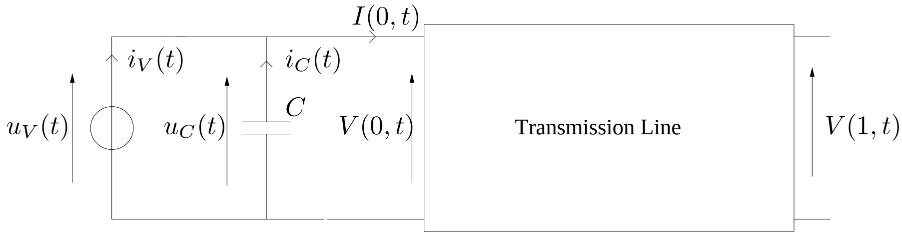

FIG. 5.1. *Electrical circuit with transmission line*

for some constants $G_T, R_T \geq 0$, $C_T, L_T > 0$. Further, due to element relations and the Kirchhoff laws [6], we get the equations

$$\begin{aligned} C\dot{u}_C(t) &= i_C(t) \\ 0 &= u_C(t) - u_V(t) \\ 0 &= -i_V(t) - i_C(t) + I(0,t) \\ 0 &= -u_C(t) + V(0,t) \\ 0 &= I(1,t). \end{aligned}$$

Equivalently, we model that system with an abstract differential algebraic system $E\dot{x}(t) = Ax(t) + q(t)$. The state x and the inhomogeneity q are chosen to be

$$x(t) = \begin{pmatrix} u_C(t) \\ i_C(t) \\ u_V(t) \\ V(t) \\ I(t) \end{pmatrix} \in X := \mathbb{R}^3 \times (L^2[0,1])^2, \quad q(t) = \begin{pmatrix} 0 \\ -u_V(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in Z := \mathbb{R}^5 \times (L^2[0,1])^2.$$

$V(t), I(t)$ are the spacial distributions of the voltage and current along the transmission line, i.e. $(V(t))(x) := V(x,t)$ and $(I(t))(x) := I(x,t)$. The operators E and A are given by

$$E = \begin{pmatrix} C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & L_T \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & C_0 \\ -1 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} & -R_T \end{pmatrix},$$

and the domain of A reads $D(A) = \mathbb{R}^5 \times (H[0,1])^2$. C_0 and C_1 are evaluation operators like in the previous section. It will turn out that this system has index 2 and operators of the operator chain in Theorem 4.1 read

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} C & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & L_T \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0 \\ -1 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} & -R_T \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ -C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad .$$

$$E_2 = \begin{pmatrix} C & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & L_T \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0 \\ 0 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} & -R_T \end{pmatrix},$$

It can be seen that E_1, E_2 are defined anywhere and for the domains of A_i, for $i \in \{1,2\}$ holds $D(A) = D(A_1) = D(A_2)$. A left inverse of E_2 and, consequently, the projector $W_2 = I - E_2E_2^-$ is given by

$$E_2^- = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & C & 0 & 0 & 0 & 0 & 0 \\ -1 & -C & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_T^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_T^{-1} \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the inherent abstract ordinary differential equation on the subspace $\operatorname{im} P_0P_1 = \{0\} \times (L^2[0,1])^2$ with boundary control $0 = W_2A_2x(t) = W_2q(t)$ is the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 0 \\ 0 \\ 0 \\ V(t) \\ I(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{G_T}{C_T} & -\frac{1}{C_T}\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{1}{L_T}\frac{\partial}{\partial x} & -\frac{R_T}{L_T} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ V(t) \\ I(t) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ V(t) \\ I(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u_V(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be shown that the operator A_T being a restriction of

$$\mathfrak{A}_T = \begin{pmatrix} -\frac{G_T}{C_T} & -\frac{1}{C_T}\frac{\partial}{\partial x} \\ -\frac{1}{L_T}\frac{\partial}{\partial x} & -\frac{R_T}{L_T} \end{pmatrix}$$

to the space

$$D(A_T) = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2[0,1]^2 : \begin{pmatrix} \frac{\partial}{\partial x} f_1 \\ \frac{\partial}{\partial x} f_2 \end{pmatrix} \in L^2[0,1]^2 \text{ and } 0 = V(0) = I(1) \right\} \subset L^2[0,1]^2$$

generates a strongly continuous semigroup $T_T(\cdot)$. For the proof, we refer to [23, 25]. The computation of $T_T(\cdot)$ can e.g. be performed by an inverse Laplace transform of the resolvent $(sI - A_T)^{-1}$ (see [11]). Hence, $P_0P_1E_2^- A_2$ with domain

$$D(A) \cap \ker W_2A_2 \cap \operatorname{im} P_0P_1 = \{0\} \times D(A_T)$$

is a generator of a strongly continuous semigroup on the space $\operatorname{im} P_0P_1$. With the method of [28], we get according to formula (4.12b)

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix} = \int_0^t T_T(t-s) \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix} u_V(s)ds,$$

where $\delta_0 \in \mathcal{D}'([0,1], \mathbb{R})$ is the Dirac delta distribution.
The (hidden) algebraic relations $Q_0Q_1\dot{x}(t) = Q_0x + Q_0P_1E_2^- A_2x(t) - Q_0P_1E_2^- q(t)$ and $0 = Q_1x(t) - Q_1E_2^- A_2x(t) - Q_1E_2^- q(t)$ read

$$\begin{pmatrix} 0 \\ C\dot{u}_C(t) \\ -C\dot{u}_C(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i_C(t) \\ i_V(t) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ C_0I(t) \\ 0 \\ 0 \end{pmatrix} \tag{5.1a}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_C(t) \\ -Cu_C(t) \\ Cu_C(t) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} u_V(t) \\ -Cu_V(t) \\ Cu_V(t) \\ 0 \\ 0 \end{pmatrix}. \tag{5.1b}$$

The formula (5.1b) implies $u_C(t) = u_V(t)$. Plugging that into (5.1a), we get

$$i_C(t) = C\dot{u}_V(t)$$
$$i_V(t) = -C\dot{u}_V(t) + I(0,t) = -C\dot{u}_V(t) + \begin{pmatrix} 0 & C_0 \end{pmatrix} \int_0^t T_T(t-s) \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix} u_V(s)ds.$$

Hence, the decoupling of the abstract differential algebraic system which modelled the given circuit was helpful for the determination of the solution trajectory. However, although by a more inspired way in writing down the circuit equations, a solution can be obtained without the decoupling procedure in this case. Nevertheless, for more complicated examples, the decoupling seems to be a reasonable method for getting inside the solvability of ADAE's and the structure of its solution. The ADAE index turned out to be 2. The perturbation index also equals 2 and it can be shown that the resolvent fulfills

$$(sE - A)^{-1} \in s \cdot \mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X) \text{ for all } \omega < 0.$$

Further, it can be seen that we cannot achieve a complete decoupling with the proposed method in this example since, here, the coupling term K contains some point

evaluation and thus, the preliminaries of Theorem 4.5 are not fulfilled.
The example consisted of a PDE whose boundaries where coupled with some finite dimensional DAE's. This is the reason why E_i and A_i for $i = 0, 1, 2$ have some block structure with matrices of the upper left part, which are mainly responsible for the kernels of the E_i. Due to that fact, the projectors Q_i can be obtained by numerical computations, in principle.

6. Conclusions. In this work, we have developed a framework for analyzing linear constant coefficient abstract differential algebraic systems. Solvability criteria which are mainly based on Laplace transform methods have been presented. Motivated by the famous Kronecker normal form, we educed a method for decoupling of infinite dimensional differential algebraic systems. It turned out that a complete decoupling, i.e. a representation of the system, where the (hidden) algebraic and the differential conditions are completely independent, is not possible in each case of practical relevance. We established criteria for systems possessing such a complete decoupling. Another difference to the finite dimensional case is the appearance of a third relation, which has been interpreted as a boundary term. The advantage of a decoupling is that one can filter out an inherent abstract ordinary differential equation and the computation of a solution of the abstract differential algebraic system is led back to the determination of the semigroup, which the operator appearing in the inherent ODE generates. We exposed that this inherent semigroup is - up to similarity - an invariant of the system. Especially, the strong continuity of an inherent semigroup is a property of the system and not of the particular decoupling form, we choose. The main intention of the authors for developing this theory is to deal with coupled systems of partial differential and differential algebraic equations and analyze their properties.

Appendix A. Here, we complete the proof of Theorem 4.1 by showing that E_ν possesses a bounded left inverse. We give a characterization whether such a bounded left inverse of E_ν^- exists. Afterwards, we will show that the preliminaries of the following Lemma are fulfilled by the assumptions made in Theorem 4.1.

THEOREM 6.1. *Let E_ν be injective and let $\operatorname{im} E_\nu$ be closed. Then, there exists a bounded projector W_ν along $\operatorname{im} E_\nu$ and a bounded $E_\nu \in L_b(Z, X)$ satisfying $E_\nu E_\nu^- = I - W_\nu$ and $E_\nu^- E_\nu = I$.*

Proof. The existence of the projector $W_\nu \in L_b(Z)$ with $\ker W_\nu = \operatorname{im} E_\nu$ is obvious, if $\operatorname{im} E_\nu$ is closed and clearly, we have that $Z = \ker W_\nu \oplus \operatorname{im} E_\nu$. Furthermore, due to its injectivity, E_ν is a bijection from X onto $\operatorname{im} I - W_i$ and both spaces X, $\operatorname{im} I - W_i$ are complete. Hence, by the Inverse Mapping Theorem [27], there exists a bounded inverse $E_\nu^- : \operatorname{im} I - W_\nu \to X$. For arbitrary $z \in Z$, we set $E_\nu^- z := E_\nu^-(I - W_\nu)z$ and therefore, E_ν^- has been boundedly extended to the whole space Z. Then for all $x \in X$, $z \in Z$, we have

$$\begin{aligned} E_\nu^- E_\nu x &= E_\nu^- E_\nu x = x \\ E_\nu E_\nu^- z &= E_\nu E_i^-(I - W_\nu)z = (I - W_\nu)z. \end{aligned}$$

∎

From the previous result, it suffices to show that $\operatorname{im} E_\nu$ is closed. This is done below.

THEOREM 6.2. *Let the preliminaries of Theorem 4.1 be given. Then $\operatorname{im} E_\nu$ is closed.*

Proof. By induction on i, we show that

$$\operatorname{im} E_i = \operatorname{im} E + A\left(\sum_{k=0}^{i-1} N_k\right) \quad \text{for } i = 0, \ldots, \nu.$$

By the assumption that $\operatorname{im} E + A\left(\sum_{k=0}^{\nu-1} N_k\right)$ is closed, we get that $\operatorname{im} E_\nu$ is closed. Since $\operatorname{im} E = \operatorname{im} E + A\left(\sum_{k=0}^{-1} N_i\right)$ the induction start holds trivially.
Further, for $i > 0$, we obtain

$$E_i x = E_i(P_{i-1}x + Q_{i-1}x) = E_{i-1}P_{i-1}x + A_{i-1}Q_{i-1}x,$$

and hence, we get

$$\begin{aligned}\operatorname{im} E_i &= \operatorname{im} E_{i-1} + \operatorname{im} AQ_{i-1} \\ &= \operatorname{im} E_{i-1} + AN_{i-1} \\ &= \operatorname{im} E + A\left(\sum_{k=0}^{i-2} N_k\right) + AN_{i-1} \\ &= \operatorname{im} E + A\left(\sum_{k=0}^{i-1} N_k\right).\end{aligned}$$

□

Appendix B. Before Lemma 4.6 is shown, we present three Lemmas which are essential for that proof. The first and the last lemma are proven in [18] for matrices. Since the proofs only involve symbolic matrix calculations, the results also covers the case of bounded operators and hence, we refer to that reference for the proof.

LEMMA 6.3 ([18],Lemma A.1). *Let*

$$\bar{Q}_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j, \quad j = 0, \ldots, \nu - 1$$

be valid. Then the chain $\bar{Q}_0, \ldots, \bar{Q}_{\nu-1}$ *is an appropriate operator chain for* (E, A). *Defining Let* $\bar{E}_0 = E$, $\bar{A}_0 = A$, $\bar{E}_i := \bar{E}_{i-1} - \bar{A}_{i-1}\bar{Q}_i$, $\bar{A}_i := \bar{A}_{i-1}\bar{P}_i$. *Then the projectors* $\bar{Q}_j$ *are an appropriate chain for decoupling the pair* (E, A), *i.e.* $\operatorname{im} \bar{Q}_i = \ker \bar{E}_i$, $\bar{Q}_i\bar{Q}_j = 0$ *for* $i > j$. *Moreover, for* $j = 0, \cdots, \nu - 1$, *we have*

$$\begin{aligned}\bar{E}_j &= E_j F_j, \\ \textit{for } F_j &:= I + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{j-1} P_{j-1}.\end{aligned}$$

Furthermore, it holds

$$F_j^{-1} := I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{j-1} P_{j-1} \tag{6.1}$$

$$\bar{Q}_i \bar{Q}_j = Q_i \bar{Q}_j \quad \textit{for all } i < j. \tag{6.2}$$

LEMMA 6.4. *Let* $Q_0, \ldots, Q_{\nu-1}$ *be a decoupling chain for the operator pair* (E, A) *and let* $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$ *for* $i = k, \ldots, \nu - 1$. *Then it holds*

$$Q_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j \quad \textit{for } i = k, \ldots, \nu - 1.$$

Proof. For $j = \nu - 1$, it holds

$$0 = Q_{\nu-1}E_\nu^- A_\nu = \bar{Q}_{\nu-1}P_{\nu-1}$$

and thus $\ker \bar{Q}_{\nu-1} = \ker Q_{\nu-1}$, which implies $\bar{Q}_{\nu-1} = Q_{\nu-1}$. Assuming that $\bar{Q}_j = Q_j$ for $j > k$ and $Q_k P_{k+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$, we get

$$\begin{aligned} 0 &= \bar{Q}_k P_k P_{k+1} \cdots P_{\nu-1} \\ &= \bar{Q}_k P_k \bar{P}_{k+1} \cdots \bar{P}_{\nu-1} \\ &= \bar{Q}_k P_k - \bar{Q}_k (P_k \bar{Q}_{k+1} + P_k \bar{P}_{k+1} \bar{Q}_{k+2} + \cdots + P_k \bar{P}_{k+1} \cdots \bar{P}_{\nu-1} \bar{Q}_{\nu-1}) \\ &= \bar{Q}_k P_k - \bar{Q}_k (\bar{P}_k \bar{Q}_{k+1} + \bar{P}_k \bar{P}_{k+1} \bar{Q}_{k+2} + \cdots + \bar{P}_k \bar{P}_{k+1} \cdots \bar{P}_{\nu-1} \bar{Q}_{\nu-1}) \\ &= \bar{Q}_k P_k, \end{aligned}$$

and by the same argumentation as above, we get $\bar{Q}_k = Q_k$. The second last equality holds due to the relation (6.2). □

Lemma 6.5 ([18],Lemma A.2). *For $j = k, \ldots, \nu - 1$, let*

$$Q_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j.$$

Then we have

$$Q_j = \bar{Q}_j = -\bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\nu-1} \bar{E}_\nu^- \bar{A}_j \quad \textit{for } j = k, \ldots, \nu - 1,$$

and, additionally

$$\bar{Q}_{k-1} = -\bar{Q}_{k-1} \bar{P}_k \cdots \bar{P}_{\nu-1} \bar{E}_\nu^- \bar{A}_{k-1}.$$

This section concludes with the proof of Lemma 4.6.

Proof. [Lemma 4.6]
Since, by Lemma 6.3, the equality between $\bar{E}_\nu$ and $E_\nu F_\nu$ holds, we choose a left inverse

$$\bar{E}_\nu^- = F_\nu^{-1} E_\nu^- = (I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{\nu-1} P_{\nu-1}) E_\nu^-$$

and then, we calculate

$$\bar{P}_{\Sigma\nu} \bar{E}_\nu^- = \bar{P}_{\Sigma\nu} (I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{\nu-1} P_{\nu-1}) E_\nu^- = \bar{P}_{\Sigma\nu} E_\nu^-.$$

Due to [19], Theorem 2.3, we have

$$\operatorname{im} Q_{\Sigma\nu} = \bigoplus_{i=0}^{\nu-1} \operatorname{im} Q_i = \bigoplus_{i=0}^{\nu-1} \operatorname{im} \bar{Q}_i = \operatorname{im} \bar{Q}_{\Sigma\nu}.$$

and thus, the relations $\bar{Q}_{\Sigma\nu} Q_{\Sigma\nu} = Q_{\Sigma\nu}$, $Q_{\Sigma\nu} \bar{Q}_{\Sigma\nu} = \bar{Q}_{\Sigma\nu}$, $P_{\Sigma\nu} \bar{P}_{\Sigma\nu} = P_{\Sigma\nu}$ and $\bar{P}_{\Sigma\nu} P_{\Sigma\nu} = \bar{P}_{\Sigma\nu}$ are valid. Hence,

$$\begin{aligned} \bar{P}_{\Sigma\nu} \bar{E}_\nu^- \bar{A}_\nu &= \bar{P}_{\Sigma\nu} F_\nu^{-1} E_\nu^- A \bar{P}_{\Sigma\nu} \\ &= \bar{P}_{\Sigma\nu} (I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{j-1} P_{j-1}) E_\nu^- A \bar{P}_{\Sigma\nu} \\ &= \bar{P}_{\Sigma\nu} E_\nu^- A \bar{P}_{\Sigma\nu} \\ &= \bar{P}_{\Sigma\nu} (P_{\Sigma\nu} E_\nu^- A) \bar{P}_{\Sigma\nu} \\ &= \bar{P}_{\Sigma\nu} (P_{\Sigma\nu} E_\nu^- A_\nu) \bar{P}_{\Sigma\nu}. \end{aligned}$$

The last equality was already shown in (4.2d). Using the relations above, we can write

$$(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l = \bar{P}_{\Sigma\nu}(P_{\Sigma\nu}E_\nu^-A_\nu)^l\bar{P}_{\Sigma\nu}$$

for $l\in\mathbb{N}$.
Now we show that $\bar{K}(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l$ is bounded for $l=0,\ldots,k-1$.
Using the relations $Q_i\bar{Q}_j=0$ for $i>j$ and $Q_i\bar{Q}_j=\bar{Q}_i\bar{Q}_j$ for $i<j$ from Lemma 6.5. Furthermore, we need the equality (4.2b) being

$$Q_iP_{\Sigma\nu}E_\nu^-A = Q_i - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}.$$

Then, we compute

$$\begin{aligned}
&-\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\
&= -\bar{Q}_i\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu\\
&= Q_iP_{\Sigma\nu}E_\nu^-A_i\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu\\
&= Q_iP_{\Sigma\nu}E_\nu^-AP_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu\\
&= (Q_i - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu})P_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu\\
&= Q_iP_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}P_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu\\
&= Q_i\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu\\
&= \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu.
\end{aligned}$$

Hence , we have the equation

$$2\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu = Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu.$$

Using that, we get

$$\begin{aligned}
&2\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^{l-1}\\
&= Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^{l-1}\\
&= Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^{l}\\
&= Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}(P_{\Sigma\nu}E_\nu^-A_\nu)^{l}\bar{P}_{\Sigma\nu}\\
&= Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu(P_{\Sigma\nu}E_\nu^-A_\nu)^{l}\bar{P}_{\Sigma\nu}.
\end{aligned}$$

By assumption, $Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu(P_{\Sigma\nu}E_\nu^-A_\nu)^l$ is bounded for $l=0,\ldots,k$. This implies the boundedness of $\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l$ for $l=0,\ldots,k-1$.

Having K as in (4.15), we derive that $\bar{K}$ equals

$$-\begin{pmatrix}\bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \vdots\\ \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \bar{Q}_{i+1}\bar{P}_{i+2}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \vdots\\ \bar{Q}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\end{pmatrix} = -\begin{pmatrix}\bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \vdots\\ \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ Q_{i+1}P_{i+1}\cdots P_{\nu-1}\\ \vdots\\ Q_{\nu-1}P_{\nu-1}\end{pmatrix} = -\begin{pmatrix}\bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ \vdots\\ \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

The first equality holds, since, by Lemma 6.4, we get $\bar{Q}_j=Q_j$ for $j=i,\ldots,\nu-1$ whereas the second is trivial. Then, Lemma 6.5 implies $\bar{Q}_i=\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_i$, and hence $\bar{Q}_i \bar{P}_{i+1} \cdots \bar{P}_{\nu-1} \bar{E}_\nu^- \bar{A}_\nu = 0$. Therefore, we conclude that equation (4.16) holds and the proof is complete. ☐

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