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Lena Scholz

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The Signature Method for DAEs arising in the Modeling of Electrical Circuits

Lena Scholz[†]

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Abstract

We consider the Signature Method (Σ-method) for the structural analysis of differential-algebraic equations (DAEs) that arise in the modeling and simulation of electrical circuits. Different formulations of the set of model equations are considered. For some formulations we show that the structural approach may fail for certain circuit topologies, while other formulations are better suited for a structural analysis. The results are illustrated by a number of examples.

Keywords: Differential-algebraic equation; structural analysis; signature method; modified nodal analysis; MNA; modified loop analysis; MLA; branch-oriented model.

AMS(MOS) subject classification: 34A09, 65L80, 94C05

1 Introduction

Modeling and simulation of dynamical systems is an important issue in the development of technical innovations. More and more equation-based object-oriented modeling environments such as Dymola, MapleSim or 20sim are used as tools for modeling and simulation of multi-physical systems. An important problem class are electrical circuits or electrical components that are embedded in multi-physical applications. Circuit equations typically lead to large-scale systems of differential-algebraic equations (DAEs). For such DAE systems it is well-known that, due to the occurrence of hidden constraints, an *index reduction* or *regularization* is required for a robust numerical integration, e.g., to avoid drift-off, instabilities or artificial oscillations, see [1, 6, 7, 16]. In most modeling environments a structural analysis of the model equations is used to determine a formulation that is suited for the numerical integration. Here, usually *Pantelides Algorithm* [10] or the *Signature Method* (Σ*-method*) [12] are used in combination with the dummy derivative approach [8]. These structural methods are powerful tools since they are computationally very efficient. In [12] it has been shown that the structural analysis based on the Σ-method works reliable for many classes of DAEs. However, it is also known that structural approaches may fail for certain problems [14, 18]. In this paper we will consider the structural analysis for commonly used circuit equations. We will present a number of examples that show that the structural approach may fail for certain circuit topologies or certain formulations of the model equations. Fortunately, an appropriate formulation of the circuit equations allows the secure application of the Σ-method.

The paper is organized as follows. In Section 2 we collect some preliminaries. Then, in Section 3, we consider different formulations of the model equations for electrical circuits. Next, in Section 4, we recapitulate the basic ideas of the Σ-method and apply the approach to the different circuit equations. We will see that for some formulations the Σ-method may fail depending on the topology of the circuit, while other formulations are better suited for structural approaches. We

[2]Institut für Mathematik, Ma 4-5, TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, FRG; email: lscholz@math.tu-berlin.de

end with some concluding remarks in Section 5. Some additional graph theoretical results are given in the Appendix A.

2 Preliminaries

In general, a DAE is given as a nonlinear system

$$F(t, x, \dot{x}) = 0, \tag{1}$$

where $F : \mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \to \mathbb{R}^n$ is a continuous function, $\mathbb{D}_x, \mathbb{D}_{\dot{x}} \subset \mathbb{R}^n$ are open, $\mathbb{I} = [t_0, t_f] \subset \mathbb{R}$ and $x : \mathbb{I} \to \mathbb{D}_x$ is a continuously differentiable unknown function. For a differentiable time depending function x, the i-th (total) derivative of x with respect to t is denoted by $x^{(i)}(t) = \mathrm{d}^i x(t)/\mathrm{d}t^i$ for $i \in \mathbb{N}$, using the convention $x^{(1)}(t) = \dot{x}(t)$, and $x^{(2)}(t) = \ddot{x}(t)$.
For a matrix $A \in \mathbb{R}^{m,n}$, $\operatorname{im} A$ denotes the image of A, $\ker A$ denotes the kernel of A, and $\operatorname{rank} A$ denotes the rank of A. Furthermore, a square matrix A that is positive (semi-)definite is denoted by $A > (\geq) 0$.

Definition 1. *A function $x : \mathbb{I} \to \mathbb{D}_x$ is said to be a* solution of the DAE (1) *if it is continuously differentiable for all $t \in \mathbb{I}$ and* (1) *is fulfilled pointwise for all $t \in \mathbb{I}$. A function $x : \mathbb{I} \to \mathbb{D}_x$ is called a* solution of the initial value problem (1) *and $x(t_0) = x_0$ with $x_0 \in \mathbb{D}_x$ if it is a solution of* (1) *and satisfies the initial condition $x(t_0) = x_0$. An initial value $x_0 \in \mathbb{D}_x$ is called* consistent*, if the initial value problem* (1) *and $x(t_0) = x_0$ has a solution.*

The original equation (1) and its derivatives up to order ℓ can be gathered into a so-called *derivative array*

$$\mathcal{F}_\ell(t, x, \dots, x^{(\ell+1)}) = \begin{bmatrix} F(t, x, \dot{x}) \\ \frac{\mathrm{d}}{\mathrm{d}t} F(t, x, \dot{x}) \\ \vdots \\ (\frac{\mathrm{d}}{\mathrm{d}t})^\ell F(t, x, \dot{x}) \end{bmatrix}. \tag{2}$$

Definition 2 ([9])**.** *Suppose that* (1) *is solvable. The smallest integer ν_d (if it exists) such that $\dot{x}$ is uniquely determined by $\mathcal{F}_{\nu_d}(t, x, \dot{x}, \dots, x^{(\nu_d+1)}) = 0$ as a function of x and t, i.e., $\dot{x} = \Phi(t, x)$, for all consistent values is called the* differentiation index (d-index) *of* (1).[1]

If the d-index is well-defined, one can extract a so-called *underlying ODE* $\dot{x}(t) = \Phi(x(t), t)$ from the derivative array with the property that every solution of the DAE (1) also solves the underlying ODE.
In the following, we will also need some basic results of graph theory. Let $G = (V, B, \Psi)$ denote a *directed graph* with $V = \{v_1, v_2, \dots\}$ the set of nodes (or vertices) and $B = \{b_1, b_2, \dots\}$ the set of branches (or edges), and incidence map $\Psi : B \to V \times V$ that maps every branch $b \in B$ onto some ordered pair of nodes (v_i, v_j). A directed graph G is *connected* if for every pair of nodes there exists a path between them. A *subgraph* $G' := (V', B', \Psi|_{B'})$ of a connected graph G is a graph such that $V' \subset V$, $B' \subset B|_{V'} := \{b \in B \mid \Psi(b) \in V' \times V'\}$. A *loop* is a simple path $(b_{j_0}, b_{j_1}, \dots, b_{j_p})$ in the directed graph G such that $b_{j_0} = b_{j_p}$. A *cutset* is a set B_c of branches of a connected graph G such that the graph G_c that results when the branches in B_c are deleted from G is disconnected, and adding any branch in B_c to G_c would result again in a connected graph.
A *(spanning) tree* in a connected graph is a connected subgraph which contains all nodes and has no loops. We will also use the term tree to refer to the set of branches contained in this subgraph. Once a tree has been chosen the branches in the tree are called *twigs*, whereas the remaining ones are called *links*. The set of links defines the *cotree*. Let n_η be the number of nodes and n_b denote the number of branches in the connected graph. Then, any tree defines $n_\eta - 1$ twigs and $n_b - n_\eta + 1$ links. Due to Lemma 27 there exists no loops just defined by twigs, and no cutsets just defined by links. For more details we refer to [3, 15] and to Appendix A.

[1] Here, $x, \dot{x}, \dots, x^{(\ell)}$ are considered locally as independent algebraic variables.

3 Model Equations for Electrical Circuits

We consider lumped electrical circuits containing (possibly nonlinear) resistors, capacitors, and inductors, as well as voltage sources and current sources. The modeling of the dynamical behavior of such electrical circuits is based on Kirchhoff's laws together with the constitutive relations for the electrical components. However, there are different ways to set up the set of model equations. We will present four different approaches to model electrical circuits in this section.

3.1 The Modified Nodal Analysis

A common way for the modeling of electrical circuits is the *Modified Nodal Analysis (MNA)* [22]. The circuit is modeled as a directed graph whose branches correspond to the circuit elements and whose nodes correspond to the interconnections of these elements. The topological structure of such a graph with n_η nodes and n_b branches can be described by an incidence matrix $A_0 \in \mathbb{R}^{n_\eta,n_b}$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i, \\ -1 & \text{if branch } j \text{ enters node } i, \\ 0 & \text{if branch } j \text{ is not incident with node } i. \end{cases}$$

If the network graph is connected the rows of A_0 are linearly dependent and we can choose one arbitrary node as reference node. By eliminating the corresponding row in the incidence matrix we obtain the *reduced incidence matrix* $A \in \mathbb{R}^{n_\eta-1,n_b}$ that then has full row rank. Kirchhoff's current law and Kirchhoff's voltage law takes the form

$$A\imath = 0, \quad \nu = A^T\eta, \tag{3}$$

where $\imath$ denotes the vector of all branch currents, ν denotes the vector of all branch voltages, and η denotes the vector of all node potentials (excepting the reference node for which $\eta_0 = 0$). Assuming that the branches are ordered by the type of component, we can split A into element-related incidence matrices

$$A = \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{L}} & A_{\mathcal{R}} & A_{\mathcal{V}} & A_{I} \end{bmatrix}, \tag{4}$$

such that $A_{\mathcal{C}} \in \mathbb{R}^{n_\eta-1,n_{\mathcal{C}}}$, $A_{\mathcal{L}} \in \mathbb{R}^{n_\eta-1,n_{\mathcal{L}}}$, $A_{\mathcal{R}} \in \mathbb{R}^{n_\eta-1,n_{\mathcal{R}}}$, $A_{\mathcal{V}} \in \mathbb{R}^{n_\eta-1,n_{\mathcal{V}}}$, and $A_I \in \mathbb{R}^{n_\eta-1,n_I}$. Here, $n_{\mathcal{V}}$ denotes the number of voltage sources, n_I the number of current sources, $n_{\mathcal{C}}$ the number of capacitors, $n_{\mathcal{L}}$ the number of inductors, and $n_{\mathcal{R}}$ the number of resistors in the circuit, respectively. The vectors ν and $\imath$ are split accordingly into

$$\nu = \begin{bmatrix} \nu_{\mathcal{C}}^T & \nu_{\mathcal{L}}^T & \nu_{\mathcal{R}}^T & \nu_{\mathcal{V}}^T & \nu_I^T \end{bmatrix}^T, \quad \imath = \begin{bmatrix} \imath_{\mathcal{C}}^T & \imath_{\mathcal{L}}^T & \imath_{\mathcal{R}}^T & \imath_{\mathcal{V}}^T & \imath_I^T \end{bmatrix}^T.$$

Furthermore, we use the constitutive element relations

$$\nu_{\mathcal{L}} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\imath_{\mathcal{L}}), \quad \imath_{\mathcal{C}} = \frac{\mathrm{d}}{\mathrm{d}t}q(\nu_{\mathcal{C}}), \quad \imath_{\mathcal{R}} = g(\nu_{\mathcal{R}}), \quad \imath_I = I_s(t), \quad \nu_{\mathcal{V}} = \mathcal{V}_s(t) \tag{5}$$

for inductors, capacitors, resistors, current and voltage sources, where $g : \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}$ is the conductance function, $q : \mathbb{R}^{n_{\mathcal{C}}} \to \mathbb{R}^{n_{\mathcal{C}}}$ is the charge function and $\phi : \mathbb{R}^{n_{\mathcal{L}}} \to \mathbb{R}^{n_{\mathcal{L}}}$ is the flux function. Here, we restrict to the case of independent current and voltage sources described by the source functions $I_s(t)$ and $\mathcal{V}_s(t)$, respectively. In general, also controlled sources are possible, see [4]. Altogether, we get a DAE system of the form

$$\begin{aligned} A_{\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}q(A_{\mathcal{C}}^T\eta) + A_{\mathcal{L}}\imath_{\mathcal{L}} + A_{\mathcal{R}}g(A_{\mathcal{R}}^T\eta) + A_{\mathcal{V}}\imath_{\mathcal{V}} + A_I I_s &= 0, && \text{(6a)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(\imath_{\mathcal{L}}) - A_{\mathcal{L}}^T\eta &= 0, && \text{(6b)} \\ A_{\mathcal{V}}^T\eta - \mathcal{V}_s &= 0. && \text{(6c)} \end{aligned}$$

The system (6) consists of $(n_\eta - 1) + n_{\mathcal{L}} + n_{\mathcal{V}}$ equations in the $(n_\eta - 1) + n_{\mathcal{L}} + n_{\mathcal{V}}$ unknowns $[\eta, \imath_{\mathcal{L}}, \imath_{\mathcal{V}}]$ and is also known as the *MNA equations* of the electrical circuit. Note that in (6) we have omitted the dependency on time for better readability. For details on the constitutive element relations and on the derivation of the MNA equations, see also [4, 5, 13, 15].

We say that the DAE system (6) is *well-posed* if it satisfies the following assumptions.

(A1) The circuit contains no $\mathcal{V}$-loops, i.e., $A_{\mathcal{V}}$ has full column rank.

(A2) The circuit contains no $\mathcal{I}$-cutsets, i.e., $[A_{\mathcal{C}}\ A_{\mathcal{L}}\ A_{\mathcal{R}}\ A_{\mathcal{V}}]$ has full row rank.

(A3) The charge function $q : \mathbb{R}^{n_{\mathcal{C}}} \to \mathbb{R}^{n_{\mathcal{C}}}$ is continuously differentiable and the Jacobian

$$\mathcal{C}(\nu_{\mathcal{C}}) := \frac{\partial}{\partial \nu_{\mathcal{C}}} q(\nu_{\mathcal{C}})$$

is symmetric and pointwise positive definite.

(A4) The flux function $\phi : \mathbb{R}^{n_{\mathcal{L}}} \to \mathbb{R}^{n_{\mathcal{L}}}$ is continuously differentiable and the Jacobian

$$\mathcal{L}(\imath_{\mathcal{L}}) := \frac{\partial}{\partial \imath_{\mathcal{L}}} \phi(\imath_{\mathcal{L}}),$$

is symmetric and pointwise positive definite.

(A5) The conductance function $g : \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}$ is continuously differentiable and the Jacobian

$$\mathcal{G}(\nu_{\mathcal{R}}) := \frac{\partial}{\partial \nu_{\mathcal{R}}} g(\nu_{\mathcal{R}}),$$

is symmetric and pointwise positive definite.

A $\mathcal{V}$-loop is defined as a loop in the circuit graph that consists only of branches corresponding to voltage sources. In the same way, a $\mathcal{CV}$-loop means a loop that consists only of branches corresponding to capacitances and/or voltage sources. Likewise, an $\mathcal{I}$-cutset is a cutset in the circuit graph that consists only of branches corresponding to current sources, and an $\mathcal{LI}$-cutset is a cutset that consists only of branches corresponding to inductances and/or current sources. Assumption **(A1)** implies that there are no short-circuits. In a similar manner, the occurrence of $\mathcal{I}$-cutsets may lead to contradictions in the Kirchhoff laws (source functions may not sum up to zero), which is excluded by assumption **(A2)**. The assumptions **(A3)**, **(A4)** and **(A5)** imply that all circuit elements are passive, i.e., they do not generate energy.
Due to the special structure of (6), it is possible to determine the index by graph theoretical considerations.

Theorem 3 ([4, 13])**.** *Consider an electrical circuit with circuit equations as in* (6)*. Assume that the assumptions* **(A1)**-**(A5)** *hold.*

1. *The following statements are equivalent:*
   - *the MNA equations* (6) *are of d-index* $\nu_d = 0$*;*
   - *the circuit contains neither voltage sources nor* $\mathcal{RLI}$*-cutsets;*
   - $n_{\mathcal{V}} = 0$ *and* $\operatorname{rank} A_{\mathcal{C}} = n_\eta - 1$.
2. *The following statements are equivalent:*
   - *the MNA equations* (6) *are of d-index* $\nu_d = 1$*;*
   - *the circuit contains neither* $\mathcal{LI}$*-cutsets nor* $\mathcal{CV}$*-loops (except for pure* $\mathcal{C}$*-loops);*
   - $\operatorname{rank}[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{V}}] = n_\eta - 1$ *and* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] = \ker A_{\mathcal{C}} \times \{0\}$.
3. *The following statements are equivalent:*

- *the MNA equations* (6) *are of d-index* $\nu_d = 2$;
- *the circuit contains* $\mathcal{LI}$*-cutsets or* $\mathcal{CV}$*-loops which are no pure* $\mathcal{C}$*-loops;*
- $\operatorname{rank}[A_\mathcal{C}, A_\mathcal{R}, A_\mathcal{V}] < n_\eta - 1$ *or* $\ker[A_\mathcal{C}, A_\mathcal{V}] \neq \ker A_\mathcal{C} \times \{0\}$.

In the case that $\nu_d \geq 1$ the MNA equations (6) contain hidden constraints that can be revealed by differentiating certain parts of the system, see [4]. In particular, the source functions that belong to $\mathcal{CV}$-loops or $\mathcal{LI}$-cutsets have to be differentiable if the DAE has d-index $\nu_d = 2$. Note that under the given assumptions, the MNA equations (6) will always have a d-index $\nu_d \leq 2$.

Remark 4. The formulation using the MNA equations (6) belong to the class of *nodal methods* that are characterized by the use of node potentials as fundamental modal variables together with some branch variables. The MNA equations (6) are often used in circuit simulation programs (e.g. in SPICE or TITAN), because their compact form allows for efficient numerical computations [4, 5, 19].

3.2 The Modified Loop Analysis

An alternative way to model electrical circuits is the *Modified Loop Analysis* (MLA) [23]. Here, in order to describe the topology of the circuit graph, instead of the incidence matrix A one uses the so-called *loop matrix* $B_0 \in \mathbb{R}^{n_\ell, n_b}$ with entries defined as follows

$$b_{ij} = \begin{cases} 1 & \text{if branch } j \text{ belongs to loop } i \text{ and has the same orientation,} \\ -1 & \text{if branch } j \text{ belongs to loop } i \text{ and has the contrary orientation,} \\ 0 & \text{if branch } j \text{ does not belong to loop } i. \end{cases}$$

Here, n_ℓ denotes the number of all oriented loops in the directed graph. By removing all linearly dependent rows in the loop matrix we get the *reduced loop matrix* $B \in \mathbb{R}^{n_b - n_\eta + 1, n_b}$ of full row rank (for details see Appendix A). Now, Kirchhoff's current law and Kirchhoff's voltage law takes the form

$$B\nu = 0, \quad \imath = B^T j,$$

where $j(t) \in \mathbb{R}^{n_b - n_\eta + 1}$ is the vector of all loop currents. If we split B into element-related incidence matrices

$$B = \begin{bmatrix} B_\mathcal{C} & B_\mathcal{L} & B_\mathcal{R} & B_\mathcal{V} & B_\mathcal{I} \end{bmatrix}$$

and insert the constitutive element relations we get a DAE system of the following form

$$\begin{aligned} B_\mathcal{L} \mathcal{L}(B_\mathcal{L}^T j) B_\mathcal{L}^T \frac{\mathrm{d}}{\mathrm{d}t} j + B_\mathcal{R} r(B_\mathcal{R}^T j) + B_\mathcal{C} \nu_\mathcal{C} + B_\mathcal{I} \nu_\mathcal{I} + B_\mathcal{V} \mathcal{V}_s &= 0, \\ \mathcal{C}(\nu_\mathcal{C}) \frac{\mathrm{d}}{\mathrm{d}t} \nu_\mathcal{C} - B_\mathcal{C}^T j &= 0, \\ B_\mathcal{I}^T j + \mathcal{I}_s &= 0. \end{aligned} \tag{7}$$

Here, $r : \mathbb{R}^{n_\mathcal{R}} \to \mathbb{R}^{n_\mathcal{R}}$ denotes the resistance function [2] and system (7) consists of $n_b - n_\eta + 1 + n_\mathcal{C} + n_\mathcal{I}$ equations in the $n_b - n_\eta + 1 + n_\mathcal{C} + n_\mathcal{I}$ unknowns $[j, \nu_\mathcal{C}, \nu_\mathcal{I}]$. The equations (7) are also known as the *MLA equations*. Again, the index of the MLA equations (7) can be determined based on the topology of the circuit.

Theorem 5 ([13])**.** *Consider an electrical circuit with MLA equations* (7). *Assume that* **(A1)**-**(A5)** *hold.*

1. *The following statements are equivalent:*
   - *the MLA equations* (7) *are of d-index* $\nu_d = 0$;

[2] If the resistance function r is continuously differentiable with $\mathcal{R} = \frac{\partial}{\partial \imath_\mathcal{R}} r$, then $\mathcal{R}$ is the pointwise inverse of $\mathcal{G}$.

- *the circuit contains neither current sources nor* $\mathcal{CRV}$*-loops;*
- $n_{\scriptscriptstyle I} = 0$ *and* $\operatorname{rank} B_{\mathcal{L}} = n_b - n_\eta + 1$.

2. *The following statements are equivalent:*

   - *the MLA equations* (7) *are of d-index* $\nu_d = 1$*;*
   - *the circuit contains neither* $\mathcal{CV}$*-loops nor* $\mathcal{L}I$*-cutsets (except for pure* $\mathcal{L}$*-cutsets);*
   - $\operatorname{rank}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_I] = n_b - n_\eta + 1$ *and* $\ker[B_{\mathcal{L}}, B_I] = \ker B_{\mathcal{L}} \times \{0\}$.

3. *The following statements are equivalent:*

   - *the MLA equations* (7) *are of d-index* $\nu_d = 2$*;*
   - *the circuit contains* $\mathcal{CV}$*-loops or* $\mathcal{L}I$*-cutsets which are no pure* $\mathcal{L}$*-cutsets;*
   - $\operatorname{rank}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_I] < n_b - n_\eta + 1$ *or* $\ker[B_{\mathcal{L}}, B_I] \neq \ker B_{\mathcal{L}} \times \{0\}$.

Remark 6. Note that for a given electrical circuit the MNA equations (6) and the MLA equations (7) may have different d-index depending on the topology of the circuit, in particular, if the circuit contains pure $\mathcal{C}$-loops or pure $\mathcal{L}$-cutsets, cf. Example 6.

3.3 Branch-Oriented Model Equations

In branch-oriented model formulations Kirchhoff's current and voltage laws are stated as

$$A\imath = 0, \tag{8}$$

$$B\nu = 0. \tag{9}$$

Together with the constitutive element relations (5) we obtain a DAE system of the form

$$\mathcal{C}(\nu_{\mathcal{C}})\frac{\mathrm{d}}{\mathrm{d}t}\nu_{\mathcal{C}} = \imath_{\mathcal{C}}, \tag{10a}$$

$$\mathcal{L}(\imath_{\mathcal{L}})\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}} = \nu_{\mathcal{L}}, \tag{10b}$$

$$0 = A_{\mathcal{C}}\imath_{\mathcal{C}} + A_{\mathcal{R}}\imath_{\mathcal{R}} + A_{\mathcal{L}}\imath_{\mathcal{L}} + A_I\imath_I + A_{\mathcal{V}}\imath_{\mathcal{V}}, \tag{10c}$$

$$0 = B_{\mathcal{C}}\nu_{\mathcal{C}} + B_{\mathcal{R}}\nu_{\mathcal{R}} + B_{\mathcal{L}}\nu_{\mathcal{L}} + B_I\nu_I + B_{\mathcal{V}}\nu_{\mathcal{V}}, \tag{10d}$$

$$0 = \imath_{\mathcal{R}} - g(\nu_{\mathcal{R}}), \tag{10e}$$

$$0 = \imath_I - I_s, \tag{10f}$$

$$0 = \nu_{\mathcal{V}} - \mathcal{V}_s, \tag{10g}$$

which consists of $2n_b = 2(n_{\scriptscriptstyle \mathcal{C}} + n_{\scriptscriptstyle \mathcal{R}} + n_{\scriptscriptstyle \mathcal{L}} + n_{\scriptscriptstyle I} + n_{\scriptscriptstyle \mathcal{V}})$ equations in the unknown branch currents $\imath_*$ and branch voltages ν_* for $* \in \{\mathcal{C}, \mathcal{R}, \mathcal{L}, I, \mathcal{V}\}$. We will call (10) the *branch-oriented model equations* of the electrical circuit.

Theorem 7. *Consider an electrical circuit with branch-oriented model equations* (10)*. Assume that* **(A1)-(A5)** *hold.*

1. *The following statements are equivalent:*

   - *the branch-oriented model equations* (10) *are of d-index* $\nu_d = 1$*;*
   - $\operatorname{rank}[A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{V}}] = n_\eta - 1$ *and* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] = \{0\}$*;*
   - $\operatorname{rank}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_I] = n_b - n_\eta + 1$ *and* $\ker[B_{\mathcal{L}}, B_I] = \{0\}$*;*
   - *the circuit contains neither* $\mathcal{L}I$*-cutsets nor* $\mathcal{CV}$*-loops (including pure* $\mathcal{C}$*-loops and pure* $\mathcal{L}$*-cutsets).*

2. *The following statements are equivalent:*

   - *the branch-oriented model equations* (10) *are of d-index* $\nu_d = 2$*;*

- $\operatorname{rank}[A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{V}}] < n_\eta - 1$ *or* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] \neq \{0\}$*;*
- $\operatorname{rank}[B_{\mathcal{L}}, B_{\mathcal{R}}, B_{I}] < n_b - n_\eta + 1$ *or* $\ker[B_{\mathcal{L}}, B_{I}] \neq \{0\}$*;*
- *the circuit contains $\mathcal{L}I$-cutsets or $\mathcal{CV}$-loops (including pure $\mathcal{C}$-loops and pure $\mathcal{L}$-cutsets).*

Proof. For the geometric index the proof is given in [15]. Except for differences in the smoothness requirements (which do not apply here) the geometric index is equal to the differentiation index [2, 9]. □

Next, we introduce the *cutset matrix* Q_0 of a connected directed graph. The removal of any cutset in the graph results in a directed graph with two connected components C_1 and C_2. Given a branch in the cutset, one terminal node must be in C_1 and the other one in C_2. Thus, we can define two different orientations in this cutset: from C_1 to C_2 or from C_2 to C_1. With this the cutset matrix $Q_0 = [q_{ij}]$ can be defined as follows

$$q_{ij} = \begin{cases} 1 & \text{if branch } j \text{ is in cutset } i \text{ with the same orientation,} \\ -1 & \text{if branch } j \text{ is in cutset } i \text{ with the opposite orientation,} \\ 0 & \text{if branch } j \text{ is not in cutset } i. \end{cases}$$

Again, $n_\eta - 1$ linearly independent rows of Q_0 define a reduced cutset matrix $Q \in \mathbb{R}^{n_\eta - 1, n_b}$. Furthermore, it holds that $BA^T = BQ^T = 0$ and $\operatorname{im} B^T = \ker A = \ker Q$, see Theorem 24. Thus, we have that $Q = MA$ for some nonsingular matrix M and Kirchhoff's current law (8) can be replaced by the relation $Q\imath = 0$.
In any circuit graph, we can always choose a tree such that all voltage sources correspond to twigs and all current sources correspond to links. This choice of a tree leads to a set of fundamental cutsets and fundamental loops (each one uniquely defined by a twig or a link, respectively), and choosing the orientations of these cutsets and loops coherently with the orientations of the corresponding twig or link, the matrices Q and B take the form

$$Q = \begin{bmatrix} I_r & -F^T \end{bmatrix}, \quad B = \begin{bmatrix} F & I_s \end{bmatrix}$$

for a certain matrix $F \in \mathbb{R}^{s,r}$ with $s = n_b - n_\eta + 1$ and $r = n_\eta - 1$. This results from the orthogonality property $BQ^T = 0$, see Theorem 24 and the details in Appendix A. The first block matrix in B and Q is associated with the twigs of the tree, whereas the second block is associated with the links.
With this Kirchhoff's current law (8) and Kirchhoff's voltage law (9) can be represented as

$$\imath_{\mathbf{1}} = F^T \imath_{\mathbf{2}}, \tag{11a}$$
$$\nu_{\mathbf{2}} = -F \nu_{\mathbf{1}}, \tag{11b}$$

where the subscript **1** denotes the tree elements while the subscript **2** denotes the cotree elements. We can split $\nu_{\mathbf{1}}$ and $\nu_{\mathbf{1}}$ into

$$\nu_{\mathbf{1}} = [\nu_{\mathcal{C}\mathbf{1}}^T, \nu_{\mathcal{L}\mathbf{1}}^T, \nu_{\mathcal{V}}^T, \nu_{\mathcal{R}\mathbf{1}}^T]^T,$$

consisting of twig capacitors, twig inductors, twig resistors, and voltage sources, and

$$\nu_{\mathbf{2}} = [\nu_{\mathcal{L}\mathbf{2}}^T, \nu_{\mathcal{C}\mathbf{2}}^T, \nu_{I}^T, \nu_{\mathcal{R}\mathbf{2}}^T]^T,$$

consisting of link inductors, link capacitors, link resistors, and current sources. Note that the tree is chosen in such a way that voltage sources always belong to the tree elements while current sources always belong to the cotree elements. A similar splitting can be done for the vectors $\imath_{\mathbf{1}}$ and $\imath_{\mathbf{2}}$ into

$$\imath_{\mathbf{1}} = \begin{bmatrix} \imath_{\mathcal{C}\mathbf{1}}^T & \imath_{\mathcal{L}\mathbf{1}}^T & \imath_{\mathcal{V}}^T & \imath_{\mathcal{R}\mathbf{1}}^T \end{bmatrix}^T, \quad \imath_{\mathbf{2}} = \begin{bmatrix} \imath_{\mathcal{L}\mathbf{2}}^T & \imath_{\mathcal{C}\mathbf{2}}^T & \imath_{I}^T & \imath_{\mathcal{R}\mathbf{2}}^T \end{bmatrix}^T,$$

and the matrix F into

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}. \tag{12}$$

A *proper tree* in a connected circuit graph is a tree which contains all voltage sources and all capacitances as well as (possibly) some resistors, but neither current sources nor inductors. Likewise, a *normal tree* in a connected circuit graph is a tree which contains all voltage sources, no current sources, as many capacitors as possible, and as few inductors as possible; it may also contain some resistors. It can be shown that a connected circuit graph has neither $\mathcal{CV}$-loops nor $\mathcal{LI}$-cutsets if and only if it contains a proper tree, see [15]. Moreover, in a normal tree the fundamental cutsets defined by each twig inductor only has link inductors and currents sources. Analogously, in a normal tree the fundamental loops defined by each link capacitor only involve twig capacitors and voltage sources. Thus, for a normal tree we have $F_{22} = 0$, $F_{24} = 0$ and $F_{42} = 0$ in (12). Summarizing these results we get from (10) by replacing (10c),(10d) by (11a),(11b) that

$$\mathcal{C}(\nu_{\mathcal{C}\mathbf{1}}, \nu_{\mathcal{C}\mathbf{2}}) \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \nu_{\mathcal{C}\mathbf{1}} \\ \nu_{\mathcal{C}\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \imath_{\mathcal{C}\mathbf{1}} \\ \imath_{\mathcal{C}\mathbf{2}} \end{bmatrix}, \tag{13a}$$

$$\mathcal{L}(\imath_{\mathcal{L}\mathbf{1}}, \imath_{\mathcal{L}\mathbf{2}}) \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \imath_{\mathcal{L}\mathbf{1}} \\ \imath_{\mathcal{L}\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \nu_{\mathcal{L}\mathbf{1}} \\ \nu_{\mathcal{L}\mathbf{2}} \end{bmatrix}, \tag{13b}$$

$$\begin{bmatrix} \nu_{\mathcal{L}\mathbf{2}} \\ \nu_{\mathcal{C}\mathbf{2}} \\ \nu_{\mathcal{I}} \\ \nu_{\mathcal{R}\mathbf{2}} \end{bmatrix} = - \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & 0 & F_{23} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & 0 & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} \nu_{\mathcal{C}\mathbf{1}} \\ \nu_{\mathcal{L}\mathbf{1}} \\ \nu_{\mathcal{V}} \\ \nu_{\mathcal{R}\mathbf{1}} \end{bmatrix}, \tag{13c}$$

$$\begin{bmatrix} \imath_{\mathcal{C}\mathbf{1}} \\ \imath_{\mathcal{L}\mathbf{1}} \\ \imath_{\mathcal{V}} \\ \imath_{\mathcal{R}\mathbf{1}} \end{bmatrix} = \begin{bmatrix} F_{11}^T & F_{21}^T & F_{31}^T & F_{41}^T \\ F_{12}^T & 0 & F_{32}^T & 0 \\ F_{13}^T & F_{23}^T & F_{33}^T & F_{43}^T \\ F_{14}^T & 0 & F_{34}^T & F_{44}^T \end{bmatrix} \begin{bmatrix} \imath_{\mathcal{L}\mathbf{2}} \\ \imath_{\mathcal{C}\mathbf{2}} \\ \imath_{\mathcal{I}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix}, \tag{13d}$$

$$0 = \begin{bmatrix} \imath_{\mathcal{R}\mathbf{1}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix} - g(\nu_{\mathcal{R}\mathbf{1}}, \nu_{\mathcal{R}\mathbf{2}}), \tag{13e}$$

$$0 = \imath_{\mathcal{I}} - \mathcal{I}_s, \tag{13f}$$

$$0 = \nu_{\mathcal{V}} - \mathcal{V}_s. \tag{13g}$$

Note that the DAEs (10) and (13) have the same d-index, since the performed transformations do not change the analytical properties of the system.

3.4 Port-Hamiltonian Circuit Equations

If we consider the electrical circuit as a power-based network model of interconnected subsystems that mutually influence each other via energy flow, this directly leads us to the starting point of port-Hamiltonian systems theory, see [11, 21]. The formulation of a physical system as *port-Hamiltonian system* has many advantages as e.g. the preservation of energy, the preservation of passivity or stability, see e.g. [21].
Here, we consider *linear port-Hamiltonian DAEs (pHDAE)* of the form

$$\begin{aligned} E\dot{x} &= (J - R)Qx + KEx + Bu, \\ y &= B^T Q x, \end{aligned} \tag{14}$$

where $J, R, K \in C(\mathbb{I}, \mathbb{R}^{n,n})$, $E, Q \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ with $R = R^T \geq 0$, and $E^T Q = Q^T E \geq 0$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q^T E) = Q^T (EK - JQ) + (K^T E^T - Q^T J^T) Q, \tag{15}$$

as well as $B \in C(\mathbb{I}, \mathbb{R}^{n,p})$. Note that also more general formulations are possible, see [20]. Here, u denotes the p-dimensional input of the system and y denotes the p-dimensional output of the system; together they define the (external) ports. The matrix J can be seen as the interconnection matrix, R is the resistance matrix, and $E^T Q$ describes the total energy of the system represented by the *Hamiltonian* $\mathcal{H}(x) = \frac{1}{2} x^T E^T Q x$. The matrix K is required to describe equivalence transformations in the time-varying setting, for constant coefficients one may set K to zero.

Remark 8. Kirchhoff's current law and Kirchhoff's voltage law define a (separable) *Dirac structure*

$$\mathcal{D} = \{(\imath, \nu) | A\imath = 0,\ \nu = A^T \eta \text{ for some } \lambda \in \mathbb{R}^n\}$$

that describes the underlying geometric structure of the port-Hamiltonian DAE. The currents through the electrical components are the flows and the voltages across the electrical components are the efforts defining the port variables of the Dirac structure.

In order to formulate the circuit equations as a pHDAE, we start by considering the system of equations consisting of Kirchhoff laws (3) together with the constitutive element relations (5). We insert the relations $\nu_{\mathcal{V}} = \mathcal{V}_s$, $\nu_{\mathcal{L}} = A_{\mathcal{L}}^T \eta$ and $\imath_I = I_s$, and, assuming linear element relations for the resistances, capacitances and inductances, a reordering of the equations yields

$$\mathcal{C}(t) \frac{\mathrm{d}}{\mathrm{d}t} \nu_{\mathcal{C}} = \imath_{\mathcal{C}}, \tag{16a}$$

$$\mathcal{L}(t) \frac{\mathrm{d}}{\mathrm{d}t} \imath_{\mathcal{L}} = A_{\mathcal{L}}^T \eta, \tag{16b}$$

$$0 = -\nu_{\mathcal{C}} + A_{\mathcal{C}}^T \eta, \tag{16c}$$

$$0 = -\nu_{\mathcal{R}} + A_{\mathcal{R}}^T \eta, \tag{16d}$$

$$0 = A_{\mathcal{V}}^T \eta - \mathcal{V}_s(t), \tag{16e}$$

$$0 = \imath_{\mathcal{R}} - \mathcal{G}(t) \nu_{\mathcal{R}}, \tag{16f}$$

$$0 = -A_{\mathcal{L}} \imath_{\mathcal{L}} - A_{\mathcal{C}} \imath_{\mathcal{C}} - A_{\mathcal{R}} \imath_{\mathcal{R}} - A_{\mathcal{V}} \imath_{\mathcal{V}} - A_I I_s(t), \tag{16g}$$

$$0 = -\nu_I + A_I^T \eta. \tag{16h}$$

Considering the last equation (16h) as an output equation, we obtain a pHDAE of the form (14) with

$$E = \left[\begin{array}{cc|ccccc} \mathcal{C}(t) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}(t) & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad R = \left[\begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{G}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$J = \left[\begin{array}{cc|ccccc} 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{\mathcal{L}}^T \\ \hline -I & 0 & 0 & 0 & 0 & 0 & A_{\mathcal{C}}^T \\ 0 & 0 & 0 & 0 & 0 & -I & A_{\mathcal{R}}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{\mathcal{V}}^T \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & -A_{\mathcal{L}} & -A_{\mathcal{C}} & -A_{\mathcal{R}} & -A_{\mathcal{V}} & 0 & 0 \end{array}\right], \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -I & 0 \\ 0 & 0 \\ 0 & -A_I \end{bmatrix}, \quad Q = I,$$

and with state variables

$$x = [\nu_{\mathcal{C}}^T, \imath_{\mathcal{L}}^T, \imath_{\mathcal{C}}^T, \imath_{\mathcal{R}}^T, \imath_{\mathcal{V}}^T, \nu_{\mathcal{R}}^T, \eta^T]^T,$$

input variables

$$u = \begin{bmatrix} \mathcal{V}_s \\ I_s \end{bmatrix},$$

and output variables

$$y = \begin{bmatrix} -\imath_{\mathcal{V}} \\ -\nu_I \end{bmatrix}.$$

The Hamiltonian is given by

$$\mathcal{H}(x) = \frac{1}{2} x^T E^T Q x = \frac{1}{2} \nu_{\mathcal{C}}^T \mathcal{C}(t) \nu_{\mathcal{C}} + \frac{1}{2} \imath_{\mathcal{L}}^T \mathcal{L}(t) \imath_{\mathcal{L}}$$

and describes the total energy of the system. In the following, we call system (16) the *port-Hamiltonian Circuit equations (pHC equations).*

Remark 9. Condition (15) is met by defining $K = diag(K_{11}, K_{22}, 0)$, where $K_{11} = K_{11}^T$ and $K_{22} = K_{22}^T$ are given by the unique solutions of the matrix Lyapunov equations

$$\begin{aligned} CK_{11} + K_{11}C &= \dot{C}, \\ LK_{22} + K_{22}L &= \dot{L}. \end{aligned}$$

Since we can always find a variable transformation for x that eliminates K in (14), we can omit these parts in the formulation of the port-Hamiltonian circuit equations (16).

Theorem 10. *Consider an electrical circuit with pHC equations* (16). *Assume that* **(A1)**-**(A5)** *hold.*

1. *The following statements are equivalent:*
   - *the pHC equations* (16) *are of d-index* $\nu_d = 1$*;*
   - $\operatorname{rank}[A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{V}}] = n_\eta - 1$ *and* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] = \{0\}$*;*
   - *the circuit contains neither* $\mathcal{L}I$*-cutsets nor* $\mathcal{CV}$*-loops (including pure* $\mathcal{C}$*-loops).*
2. *The following statements are equivalent:*
   - *the pHC equations* (16) *are of d-index* $\nu_d = 2$*;*
   - $\operatorname{rank}[A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{V}}] < n_\eta - 1$ *or* $\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] \neq \{0\}$*;*
   - *the circuit contains* $\mathcal{L}I$*-cutsets or* $\mathcal{CV}$*-loops.*

In order to prove Theorem 10 we make use of the following Lemma.

Lemma 11. *Let* $A \in \mathbb{R}^{n_1,m}$*,* $B \in \mathbb{R}^{n_1,n_2}$*,* $C \in \mathbb{R}^{m,m}$ *be symmetric positive definite. Then for*

$$M = \begin{bmatrix} ACA^T & B \\ -B^T & 0 \end{bmatrix}$$

it holds that $\ker M = \ker [A, B]^T \times \ker B$*. In particular,* M *is invertible if and only if* $\ker [A, B]^T = \{0\}$ *and* $\ker B = \{0\}$*.*

Proof. For $v \in \ker [A, B]^T \times \ker B$ it follows immediately that $v \in \ker M$. For the converse, let $v \in \ker M$ and partition $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ according to the block structure of M. Then

$$v^T M v = v_1^T ACA^T v_1 + v_1 B v_2 - v_2^T B^T v_1 = 0 \quad \iff \quad v_1^T ACA^T v_1 = 0.$$

Thus, $v_1 \in \ker A^T$ and from $Mv = 0$ we get that $Bv_2 = 0$ and $-B^T v_1 = 0$, and consequently $v \in \ker [A, B]^T \times \ker B$. □

In the following, we denote by $Q_{\mathcal{C}}$, $Q_{\mathcal{V}-\mathcal{C}}$, $Q_{\mathcal{R}-\mathcal{CV}}$, $Q_{\mathcal{V}}$, $\bar{Q}_{\mathcal{C}}$, and $\bar{Q}_{\mathcal{V}-\mathcal{C}}$ projectors onto $\ker A_{\mathcal{C}}^T$, $\ker A_{\mathcal{V}}^T Q_{\mathcal{C}}$, $\ker A_{\mathcal{R}}^T Q_{\mathcal{C}} Q_{\mathcal{V}-\mathcal{C}}$, $\ker A_{\mathcal{V}}^T$, $\ker A_{\mathcal{C}}$, and $\ker Q_{\mathcal{C}}^T A_{\mathcal{V}}$, respectively. The complementary projectors will be denoted by $P := I - Q$, with the corresponding sub-index. In order to shorten notations, we use the abbreviation $Q_{\mathcal{CRV}} := Q_{\mathcal{C}} Q_{\mathcal{V}-\mathcal{C}} Q_{\mathcal{R}-\mathcal{CV}}$.

Lemma 12. *[4] If $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{G}$ are positive definite, then the matrices*

$$
\begin{aligned}
H_1 &:= A_\mathcal{C}\mathcal{C}A_\mathcal{C}^T + Q_\mathcal{C}^T Q_\mathcal{C} \\
H_2 &:= Q_\mathcal{C}^T A_\mathcal{V} A_\mathcal{V}^T Q_\mathcal{C} + Q_{\mathcal{V}-\mathcal{C}}^T Q_{\mathcal{V}-\mathcal{C}} \\
H_3 &:= A_\mathcal{V}^T Q_\mathcal{C} Q_\mathcal{C}^T A_\mathcal{V} + \bar{Q}_{\mathcal{V}-\mathcal{C}}^T \bar{Q}_{\mathcal{V}-\mathcal{C}} \\
H_4 &:= Q_{\mathcal{V}-\mathcal{C}}^T Q_\mathcal{C}^T A_\mathcal{R}\mathcal{G}A_\mathcal{R}^T Q_\mathcal{C} Q_{\mathcal{V}-\mathcal{C}} + Q_{\mathcal{R}-\mathcal{C}\mathcal{V}}^T Q_{\mathcal{R}-\mathcal{C}\mathcal{V}} \\
H_5 &:= Q_{\mathcal{C}\mathcal{R}\mathcal{V}}^T A_\mathcal{L}\mathcal{L}^{-1}A_\mathcal{L}^T Q_{\mathcal{C}\mathcal{R}\mathcal{V}} + P_{\mathcal{C}\mathcal{R}\mathcal{V}}^T P_{\mathcal{C}\mathcal{R}\mathcal{V}}^T \\
H_6 &:= \bar{Q}_{\mathcal{V}-\mathcal{C}}^T A_\mathcal{V}^T H_1^{-1} A_\mathcal{V} \bar{Q}_{\mathcal{V}-\mathcal{C}} + \bar{P}_{\mathcal{V}-\mathcal{C}}^T \bar{P}_{\mathcal{V}-\mathcal{C}} \\
H_7 &:= A_\mathcal{C}^T A_\mathcal{C} + \bar{Q}_\mathcal{C}^T \bar{Q}_\mathcal{C}
\end{aligned}
$$

are nonsingular.

Proof of Theorem 10. Omitting the output equations, the pHC equations (16) are of the form

$$
\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & A_{12} \\ -A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f_2 \end{bmatrix}, \tag{17}
$$

where E_{11} is (pointwise) nonsingular. The DAE (17) is of d-index 1 if and only if A_{22} is invertible. In this case, differentiation of the constraints yields

$$
\begin{aligned}
& 0 = -A_{12}^T \dot{x}_1 + A_{22}\dot{x}_2 + \dot{f}_2 \\
\implies \quad & \dot{x}_2 = A_{22}^{-1}A_{12}^T \dot{x}_1 - A_{22}^{-1}\dot{f}_2 = A_{22}^{-1}A_{12}^T E_{11}^{-1} A_{12} x_2 - A_{22}^{-1}\dot{f}_2.
\end{aligned}
$$

It holds that

$$
A_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & A_\mathcal{C}^T \\ 0 & 0 & 0 & -I & A_\mathcal{R}^T \\ 0 & 0 & 0 & 0 & A_\mathcal{V}^T \\ 0 & I & 0 & -\mathcal{G} & 0 \\ -A_\mathcal{C} & -A_\mathcal{R} & -A_\mathcal{V} & 0 & 0 \end{bmatrix} \sim \left[\begin{array}{cc|ccc} 0 & -I & 0 & 0 & 0 \\ I & -\mathcal{G} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & A_\mathcal{C}^T \\ 0 & 0 & 0 & 0 & A_\mathcal{V}^T \\ 0 & 0 & -A_\mathcal{C} & -A_\mathcal{V} & -A_R\mathcal{G}A_R^T \end{array}\right]
$$

by basic row and column transformations. Due to Lemma 11, the matrix A_{22} is invertible if and only if

$$
\ker[A_R, A_C, A_V]^T = \{0\} \quad \text{and} \quad \ker[A_C, A_V] = \{0\},
$$

i.e., if and only if the matrix $[A_\mathcal{C}\ A_\mathcal{R}\ A_\mathcal{V}]$ has full row rank and $\begin{bmatrix} A_\mathcal{C}^T \\ A_\mathcal{V}^T \end{bmatrix}$ has full column rank. Due to Lemma 26 the first condition is equivalent to the absence of $\mathcal{L}\mathcal{I}$-cutsets in the circuit, while the second condition is equivalent to the absence of $\mathcal{C}\mathcal{V}$-loops (including pure $\mathcal{C}$-loops).

If the matrix A_{22} is not invertible, then there are hidden constraints contained in the system. We follow the lines of the proof of Theorem 5.1 in [4]. If we differentiate the algebraic constraints in (16) we get

$$
\begin{aligned}
0 &= -\frac{\mathrm{d}}{\mathrm{d}t}\nu_\mathcal{C} + A_\mathcal{C}^T \frac{\mathrm{d}}{\mathrm{d}t}\eta, \\
0 &= -\frac{\mathrm{d}}{\mathrm{d}t}\nu_\mathcal{R} + A_\mathcal{R}^T \frac{\mathrm{d}}{\mathrm{d}t}\eta, \\
0 &= A_\mathcal{V}^T \frac{\mathrm{d}}{\mathrm{d}t}\eta + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}_s, \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\imath_\mathcal{R} - \mathcal{G}\frac{\mathrm{d}}{\mathrm{d}t}\nu_\mathcal{R}, \\
0 &= -A_\mathcal{C}\frac{\mathrm{d}}{\mathrm{d}t}\imath_\mathcal{C} - A_\mathcal{R}\frac{\mathrm{d}}{\mathrm{d}t}\imath_\mathcal{R} - A_\mathcal{I}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_s - A_\mathcal{V}\frac{\mathrm{d}}{\mathrm{d}t}\imath_\mathcal{V} - A_\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}t}\imath_\mathcal{L}.
\end{aligned}
$$

Inserting the expression for $\frac{\mathrm{d}}{\mathrm{d}t}\nu_{\mathcal{C}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}}$ from (16a) and (16b), as well as $\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{R}} = \mathcal{G}A_{\mathcal{R}}^T\frac{\mathrm{d}}{\mathrm{d}t}\eta$ yields

$$
\begin{aligned}
0 &= -\mathcal{C}^{-1}\imath_{\mathcal{C}} + A_{\mathcal{C}}^T\frac{\mathrm{d}}{\mathrm{d}t}\eta,\\
0 &= A_{\mathcal{V}}^T\frac{\mathrm{d}}{\mathrm{d}t}\eta + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}_s,\\
0 &= -A_{\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{C}} - A_{\mathcal{R}}GA_{\mathcal{R}}^T\frac{\mathrm{d}}{\mathrm{d}t}\eta - A_I\frac{\mathrm{d}}{\mathrm{d}t}I_s - A_{\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} - A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T\eta.
\end{aligned}
\tag{18}
$$

In order to extract the underlying ODE, we need to derive expressions for $\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{C}}$, $\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\eta$ from (18). Then we also get representations for $\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{R}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\nu_{\mathcal{R}}$ from the above relations. We consider the following splitting

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\eta &= P_{\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\eta + Q_{\mathcal{C}}P_{\mathcal{V}-\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\eta + Q_{\mathcal{C}}Q_{\mathcal{V}-\mathcal{C}}P_{\mathcal{R}-\mathcal{C}\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\eta + Q_{\mathcal{C}\mathcal{R}\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\eta\\
&=: \frac{\mathrm{d}}{\mathrm{d}t}\eta_1 + \frac{\mathrm{d}}{\mathrm{d}t}\eta_2 + \frac{\mathrm{d}}{\mathrm{d}t}\eta_3 + \frac{\mathrm{d}}{\mathrm{d}t}\eta_4 = \dot\eta_1 + \dot\eta_2 + \dot\eta_3 + \dot\eta_4,\\
\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} &= \bar P_{\mathcal{V}-\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} + \bar Q_{\mathcal{V}-\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} =: \frac{\mathrm{d}}{\mathrm{d}t}\imath_1 + \frac{\mathrm{d}}{\mathrm{d}t}\imath_2,\\
\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{C}} &= \bar P_{\mathcal{C}-\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{C}} + \bar Q_{\mathcal{C}-\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{C}} =: \frac{\mathrm{d}}{\mathrm{d}t}\imath_3 + \frac{\mathrm{d}}{\mathrm{d}t}\imath_4,
\end{aligned}
$$

giving

$$0 = -\mathcal{C}^{-1}\imath_{\mathcal{C}} + A_{\mathcal{C}}^T\dot\eta_1, \tag{19a}$$

$$0 = A_{\mathcal{V}}^T(\dot\eta_1 + \dot\eta_2) + \dot{\mathcal{V}}_s, \tag{19b}$$

$$0 = -A_{\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{C}} - A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T(\dot\eta_1 + \dot\eta_2 + \dot\eta_3) - A_I\dot I_s - A_{\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} - A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T\eta. \tag{19c}$$

Multiplication of (19c) with $Q_{\mathcal{C}}^T$, $Q_{\mathcal{V}-\mathcal{C}}^TQ_{\mathcal{C}}^T$ and $Q_{\mathcal{C}\mathcal{R}\mathcal{V}}^T$ yields

$$0 = -Q_{\mathcal{C}}^T\left[A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T(\dot\eta_1 + \dot\eta_2 + \dot\eta_3) + A_I\dot I_s + A_{\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} + A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T\eta\right], \tag{20a}$$

$$0 = -Q_{\mathcal{V}-\mathcal{C}}^TQ_{\mathcal{C}}^T\left[A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T(\dot\eta_1 + \dot\eta_2 + \dot\eta_3) + A_I\dot I_s + A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T\eta\right], \tag{20b}$$

$$0 = -Q_{\mathcal{C}\mathcal{R}\mathcal{V}}^T\left[A_I\dot I_s + A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T\eta\right]. \tag{20c}$$

If we multiply (19a) with $H_1^{-1}A_{\mathcal{C}}\mathcal{C}$ we get

$$\dot\eta_1 = P_{\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\eta = H_1^{-1}A_{\mathcal{C}}\imath_{\mathcal{C}}, \tag{21}$$

and inserting (21) into (19b) gives

$$0 = A_{\mathcal{V}}^T(H_1^{-1}A_{\mathcal{C}}\imath_{\mathcal{C}} + \dot\eta_2) + \dot{\mathcal{V}}_s. \tag{22}$$

Moreover, multiplication with $H_2^{-1}Q_{\mathcal{C}}^TA_{\mathcal{V}}$ yields

$$0 = H_2^{-1}Q_{\mathcal{C}}^TA_{\mathcal{V}}A_{\mathcal{V}}^T(H_1^{-1}A_{\mathcal{C}}\imath_{\mathcal{C}} + \dot\eta_2) + \dot{\mathcal{V}}_s,$$

giving $P_{\mathcal{V}-\mathcal{C}}\frac{\mathrm{d}}{\mathrm{d}t}\eta = f(\imath_{\mathcal{C}}, \dot{\mathcal{V}}_s)$ as a function depending on $\imath_{\mathcal{C}}$ and $\dot{\mathcal{V}}_s$, such that $\dot\eta_2 = Q_{\mathcal{C}}f(\imath_{\mathcal{C}}, \dot{\mathcal{V}}_s)$. Inserting the expressions for $\dot\eta_1$ and $\dot\eta_2$ into (20b) and multiplication of the resulting system with H_4^{-1} gives a representation of $P_{\mathcal{R}-\mathcal{C}\mathcal{V}}\frac{\mathrm{d}}{\mathrm{d}t}\eta$ such that the multiplication with $Q_{\mathcal{C}}Q_{\mathcal{V}-\mathcal{C}}$ gives $\dot\eta_3 = g(\eta, \imath_{\mathcal{C}}, \dot{\mathcal{V}}_s, \dot I_s)$ as function of $\eta, \imath_{\mathcal{C}}, \dot{\mathcal{V}}_s, \dot I_s$. Furthermore, inserting the expressions for $\dot\eta_1$, $\dot\eta_2$

and $\dot{\eta}_3$ into (20a) and multiplying the resulting equation with $H_3^{-1}A_{\mathcal{V}}^T Q_C$ yields a representation for $\bar{P}_{\mathcal{V}-C}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{V}} = \frac{\mathrm{d}}{\mathrm{d}t}\imath_1$ as function of $\eta, \imath_C, \dot{\mathcal{V}}_s, I_s$.
Multiplication of (22) with $\bar{Q}_{\mathcal{V}-C}^T$ yields

$$0 = \bar{Q}_{\mathcal{V}-C}^T A_{\mathcal{V}}^T H_1^{-1} A_C \imath_C + \bar{Q}_{\mathcal{V}-C}^T \dot{\mathcal{V}}_s. \tag{23}$$

If we differentiate (20c) and (23) a second time, we get

$$0 = -Q_{C\mathcal{R}\mathcal{V}}^T \left[A_I \ddot{I}_s + A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T(\dot{\eta}_1 + \dot{\eta}_2 + \dot{\eta}_3 + \dot{\eta}_4) \right], \tag{24}$$

$$0 = \bar{Q}_{\mathcal{V}-C}^T A_{\mathcal{V}}^T H_1^{-1} A_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C + \bar{Q}_{\mathcal{V}-C}^T \ddot{\mathcal{V}}_s. \tag{25}$$

Inserting the expressions for $\dot{\eta}_1$, $\dot{\eta}_2$ and $\dot{\eta}_3$ and multiplying (24) with H_5^{-1} gives

$$0 = -H_5^{-1} Q_{C\mathcal{R}\mathcal{V}}^T A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T \dot{\eta}_4 + \tilde{f}(\imath_C, \eta, \dot{\mathcal{V}}_s, \dot{I}_s, \ddot{I}_s),$$

which gives an expression for $\dot{\eta}_4$ as function of $\imath_C, \eta, \dot{\mathcal{V}}_s, \dot{I}_s, \ddot{I}_s$. Multiplication of (25) with H_6^{-1} yields

$$0 = H_6^{-1}\bar{Q}_{\mathcal{V}-C}^T A_{\mathcal{V}}^T H_1^{-1} A_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C + H_6^{-1}\bar{Q}_{\mathcal{V}-C}^T \ddot{\mathcal{V}}_s,$$

and by inserting $A_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C$ from (19c) we get $\frac{\mathrm{d}}{\mathrm{d}t}\imath_2$ as function of $\imath_C, \eta, \dot{\mathcal{V}}_s, \ddot{\mathcal{V}}_s, \dot{I}_s$. Inserting the obtained expression for $\dot{\eta}_1$, $\dot{\eta}_2$, $\dot{\eta}_3$, $\frac{\mathrm{d}}{\mathrm{d}t}\imath_1$ and $\frac{\mathrm{d}}{\mathrm{d}t}\imath_2$ into (19c) yields

$$A_C(\bar{P}_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C + \bar{Q}_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C) = F(\eta, \imath_C, \dot{\mathcal{V}}_s, \dot{I}_s, \ddot{\mathcal{V}}_s).$$

Finally, multiplication with $H_7^{-1}A_C^T$ gives

$$\bar{P}_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C = H_7^{-1}A_C^T F(\eta, \imath_C, \dot{\mathcal{V}}_s, \dot{I}_s, \ddot{\mathcal{V}}_s),$$

and from the derivatives of (19a) and (21), and inserting the expression for $\bar{P}_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C$ we get

$$\bar{Q}_C \frac{\mathrm{d}}{\mathrm{d}t}\imath_C = [\mathcal{C}A_C^T H_1^{-1} A_C - I] H_7^{-1}A_C^T F(\eta, \imath_C, \dot{\mathcal{V}}_s, \dot{I}_s, \ddot{\mathcal{V}}_s).$$

□

Remark 13. Equation (20c) is a hidden constraint on η (in particular on η_4) and corresponds to the hidden constraints of the MNA equations (29) in case of the existence of $\mathcal{L}I$-cutsets, see [4]. In particular, $Q_{C\mathcal{R}\mathcal{V}} = 0$ if and only if the circuit does not contain $\mathcal{L}I$-cutsets. Equation (23) poses a hidden constraint on $\imath_C$ and it holds that $\bar{Q}_{\mathcal{V}-C} = 0$ if and only if the circuit contains no $\mathcal{C}\mathcal{V}$-loops.

Remark 14. In contrast to the MNA formulation (6) or the MLA formulation (7) the pHC formulation (16) will never be a DAE of d-index $\nu_d = 0$, i.e., it will never take the form of an ODE. Otherwise, the conditions for d-index $\nu_d = 1$ and $\nu_d = 2$ are similar to the conditions for the MNA equation (6) with the exception of pure $\mathcal{C}$-loops. If a circuit contains no $\mathcal{L}I$-cutsets and only pure $\mathcal{C}$-loops the MNA equations (6) are of d-index $\nu_d = 1$ while the pHC equations (16) are of d-index $\nu_d = 2$ (cf. Example 6).

4 Structural Analysis of the Circuit Equations

For the analysis and regularization of DAEs structural approaches are widely used in equation-based modeling environments. In this paper, we focus on the *Signature method* (Σ*-method*) [12] for the structural analysis of DAEs. Another popular structural method is the *Pantelides algorithm* [10] that is related to the Σ-method, see [12]. First, we will review the basic steps of the Σ-method. More details can be found in [12]. Then, in Sections 4.2,4.3,4.4 and 4.5 we apply the Σ-method to the different formulations of the circuit equations derived in the previous sections. Although, the four formulations describe the same dynamical behavior of the circuit, there are small differences in the analytical properties (as e.g. the index of the system) and, as we will see, there may be huge differences when it comes to the applicability of the Σ-method. Some of the formulations are suited for a structural analysis while others are not depending on the topology of the circuit.

4.1 The Signature Method for DAEs

The Σ-method can be applied to regular nonlinear DAEs of arbitrary high order p of the form

$$F(t, x, \dot{x}, \ldots, x^{(p)}) = 0, \tag{26}$$

with $F : \mathbb{I} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth. We denote by F_i the ith component of the vector-valued function F and by x_j the jth component of the vector x. Then, the Σ-method consists of the following steps:

1. Building the *signature matrix* $\Sigma = [\sigma_{ij}]_{i,j=1,\ldots,n}$ with

$$\sigma_{ij} = \begin{cases} \text{highest order of derivative of } x_j \text{ in } F_i, \\ -\infty \text{ if } x_j \text{ does not occur in } F_i. \end{cases}$$

2. Finding a *highest value transversal (HVT)* of Σ, i.e., a *transversal* T of Σ

$$T = \{(1, j_1), (2, j_2), \ldots, (n, j_n)\},$$

   where $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$, with maximal *value*

$$\mathrm{Val}(T) := \sum_{(i,j)\in T} \sigma_{ij}.$$

3. Computing the *offset vectors* $c = [c_i]_{i=1,\ldots,n}$ and $d = [d_j]_{j=1,\ldots,n}$ with $c_i \geq 0$, $d_j \geq 0$ such that

$$d_j - c_i \geq \sigma_{ij} \quad \text{for all} \quad i, j = 1, \ldots, n, \tag{27a}$$
$$d_j - c_i = \sigma_{ij} \quad \text{for all} \quad (i, j) \in T. \tag{27b}$$

4. Forming the Σ-*Jacobian* $\mathfrak{J} = [\mathfrak{J}_{ij}]_{i,j=1,\ldots,n}$, with

$$\mathfrak{J}_{ij} := \begin{cases} \frac{\partial F_i}{\partial x_j^{(\sigma_{ij})}} & \text{if } d_j - c_i = \sigma_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

5. Building the *reduced derivative array*

$$\mathcal{F}(t,\mathcal{X}) = \begin{bmatrix} F_1(t,x,\dot{x},\ldots,x^{(p)}) \\ \frac{\mathrm{d}}{\mathrm{d}t}F_1(t,x,\dot{x},\ldots,x^{(p)}) \\ \vdots \\ \frac{\mathrm{d}^{c_1}}{\mathrm{d}t^{c_1}}F_1(t,x,\dot{x},\ldots,x^{(p)}) \\ \vdots \\ F_n(t,x,\dot{x},\ldots,x^{(p)}) \\ \frac{\mathrm{d}}{\mathrm{d}t}F_n(t,x,\dot{x},\ldots,x^{(p)}) \\ \vdots \\ \frac{\mathrm{d}^{c_n}}{\mathrm{d}t^{c_n}}F_n(t,x,\dot{x},\ldots,x^{(p)}) \end{bmatrix} = 0, \tag{28}$$

with

$$\mathcal{X} = \begin{bmatrix} x_1 & \dot{x}_1 & \ldots & x_1^{(d_1)} & \ldots & x_n & \dot{x}_n & \ldots & x_n^{(d_n)} \end{bmatrix}^T.$$

6. Success check: if $\mathcal{F}(t,\mathcal{X}) = 0$, considered locally as an algebraic system, has a solution $(t^*,\mathcal{X}^*) \in \mathbb{I} \times \mathbb{R}^{n+\sum_{i=1}^n d_i}$ and $\mathfrak{J}$ is nonsingular at $(t^*,\mathcal{X}^*)$, then $(t^*,\mathcal{X}^*)$ is a consistent point and the method succeeds.

If the Σ-method succeeds, it allows to determine the *structural index* of the DAE as

$$\nu_S := \max_i c_i + \begin{cases} 0 & \text{if all } d_j > 0, \\ 1 & \text{if some } d_j = 0, \end{cases}$$

and $\mathrm{Val}(\Sigma)$, defined as the value of the highest value transversal T, corresponds to the number of degrees of freedom of the system. We call $\mathfrak{J}$ the Σ-Jacobian since it is in general not equal to the Jacobian $\frac{\partial F}{\partial x}$ or $\frac{\partial F}{\partial \dot{x}}$, but defined by the offset vectors. Note that the success check of the Σ-method is performed locally at a fixed point $(t^*,\mathcal{X}^*)$, such that the result may hold only locally in a neighborhood of a consistent point. The HVT defines a mapping of maximal value between variables and equations, but it is usually not uniquely determined. A HVT, as well as the offset vectors, can be computed by solving a linear assignment problem (LAP), see [12]. That means, Σ is the matrix of the LAP, where each assignment is specified by a transversal. This LAP (as a special kind of a linear programming problem) also has a dual problem, and the offset vectors c and d are the corresponding solutions of the dual problem. Note that the offset vectors c and d are not uniquely defined by the conditions (27), since for any feasible solution c and d, also the vectors $[c_i + \theta]_i$ and $[d_j + \theta]_j$ form a solution for any $\theta > 0$. However, since there exists a unique element-wise smallest solution of the dual problem, these so-called *canonical offsets* are uniquely determined and independent of the chosen HVT, see [12, Theorem 3.6].
The crucial step in the Σ-method is the success check, i.e., the verification of regularity of the Σ-Jacobian at a consistent point. Systems for which the Σ-Jacobian is singular for all points $(t,\mathcal{X})$ that solve the enlarged system (28), or systems for which there exists no HVT, are called *structurally singular*. Accordingly, we call systems for which the Σ-method succeeds *structurally regular*.
It has been shown in [12] that the Σ-method works successfully for certain (structured) classes of DAE systems, among others for systems in Hessenberg form including semi-explicit systems of d-index 2 and the equations of motion of constrained multibody systems of d-index 3. For such systems the Σ-method succeeds (locally at a consistent point) with $\nu_S = \nu_d$. In general, if the Σ-method succeeds, the structural index gives an upper bound for the d-index of the system, $\nu_d \leq \nu_S$, see [12]. However, there are also cases where the Σ-method fails, see [14, 18]. In particular, this can happen for circuit equations as we will see in the following sections.

4.2 The Signature Method for the MNA equations

We want to apply the Σ-method to the MNA equations (6). To start with we consider the following two examples.

Example 1. Consider the circuit given in Figure 1 consisting of a current source with source function $I_s(t)$, two inductors with inductances $\mathcal{L}_1$ and $\mathcal{L}_2$, a capacitor with capacitance $\mathcal{C}$ and a resistor with conductance $\mathcal{G}$. The directed graph corresponding to the circuit is given in Figure 2.

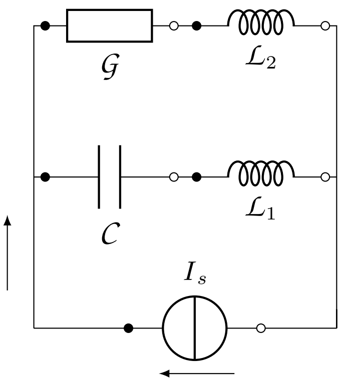

Figure 1: $\mathcal{LRCI}$-circuit

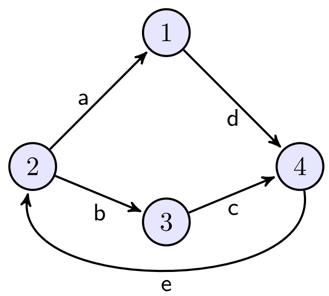

Figure 2: Graph for $\mathcal{LRCI}$-circuit

(a) If we select node 1 as reference node, the reduced incidence matrix is given by

$$A = [A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}, A_I] = \left[\begin{array}{c|c|cc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

and the MNA equations are given by

$$\begin{aligned} 0 &= \mathcal{C}(\dot{\eta}_2 - \dot{\eta}_3) + \mathcal{G}\eta_2 - I_s, \\ 0 &= -\mathcal{C}(\dot{\eta}_2 - \dot{\eta}_3) + \imath_{\mathcal{L}_1}, \\ 0 &= -\imath_{\mathcal{L}_1} - \imath_{\mathcal{L}_2} + I_s, \\ 0 &= \mathcal{L}_1 \dot{\imath}_{\mathcal{L}_1} - (\eta_3 - \eta_4), \\ 0 &= \mathcal{L}_2 \dot{\imath}_{\mathcal{L}_2} + \eta_4, \end{aligned}$$

with unknowns $x = [\eta_2, \eta_3, \eta_4, \imath_{\mathcal{L}_1}, \imath_{\mathcal{L}_2}]^T$. The signature matrix corresponding to this DAE system is given by

$$\Sigma = \left[\begin{array}{ccc|cc} \boxed{1} & 1 & - & - & - \\ 1 & \boxed{1} & - & 0 & - \\ - & - & - & \boxed{0} & 0 \\ \hline - & 0 & \boxed{0} & 1 & - \\ - & - & 0 & - & \boxed{1} \end{array}\right],$$

where one possible HVT is marked by gray boxes. Here, the entry $-$ stands for $-\infty$. The canonical offset vectors are $c = [0, 0, 1, 0, 0]$ and $d = [1, 1, 0, 1, 1]$, and $\mathrm{Val}(\Sigma) = 3$. The corresponding Σ-Jacobian is given by

$$\mathfrak{J} = \left[\begin{array}{ccc|cc} \mathcal{C} & -\mathcal{C} & 0 & 0 & 0 \\ -\mathcal{C} & \mathcal{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ \hline 0 & 0 & 1 & \mathcal{L}_1 & 0 \\ 0 & 0 & 1 & 0 & \mathcal{L}_2 \end{array}\right],$$

and we see immediately that the success check of the Σ-method fails, since $\mathfrak{J}$ is singular.

(b) If we select node 2 as reference node, the reduced incidence matrix is given by

$$A = [A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}, A_I] = \left[\begin{array}{c|c|cc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

and, in this case, the MNA equations take the form

$$\begin{aligned} 0 &= \mathcal{G}\eta_1 + \imath_{\mathcal{L}_2}, \\ 0 &= \mathcal{C}\dot{\eta}_3 + \imath_{\mathcal{L}_1}, \\ 0 &= -\imath_{\mathcal{L}_1} - \imath_{\mathcal{L}_2} + I_s, \\ 0 &= \mathcal{L}_1 \dot{\imath}_{\mathcal{L}_1} - \eta_3 + \eta_4, \\ 0 &= \mathcal{L}_2 \dot{\imath}_{\mathcal{L}_2} - \eta_1 + \eta_4, \end{aligned}$$

with unknowns $x = [\eta_1, \eta_3, \eta_4, \imath_{\mathcal{L}_1}, \imath_{\mathcal{L}_2}]^T$. The signature matrix and Σ-Jacobian are given by

$$\Sigma = \left[\begin{array}{ccc|cc} \boxed{0} & - & - & - & 0 \\ - & \boxed{1} & - & 0 & - \\ - & - & - & \boxed{0} & 0 \\ \hline - & 0 & \boxed{0} & 1 & - \\ 0 & - & 0 & - & \boxed{1} \end{array}\right], \quad \mathfrak{J} = \left[\begin{array}{ccc|cc} \mathcal{G} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ \hline 0 & 0 & 1 & \mathcal{L}_1 & 0 \\ -1 & 0 & 1 & 0 & \mathcal{L}_2 \end{array}\right]$$

with canonical offsets $c = [0,0,1,0,0]$ and $d = [0,1,0,1,1]$, and $\mathrm{Val}(\Sigma) = 2$. Now, the Σ-method succeeds, since $\mathfrak{J}$ is regular, and the structural index is determined to be $\nu_S = 2$.

(c) If we select node 3 as reference node, the MNA equations take the form

$$\begin{aligned} 0 &= -\mathcal{G}(-\eta_1 + \eta_2) + \imath_{\mathcal{L}_2}, \\ 0 &= \mathcal{C}\dot{\eta}_2 + \mathcal{G}(-\eta_1 + \eta_2) - I_s, \\ 0 &= -\imath_{\mathcal{L}_1} - \imath_{\mathcal{L}_2} + I_s, \\ 0 &= \mathcal{L}_1 \dot{\imath}_{\mathcal{L}_1} + \eta_4, \\ 0 &= \mathcal{L}_2 \dot{\imath}_{\mathcal{L}_2} - \eta_1 + \eta_4, \end{aligned}$$

with unknowns $x = [\eta_1, \eta_2, \eta_4, \imath_{\mathcal{L}_1}, \imath_{\mathcal{L}_2}]^T$. The signature matrix and Σ-Jacobian are given by

$$\Sigma = \left[\begin{array}{ccc|cc} \boxed{0} & 0 & - & - & 0 \\ 0 & \boxed{1} & - & - & - \\ - & - & - & \boxed{0} & 0 \\ \hline - & - & \boxed{0} & 1 & - \\ 0 & - & 0 & - & \boxed{1} \end{array}\right], \quad \mathfrak{J} = \left[\begin{array}{ccc|cc} \mathcal{G} & 0 & 0 & 0 & 0 \\ -\mathcal{G} & \mathcal{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ \hline 0 & 0 & 1 & \mathcal{L}_1 & 0 \\ -1 & 0 & 1 & 0 & \mathcal{L}_2 \end{array}\right]$$

with canonical offsets $c = [0,0,1,0,0]$ and $d = [0,1,0,1,1]$ and $\mathrm{Val}(\Sigma) = 2$. Again, the Σ-method succeeds with $\nu_S = 2$.

(d) If we select node 4 as reference node, the MNA equations take the form

$$\begin{aligned} 0 &= -\mathcal{G}(-\eta_1 + \eta_2) + \imath_{\mathcal{L}_2}, \\ 0 &= \mathcal{C}(\dot{\eta}_2 - \dot{\eta}_3) + \mathcal{G}(-\eta_1 + \eta_2) - I_s, \\ 0 &= -\mathcal{C}(\dot{\eta}_2 - \dot{\eta}_3) + \imath_{\mathcal{L}_1}, \\ 0 &= \mathcal{L}_1 \dot{\imath}_{\mathcal{L}_1} - \eta_3, \\ 0 &= \mathcal{L}_2 \dot{\imath}_{\mathcal{L}_2} - \eta_1, \end{aligned}$$

with unknowns $x = [\eta_1, \eta_2, \eta_3, \imath_{\mathcal{L}_1}, \imath_{\mathcal{L}_2}]^T$. The signature matrix and Σ-Jacobian are given by

$$\Sigma = \left[\begin{array}{ccc|cc} 0 & 0 & - & - & 0 \\ 0 & 1 & 1 & - & - \\ - & 1 & 1 & 0 & - \\ \hline - & - & 0 & 1 & - \\ 0 & - & - & - & 1 \end{array}\right], \quad \mathfrak{J} = \left[\begin{array}{ccc|cc} \mathcal{G} & 0 & 0 & 0 & 0 \\ -\mathcal{G} & \mathcal{C} & -\mathcal{C} & 0 & 0 \\ 0 & -\mathcal{C} & \mathcal{C} & 0 & 0 \\ \hline 0 & 0 & 0 & \mathcal{L}_1 & 0 \\ -1 & 0 & 0 & 0 & \mathcal{L}_2 \end{array}\right]$$

with canonical offsets $c = [0,0,0,0,0]$ and $d = [0,1,1,1,1]$ and $\mathrm{Val}(\Sigma) = 4$. In this case, the success check fails due to the singularity of the Σ-Jacobian.

◁

In Example 1 we can observe that failure or success of the Σ-method can depend on the selection of the reference node. If we select node 1 or node 4, the Σ-method fails due to a singular Σ-Jacobian, if we select node 2 or node 3, the Σ-method succeeds with $\nu_S = 2$. Note that due to the occurrence of the $\mathcal{L}\mathcal{I}$-cutset in the circuit given in Example 1, the MNA equations have d-index $\nu_d = 2$ and this analytical property is independent of the chosen reference node.

Example 2. Consider the circuit given in Figure 3 with corresponding circuit graph given in Figure 4. If we select node 1 as reference node, we get the reduced incidence matrix

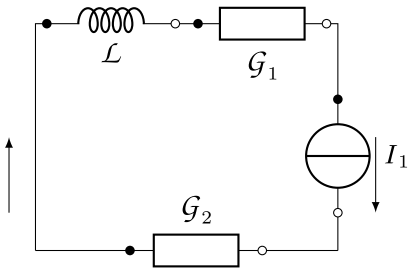

Figure 3: $\mathcal{RLI}$-circuit

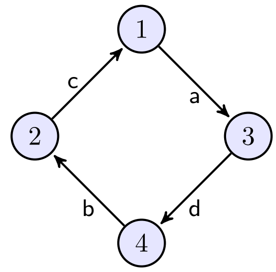

Figure 4: Graph for $\mathcal{RLI}$-circuit

$$A = [A_{\mathcal{R}}, A_{\mathcal{L}}, A_{\mathcal{I}}] = \left[\begin{array}{cc|c|c} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

and the MNA equations take the form

$$\begin{aligned} 0 &= -\mathcal{G}_2(-\eta_2 + \eta_4) + \imath_{\mathcal{L}}, \\ 0 &= \mathcal{G}_1 \eta_3 + \mathcal{I}_s, \\ 0 &= \mathcal{G}_2(-\eta_2 + \eta_4) - \mathcal{I}_s, \\ 0 &= \mathcal{L} i_{\mathcal{L}} - \eta_2. \end{aligned}$$

The corresponding signature matrix and Σ-Jacobian are given by

$$\Sigma = \left[\begin{array}{ccc|c} 0 & - & 0 & 0 \\ - & 0 & - & - \\ 0 & - & 0 & - \\ \hline 0 & - & - & 1 \end{array}\right], \quad \mathfrak{J} = \left[\begin{array}{ccc|c} \mathcal{G}_2 & 0 & -\mathcal{G}_2 & 0 \\ 0 & \mathcal{G}_1 & 0 & 0 \\ -\mathcal{G}_2 & 0 & \mathcal{G}_2 & 0 \\ \hline -1 & 0 & 0 & \mathcal{L} \end{array}\right]$$

with canonical offsets $c = [0,0,0,0]$ and $d = [0,0,0,1]$ and $\mathrm{Val}(\Sigma) = 1$. The success check of the Σ-method fails due to singularity of the Σ-Jacobian. If we select node 2,3 or 4 as reference node, the Σ-method also fails due to singularity of $\mathfrak{J}$. ◁

Example 2 shows that there are cases where the Σ-method will always fail, independently of the selection of the reference node. Note again, that the circuit given in Example 2 contains an $\mathcal{LI}$-cutset and, thus, the MNA equations have d-index $\nu_d = 2$.

In order to explain the problems that show up in the examples we first rewrite the MNA equations (6) in the following form

$$\begin{aligned} A_{\mathcal{C}}\mathcal{C}(A_{\mathcal{C}}^T\eta)A_{\mathcal{C}}^T\frac{\mathrm{d}}{\mathrm{d}t}\eta + A_{\mathcal{L}}\imath_{\mathcal{L}} + A_{\mathcal{R}}g(A_{\mathcal{R}}^T\eta) + A_{\mathcal{V}}\imath_{\mathcal{V}} + A_I I_s = 0,\\ \mathcal{L}(\imath_{\mathcal{L}})\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}} - A_{\mathcal{L}}^T\eta = 0,\\ A_{\mathcal{V}}^T\eta - \mathcal{V}_s = 0, \end{aligned} \tag{29}$$

using the definition of the Jacobians $\mathcal{C}(\nu_{\mathcal{C}})$ and $\mathcal{L}(\imath_{\mathcal{L}})$ in **(A3)** and **(A4)**. For such a system, we get a signature matrix of the form

$$\Sigma = \begin{bmatrix} \Sigma_{\mathfrak{C}\mathfrak{G}} & \Sigma_{A_{\mathcal{L}}} & \Sigma_{A_{\mathcal{V}}} \\ \Sigma_{A_{\mathcal{L}}}^T & \Sigma_{\mathcal{L}} & - \\ \Sigma_{A_{\mathcal{V}}}^T & - & - \end{bmatrix}, \tag{30}$$

where $\Sigma_{\mathfrak{C}\mathfrak{G}} = \Sigma_{\mathfrak{C}\mathfrak{G}}^T$ is of size $n_\eta - 1 \times n_\eta - 1$ with entries in $\{-\infty, 0, 1\}$, $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{L}}^T$ is of size $n_{\mathcal{L}} \times n_{\mathcal{L}}$ with entries in $\{-\infty, 0, 1\}$, $\Sigma_{A_{\mathcal{L}}}$ is of size $n_\eta - 1 \times n_{\mathcal{L}}$ with entries in $\{-\infty, 0\}$ and $\Sigma_{A_{\mathcal{V}}}$ is of size $n_\eta - 1 \times n_{\mathcal{V}}$ with entries in $\{-\infty, 0\}$. The corresponding Σ-Jacobian has a block structure according to (30) with

$$\mathfrak{J} = \begin{bmatrix} \mathfrak{J}_{11} & \mathfrak{J}_{12} & \mathfrak{J}_{13} \\ \mathfrak{J}_{21} & \mathfrak{J}_{22} & 0 \\ \mathfrak{J}_{31} & 0 & 0 \end{bmatrix}. \tag{31}$$

Note that due to the symmetry of Σ, also the positions of the HVT will be symmetric on Σ, i.e., if $(i, j) \in T$, then also $(j, i) \in T$. In contrast to (30) the Σ-Jacobian (31) is not symmetric. If $n_{\mathcal{L}} = 0$ or $n_{\mathcal{V}} = 0$, then some blocks in (30) and (31) may be void. Furthermore, under assumption **(A4)** we know that

$$\Sigma_{\mathcal{L}} = \begin{bmatrix} 1 & & \leq 1 \\ & \ddots & \\ \leq 1 & & 1 \end{bmatrix},$$

i.e., there are ones on the diagonal and entries ≤ 1 (i.e., $-\infty$, 0 or 1) on the off-diagonal of $\Sigma_{\mathcal{L}}$.

Case 1: no capacitances and no voltage sources. At first we assume that $n_{\mathcal{C}} = 0$ and $n_{\mathcal{V}} = 0$ (cf. Example 2). In this case, the MNA equations (29) take the form

$$\begin{aligned} A_{\mathcal{R}}g(A_{\mathcal{R}}^T\eta) + A_{\mathcal{L}}\imath_{\mathcal{L}} + A_I I_s = 0,\\ \mathcal{L}(\imath_{\mathcal{L}})\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}} - A_{\mathcal{L}}^T\eta = 0, \end{aligned} \tag{32}$$

and the signature matrix (30) reduces to

$$\Sigma = \begin{bmatrix} \Sigma_{\mathfrak{G}} & \Sigma_{A_{\mathcal{L}}} \\ \Sigma_{A_{\mathcal{L}}}^T & \Sigma_{\mathcal{L}} \end{bmatrix}, \tag{33}$$

with $\Sigma_{\mathfrak{G}} = \Sigma_{\mathfrak{G}}^T$ of size $n_\eta - 1 \times n_\eta - 1$ and entries in $\{-\infty, 0\}$.
If we assume that we can find a HVT on the diagonal of (33), we get the canonical offsets $c = [0, \dots, 0]$ and $d = [0, \dots, 0, 1 \dots, 1]$ and the Σ-Jacobian

$$\mathfrak{J} = \begin{bmatrix} A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T & 0 \\ -A_{\mathcal{L}}^T & \mathcal{L} \end{bmatrix}.$$

Thus, $\mathfrak{J}$ is nonsingular if and only if $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$ is nonsingular, and in this case the Σ-method will succeed with structural index $\nu_S = 1$. Due to **(A5)** the matrix $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$ is nonsingular if and only if $A_{\mathcal{R}}^T$ has full column rank. In particular, the following statements are equivalent (see [4, 13]):

- $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$ is nonsingular;
- $\ker A_{\mathcal{R}}^T = \{0\}$;
- the graph corresponding to the circuit contains no $\mathcal{CLV}I$-cutsets. [1]

Note that $\mathcal{CLV}I$-cutsets include $\mathcal{L}I$-cutsets as special case. Thus, the Σ-method will fail for systems of the form (32) and HVT on the diagonal whenever $\nu_d > 1$. But also pure I-cutsets (which are excluded by Assumption **(A2)**) or pure $\mathcal{L}$-cutsets will lead to failure of the Σ-method in this setting.

If there exists no HVT on the diagonal (but the system is structurally well-posed), then necessarily $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$ contains zero rows/columns. In this case we can reorder the rows of $A_{\mathcal{R}}$ such that

$$\Pi_{\mathcal{R}}A_{\mathcal{R}} = \begin{array}{cc} \begin{bmatrix} \tilde{A}_{\mathcal{R}} \\ 0 \end{bmatrix} & \begin{array}{c} \tilde{r} \\ n_\eta - 1 - \tilde{r} \end{array} \end{array}$$

where $\Pi_{\mathcal{R}}$ is a permutation. Then, the corresponding permutation of (32) yields

$$\begin{aligned} \begin{bmatrix} \tilde{A}_{\mathcal{R}} \\ 0 \end{bmatrix} g(\tilde{A}_{\mathcal{R}}^T\tilde{\eta}_1) + \begin{bmatrix} \tilde{A}_{\mathcal{L},1} \\ \tilde{A}_{\mathcal{L},2} \end{bmatrix} \imath_{\mathcal{L}} + \begin{bmatrix} \tilde{A}_{I,1} \\ \tilde{A}_{I,2} \end{bmatrix} I_s &= 0, \\ \mathcal{L}(\imath_{\mathcal{L}})\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}} - \begin{bmatrix} \tilde{A}_{\mathcal{L},1}^T & \tilde{A}_{\mathcal{L},2}^T \end{bmatrix} \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{bmatrix} &= 0, \end{aligned} \tag{34}$$

where

$$\Pi_{\mathcal{R}}\eta =: \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{bmatrix}, \quad \Pi_{\mathcal{R}}A_{\mathcal{L}} =: \begin{bmatrix} \tilde{A}_{\mathcal{L},1} \\ \tilde{A}_{\mathcal{L},2} \end{bmatrix} \quad \Pi_{\mathcal{R}}A_I =: \begin{bmatrix} \tilde{A}_{I,1} \\ \tilde{A}_{I,2} \end{bmatrix}.$$

The Σ-matrix for this permuted system takes the form

$$\Sigma = \begin{array}{cc} \begin{bmatrix} \tilde{\Sigma}_{\mathfrak{G}} & - & \Sigma_{\tilde{A}_{\mathcal{L},1}} \\ - & - & \Sigma_{\tilde{A}_{\mathcal{L},2}} \\ \Sigma_{\tilde{A}_{\mathcal{L},1}}^T & \Sigma_{\tilde{A}_{\mathcal{L},2}}^T & \Sigma_{\mathcal{L}} \end{bmatrix} & \begin{array}{c} \tilde{r} \\ n_\eta - 1 - \tilde{r} \\ n_{\mathcal{L}} \end{array} \end{array}$$

Since the system is structurally well-posed there must be $\tilde{s} := n_\eta - 1 - \tilde{r}$ positions for the HVT be taken from the block $\Sigma_{\tilde{A}_{\mathcal{L},2}}$, which can w.l.o.g. assumed to be positioned on the diagonal of the

[1] Note that $A_{\mathcal{R}}^T$ can only be of full column rank if $n_\eta - 1 \leq n_{\mathcal{R}}$. If $n_\eta - 1 > n_{\mathcal{R}}$, then there will be $\mathcal{CLV}I$-cutsets in the circuit.

first $\tilde{s}$-by-$\tilde{s}$ block of $\Sigma_{\tilde{A}_{\mathcal{L},2}}$. This results in a Σ-matrix of the form

$$\left[\begin{array}{ccc|ccc|cccccc}
\boxed{0} & & \leq 0 & & & & & & & & & \\
& \ddots & & & - & & & & \leq 0 & & & \\
\leq 0 & & \boxed{0} & & & & & & & & & \\ \hline
& & & & & & \boxed{0} & & & & & \\
& - & & & - & & & \ddots & & & \leq 0 & \\
& & & & & & & & \boxed{0} & & & \\ \hline
& & & \boxed{0} & & & 1 & & & & & \\
& & & & \ddots & & & \ddots & & & \leq 1 & \\
& \leq 0 & & & & \boxed{0} & & & 1 & & & \\
& & & & & & & & & \boxed{1} & & \\
& & & & \leq 0 & & & \leq 1 & & & \ddots & \\
& & & & & & & & & & & \boxed{1}
\end{array}\right]$$

with canonical offsets

$$c = \left[\ 0,\dots,0 \mid 1,\dots,1 \mid 0,\dots,0\ \right], \quad d = \left[\ 0,\dots,0 \mid 0,\dots,0 \mid 1,\dots,1\ \right],$$

and the Σ-Jacobian takes the form

$$\mathfrak{J} = \begin{bmatrix} \tilde{A}_{\mathcal{R}}\mathcal{G}\tilde{A}_{\mathcal{R}}^T & 0 & 0 \\ 0 & 0 & -\tilde{A}_{\mathcal{L},2} \\ -\tilde{A}_{\mathcal{L},1}^T & -\tilde{A}_{\mathcal{L},2}^T & \mathcal{L} \end{bmatrix}.$$

We see that $\mathfrak{J}$ can only be nonsingular if $\tilde{A}_{\mathcal{R}}\mathcal{G}\tilde{A}_{\mathcal{R}}^T$ is nonsingular. As before, $\tilde{A}_{\mathcal{R}}\mathcal{G}\tilde{A}_{\mathcal{R}}^T$ is nonsingular if and only if $\ker \tilde{A}_{\mathcal{R}}^T = \{0\}$ or, equivalently, if the subgraph consisting of the nodes corresponding to $\tilde{A}_{\mathcal{R}}$ contains no $\mathcal{CLVI}$-cutsets.

Case 2: no resistors and no voltage sources. Next we assume that $n_{\mathcal{R}} = 0$ and $n_{\mathcal{V}} = 0$. In this case, the MNA equations (29) take the form

$$\begin{aligned} A_C\mathcal{C}(A_C^T\eta)A_C^T\frac{d}{dt}\eta + A_{\mathcal{L}}i_{\mathcal{L}} + A_I I_s &= 0, \\ \mathcal{L}(i_{\mathcal{L}})\frac{\mathrm{d}}{\mathrm{d}t}i_{\mathcal{L}} - A_{\mathcal{L}}^T\eta &= 0, \end{aligned} \tag{35}$$

and the signature matrix (30) reduces to

$$\Sigma = \begin{bmatrix} \Sigma_{\mathfrak{C}} & \Sigma_{A_{\mathcal{L}}} \\ \Sigma_{A_{\mathcal{L}}}^T & \Sigma_{\mathcal{L}} \end{bmatrix},$$

where $\Sigma_{\mathfrak{C}} = \Sigma_{\mathfrak{C}}^T$ is of size $n_\eta - 1 \times n_\eta - 1$ with entries in $\{-\infty, 0, 1\}$. Assuming that there is a HVT on the diagonal we get

$$\Sigma = \left[\begin{array}{ccc|ccc}
\boxed{1} & & & & & \\
& \ddots & & & \leq 0 & \\
& & \boxed{1} & & & \\ \hline
& & & \boxed{1} & & \\
& \leq 0 & & & \ddots & \\
& & & & & \boxed{1}
\end{array}\right],$$

with $c = [0, \dots, 0]$, $d = [1, \dots, 1]$ and the Σ-Jacobian takes the form

$$\mathfrak{J} = \begin{bmatrix} A_C \mathcal{C} A_C^T & 0 \\ 0 & \mathcal{L} \end{bmatrix}.$$

We see that $\mathfrak{J}$ is nonsingular if and only if $A_C \mathcal{C} A_C^T$ is nonsingular. Since $\mathcal{C}$ is assumed to be symmetric positive definite, $A_C \mathcal{C} A_C^T$ will be nonsingular if and only if A_C^T has full column rank. Similar as before the following statements are equivalent (see [4, 13]):

- $A_C \mathcal{C} A_C^T$ is nonsingular;
- $\ker A_C^T = \{0\}$;
- the graph corresponding to the circuit contains no $\mathcal{RLV}I$-cutsets.[2]

Again, $\mathcal{L}I$-cutsets are included as special case. Thus, the Σ-method will fail for the MNA equations (35) whenever $\nu_d > 1$. But, again, also pure I-cutsets or pure $\mathcal{L}$-cutsets will lead to a singular Σ-Jacobian in this setting.
If there is no HVT on the diagonal, then $A_C \mathcal{C} A_C^T$ contains zero rows/columns and we can proceed in a similar manner as before using permutations of the system such that

$$\Pi_C A_C = \begin{bmatrix} \tilde{A}_C \\ 0 \end{bmatrix}. \tag{36}$$

In this case the success check will fail whenever there are $\mathcal{RLV}I$-cutsets (in particular $\mathcal{L}I$-cutsets) in the subgraph consisting of the nodes corresponding to $\tilde{A}_C$.

Remark 15. In the previous discussion we have seen that $A_\mathcal{R} \mathcal{G} A_\mathcal{R}^T$ and $A_C \mathcal{C} A_C^T$ can be singular depending on the topology of the circuit. However, from Example 1 (cases (b) and (c)) we can observe that $A_\mathcal{R} \mathcal{G} A_\mathcal{R}^T$ and $A_C \mathcal{C} A_C^T$ can both be singular and nevertheless the Σ-method can succeed with $\nu_S = \nu_d$, even for higher index DAEs. On the other hand, singular blocks in the Σ-Jacobian can also result from a combination of the matrices $A_\mathcal{R} \mathcal{G} A_\mathcal{R}^T$ and $A_C \mathcal{C} A_C^T$ as can be seen in Example 1, case (d).

The previous discussion might suggest that the failure of the Σ-method is related to the occurrence of $\mathcal{L}I$-cutsets. That this is not the case can be seen in the following example.

Example 3. Consider the circuit given in Figure 5 with corresponding directed graph given in Figure 6. If we select node 3 as reference node, the MNA equations take the form

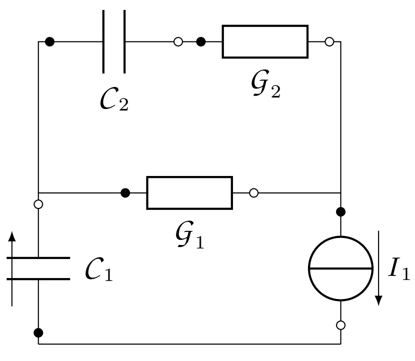

Figure 5: $\mathcal{RCI}$-circuit

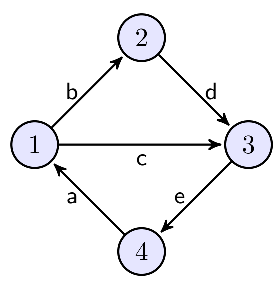

Figure 6: Graph for $\mathcal{RCI}$-circuit

$$\begin{bmatrix} \mathcal{C}_1 + \mathcal{C}_2 & -\mathcal{C}_2 & -\mathcal{C}_1 \\ -\mathcal{C}_2 & \mathcal{C}_2 & 0 \\ -\mathcal{C}_1 & 0 & \mathcal{C}_1 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_4 \end{bmatrix} + \begin{bmatrix} \mathcal{G}_1 & 0 & 0 \\ 0 & \mathcal{G}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} I_s = 0,$$

[2]Similar as before, A_C^T can only be of full column rank if $n_\eta - 1 \leq n_C$.

and the signature matrix and Σ-Jacobian are given by

$$\Sigma = \begin{bmatrix} \boxed{1} & 1 & 1 \\ 1 & \boxed{1} & - \\ 1 & - & \boxed{1} \end{bmatrix}, \quad \mathfrak{J} = \begin{bmatrix} \mathcal{C}_1 + \mathcal{C}_2 & -\mathcal{C}_2 & -\mathcal{C}_1 \\ -\mathcal{C}_2 & \mathcal{C}_2 & 0 \\ -\mathcal{C}_1 & 0 & \mathcal{C}_1 \end{bmatrix}.$$

The success check fails, since $\mathfrak{J}$ is singular. If we choose one of the other nodes as reference node, then the Σ-method succeeds with structural index $\nu_S = 1 = \nu_d$. ◁

In Example 3 we have $n_{\mathcal{V}} = 0$ and $n_{\mathcal{L}} = 0$ such that (30) reduces to $\Sigma = \Sigma_{\mathfrak{C}\mathfrak{G}}$. Now, if $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ is singular (in Example 3 due to the existence of an $\mathcal{R}I$-cutset), and as reference a node is chosen that is not adjacent to a capacitor, we get $\mathfrak{J} = A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ and the Σ-method fails. Example 3 shows that the Σ-method can fail also for MNA equations (29) of d-index $\nu_d = 1$.
We might consider more general forms of the MNA equations (29), but still, combining the ideas of Case 1 and Case 2 above, singular blocks of the form $\tilde{A}_R G \tilde{A}_R^T$ or $\tilde{A}_C C \tilde{A}_C^T$ can occur, corresponding to the occurrence of $\mathcal{CLV}I$-cutsets or $\mathcal{RLV}I$-cutsets in certain (sub)graphs. A complete characterization of all circuit configurations that lead to failure of the Σ-method for the MNA equations (29) has proven to be quite cumbersome. At least we can formulate the following result.

Theorem 16. *If the MNA equations* (29) *have d-index $\nu_d = 0$, then the Σ-method succeeds with $c = [0, \dots, 0]$, $d = [1, \dots, 1]$ and structural index $\nu_S = 0$.*

Proof. If the MNA equations (29) have d-index $\nu_d = 0$, then from Theorem 3 we know that $n_{\mathcal{V}} = 0$ and $\operatorname{rank} A_{\mathcal{C}} = n_\eta - 1$. Thus, the matrix $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ is regular and, consequently, also the Σ-Jacobian

$$\mathfrak{J} = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 \\ 0 & \mathcal{L} \end{bmatrix}$$

is regular. As a result the Σ-method succeeds with $c = [0, \dots, 0]$ and $d = [1, \dots, 1]$, such that $\nu_S = \max_i c_i = 0$. □

In the previous examples we have observed that failure or success of the Σ-method for the MNA equations (29) can depend on the selection of the reference node. However, in some cases the Σ-method always fails independently of the chosen reference (see Example 2, where $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$ is a singular block in $\mathfrak{J}$ for all possible choices of the reference node). Note again, that the selection of the reference node does not influence the analytical properties of the system (e.g. the index or the hidden constraints), and, in particular, the regularity of the matrices $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ and $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$ is independent of the selection of the reference node. Can we nevertheless give a characterization for a "good" choice of a reference node?
Consider $Y = [y_{ij}] = A_{\mathcal{K}}\mathcal{K}A_{\mathcal{K}}^T$ with $\mathcal{K} = \operatorname{diag}(\mathcal{K}_1, \dots, \mathcal{K}_\ell) > 0$ and incidence matrix $A_{\mathcal{K}}$. Then, we have $y_{kk} = \sum_{j=1}^{\ell} a_{kj}^2 \mathcal{K}_j$ with

$$a_{kj}^2 = \begin{cases} 1 & \text{if branch } j \text{ is adjacent to node } k, \\ 0 & \text{else,} \end{cases}$$

i.e., for all k the diagonal entry y_{kk} of Y is the sum of the $\mathcal{K}_i$'s of all branches that are adjacent to node k . Furthermore, we have $y_{ki} = \sum_{j=1}^{\ell} a_{kj} a_{ij} \mathcal{K}_j$, with

$$a_{kj} a_{ij} = \begin{cases} -1 & \text{if branch } j \text{ connects the nodes } k \text{ and } i, \\ 0 & \text{else,} \end{cases}$$

for $k \neq i$, i.e., the off-diagonal entry y_{ki} of Y is the negative sum of the $\mathcal{K}_i$'s of all branches that connect node k with node i. If we replace Y by $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ or $A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T$, respectively, we see that the characteristic values of components that are connected to the reference node appear only on the diagonal of Y, while the characteristic values of elements that are not connected to

the reference node appear both on the diagonal and off-diagonal terms. These observations (and numerous examples) show that the reference node should be adjacent to a capacitance or, if no capacitances are present, adjacent to a resistance in the circuit. If the reference node is adjacent to a capacitance, then $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ contains at least one zero-row and by permutation with $\Pi_{\mathcal{C}}$ as in (36) this row is removed from $\tilde{A}_{\mathcal{C}}$. However, this is no guarantee for the success of the Σ-method as Example 2 shows, since the remaining part $\tilde{A}_{\mathcal{C}}\mathcal{C}\tilde{A}_{\mathcal{C}}^T$ can still be singular. In particular, $A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T$ is only nonsingular if there exists a capacitive tree in the circuit graph.
A simple check in the case that $n_{\mathcal{V}} = 0$ can be performed as follows. We permute $[A_{\mathcal{C}}, A_{\mathcal{R}}]$ as

$$\Pi[A_{\mathcal{C}}, A_{\mathcal{R}}] = \begin{bmatrix} \tilde{A}_{\mathcal{C}} & \tilde{A}_{\mathcal{R},1} \\ 0 & \tilde{A}_{\mathcal{R},2} \\ 0 & 0 \end{bmatrix} \begin{array}{l} \tilde{n}_1 \\ \tilde{n}_2 \\ n_\eta - 1 - \tilde{n}_1 - \tilde{n}_2 \end{array}$$

where Π is a permutation, $\tilde{A}_{\mathcal{C}}$ is of size $\tilde{n}_1 \times n_{\mathcal{C}}$, with $\tilde{n}_1$ minimal, $\bar{A}_{\mathcal{R},1}$ is of size $\tilde{n}_1 \times n_{\mathcal{R}}$, and $\bar{A}_{\mathcal{R},2}$ is of size $\tilde{n}_2 \times n_{\mathcal{R}}$ with $\tilde{n}_2$ minimal. The $\tilde{n}_1$ nodes corresponding to the first block row are the nodes that are directly connected to a capacitance, the $\tilde{n}_2$ nodes corresponding to the second block row are the nodes that are directly connected to a resistance, but not connected to a capacitance, and the $n_\eta - 1 - \tilde{n}_1 - \tilde{n}_2$ nodes corresponding to the last block row are the nodes that are neither connected to a capacitance nor connected to a resistance. Then, if $n_{\mathcal{V}} = 0$, the Σ-method will fail if $\tilde{A}_{\mathcal{C}}\mathcal{C}\tilde{A}_{\mathcal{C}}^T$ **or** $\tilde{A}_{\mathcal{R},2}\mathcal{G}\tilde{A}_{\mathcal{R},2}^T$ is singular, i.e., if

$$\operatorname{rank} \tilde{A}_{\mathcal{C}} < \tilde{n}_1 \quad or \quad \operatorname{rank} \bar{A}_{\mathcal{R},2} < \tilde{n}_2.$$

4.3 The Signature Method for the MLA equations

In this section we apply the Σ-method to the MLA equations (7). Again, we start by considering the two examples from Section 4.2.

Example 4. We consider again the circuit given in Example 1. The oriented loops contained in the graph are depicted in Figure 7. A reduced loop matrix is given by

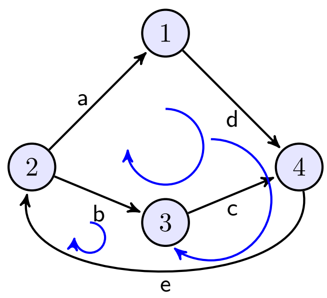

Figure 7: Loops in the $\mathcal{LRCI}$-circuit

$$B = \left[\begin{array}{c|c|cc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc} B_{\mathcal{R}} & B_{\mathcal{C}} & B_{\mathcal{L}} & B_{\mathcal{I}} \end{array}\right]$$

and the MLA equations (7) take the form

$$\begin{aligned} (\mathcal{L}_1 + \mathcal{L}_2)\frac{\mathrm{d}}{\mathrm{d}t}j_1 - \mathcal{L}_1\frac{\mathrm{d}}{\mathrm{d}t}j_2 + \mathcal{R}j_1 - \nu_{\mathcal{C}} &= 0, \\ -\mathcal{L}_1\frac{\mathrm{d}}{\mathrm{d}t}j_1 + \mathcal{L}_1\frac{\mathrm{d}}{\mathrm{d}t}j_2 + \nu_{\mathcal{C}} + \nu_{\mathcal{I}} &= 0, \\ \mathcal{C}\frac{\mathrm{d}}{\mathrm{d}t}\nu_{\mathcal{C}} + j_1 - j_2 &= 0, \\ j_2 + I_s &= 0. \end{aligned}$$

Thus, the Σ-matrix and Σ-Jacobian are given by

$$\Sigma = \left[\begin{array}{cc|c|c} \boxed{1} & 1 & 0 & - \\ 1 & 1 & 0 & \boxed{0} \\ \hline 0 & 0 & \boxed{1} & - \\ \hline - & \boxed{0} & - & - \end{array}\right], \quad \mathfrak{J} = \left[\begin{array}{cc|c|c} \mathcal{L}_1 + \mathcal{L}_2 & -\mathcal{L}_1 & 0 & 0 \\ -\mathcal{L}_1 & \mathcal{L}_1 & 0 & 1 \\ \hline 0 & 0 & \mathcal{C} & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}\right].$$

We see that $\mathfrak{J}$ is nonsingular and, thus, the Σ-method succeeds with $\nu_S = 2 = \nu_d$. ◁

Example 5. We consider again the circuit given in Example 2. In this case, the circuit graph contains only one loop and the loop matrix is given by

$$B_0 = B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

The MLA equations (7) take the form

$$\begin{aligned} \mathcal{L}\frac{\mathrm{d}}{\mathrm{d}t}j + (\mathcal{R}_1 + \mathcal{R}_2)j + \nu_I &= 0, \\ j + I_s &= 0, \end{aligned}$$

and the Σ-matrix and Σ-Jacobian for this system are given by

$$\Sigma = \begin{bmatrix} 1 & \boxed{0} \\ \boxed{0} & - \end{bmatrix}, \quad \mathfrak{J} = \begin{bmatrix} \mathcal{L} & 1 \\ 1 & 0 \end{bmatrix}.$$

We see that $\mathfrak{J}$ is nonsingular such that the Σ-method succeeds with $\nu_S = 2 = \nu_d$. ◁

We see that, if we use the MLA equations (7) instead of the MNA equations (6) to describe the behavior of the electrical circuits given in Example 1 and 2, the Σ-method succeeds with $\nu_S = 2 = \nu_d$.
A natural question that arises is: Will the Σ-method always succeed for the MLA equations? Unfortunately, this is not the case. For the MLA equations (7) we get a signature matrix of the form

$$\Sigma = \begin{bmatrix} \Sigma_{\mathfrak{L}\mathfrak{R}} & \Sigma_{B_{\mathcal{C}}} & \Sigma_{B_I} \\ \Sigma_{B_{\mathcal{C}}}^T & \Sigma_{\mathcal{C}} & - \\ \Sigma_{B_I}^T & - & - \end{bmatrix},$$

where $\Sigma_{\mathfrak{L}\mathfrak{R}} = \Sigma_{\mathfrak{L}\mathfrak{R}}^T$ is of size $n_b - n_\eta + 1 \times n_b - n_\eta + 1$ with entries in $\{-\infty, 0, 1\}$, $\Sigma_{\mathcal{C}} = \Sigma_{\mathcal{C}}^T$ is of size $n_{\mathcal{C}} \times n_{\mathcal{C}}$ with entries in $\{-\infty, 0, 1\}$, $\Sigma_{B_{\mathcal{C}}}$ is of size $n_b - n_\eta + 1 \times n_{\mathcal{C}}$ with entries in $\{-\infty, 0\}$, and Σ_{B_I} is of size $n_b - n_\eta + 1 \times n_I$ with entries in $\{-\infty, 0\}$. Due to Assumption **(A3)** we know that

$$\Sigma_{\mathcal{C}} = \begin{bmatrix} 1 & & \leq 1 \\ & \ddots & \\ \leq 1 & & 1 \end{bmatrix}$$

with ones on the diagonal and entries ≤ 1 on the off-diagonal.
Hence, the signature matrix has a similar structure as the signature matrix (30) for the MNA equations and therefore also similar problems will arise. Analogously as in Section 4.2 Assumption (**A1**) means that the matrix $[B_{\mathcal{L}}\ B_{\mathcal{R}}\ B_{\mathcal{C}}\ B_I]$ has full row rank. In the same way Assumption (**A2**) means that the matrix B_I has full column rank. Moreover, the following statements are equivalent (see [4, 13]):

- $B_{\mathcal{L}}\mathcal{L}B_{\mathcal{L}}^T$ is nonsingular;
- $\ker B_{\mathcal{L}}^T = \{0\}$;
- the graph corresponding to the circuit does not contain $\mathcal{C}\mathcal{R}\mathcal{V}I$-loops.

Analogously, the following statements are equivalent (see [4, 13]):

- $B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T$ is nonsingular;
- $\ker B_{\mathcal{R}}^T = \{0\}$;
- the graph corresponding to the circuit does not contain $\mathcal{CLVI}$-loops.

Note that $\mathcal{CRVI}$-loops and $\mathcal{CLVI}$-loops include $\mathcal{CV}$-loops (but also pure $\mathcal{C}$-loops) as special cases. So again, certain topological configurations lead to singular blocks $B_{\mathcal{L}}\mathcal{L}B_{\mathcal{L}}^T$ or $B_{\mathcal{R}}\mathcal{R}B_{\mathcal{R}}^T$. In Example 4 the Σ-Jacobian is of the form

$$\mathfrak{J} = \begin{bmatrix} B_{\mathcal{L}}\mathcal{L}B_{\mathcal{L}}^T & 0 & B_I \\ 0 & \mathcal{C} & 0 \\ B_I^T & 0 & 0 \end{bmatrix},$$

and in Example 5 it is of the form

$$\mathfrak{J} = \begin{bmatrix} B_{\mathcal{L}}\mathcal{L}B_{\mathcal{L}}^T & B_I \\ B_I^T & 0 \end{bmatrix},$$

and both are nonsingular. However, we can easily construct an example where the Σ-method fails for the MLA equations (7).

Example 6. Consider the circuit given in Figure 8. The graph together with the loops corre-

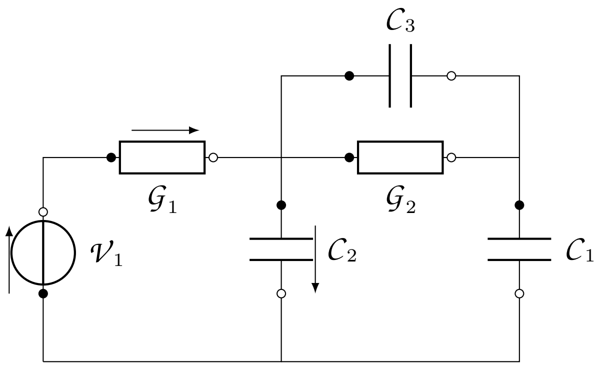

Figure 8: $\mathcal{RC}$-circuit

sponding to the circuit is given in Figure 9. A reduced loop matrix is given by

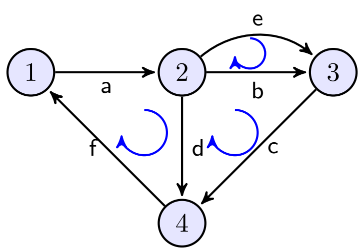

Figure 9: Graph of $\mathcal{RC}$-circuit with loops

$$B = \left[\begin{array}{cc|ccc|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} B_{\mathcal{R}} & B_{\mathcal{C}} & B_{\mathcal{V}} \end{bmatrix}$$

and the MLA equations (7) take the form

$$\begin{bmatrix} \mathcal{R}_1 & 0 & 0 \\ 0 & \mathcal{R}_2 & -\mathcal{R}_2 \\ 0 & -\mathcal{R}_2 & \mathcal{R}_2 \end{bmatrix} j + \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nu_C + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathcal{V}_s = 0,$$

$$\begin{bmatrix} \mathcal{C}_1 & 0 & 0 \\ 0 & \mathcal{C}_2 & 0 \\ 0 & 0 & \mathcal{C}_3 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \nu_C - \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} j = 0.$$

Since the circuit contains a $\mathcal{C}$-loop the MLA equations are of d-index $\nu_d = 2$. The corresponding Σ-matrix is given by

$$\Sigma = \left[\begin{array}{ccc|ccc} \boxed{0} & - & - & - & 0 & - \\ - & \boxed{0} & 0 & 0 & 0 & - \\ - & 0 & \boxed{0} & - & - & 0 \\ \hline - & 0 & - & \boxed{1} & - & - \\ 0 & 0 & - & - & \boxed{1} & - \\ - & - & 0 & - & - & \boxed{1} \end{array}\right]$$

with Σ-Jacobian

$$\mathfrak{J} = \left[\begin{array}{ccc|ccc} \mathcal{R}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}_2 & -\mathcal{R}_2 & 0 & 0 & 0 \\ 0 & -\mathcal{R}_2 & \mathcal{R}_2 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & \mathcal{C}_1 & 0 & 0 \\ 1 & -1 & 0 & 0 & \mathcal{C}_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \mathcal{C}_3 \end{array}\right]$$

which is singular. $\triangleleft$

In Example 6 the Σ-Jacobian is of the form

$$\mathfrak{J} = \begin{bmatrix} B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T & 0 \\ -B_{\mathcal{C}}^T & \mathcal{C} \end{bmatrix},$$

and the block $B_{\mathcal{R}} \mathcal{R} B_{\mathcal{R}}^T$ is singular due to presence of the $\mathcal{C}$-loop. Note that, if we choose for B the fundamental loop matrix for the fundamental loops that are defined according to the normal tree $T = \{c, d, f\}$ in the graph depicted in Figure 9, then the Σ-method succeeds with nonsingular Σ-Jacobian and $\nu_S = \nu_d = 2$.

Remark 17. If we use the MNA equations (6) to describe the behavior of the circuit given in Example 6, then the corresponding DAE system is of d-index $\nu_d = 1$, and the Σ-method succeeds for all choices of the reference node. However, for node 1 as reference the Σ-methods determines the structural index $\nu_S = 2 > \nu_d$, while for all other nodes the Σ-method succeeds with $\nu_S = \nu_d = 1$. Both, $A_{\mathcal{C}} \mathcal{C} A_{\mathcal{C}}^T$ and $A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^T$ are singular (there is a $\mathcal{C}\mathcal{V}$-cutset and a $\mathcal{R}\mathcal{V}$-cutset), however $\mathfrak{J}$ is regular for all cases.

We get the corresponding result of Theorem 16 for the MLA equations.

Theorem 18. *If the MLA equations* (7) *have d-index* $\nu_d = 0$*, then the* Σ*-method succeeds with* $c = [0, \dots, 0]$*,* $d = [1, \dots, 1]$ *and structural index* $\nu_S = 0$*.*

Proof. Analogously to the proof of Theorem 16. $\square$

4.4 The Signature Method for the Branch-Oriented Model Equations

In the previous sections we have seen that certain circuit configurations lead to failure of the Σ-method for the MNA equations while other configurations lead to failure of the Σ-method for the MLA equations. For more complex circuit examples one could construct configurations where the Σ-method fails for both formulations. In this section we apply the Σ-method to the branch-oriented model equations (13). The key result is formulated in the following theorem.

Theorem 19. *Consider the branch-oriented model equations* (13) *and let Assumptions* **(A1)**-**(A5)** *hold. Then the* Σ*-method applied to system* (13) *always succeeds with nonsingular* Σ*-Jacobian and structural index* $\nu_S = \nu_d$.

Proof. We can rearrange the equations (13) as

$$\mathcal{C}(\nu_{\mathcal{C}\mathbf{1}}, \nu_{\mathcal{C}\mathbf{2}}) \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \nu_{\mathcal{C}\mathbf{1}} \\ \nu_{\mathcal{C}\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \imath_{\mathcal{C}\mathbf{1}} \\ \imath_{\mathcal{C}\mathbf{2}} \end{bmatrix}, \tag{37a}$$

$$\begin{bmatrix} \imath_{\mathcal{R}\mathbf{1}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix} = g(\nu_{\mathcal{R}\mathbf{1}}, \nu_{\mathcal{R}\mathbf{2}}), \tag{37b}$$

$$0 = \nu_{\mathcal{V}} - \mathcal{V}_s(t), \tag{37c}$$

$$\mathcal{L}(\imath_{\mathcal{L}\mathbf{1}}, \imath_{\mathcal{L}\mathbf{2}}) \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \imath_{\mathcal{L}\mathbf{1}} \\ \imath_{\mathcal{L}\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \nu_{\mathcal{L}\mathbf{1}} \\ \nu_{\mathcal{L}\mathbf{2}} \end{bmatrix}, \tag{37d}$$

$$0 = \imath_{\mathcal{I}} - \mathcal{I}_s(t), \tag{37e}$$

$$\begin{bmatrix} \nu_{\mathcal{L}\mathbf{2}} \\ \nu_{\mathcal{C}\mathbf{2}} \\ \nu_{\mathcal{I}} \\ \nu_{\mathcal{R}\mathbf{2}} \end{bmatrix} = - \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & 0 & F_{23} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & 0 & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} \nu_{\mathcal{C}\mathbf{1}} \\ \nu_{\mathcal{L}\mathbf{1}} \\ \nu_{\mathcal{V}} \\ \nu_{\mathcal{R}\mathbf{1}} \end{bmatrix}, \tag{37f}$$

$$\begin{bmatrix} \imath_{\mathcal{C}\mathbf{1}} \\ \imath_{\mathcal{L}\mathbf{1}} \\ \imath_{\mathcal{V}} \\ \imath_{\mathcal{R}\mathbf{1}} \end{bmatrix} = \begin{bmatrix} F_{11}^T & F_{21}^T & F_{31}^T & F_{41}^T \\ F_{12}^T & 0 & F_{32}^T & 0 \\ F_{13}^T & F_{23}^T & F_{33}^T & F_{43}^T \\ F_{14}^T & 0 & F_{34}^T & F_{44}^T \end{bmatrix} \begin{bmatrix} \imath_{\mathcal{L}\mathbf{2}} \\ \imath_{\mathcal{C}\mathbf{2}} \\ \imath_{\mathcal{I}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix} \tag{37g}$$

with variables

$$x = [\nu_{\mathcal{C}\mathbf{1}}, \nu_{\mathcal{C}\mathbf{2}}, \nu_{\mathcal{R}\mathbf{1}}, \nu_{\mathcal{R}\mathbf{2}}, \nu_{\mathcal{V}}, \nu_{\mathcal{L}\mathbf{1}}, \nu_{\mathcal{L}\mathbf{2}}, \nu_{\mathcal{I}}, \imath_{\mathcal{C}\mathbf{1}}, \imath_{\mathcal{C}\mathbf{2}}, \imath_{\mathcal{R}\mathbf{1}}, \imath_{\mathcal{R}\mathbf{2}}, \imath_{\mathcal{V}}, \imath_{\mathcal{L}\mathbf{1}}, \imath_{\mathcal{L}\mathbf{2}}, \imath_{\mathcal{I}}].$$

If the DAE (13) is of d-index $\nu_d = 1$, the normal tree used for the formulation of (13) is actually a proper tree, such that

$$\nu_{\mathbf{1}} = [\nu_{\mathcal{C}}^T, \nu_{\mathcal{V}}^T, \nu_{\mathcal{R}\mathbf{1}}^T]^T$$

consists of the branch voltages for all capacitors, voltage sources and twig resistors, and

$$\nu_{\mathbf{2}} = [\nu_{\mathcal{L}}^T, \nu_{\mathcal{I}}^T, \nu_{\mathcal{R}\mathbf{2}}^T]^T$$

consists of the branch voltages for all inductors, current sources and link resistors. Thus, system (37) reduces to

$$\begin{aligned}
\mathcal{C}(\nu_{\mathcal{C}}) \frac{d}{dt} \nu_{\mathcal{C}} &= \imath_{\mathcal{C}}, \\
\begin{bmatrix} \imath_{\mathcal{R}\mathbf{1}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix} &= g(\nu_{\mathcal{R}\mathbf{1}}, \nu_{\mathcal{R}\mathbf{2}}), \\
0 &= \nu_{\mathcal{V}} - \mathcal{V}_s(t), \\
\mathcal{L}(\imath_{\mathcal{L}}) \frac{\mathrm{d}}{\mathrm{d}t} \imath_{\mathcal{L}} &= \nu_{\mathcal{L}}, \\
0 &= \imath_{\mathcal{I}} - \mathcal{I}_s(t), \\
\begin{bmatrix} \nu_{\mathcal{L}} \\ \nu_{\mathcal{I}} \\ \nu_{\mathcal{R}\mathbf{2}} \end{bmatrix} &= - \begin{bmatrix} F_{11} & F_{13} & F_{14} \\ F_{31} & F_{33} & F_{34} \\ F_{41} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} \nu_{\mathcal{C}} \\ \nu_{\mathcal{V}} \\ \nu_{\mathcal{R}\mathbf{1}} \end{bmatrix}, \\
\begin{bmatrix} \imath_{\mathcal{C}} \\ \imath_{\mathcal{V}} \\ \imath_{\mathcal{R}\mathbf{1}} \end{bmatrix} &= \begin{bmatrix} F_{11}^T & F_{31}^T & F_{41}^T \\ F_{13}^T & F_{33}^T & F_{43}^T \\ F_{14}^T & F_{34}^T & F_{44}^T \end{bmatrix} \begin{bmatrix} \imath_{\mathcal{L}} \\ \imath_{\mathcal{I}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix},
\end{aligned}$$

with corresponding Σ-matrix of the form

$$
\left[\begin{array}{cccccc|cccccc}
\Sigma_{\mathcal{C}} & - & - & - & - & - & \mathcal{O}_{\mathcal{C}} & - & - & - & - & - \\
- & \Sigma_{22} & \Sigma_{23} & - & - & - & - & \mathcal{O}_{\mathcal{R}1} & - & - & - & - \\
- & \Sigma_{32} & \Sigma_{33} & - & - & - & - & - & \mathcal{O}_{\mathcal{R}2} & - & - & - \\
- & - & - & \mathcal{O}_{\mathcal{V}} & - & - & - & - & - & - & - & - \\
- & - & - & - & \mathcal{O}_{\mathcal{L}} & - & - & - & - & - & \Sigma_{\mathcal{L}} & - \\
- & - & - & - & - & - & - & - & - & - & - & \mathcal{O}_{I} \\
\hline
\Sigma_{71} & \Sigma_{72} & - & \Sigma_{74} & \mathcal{O}_{\mathcal{L}} & - & - & - & - & - & - & - \\
\Sigma_{81} & \Sigma_{82} & - & \Sigma_{84} & - & \mathcal{O}_{I} & - & - & - & - & - & - \\
\Sigma_{91} & \Sigma_{92} & \mathcal{O}_{\mathcal{R}2} & \Sigma_{94} & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & \mathcal{O}_{\mathcal{C}} & - & \Sigma_{91}^T & - & \Sigma_{71}^T & \Sigma_{81}^T \\
- & - & - & - & - & - & - & - & \Sigma_{94}^T & \mathcal{O}_{\mathcal{V}} & \Sigma_{74}^T & \Sigma_{84}^T \\
- & - & - & - & - & - & - & \mathcal{O}_{\mathcal{R}1} & \Sigma_{92}^T & - & \Sigma_{72}^T & \Sigma_{82}^T
\end{array}\right].
$$

Here, $\mathcal{O}_*$ for $* \in \{\mathcal{V}, \mathcal{L}, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, I\}$ are square blocks of appropriate size with 0-entries on the diagonal and $-\infty$-entries elsewhere. Moreover, $\Sigma_{\mathcal{L}}$ and $\Sigma_{\mathcal{C}}$ are defined as before. Consequently, the positions for the HVT can be taken from the diagonal entries of the marked blocks and we get the canonical offsets $c = [0]$ and

$$
d = [\underbrace{1, \dots, 1}_{n_{\mathcal{C}}}, \underbrace{0, \dots, 0}_{2n_{\mathcal{R}}+2n_{\mathcal{V}}+n_{\mathcal{L}}+n_I+n_{\mathcal{C}}}, \underbrace{1, \dots, 1}_{n_{\mathcal{L}}}, \underbrace{0, \dots, 0}_{n_I}].
$$

The corresponding Σ-Jacobian is given by

$$
\mathfrak{J} = \left[\begin{array}{cccccc|cccccc}
\mathcal{C} & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{G}_{11} & \mathcal{G}_{12} & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\
0 & \mathcal{G}_{22} & \mathcal{G}_{22} & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & \mathcal{L} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
\hline
0 & F_{14} & 0 & F_{13} & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & F_{34} & 0 & F_{33} & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & F_{44} & I & F_{43} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -F_{41}^T & 0 & 0 & -F_{31}^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -F_{43}^T & I & 0 & -F_{33}^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -F_{44}^T & 0 & 0 & -F_{34}^T
\end{array}\right],
$$

which is nonsingular if and only if the matrix

$$
M = \left[\begin{array}{cc|cc}
\mathcal{G}_{11} & \mathcal{G}_{12} & -I & 0 \\
\mathcal{G}_{21} & \mathcal{G}_{22} & 0 & -I \\
\hline
F_{44} & I & 0 & 0 \\
0 & 0 & I & -F_{44}^T
\end{array}\right]
$$

is nonsingular. By permutation and Gaussian elimination the matrix M can be brought into the form

$$
M = \left[\begin{array}{cc|cc}
I & 0 & 0 & -F_{44}^T \\
0 & I & F_{44} & 0 \\
\hline
-\mathcal{G}_{11}^{-1} & \mathcal{G}_{11}^{-1}\mathcal{G}_{12} & I & 0 \\
-\mathcal{G}_{21}\mathcal{G}_{11}^{-1} & \mathcal{G}_{21}\mathcal{G}_{11}^{-1}\mathcal{G}_{12} - \mathcal{G}_{22} & 0 & I
\end{array}\right] =: \begin{bmatrix} I & \tilde{F} \\ \tilde{G} & I \end{bmatrix},
$$

which is in the form of the matrix in Lemma 20 with skew-symmetric matrix $\tilde{F}$ and negative definite matrix $\tilde{G}$ (since $\mathcal{G}$ is symmetric positive definite due to Assumption (**A5**), see Lemma

21). Thus, $\mathfrak{J}$ is nonsingular and the Σ-method succeeds with $\nu_S = \nu_d = 1$.

If the DAE (13) is of d-index $\nu_d = 2$, the Σ-matrix for (37) takes the form as in (39), where, again, the positions for the HVT can be taken from the diagonal entries of the marked blocks. Since we have chosen a normal tree in the formulation of (37), and $F_{22} = 0$, $F_{42} = 0$ as well as $F_{24} = 0$, this choice for the HVT is unique (up to permutations within each block). The canonical offset vectors are given by

$$c = [\underbrace{0\dots0}_{n_\mathcal{C}+n_\mathcal{R}}, \underbrace{\leq 1 \cdots \leq 1}_{n_\mathcal{V}}, \underbrace{0\dots0}_{n_\mathcal{L}}, \underbrace{\leq 1 \cdots \leq 1}_{n_\mathcal{I}}, \underbrace{0\dots0}_{n_{\mathcal{L}2}}, \underbrace{1\dots1}_{n_{\mathcal{C}2}}, \underbrace{0\dots0}_{n_\mathcal{I}+n_{\mathcal{R}2}+n_{\mathcal{C}1}}, \underbrace{1\dots1}_{n_{\mathcal{L}1}}, \underbrace{0\dots0}_{n_\mathcal{V}+n_{\mathcal{R}1}}],$$
$$d = [\underbrace{1\dots1}_{n_\mathcal{C}}, \underbrace{0\dots0}_{n_\mathcal{R}}, \underbrace{\leq 1 \cdots \leq 1}_{n_\mathcal{V}}, \underbrace{0\dots0}_{n_\mathcal{L}+n_\mathcal{I}+n_\mathcal{C}+n_\mathcal{R}+n_\mathcal{V}}, \underbrace{1\dots1}_{n_\mathcal{L}}, \underbrace{\leq 1 \cdots \leq 1}_{n_\mathcal{I}}].$$

The entries with values ≤ 1 depend on the blocks F_{23} and F_{32}^T, however, these particular values do not influence the result of the Σ-method, since the corresponding Σ-Jacobian has the form

$$\mathfrak{J} = \left[\begin{array}{cccccccc|cccccccc}
\mathcal{C}_{11} & \mathcal{C}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{C}_{21} & \mathcal{C}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathcal{G}_{11} & \mathcal{G}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathcal{G}_{21} & \mathcal{G}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{L}_{11} & \mathcal{L}_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{L}_{21} & \mathcal{L}_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
\hline
0 & 0 & F_{14} & 0 & * & F_{12} & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{21} & I & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & F_{34} & 0 & * & F_{32} & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & F_{44} & I & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -F_{21}^T & 0 & -F_{41}^T & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -F_{12}^T & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -F_{23}^T & 0 & -F_{43}^T & I & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -F_{44}^T & 0 & 0 & 0 & *
\end{array}\right],$$

which is nonsingular due to Assumptions (**A3**)-(**A5**) no matter which entries occur in the blocks denoted with $*$. Thus, the Σ-method succeeds with structural index $\nu_S = \nu_d = 2$. $\square$

Lemma 20. *Consider a matrix of the form*

$$A = \begin{bmatrix} I & \tilde{F} \\ \tilde{G} & I \end{bmatrix},$$

where $\tilde{F}$ is skew-symmetric and $\tilde{G}$ is definite. Then A is nonsingular.

Proof. The matrix A is nonsingular if and only if the Schur complement $I - \tilde{G}\tilde{F}$ is nonsingular. Assume that

$$(I - \tilde{G}\tilde{F})v = 0 \tag{38}$$

for some vector v. Then

$$v^T \tilde{F}^T (I - \tilde{G}\tilde{F})v = v^T \tilde{F}^T v - v^T \tilde{F}^T \tilde{G}\tilde{F} v = -v^T \tilde{F}^T \tilde{G}\tilde{F} v = 0$$

due to the skew-symmetry of $\tilde{F}$. Since $\tilde{G}$ is definite we get $\tilde{F}v = 0$ and from (38) it follows that $v = 0$. $\square$

$$
\Sigma = \left[\begin{array}{cccccccc|cccccccc}
\Sigma_{\mathcal{C}_{11}} & \Sigma_{\mathcal{C}_{12}} & - & - & - & - & - & - & \mathcal{O}_{\mathcal{C}_1} & - & - & - & - & - & - & - \\
\Sigma_{\mathcal{C}_{21}} & \Sigma_{\mathcal{C}_{22}} & - & - & - & - & - & - & - & \mathcal{O}_{\mathcal{C}_2} & - & - & - & - & - & - \\
- & - & \Sigma_{\mathcal{G}_{11}} & \Sigma_{\mathcal{G}_{12}} & - & - & - & - & - & - & \mathcal{O}_{\mathcal{R}_1} & - & - & - & - & - \\
- & - & \Sigma_{\mathcal{G}_{21}} & \Sigma_{\mathcal{G}_{22}} & - & - & - & - & - & - & - & \mathcal{O}_{\mathcal{R}_2} & - & - & - & - \\
- & - & - & - & \mathcal{O}_{\mathcal{V}} & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & \mathcal{O}_{\mathcal{L}_1} & - & - & - & - & - & - & - & \Sigma_{\mathcal{L}_{11}} & \Sigma_{\mathcal{L}_{12}} & - \\
- & - & - & - & - & - & \mathcal{O}_{\mathcal{L}_2} & - & - & - & - & - & - & \Sigma_{\mathcal{L}_{21}} & \Sigma_{\mathcal{L}_{22}} & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & \mathcal{O}_I \\
\hline
\Sigma_{F_{11}} & - & \Sigma_{F_{14}} & - & \Sigma_{F_{13}} & \Sigma_{F_{12}} & \mathcal{O}_{\mathcal{L}_2} & - & - & - & - & - & - & - & - & - \\
\Sigma_{F_{21}} & \mathcal{O}_{\mathcal{C}_2} & - & - & \Sigma_{F_{23}} & - & - & - & - & - & - & - & - & - & - & - \\
\Sigma_{F_{31}} & - & \Sigma_{F_{34}} & - & \Sigma_{F_{33}} & \Sigma_{F_{32}} & - & \mathcal{O}_I & - & - & - & - & - & - & - & - \\
\Sigma_{F_{41}} & - & \Sigma_{F_{44}} & \mathcal{O}_{\mathcal{R}_2} & \Sigma_{F_{43}} & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & \mathcal{O}_{\mathcal{C}_1} & \Sigma_{F^T_{21}} & - & \Sigma_{F^T_{41}} & - & - & \Sigma_{F^T_{11}} & \Sigma_{F^T_{31}} \\
- & - & - & - & - & - & - & - & - & - & - & - & - & \mathcal{O}_{\mathcal{L}_1} & \Sigma_{F^T_{12}} & \Sigma_{F^T_{32}} \\
- & - & - & - & - & - & - & - & - & \Sigma_{F^T_{23}} & - & \Sigma_{F^T_{43}} & \mathcal{O}_{\mathcal{V}} & - & \Sigma_{F^T_{13}} & \Sigma_{F^T_{33}} \\
- & - & - & - & - & - & - & - & - & - & \mathcal{O}_{\mathcal{R}_1} & \Sigma_{F^T_{44}} & - & - & \Sigma_{F^T_{14}} & \Sigma_{F^T_{34}}
\end{array}\right] \tag{39}
$$

Lemma 21. *Let* $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ *be a symmetric positive definite matrix. Then the matrix*

$$\tilde{G} = \begin{bmatrix} G_{11}^{-1} & -G_{11}^{-1}G_{12} \\ G_{21}G_{11}^{-1} & G_{22} - G_{21}G_{11}^{-1}G_{12} \end{bmatrix}$$

is again positive definite.

Proof. Since G is symmetric positive definite we have that G_{11} is symmetric positive definite, and in particular G_{11}^{-1} exists. Thus, from a relation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

we get

$$\begin{aligned} v &= G_{11}^{-1}(x - G_{12}w), \\ y &= G_{21}G_{11}^{-1}(x - G_{12}w) + G_{22}w. \end{aligned}$$

Thus, for arbitrary vectors v and w we get

$$\begin{aligned} \begin{bmatrix} v^T & w^T \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} &= v^T x + w^T y = x^T v + w^T y \\ &= \begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} G_{11}^{-1} & -G_{11}^{-1}G_{12} \\ G_{21}G_{11}^{-1} & G_{22} - G_{21}G_{11}^{-1}G_{12} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \end{aligned}$$

and positive definiteness of G transfers to $\tilde{G}$. □

In contrast to the formulation of the circuit equations using the MNA equations (6) or the MLA equations (7), the Σ-method for the branch-oriented model equations (13) will always succeed.

Example 7. We consider again the circuit given in Example 2. The only proper tree for the graph depicted in Figure 4 is given by the set of branches $\{a, b, c\}$ and the branch d is the only link. Based on this tree the fundamental cutset matrix and fundamental loop matrix are given by

$$Q = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right], \quad B = \left[\begin{array}{c|ccc} 1 & 1 & 1 & 1 \end{array} \right],$$

i.e., we have $F = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Thus, the branch-oriented model equations (13) take the form

$$\begin{aligned} \begin{bmatrix} \imath_{\mathcal{R}\mathbf{1}} \\ \imath_{\mathcal{R}\mathbf{2}} \end{bmatrix} &= \begin{bmatrix} \mathcal{G}_{\mathbf{1}} & 0 \\ 0 & \mathcal{G}_{\mathbf{2}} \end{bmatrix} \begin{bmatrix} \nu_{\mathcal{R}\mathbf{1}} \\ \nu_{\mathcal{R}\mathbf{2}} \end{bmatrix}, \\ \mathcal{L}\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}} &= \nu_{\mathcal{L}}, \\ 0 &= \imath_I - I_s(t), \\ \nu_I &= -\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \nu_{\mathcal{L}} \\ \nu_{\mathcal{R}_1} \\ \nu_{\mathcal{R}_2} \end{bmatrix}, \\ \begin{bmatrix} \imath_{\mathcal{L}} \\ \imath_{\mathcal{R}_1} \\ \imath_{\mathcal{R}_2} \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \imath_I. \end{aligned}$$

The corresponding Σ-matrix and Σ-Jacobian are given by

$$\Sigma = \left[\begin{array}{cccc|cccc}
0 & - & - & - & 0 & - & - & - \\
- & 0 & - & - & - & 0 & - & - \\
- & - & 0 & - & - & - & 1 & - \\
- & - & - & - & - & - & - & 0 \\
\hline
0 & 0 & 0 & 0 & - & - & - & - \\
- & - & - & - & - & - & 0 & 0 \\
- & - & - & - & 0 & - & - & 0 \\
- & - & - & - & - & 0 & - & 0
\end{array}\right],$$

$$\mathfrak{J} = \left[\begin{array}{cccc|cccc}
\mathcal{G}_1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & \mathcal{G}_2 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & \mathcal{L} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{array}\right],$$

and $\mathfrak{J}$ is nonsingular such that the Σ-method succeeds with $\nu_S = 2$. ◁

4.5 The Signature Method for the pHC equations

Finally, we apply the Σ-method to the pHC equations (16). We consider the pHC equations (16) given in the form

$$\mathcal{C}(t)\frac{d}{dt}\nu_{\mathcal{C}} - \imath_{\mathcal{C}} = 0, \tag{40a}$$

$$\imath_{\mathcal{R}} - \mathcal{G}(t)\nu_{\mathcal{R}} = 0, \tag{40b}$$

$$-\nu_{\mathcal{C}} + A_{\mathcal{C}}^T\eta = 0, \tag{40c}$$

$$-\nu_{\mathcal{R}} + A_{\mathcal{R}}^T\eta = 0, \tag{40d}$$

$$A_{\mathcal{V}}^T\eta - \mathcal{V}_s(t) = 0, \tag{40e}$$

$$\mathcal{L}(t)\frac{\mathrm{d}}{\mathrm{d}t}\imath_{\mathcal{L}} - A_{\mathcal{L}}^T\eta = 0, \tag{40f}$$

$$-A_{\mathcal{C}}\imath_{\mathcal{C}} - A_{\mathcal{R}}\imath_{\mathcal{R}} - A_I I_s(t) - A_{\mathcal{V}}\imath_{\mathcal{V}} - A_{\mathcal{L}}\imath_{\mathcal{L}} = 0, \tag{40g}$$

with symmetric and pointwise positive definite matrices $\mathcal{C}(t)$, $\mathcal{G}(t)$ and $\mathcal{L}(t)$ due to Assumptions **(A3)-(A5)**. For such a system we get a Σ-matrix of the form

$$\Sigma_{pHC} = \left[\begin{array}{cc|cccc|c}
\Gamma_{\mathcal{C}} & - & \mathcal{O}_{\mathcal{C}} & - & - & - & - \\
- & \Gamma_{\mathcal{R}} & - & \mathcal{O}_{\mathcal{R}} & - & - & - \\
\hline
\mathcal{O}_{\mathcal{C}} & - & - & - & - & - & \Sigma_{A_{\mathcal{C}}}^T \\
- & \mathcal{O}_{\mathcal{R}} & - & - & - & - & \Sigma_{A_{\mathcal{R}}}^T \\
- & - & - & - & - & - & \Sigma_{A_{\mathcal{V}}}^T \\
- & - & - & - & - & \Sigma_{\mathcal{L}} & \Sigma_{A_{\mathcal{L}}}^T \\
\hline
- & - & \Sigma_{A_{\mathcal{C}}} & \Sigma_{A_{\mathcal{R}}} & \Sigma_{A_{\mathcal{V}}} & \Sigma_{A_{\mathcal{L}}} & -
\end{array}\right], \tag{41}$$

where the blocks $\mathcal{O}_{\mathcal{C}}$ and $\mathcal{O}_{\mathcal{R}}$ of size $n_{\mathcal{C}} \times n_{\mathcal{C}}$ and $n_{\mathcal{R}} \times n_{\mathcal{R}}$, respectively, have 0-diagonal and entries $-\infty$ on the off-diagonal, the block $\Gamma_{\mathcal{C}}$ of size $n_{\mathcal{C}} \times n_{\mathcal{C}}$ has 1-diagonal and entries ≤ 1 on the off-diagonal, and the block $\Gamma_{\mathcal{R}}$ of size $n_{\mathcal{R}} \times n_{\mathcal{R}}$ has 0-diagonal and entries ≤ 0 on the off-diagonal. The other blocks are defined as before. In this case we get the following result.

Theorem 22. *Consider an electrical circuit that contains neither $\mathcal{L}I$-cutsets nor $\mathcal{C}\mathcal{V}$-loops (including pure $\mathcal{C}$-loops) and let Assumptions* **(A1)**-**(A5)** *hold. Then the pHC equations* (40) *are of d-index $\nu_d = 1$ and the Σ-method succeeds with structural index $\nu_S = 1$.*

Proof. Due to **(A2)** the matrix $[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{V}}, A_{\mathcal{L}}]$ has full row rank $n_\eta - 1$, and because of **(A1)** the matrix $A_{\mathcal{V}}$ has full column rank $n_{\mathcal{V}}$. Thus, we can rearrange rows and columns such that

$$\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{V}} & A_{\mathcal{L}} \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} A_{\mathcal{V}_1} & A_{\mathcal{C}_1} & A_{\mathcal{R}_1} & A_{\mathcal{L}_1} \\ A_{\mathcal{V}_2} & A_{\mathcal{C}_2} & A_{\mathcal{R}_2} & A_{\mathcal{L}_2} \end{bmatrix} \end{array} \begin{array}{c} n_{\mathcal{V}} \\ n_\eta - 1 - n_{\mathcal{V}} \end{array}$$

where $A_{\mathcal{V}_1}$ is regular and $\begin{bmatrix} A_{\mathcal{C}_2} & A_{\mathcal{R}_2} & A_{\mathcal{L}_2} \end{bmatrix}$ has full row rank $n_\eta - 1 - n_{\mathcal{V}}$. Here, the assumption that $n_\eta - 1 \geq n_{\mathcal{V}}$ is reasonable, since otherwise there would be $\mathcal{V}$-loops in the circuit contradicting **(A1)**. As a consequence, we can always find $n_\eta - 1$ positions for the HVT in the last block row and block column of (41). Moreover, $n_{\mathcal{V}}$ position can be chosen from the blocks $\Sigma_{A_{\mathcal{V}}}$ and $\Sigma_{A_{\mathcal{V}}}^T$, such that the existence of a HVT is guaranteed in any case. If the pHC equations (40) are of d-index $\nu_d = 1$, we have

$$\operatorname{rank}[A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{V}}] = n_\eta - 1$$

and

$$\ker[A_{\mathcal{C}}, A_{\mathcal{V}}] = \{0\}$$

due to Theorem 10. Moreover, we have that $n_{\mathcal{C}} + n_{\mathcal{V}} \leq n_\eta - 1$ since the circuit contains no $\mathcal{CV}$-loops and also no $\mathcal{V}$-loops or $\mathcal{C}$-loops. As a consequence the $n_\eta - 1$ positions for the HVT in the last block row can be picked from $\left[\Sigma_{A_{\mathcal{C}}}, \Sigma_{A_{\mathcal{R}}}, \Sigma_{A_{\mathcal{V}}}\right]$ only. If $n_{\mathcal{V}} > 0$, then $n_{\mathcal{V}}$ position of the HVT in the last block row are fixed in $\Sigma_{A_{\mathcal{V}}}$, and the remaining $n_\eta - 1 - n_{\mathcal{V}}$ positions can be chosen form $\left[\Sigma_{A_{\mathcal{C}}}, \Sigma_{A_{\mathcal{R}}}\right]$. In particular, $n_{\mathcal{C}}$ positions can be chosen from $\Sigma_{A_{\mathcal{C}}}$. If $n_\eta - 1 > n_{\mathcal{C}} + n_{\mathcal{V}}$, then the remaining $n_\eta - 1 - n_{\mathcal{V}} - n_{\mathcal{C}}$ positions for the last block row have to be taken from $\Sigma_{A_{\mathcal{R}}}$. Due to the symmetry of (41) the transposed entries can be picked as positions for the HVT in the last block column. The remaining positions for the HVT can be picked from the diagonal of $\Gamma_{\mathcal{C}}$, $\Gamma_{\mathcal{R}}$, $\mathcal{O}_{\mathcal{R}}$ and $\Sigma_{\mathcal{L}}$. Concluding, we get a HVT of value $\operatorname{Val}(HVT) = n_{\mathcal{C}} + n_{\mathcal{L}}$ and the canonical offset vectors are given by $c = [0]$ and

$$d = [\underbrace{1\dots1}_{n_{\mathcal{C}}}, \underbrace{0\dots0}_{n_{\mathcal{R}}}, \underbrace{0\dots0}_{n_{\mathcal{C}}}, \underbrace{0\dots0}_{n_{\mathcal{R}}}, \underbrace{0\dots0}_{n_{\mathcal{V}}}, \underbrace{1\dots1}_{n_{\mathcal{L}}}, \underbrace{0\dots0}_{n_\eta - 1}],$$

according to the block structure of (41). The resulting Σ-Jacobian is given by

$$\mathfrak{J} = \left[\begin{array}{cc|ccccc|c} \mathcal{C} & 0 & I & 0 & 0 & 0 & 0 \\ 0 & \mathcal{G} & 0 & I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & A_{\mathcal{C}}^T \\ 0 & I & 0 & 0 & 0 & 0 & A_{\mathcal{R}}^T \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{\mathcal{V}}^T \\ 0 & 0 & 0 & 0 & 0 & \mathcal{L} & A_{\mathcal{L}}^T \\ \hline 0 & 0 & A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{V}} & 0 & 0 \end{array}\right].$$

Under the d-index-1 conditions for the pHC-equations (40) this matrix is nonsingular (see also the proof of Theorem 10) and the Σ-method succeeds with $\nu_S = 1 = \nu_d$.

□

Remark 23. If the circuit contains $\mathcal{L}I$-cutsets or $\mathcal{CV}$-loops, i.e., the pHC equations (40) are of d-index $\nu_d = 2$, then the Σ-method for the pHC equations (40) might still work successfully in most of the cases. However, in this case the results (and the success) can depend on the selection of the reference node as can be seen in Example 8. In general, we can say that the Σ-method can fail if there exist nodes in the circuit graph that are incident to a twig inductor as well as to a link capacitor for an (arbitrarily chosen) normal tree. If such a node exists it should be chosen as reference node in order to prevent the failure of the Σ-method.

Example 8. Consider the circuit given in Figure 10 containing a $\mathcal{CV}$-loop and, thus, having d-index $\nu_d = 2$. The graph corresponding to the circuit is given in Figure 11. If we select node 4 as reference node, then the Σ-method fails due to a singular Σ-Jacobian. Note that node 4 is the only node that is not adjacent to a (link) capacitor. If we select node 1, 2 or 3 as reference node, the Σ-method succeeds with nonsingular Σ-Jacobian and $\nu_S = 2$. ◁

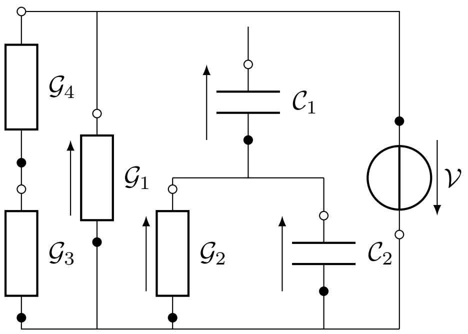

Figure 10: $\mathcal{RCV}$-circuit

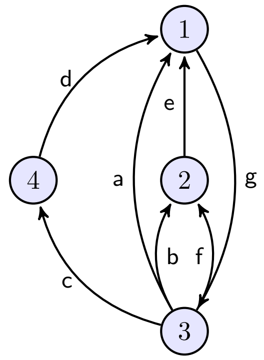

Figure 11: Graph for the $\mathcal{RCV}$-circuit

5 Conclusion

The structural analysis is a commonly used and powerful tool in the numerical treatment of DAEs. In this paper we have used the Σ-method for the structural analysis of DAEs that arise in the modeling and simulation of electrical circuits. The presented examples show that success or failure of the Σ-method for the commonly used MNA equations (6) can depend on the topology of the circuit. In the case of failure of the Σ-method the computed offset vectors will not give the required information on the index and on the system structure that can otherwise be used for regularization of the system (as proposed e.g. in [17]). As a consequence, a robust and stable numerical integration cannot be guaranteed. Moreover, the result of the Σ-method for the MNA equations (6) can also depend on the choice of the reference node. This choice does not influence the analytical properties of the system (as e.g. the index) and typically a node that has a large number of adjacent edges is chosen as reference. Our investigations suggest that a node that is adjacent to a capacitance or, if no capacitances are present, adjacent to a resistance should be chosen as reference node. However, this is no guarantee for the success of the Σ-method for the MNA equations (6) as Example 2 shows. Similar observations can be made if the Σ-method is applied to the MLA equations (7), and we have also seen that the Σ-method can give different results for the MNA and MLA equations depending on the topology of the circuit.

For the branch-oriented model equations (10) we have shown in Theorem 19 that the Σ-method always succeeds with a structural index that corresponds to the d-index of the system. Thus, with regard to automatized modeling and simulation of electrical circuit equations (also in the context of multi-physics applications), this formulation of the model equations is to be preferred if a structural analysis of the system equations is desired.

We have also considered a port-Hamiltonian formulation of the circuit equations. In this case, the Σ-method applied to the pHC equations (40) will succeed if the circuit contains neither $\mathcal{LI}$-cutsets nor $\mathcal{CV}$-loops (including pure $\mathcal{C}$-loops). If the circuit contains $\mathcal{LI}$-cutsets or $\mathcal{CV}$-loops, i.e., the pHC equations (40) are of d-index $\nu_d = 2$, the Σ-method might still work successfully in most of the cases, but the results (and the success) can depend on the selection of the reference node. In particular, the Σ-method can fail for the pHC equations (40) if there exists nodes in the circuit graph that are incident to a twig inductor as well as to a link capacitor for an arbitrarily chosen normal tree. If this is the case, such a node should be chosen as reference node in order to prevent the failure of the Σ-method.

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A Graph theoretical results

The following graph theoretical results can be found e.g. in [3, 15].

Theorem 24. *Let G be a directed graph consisting of n_η nodes, n_b branches and containing n_ℓ loops and n_q cutsets. Moreover, let $A_0 \in \mathbb{R}^{n_\eta, n_b}$ be the all-node incidence matrix, $B_0 \in \mathbb{R}^{n_\ell, n_b}$ the loop matrix and $Q_0 \in \mathbb{R}^{n_q, n_b}$ the cutset matrix with columns arranged according to the same order of branches. Furthermore, let k be the number of connected components of G. Then*

(i) $\operatorname{rank} A_0 = n_\eta - k$,

(ii) $\operatorname{rank} B_0 = n_b - n_\eta + k$,

(iii) $\operatorname{rank} Q_0 = n_\eta - k$,

(iv) $\operatorname{im} B_0^T = \ker A_0 = \ker Q_0$,

(v) $A_0 B_0^T = 0$ *and* $B_0 Q_0^T = 0$,

(vi) *G is a tree if and only if* $A_0 \in \mathbb{R}^{n_\eta, n_\eta - 1}$ *and* $\ker A_0 = \{0\}$.

Under the reasonable assumption that the electrical circuit graph is connected we can always select $n_\eta - 1$ linearly independent rows of A_0 to obtain the reduced matrix A, and $n_b - n_\eta + 1$ linearly independent rows of B_0 to obtain the reduced matrix B.

Remark 25. The all-node incidence matrix A_0 is a submatrix of the cutset matrix Q_0 since for each node of the graph the set of all branches incident to this node forms a cutset of the graph.

A subgraph $K = (V', B', \Psi|_{B'})$ of a connected graph G with $V' = V$, $B' \subset B$ is called a *spanning subgraph.* For a spanning subgraph K of a directed graph G let A_K (A_{G-K}) denote the submatrix of the incidence matrix A that is formed by the columns corresponding to branches in K (respectively the complementary graph $G - K$). Analogously, let B_K and B_{G-K} denote the corresponding loop matrices. By suitable reordering we can always get $A = [A_K \quad A_{G-K}]$ and $B = [B_K \quad B_{G-K}]$.

Lemma 26. *Let G be a connected directed graph with reduced incidence and loop matrices $A \in \mathbb{R}^{n_\eta - 1, n_b}$ and $B \in \mathbb{R}^{n_b - n_\eta + 1, n_b}$. Furthermore, let K be a spanning subgraph of G and assume that the branches of G are sorted such that*

$$A = [A_K \quad A_{G-K}], \quad B = [B_K \quad B_{G-K}].$$

1. *The following assertions are equivalent:*
   - *G does not contain K-cutsets;*
   - $\ker A_{G-K}^T = \{0\}$, *i.e., A_{G-K} has full row rank;*
   - $\ker B_K = \{0\}$, *i.e., B_K has full column rank.*

2. The following assertions are equivalent:

- *G does not contain K-loops;*
- *$\ker A_K = \{0\}$, i.e., A_K has full column rank;*
- *$\ker B_{G-K}^T = \{0\}$, i.e., B_{G-K} has full row rank.*

From Lemma 26 we get that an electrical circuit contains no $\mathcal{CV}$-loops if and only if the matrix $[A_{\mathcal{C}}\ A_{\mathcal{V}}]$ has full column rank. Similar the circuit contains no $\mathcal{LI}$-cutsets if and only if the matrix $[A_{\mathcal{R}}\ A_{\mathcal{C}}\ A_{\mathcal{V}}]$ has full row rank.

Lemma 27. *Let $G = (V, B)$ be a connected graph and let J, K be disjoint subsets of B. Then there exist a tree which contains all branches from J and no branches from K if and only if J has no loops and K has no cutsets.*

Lemma 28. *Let K be a set of $n_\eta - 1$ branches of a connected directed graph. Then A_K is nonsingular if and only if K defines a tree. In this case $\det(A_K) = \pm 1$.*

Thus, if we partition A as $A = [A_T, A_{coT}]$ where the columns of A_T corresponds to the twigs of a tree, then A_T is regular.
Let T be a tree of the connected and directed graph G, and let L be the set of all branches that do not belong to the tree (i.e., the set of all links). Then, for every $z \in L$, the set $T \cup \{z\}$ forms a loop. These are the so-called *fundamental loops* with orientations defined as the orientation of the corresponding link z. Since each tree contains $n_\eta - 1$ twigs there are exactly $n_b - n_\eta + 1$ fundamental loops. The fundamental loop matrix that only contains the fundamental loops has full row rank and by a suitable ordering of the branches takes the form

$$B = \begin{bmatrix} I & B_T \end{bmatrix},$$

where the columns of B_T corresponds to the twigs of the tree T.
On the other hand, let $b \in T$ be a twig of the tree. If we remove the branch b, then T decomposes into two separated but connected subtrees T_1 and T_2. If we denote the set of all nodes in T_i as N_i, for $i = 1, 2$, then the set of all branches of G that connect nodes from N_1 with nodes from N_2 forms a cutset of G. This cutset can be uniquely identified with the corresponding twig b of T. These cutsets are the so-called *fundamental cutsets* with orientations defined as the orientation of the corresponding twig b. Since the tree T contains $n_\eta - 1$ twigs there are $n_\eta - 1$ fundamental cutsets. The fundamental cutset matrix that only contains the fundamental cutsets can be represented by a suitable ordering of the branches as

$$Q = \begin{bmatrix} Q_L & I \end{bmatrix},$$

where the branches corresponding to the last $n_\eta - 1$ columns belong to the tree T.