

Linear algebra properties of dissipative Hamiltonian descriptor systems

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Abstract

A wide class of matrix pencils connected with dissipative Hamiltonian descriptor systems is investigated. In particular, the following properties are shown: all eigenvalues are in the closed left half plane, the nonzero finite eigenvalues on the imaginary axis are semisimple, the index is at most two, and there are restrictions for the possible left and right minimal indices. For the case that the eigenvalue zero is not semisimple, a structure-preserving method is presented that perturbs the given system into a Lyapunov stable system.

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1 Introduction

This paper deals with the linear algebra properties of matrix pencils that are associated with linear time-invariant *dissipative Hamiltonian descriptor systems* of the form

$$E\dot{x} = (J - R)Qx. \quad (1)$$

These systems arise in energy based modeling of dynamical systems and are a special case of so-called *port-Hamiltonian (pH) descriptor systems*, see, e.g., [1, 2, 13, 18, 23, 25, 26, 27]. General pH descriptor systems have the form

$$\begin{aligned} E\dot{x} &= (J - R)Qx + (B - P)u, \\ y &= (B + P)^*Qx + (S + N)u, \end{aligned} \quad (2)$$

where for $(\mathbb{F}, \star) \in \{(\mathbb{C}, *), (\mathbb{R}, \top)\}$, we have $J, R \in \mathbb{F}^{n,n}$, $E, Q \in \mathbb{F}^{n,m}$, $m \leq n$, and in addition $J^* = -J$ and $R^* = R \geq 0$, where we use the standard notation $W > 0$ ($W \geq 0$) for a real symmetric or complex Hermitian matrix W to be positive (semi-)definite.

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An important quantity of the system is the *Hamiltonian* $\mathcal{H}(x) = \frac{1}{2}x^*E^*Qx$ which describes the distribution of internal energy among the energy storage elements of the system. In order for this to make sense and to guarantee that $\mathcal{H}(x)$ is real, it is natural to assume that

$$E^*Q = Q^*E. \quad (3)$$

The matrix J is the *structure matrix* describing the energy flux among energy storage elements within the system; R is the *dissipation matrix* describing energy dissipation/loss in the system. The matrices $B \pm P \in \mathbb{F}^{n,m}$ are *port* matrices describing the manner in which energy enters and exits the system, and $S + N$, with $S = S^* \in \mathbb{F}^{m,m}$ and $N = -N^* \in \mathbb{F}^{m,m}$, describe the direct *feed-through* from input to output. In most practical examples the system has the extra property that Q^*E is positive semidefinite, in which case we speak of a pHDAE *with nonnegative Hamiltonian*.

To illustrate the broad variety of application areas where pH descriptor systems play a role, we review the following examples from the literature.

Example 1 A simple RLC network, see, e.g., [1, 4, 9], can be modeled by a DAE of the form

$$\underbrace{\begin{bmatrix} G_c C G_c^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=E} \begin{bmatrix} \dot{V} \\ \dot{I}_l \\ \dot{I}_v \end{bmatrix} = \underbrace{\begin{bmatrix} -G_r R_r^{-1} G_r^\top & -G_l & -G_v \\ G_l^\top & 0 & 0 \\ G_v^\top & 0 & 0 \end{bmatrix}}_{:=J-R} \begin{bmatrix} V \\ I_l \\ I_v \end{bmatrix}, \quad (4)$$

where $L > 0$, $C > 0$, $R_r > 0$ are real symmetric, G_v is of full column rank, and the other matrices have appropriate dimensions. (For a detailed physical interpretation of the matrices and variables we refer to [1].) Here, J and $-R$ are defined to be the skew-symmetric and symmetric parts, respectively, of the matrix on the right hand side of (4). We see that in this example we have a dissipative Hamiltonian descriptor system of the form (1) with the matrix Q being the identity, $E = E^T \geq 0$, $J = -J^T$, and $R \geq 0$.

Example 2 Space discretization of the Stokes or Oseen equation in fluid dynamics, see, e.g., [7], leads to a dissipative Hamiltonian system with pencil

$$\lambda E - (J - R) = \lambda \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ -B^\top & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -B \\ B^\top & 0 \end{bmatrix}, \quad R = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

with A being a positive semidefinite discretization of the negative Laplace operator, B being a discretized gradient, and M being a positive definite mass matrix, so $R \geq 0$, $Q = I$, and $E = E^\top \geq 0$. Note that the matrix B may be rank deficient, in which case the pencil is singular.

Example 3 Space discretization of the Euler equation describing the flow in a gas network [6] leads to a dissipative Hamiltonian descriptor system with

$$E = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -G & 0 \\ G^\top & 0 & K^\top \\ 0 & -K & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q = I,$$

where M_1, M_2, D are positive definite and the matrices N and $[G^\top \ K^\top]$ have full column rank. Thus, again we have $R \geq 0$ and a nonnegative Hamiltonian, since $Q^\top E = E^\top Q \geq 0$.

Example 4 Classical second order representations of linear damped mechanical systems, see, e.g., [32, 34], have the form

$$M\ddot{x} + D\dot{x} + Kx = f, \quad (5)$$

where $M, D, K \in \mathbb{R}^{n,n}$ are symmetric positive semidefinite matrices and represent the mass, damping, and stiffness matrices, respectively, while f is an exiting force that is assumed to be sufficiently often differentiable. First order formulation leads to a system associated with the matrix pencil

$$\lambda \underbrace{\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}}_{=:E} - \left(\underbrace{\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}}_{=:J} - \underbrace{\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}}_{=:R} \right) \underbrace{\begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}}_{=:Q}, \quad (6)$$

where we have $E^\top Q = Q^\top E \geq 0$ and $R \geq 0$. As a particular case, freely vibrating strongly damped systems of the form $M\ddot{x} + sD\dot{x} + Kx = 0$ were investigated in [30] for $s \rightarrow \infty$. In the limit case, the first order formulation of the equivalent system $\frac{1}{s}M\ddot{x} + D\dot{x} + \frac{1}{s}Kx = 0$ leads to a pencil as in (6) with $M = K = 0$. Observe that this pencil is singular if D is singular.

Example 5 Consider a mechanical system as in (5) with a position constraint $Gx = 0$. Replacing this constraint by its derivative $G\dot{x} = 0$ and incorporating it in the dynamical equation using a Lagrange multiplier y , the system can be written as

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} K & G^\top \\ G & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

which in first order formulation gives a system associated with the pencil

$$\lambda \underbrace{\begin{bmatrix} M & & & \\ & 0 & & \\ & & I & \\ & & & I \end{bmatrix}}_{=:E} - \left(\underbrace{\begin{bmatrix} & & I & \\ -I & & & I \end{bmatrix}}_{=:J} - \underbrace{\begin{bmatrix} D & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}}_{=:R} \right) \underbrace{\begin{bmatrix} I & & & \\ & I & & \\ & & K & G^\top \\ & & G & 0 \end{bmatrix}}_{=:Q}.$$

Here, both E and Q are symmetric, commuting square matrices and R is positive semidefinite, but in contrast to the previous examples $E^\top Q$ is not positive semidefinite in this case. Note also that often the system is modeled with redundant constraints which implies that G is rank deficient and then the pencil is singular.

Similar examples arise in many other applications, see e.g. [15], where the singular pencil case typically arises as a limiting case or when redundant system modeling is used.

To understand the linear algebra properties of pH descriptor systems, we analyze matrix pencils of the form

$$P(\lambda) = \lambda E - LQ \in \mathbb{F}^{n,m}[\lambda], \quad \text{with } L = J - R, \quad J = -J^*, \quad R = R^* \geq 0, \quad (7)$$

where $\mathbb{F}^{n,m}[\lambda]$ denotes the set of $n \times m$ matrix polynomials over \mathbb{F} . So far, mainly the cases of $Q = I$ and $Q > 0$ were considered in the literature, but here we will deal with the general case, allowing even that the pencil is rectangular and thus singular. If $Q = I$ and $R = 0$, then $P(\lambda)$ is an *even pencil*, i.e., the coefficient associated with λ is Hermitian/symmetric and

the second coefficient is skew-Hermitian/skew-symmetric. Canonical forms for $P(\lambda)$ in these cases are well known, see, e.g., [31]. If, in addition, E is positive semidefinite, then it easily follows from these forms that all finite eigenvalues are on the imaginary axis and semisimple. Then replacing the condition $R = 0$ by $R \geq 0$ has the effect that now all finite eigenvalues are in the closed left-half complex plane, and those on the imaginary axis are still semisimple, see, e.g., [21].

At first sight, it does not seem obvious that the structure of the pencil $P(\lambda)$ in (7) preserves any of these restrictions in the spectrum. However, it turns out that the assumption of both R and Q^*E being positive semidefinite leads to surprisingly rich linear algebraic properties of $P(\lambda)$.

The remainder of the paper is organized as follows. Section 2 reviews the Kronecker canonical form and introduces the concept of regular deflating subspaces of singular matrix pencils. It appears that the properties of the pencil $\lambda E - Q$ are crucial for understanding the pencil $P(\lambda)$. Hence, we first analyze the pencil $\lambda E - Q$ in Section 3. In the regular case a canonical form is derived in Propositions 9 and 11. In the singular case we present a condensed form (Theorem 15) and show what are the possible left and right minimal indices (Corollary 17). Sections 4 and 5 discuss the stability properties and the Kronecker canonical form of the pencil $P(\lambda) = \lambda E - LQ$. In particular, we show in Theorem 21 that if the pencil $\lambda E - Q$ does not have left minimal indices larger than zero, then the pencil $P(\lambda)$ has all finite eigenvalues in the closed left half-plane, and all eigenvalues on the imaginary axis except zero are semisimple. Furthermore, the index of the pencil (see Section 2) is at most two and its right minimal indices (if there are any) are not larger than one. If, in addition, the pencil $\lambda E - Q$ is regular, then the left minimal indices of $P(\lambda)$ are all zero (if there are any). Furthermore, we present many examples that illustrate which properties are lost if some of the assumptions on the pencil $\lambda E - Q$ are weakened. Finally, we discuss in Section 6 structure-preserving perturbations of $P(\lambda)$ that make the eigenvalue zero semisimple, but keep the Jordan structure of all other eigenvalues invariant.

2 Preliminaries

For general matrix pencils $\lambda E - A$ the structural properties are characterized via the Kronecker canonical form. Recall the following result for complex matrix pencils [11].

Theorem 6 *Let $E, A \in \mathbb{C}^{n,m}$. Then there exist nonsingular matrices $S \in \mathbb{C}^{n,n}$ and $T \in \mathbb{C}^{m,m}$ such that*

$$S(\lambda E - A)T = \text{diag}(\mathcal{L}_{\epsilon_1}, \dots, \mathcal{L}_{\epsilon_p}, \mathcal{L}_{\eta_1}^\top, \dots, \mathcal{L}_{\eta_q}^\top, \mathcal{J}_{\rho_1}^{\lambda_1}, \dots, \mathcal{J}_{\rho_r}^{\lambda_r}, \mathcal{N}_{\sigma_1}, \dots, \mathcal{N}_{\sigma_s}), \quad (8)$$

where the block entries have the following properties:

- (i) Every entry \mathcal{L}_{ϵ_j} is a bidiagonal block of size $\epsilon_j \times (\epsilon_j + 1)$, $\epsilon_j \in \mathbb{N}_0$, of the form

$$\lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}.$$

(ii) Every entry $\mathcal{L}_{\eta_j}^\top$ is a bidiagonal block of size $(\eta_j + 1) \times \eta_j$, $\eta_j \in \mathbb{N}_0$, of the form

$$\lambda \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & & \ddots & \\ & & & 1 \\ & & & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & & \ddots & \\ & & & 0 \\ & & & & 1 \end{bmatrix}.$$

(iii) Every entry $\mathcal{J}_{\rho_j}^{\lambda_j}$ is a Jordan block of size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$, of the form

$$\lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & & \lambda_j \end{bmatrix}.$$

(iv) Every entry \mathcal{N}_{σ_j} is a nilpotent block of size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$, of the form

$$\lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

The Kronecker canonical form is unique up to permutation of the blocks.

For real matrices there exists a real Kronecker canonical form which is obtained under real transformation matrices S, T . Here, the blocks $\mathcal{J}_{\rho_j}^{\lambda_j}$ with $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$ are in real Jordan canonical form instead, but the other blocks have the same structure as in the complex case.

The sizes η_j , and ϵ_i of the rectangular blocks are called the *left and right minimal indices* of $\lambda E - A$, respectively. Furthermore, a value $\lambda_0 \in \mathbb{C}$ is called a (finite) eigenvalue of $\lambda E - A$ if

$$\text{rank}(\lambda_0 E - A) < \text{nrank}(\lambda E - A) := \min_{\alpha \in \mathbb{C}} \text{rank}(\alpha E - A),$$

where $\text{nrank}(\lambda E - A)$ is called the *normal rank* of $\lambda E - A$. Furthermore, $\lambda_0 = \infty$ is said to be an eigenvalue of $\lambda E - A$ if zero is an eigenvalue of $\lambda A - E$. It is obvious from Theorem 6 that the blocks \mathcal{J}_{ρ_j} as in (iii) correspond to finite eigenvalues of $\lambda E - A$, whereas blocks \mathcal{N}_{σ_j} as in (iv) correspond to the eigenvalue ∞ . The sum of all sizes of blocks that are associated with a fixed eigenvalue $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ is called the *algebraic multiplicity* of λ_0 . The size of the largest block \mathcal{N}_{σ_j} is called the *index* ν of the pencil $\lambda E - A$, where, by convention, $\nu = 0$ if E is invertible.

The matrix pencil $\lambda E - A \in \mathbb{F}^{n,m}[\lambda]$ is called *regular* if $n = m$ and $\det(\lambda_0 E - A) \neq 0$ for some $\lambda_0 \in \mathbb{C}$, otherwise it is called *singular*. A pencil is singular if and only if it has blocks of at least one of the types \mathcal{L}_{ϵ_j} or $\mathcal{L}_{\eta_j}^\top$ in the Kronecker canonical form.

While eigenvalues of singular pencils are well-defined, the same is not true for eigenvectors. For example, the pencil

$$\lambda E - A = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \end{bmatrix} = \mathcal{J}_1^1 \oplus \mathcal{L}_1 \quad (9)$$

has the eigenvalue 1 with algebraic multiplicity one, but any vector x of the form $x = [\alpha \ \beta \ \beta]^\top$ with $\alpha, \beta \in \mathbb{F}$ satisfies $\lambda_0 E x = A x$. Hence, the equation $\lambda_0 E x = A x$ itself is not suitable (at least for the purpose of the current paper) as a generalization of the notion *eigenvector* in the singular case. We therefore introduce the following concept.

Definition 7 Let $E, A \in \mathbb{F}^{n,m}$ and let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $P(\lambda) = \lambda E - A$ with algebraic multiplicity k .

- (i) A k -dimensional subspace \mathcal{X} of \mathbb{C}^m is called a (*right*) *regular deflating subspace* of $P(\lambda)$ associated with λ_0 if there exists nonsingular matrices $X \in \mathbb{C}^{m,m}$ and $Y \in \mathbb{C}^{n,n}$ such that the first k columns of X span \mathcal{X} and such that

$$Y(\lambda E - A)X = \begin{bmatrix} R(\lambda) & 0 \\ 0 & \tilde{P}(\lambda) \end{bmatrix}, \quad (10)$$

where $R(\lambda)$ is a regular $k \times k$ pencil that has only the eigenvalue λ_0 , and where $\tilde{P}(\lambda)$ does not have λ_0 as an eigenvalue.

- (ii) A vector $x \in \mathbb{C}^m \setminus \{0\}$ is called a *regular eigenvector* of $P(\lambda)$ associated with λ_0 , if $\lambda_0 E x = A x$ and if there exists a regular deflating subspace associated with λ_0 that contains x .

Regular deflating subspaces and regular eigenvectors associated with the eigenvalue ∞ are defined in the obvious way by considering the reversed pencil $\lambda A - E$ and the eigenvalue zero.

Clearly, for each eigenvalue of a singular pencil there exists a regular eigenvector and a regular deflating subspace. For the pencil in (9) each subspace of the form $\mathcal{X} = \text{span}([\alpha \ \beta \ \beta]^\top)$ with some $\alpha, \beta \in \mathbb{F}$, $\alpha \neq 0$ is a regular deflating subspace. Indeed, with

$$X = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1/\alpha & 0 \\ -\beta/\alpha & 1 \end{bmatrix}$$

one has $Y(\lambda E - A)X = \lambda E - A$. In particular, this shows that a regular deflating subspace associated with an eigenvalue is not unique if the pencil is singular, which is in contrast with the regular case. Furthermore, it is straightforward to show that neither the subspace

$$\text{span}([0 \ 1 \ 1]^\top) \quad \text{nor} \quad \text{span}([0 \ 1 \ 1]^\top, [1 \ 0 \ 0]^\top)$$

is a regular deflating subspace. We have the following key property of regular deflating subspaces.

Lemma 8 Let $E, A \in \mathbb{F}^{n,m}$, let $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ be an eigenvalue of $\lambda E - A$, and let \mathcal{X} be an associated regular deflating subspace. Then

$$\mathcal{X} \cap \text{span} \left(\bigcup_{\lambda \in (\mathbb{C} \cup \{\infty\}) \setminus \{\lambda_0\}} \ker(\lambda E - A) \right) = \{0\},$$

with the usual convention $\infty E - A := E$.

Proof. Indeed, let X, Y be as in Definition 7 such that (10) is satisfied. Furthermore assume that $y \in \ker(\lambda_1 E - A)$ for some $\lambda_1 \in (\mathbb{C} \cup \{\infty\}) \setminus \{\lambda_0\}$ and let $X^{-1}y = \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix}^\top$ be split conformably with (10). First we show that $y_1 = 0$. Indeed,

$$0 = Y(\lambda_1 E - A)X(X^{-1}y) = \begin{bmatrix} R(\lambda_1)y_1 \\ * \end{bmatrix}.$$

In particular, $y \in \ker R(\lambda_1)$. But the pencil $R(\lambda)$ is regular and has only one eigenvalue $\lambda_0 \neq \lambda_1$. Therefore $y_1 = 0$. In other words, $PX^{-1}y = 0$, where P is the canonical projection onto the first k coordinates. Assume now that $x = \sum_{i=1}^l z_i$, with $x \in \mathcal{X}$ and $z_i \in \ker(\lambda_i E - A)$, $\lambda_i \neq \lambda_0$ for $i = 1, \dots, l$ and some $l > 0$. Then $PX^{-1}x = \sum_{i=1}^l Pz_i = 0$. But since x is a vector in \mathcal{X} , there exists a vector $v \in \mathbb{F}^k$ such that $x = X \begin{bmatrix} v^\top & 0 \end{bmatrix}^\top$, i.e., $PX^{-1}x = X^{-1}x$. Hence $x = 0$, which finishes the proof. \square

Let us mention two other concepts associated with eigenvectors and deflating subspaces in the case of singular pencils that are present in the literature. In [33], *reducing subspaces* were introduced to replace the concept of deflating subspaces in the singular case. This concept has the advantage that reducing subspaces associated with an eigenvalue are now uniquely defined. However, they are explicitly allowed to contain vectors *from the singular part* and thus a result similar to Lemma 8 cannot be obtained. For the second concept, that only refers to eigenvectors, we first discuss a different characterization of regular eigenvectors as follows. Let $\mathcal{N}_r(P)$ denote the right nullspace of $P(\lambda)$ interpreted as a matrix over the field of rational functions (thus $\mathcal{N}_r(P)$ is a subspace of the space $\mathbb{C}^m(\lambda)$ of vectors of length m with rational functions as entries) and let

$$\mathcal{N}_r(P)|_{\lambda_0} := \{z \in \mathbb{C}^m \mid z = v(\lambda_0) \text{ for some } v(\lambda) \in \mathcal{N}_r(P)\}$$

denote the *right nullspace of $P(\lambda)$ evaluated at λ_0* , see [10]. Then it can be shown that $x \in \mathbb{C}^m$ is a regular eigenvector of $P(\lambda)$ associated with λ_0 if and only if $\lambda_0 E x = A x$ and $x \notin \mathcal{N}_r(P)|_{\lambda_0}$. This construction allows to define *eigenspaces* uniquely, e.g., as being orthogonal to $\mathcal{N}_r(P)|_{\lambda_0}$, cf. [10]. However, this orthogonality property is not preserved under equivalence transformations for pencils.

3 Properties of the pencil $\lambda E - Q$

As it was announced in the introduction, in this section we discuss the properties of a (possibly rectangular) pencil $\lambda E - Q \in \mathbb{F}^{m,n}[\lambda]$ satisfying the condition (3). We will also frequently assume that, in addition, one (or both) of the conditions

$$EQ^* = QE^*, \tag{11}$$

and

$$E^*Q \geq 0 \tag{12}$$

are satisfied. Although (11) does not have a direct physical interpretation, it is satisfied by most classical examples (see the introduction) and together with (3) it forms a very strong assumption on the structure of the pencil $\lambda E - Q$. Condition (12) has a minor impact on the properties of $\lambda E - Q$, but strongly influences the properties of $P(\lambda)$, as will be seen in subsequent sections. We will not discuss pencils $\lambda E - Q$ for which only (11) is assumed,

because results for these pencils can easily be obtained from the corresponding results on pencils satisfying (3) by considering the pencil $\lambda E^* - Q^*$.

Clearly, when E and Q are square Hermitian (or real symmetric matrices) that commute (as is the case in some of our applications), then both $E^*Q = Q^*E$ and $EQ^* = QE^*$ are satisfied. It is clear that for any unitary $U \in \mathbb{F}^{m,m}$ and invertible $X \in \mathbb{F}^{n,n}$ a transformation of the form $(E, Q) \mapsto (UEX, UQX)$ preserves the conditions (3) and (12). The following result yields a canonical form for regular pencils under this transformation.

Proposition 9 *Let $\lambda E - Q \in \mathbb{F}^{n,n}[\lambda]$ be a regular pencil. Then we have $E^*Q = Q^*E$ if and only if there exist a unitary (in the case $\mathbb{F} = \mathbb{R}$ real orthogonal) matrix $U \in \mathbb{F}^{n,n}$ and an invertible $X \in \mathbb{F}^{n,n}$ such that UEX and UQX are diagonal and real, and satisfy*

$$(UEX)^2 + (UQX)^2 = I_n.$$

Moreover, U and X can be chosen such that one of the matrices UEX or UQX is nonnegative. If, in addition, we have $E^*Q \geq 0$, then U and X can be chosen such that UEX and UQX are both nonnegative.

Proof. Suppose that $E^*Q = Q^*E$ holds. Then the regularity of the pencil $\lambda E - Q$ implies that the columns of the matrix

$$L := \begin{bmatrix} E \\ Q \end{bmatrix}$$

are linearly independent and they span an n -dimensional *Lagrangian subspace*, i.e., a subspace $\mathcal{U} \subset \mathbb{F}^{2n}$ such that

$$u^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} v = 0$$

for all $u, v \in \mathcal{U}$. Orthonormalizing the columns of L , i.e., carrying out a factorization, $L = ZT$ with $T \in \mathbb{F}^{n,n}$ invertible, and

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \quad Z_1, Z_2 \in \mathbb{F}^{n,n}$$

satisfying $Z^*Z = I_n$, we obtain that the columns of $Z = LT^{-1}$ span the same Lagrangian subspace as the columns of L and according to the CS decomposition (see, e.g., [24, Theorem 2.1] for the complex case and [22, Lemma 2.1] for the real case), there exists $U, V \in \mathbb{F}^{n,n}$ unitary and such that

$$\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} E \\ Q \end{bmatrix} TV = \begin{bmatrix} C \\ S \end{bmatrix},$$

with C and S being diagonal such that $C^2 + S^2 = I_n$. The claim follows now by taking $X := TV$. The converse implication is straightforward.

Clearly, by premultiplying U with a diagonal signature matrix that switches the signs of diagonal entries if necessary then yields the desired nonnegativity of one of the matrices UEX or UQX . If, in addition, we have $E^*Q \geq 0$, then it follows that

$$(UEX)(UQX) = (UEX)^*(UQX) = X^*E^*QX \geq 0$$

and thus nonnegativity of one of the diagonal matrices UEX or UQX implies nonnegativity of the other. \square

A direct corollary of Proposition 9 gives a characterization of the index of $\lambda E - Q$.

Corollary 10 *Let $E, Q \in \mathbb{F}^{n,n}$ be such that $E^*Q = Q^*E$ and that the pencil $\lambda E - Q$ is regular. Then it is of index at most one.*

Proof. The proof follows directly from Proposition 9, since if both E and Q are transformed to be diagonal, then the regularity implies that the nilpotent part in the Kronecker canonical form is diagonal and hence the index is zero if E is nonsingular and it is one otherwise. \square

The next result shows that the assumption of regularity in Proposition 9 can be dropped, if both conditions (3) and (11) are satisfied. Note that the rectangular case $m \neq n$ is included. By a diagonal matrix $D = [d_{ij}]$ we then mean a matrix with all entries d_{ij} equal to zero unless $i = j$.

Proposition 11 *Let $E, Q \in \mathbb{F}^{m,n}$ be such that $E^*Q = Q^*E$ and $EQ^* = QE^*$. Then there exist unitary (real orthogonal), matrices $U \in \mathbb{F}^{m,m}, V \in \mathbb{F}^{n,n}$ such that UEV and UQV are diagonal and real. Furthermore, U and V can be chosen such that one of the matrices UEV and UQV is nonnegative, and if in addition $E^*Q \geq 0$ holds, then U, V can be chosen such that UEV and UQV are both nonnegative.*

Proof. The proof follows from the fact that if both conditions $E^*Q = Q^*E$ and $EQ^* = QE^*$ hold then this is equivalent to the fact that the four matrices E^*Q, Q^*E, Q^*Q and E^*E are all normal and commuting, as well as the four matrices EQ^*, QE^*, QQ^* and EE^* . Then the statement follows immediately from [20, Corollary 5] in the complex case and [20, Corollary 9] in the real case. \square

Surprisingly, the statement of Proposition 9 is in general not true for singular pencils $\lambda E - Q$ satisfying only $E^*Q = Q^*E$ as the following example shows.

Example 12 Let

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = S \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{with } S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then E^*Q is symmetric as

$$E^*Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

but by construction $\lambda E - Q$ has one singular block with left minimal index 2 in its Kronecker canonical form and thus cannot be equivalent to a pencil in diagonal form.

Although a direct generalization of Proposition 9 to the singular case is not possible, it turns out that there is a restriction in the possible Kronecker structure for pencils satisfying (3) and (12).

Proposition 13 *Let $E, Q \in \mathbb{F}^{m,n}$ be such that $E^*Q = Q^*E$. Then all right minimal indices of $\lambda E - Q$ are zero (if there are any).*

Proof. Let $\lambda E - Q = S(\lambda E_0 - Q_0)T$, where $\lambda E_0 - Q_0$ is in Kronecker canonical form. Clearly, with E^*Q also $E_0^*S^*SQ_0$ is Hermitian. Suppose that $\lambda E - Q$ has a right minimal index which is nonzero. Then $\lambda E_0 - Q_0$ contains a block of the form

$$\lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \quad (13)$$

of size $k \times (k + 1)$ with $k > 0$. Without loss of generality let this block be the first block in the Kronecker canonical form $\lambda E_0 - Q_0$. Then the k th and $(k + 1)$ st standard basis vectors $x := e_k$ and $y := e_{k+1}$ satisfy $Q_0 y = E_0 x \neq 0$ and $E_0 y = 0$. This, however, implies that

$$y^* Q_0^* S^* S E_0 x = 0 < y^* E_0^* S^* S Q_0 x$$

contradicting the fact that $E_0^* S^* S Q_0$ is Hermitian. \square

Clearly, by applying Proposition 13 to the pencil $\lambda E^* - Q^*$ we get that $E Q^* = Q E^*$ implies all left minimal indices of $\lambda E - Q$ being zero. Another immediate consequence of Proposition 13 is the following corollary.

Corollary 14 *Let $E, Q \in \mathbb{F}^{n,n}$ be such that $E^* Q = Q^* E$. If $\lambda E - Q$ is singular, then E and Q have a common nullspace.*

Proof. This follows immediately from Proposition 13, the fact that a square singular pencil must have both left and right minimal indices, and the fact that a right minimal index zero corresponds to a common nullspace. \square

With the help of these intermediate results, we are now able to derive a Kronecker-like condensed form for pencils $\lambda E - Q$ satisfying $E^* Q = Q^* E$.

Theorem 15 *Let $E, Q \in \mathbb{F}^{m,n}$ be such that $E^* Q = Q^* E$. Then there exist a unitary (real orthogonal) matrix $U \in \mathbb{F}^{m,m}$ and a nonsingular matrix $X \in \mathbb{F}^{n,n}$ such that*

$$U E X = \begin{bmatrix} E_{11} & E_{12} & 0 \\ 0 & E_{22} & 0 \end{bmatrix} \quad \text{and} \quad U Q X = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ 0 & Q_{22} & 0 \end{bmatrix} \quad (14)$$

and the following statements hold.

- (i) $E_{11}, Q_{11} \in \mathbb{F}^{n_1, n_1}$ are diagonal, real and satisfy $E_{11}^2 + Q_{11}^2 = I_{n_1}$. Furthermore, U and X can be chosen such that one of the matrices E_{11} and Q_{11} is nonnegative and if, in addition, $E^* Q \geq 0$, then U, X can be chosen such that both E_{11} and Q_{11} are nonnegative.
- (ii) $Q_{22}, E_{22} \in \mathbb{F}^{m_2, n_2}$, with $m_2 = n - n_1 > n_2$ (or $m_2 = n_2 = 0$), are such that $\lambda E_{22} - Q_{22}$ is singular having left singular blocks (blocks of the form $\mathcal{L}_{n_j}^\top$) only;
- (iii) $\text{Ker}(E_{22}) \subseteq \text{Ker}(E_{12})$ and $\text{Ker}(Q_{22}) \subseteq \text{Ker}(Q_{12})$.
- (iv) The pencil $\lambda E - Q$ has the same Kronecker canonical form as

$$\lambda \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \end{bmatrix} - \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \end{bmatrix}. \quad (15)$$

In particular, the regular part of $\lambda E - Q$ is given by $\lambda E_{11} - Q_{11}$, the left minimal indices of $\lambda E - Q$ coincide with those of $\lambda E_{22} - Q_{22}$ and $\lambda E - Q$ has exactly $n - n_1 - n_2$ right minimal indices which are all zero.

Proof. Let S, T be nonsingular matrices such that

$$S(\lambda E - Q)T = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ 0 & \mathcal{S}(\lambda) \end{bmatrix} \quad (16)$$

is in Kronecker canonical form, where $\mathcal{R}(\lambda) \in \mathbb{F}^{n_1, n_1}$ and $\mathcal{S}(\lambda) \in \mathbb{F}^{m_1, n-n_1}$ are the regular part and singular part, respectively. Without loss of generality, let the singular blocks in $\mathcal{S}(\lambda)$ be ordered in such a way that the left singular blocks (of the form $\mathcal{L}_{\eta_j}^\top$) come first. Then, since by Proposition 13 all right minimal indices of $\lambda E - Q$ are zero, we obtain that $\mathcal{S}(\lambda)$ has the form

$$\mathcal{S}(\lambda) = \begin{bmatrix} \lambda E_0 - Q_0 & 0 \end{bmatrix},$$

where $\lambda E_0 - Q_0 \in \mathbb{F}^{m_2, n_2}$ contains all singular blocks $\mathcal{L}_{\eta_j}^\top$. This implies that $m_2 > n_2$ unless there are no blocks $\mathcal{L}_{\eta_j}^\top$, i.e., $m_2 = n_2 = 0$. The zero block is of size $n - n_1 - n_2$, i.e., the pencil $\lambda E - Q$ has $n - n_1 - n_2$ right minimal indices all of which are zero. Let

$$S^{-1} = V \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}, \quad \text{with } V^*V = I,$$

be a QR decomposition of S^{-1} partitioned conformably with (16). Then, with $\lambda \tilde{E}_{11} - \tilde{Q}_{11} := R_{11}\mathcal{R}(\lambda)$ one has

$$V^*(\lambda E - Q)T = \begin{bmatrix} \lambda \tilde{E}_{11} - \tilde{Q}_{11} & R_{12}(\lambda E_0 - Q_0) & 0 \\ 0 & R_{22}(\lambda E_0 - Q_0) & 0 \end{bmatrix}.$$

Furthermore, $E^*Q = Q^*E$ implies that $\tilde{E}_{11}^*\tilde{Q}_{11} = \tilde{Q}_{11}^*\tilde{E}_{11}$. Since R , and thus also R_{11} and R_{22} are invertible, it follows that $\lambda \tilde{E}_{11} - \tilde{Q}_{11}$ is regular. Thus, by Proposition 9 there exists a unitary matrix $U_1 \in \mathbb{F}^{n_1, n_1}$ and a nonsingular matrix $X_1 \in \mathbb{F}^{n_1, n_1}$ such that $E_{11} := U_1 \tilde{E}_{11} X_1$ and $Q_{11} := U_1 \tilde{Q}_{11} X_1$ are real diagonal satisfying $E_{11}^2 + Q_{11}^2 = I_{n_1}$, with at least one of the matrices E_{11}, Q_{11} being nonnegative. Setting $U := \text{diag}(U_1, I_{m_2})V^*$, $X := T \text{diag}(X_1, I_{n-n_1})$, we obtain

$$U(\lambda E - Q)X = \begin{bmatrix} \lambda U_1(\tilde{E}_{11} - \tilde{Q}_{11})X_1 & U_1 R_{12}(\lambda E_0 - Q_0) & 0 \\ 0 & R_{22}(\lambda E_0 - Q_0) & 0 \end{bmatrix}, \quad (17)$$

which is the desired form (14). We also have already shown the first statement of (i) and statement (ii). To finish the proof of (i), note that if E^*Q is positive semidefinite, then $\tilde{E}_{11}^*\tilde{Q}_{11}$ is positive definite as well, as a principle submatrix of a matrix congruent to E^*Q . Hence, one may apply Proposition 9.

Invertibility of R_{22} together with (17) implies that statement (iii) holds. To see (iv) note that the equivalence transformation

$$\begin{bmatrix} I & 0 \\ -U_1 R_{12} R_{22}^{-1} & I \end{bmatrix} U(\lambda E - Q)X$$

transforms $\lambda E - Q$ into the pencil in (15), see also again equation (17). \square

So far we have not seen any restrictions on the left minimal indices of pencils $\lambda E - Q$ satisfying $E^*Q = Q^*E$. In fact, the left minimal indices may be arbitrary, even under the additional assumption $E^*Q \geq 0$ as long as the corresponding singular blocks are of suitable dimensions to be part of the block $\lambda E_{22} - Q_{22}$ in the condensed form of Theorem 15. To show this, it would be helpful to obtain a canonical form for pencils $\lambda E - Q$ satisfying $Q^*E = E^*Q \geq 0$. The problem in obtaining such a form lies in the fact that left-multiplication with a non-unitary invertible matrix S does not preserve the property of Q^*E being Hermitian, let alone being Hermitian positive semi-definite. In particular, if $\lambda E_0 - Q_0 = \mathcal{L}_\eta^\top$ is a Kronecker block associated with a left minimal index $\eta > 0$, then it is straightforward to check that $Q_0^*E_0$ is not a Hermitian matrix. We will therefore investigate the structure of matrices that make the pencil $\lambda E_0 - Q_0$ in Kronecker canonical form Hermitian by left-multiplication.

Proposition 16 Let $S \in \mathbb{F}^{l,k}$, and let $E_1 = SE_0$, $Q_1 = SQ_0$, where

$$\lambda E_0 - Q_0 = \mathcal{L}_{k-1}^\top = \lambda \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & \ddots & 1 & \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{k,k-1}[\lambda] \quad (18)$$

Then the following statements hold.

- (i) We have $E_1^*Q_1 = Q_1^*E_1$ if and only if $H = S^*S$ is a positive definite real Hankel matrix, i.e.,

$$H = \begin{bmatrix} a_0 & a_1 & \dots & a_{k-1} \\ a_1 & \ddots & \ddots & a_k \\ \vdots & \ddots & \ddots & \vdots \\ a_{k-1} & a_k & \dots & a_{2k-1} \end{bmatrix} \in \mathbb{R}^{k,k}, \quad H > 0. \quad (19)$$

- (ii) We have $E_1^*Q_1 = Q_1^*E_1 \geq 0$ (or, equivalently, $E_1^*Q_1 = Q_1^*E_1 > 0$) if and only if $H = S^*S$ is a positive definite real Hankel matrix such that its submatrix \tilde{H} obtained by deleting the first row and last column is also positive definite, i.e.,

$$\tilde{H} = \begin{bmatrix} a_1 & a_2 & \dots & a_{k-1} \\ a_2 & \ddots & \ddots & a_k \\ \vdots & \ddots & \ddots & \vdots \\ a_{k-1} & a_k & \dots & a_{2k-2} \end{bmatrix} > 0. \quad (20)$$

Proof. (i) Suppose that $H = S^*S$ is real and has the form given in (19). Then

$$E_1^*Q_1 = E_0^*HQ_0 = \tilde{H} = Q_0^*HE_0 = Q_1^*E_1, \quad (21)$$

where \tilde{H} has the form as in (20) (but need not be positive definite). Since this matrix is a real Hankel matrix and thus Hermitian, it follows that $E_1^*Q_1$ is Hermitian.

Conversely, assume that $E_1^*Q_1 = Q_1^*E_1$. Then it follows by comparison of elements in the equation $E_0^*HQ_0 = Q_0^*HE_0$ that H is a Hermitian Hankel matrix, or equivalently, a real Hankel matrix.

(ii) follows immediately from (21). Note that for $E_1^*Q_1$ positive semi-definiteness is equivalent to positive definiteness, since $E_1^*Q_1$ has full rank. \square

Using these preparations, we now obtain the following result showing that a pencil $\lambda E - Q$ satisfying $Q^*E = E^*Q \geq 0$ may have arbitrary left minimal indices as long as they fit the dimension restrictions.

Corollary 17 Let $n, m \in \mathbb{N}$, and let q, η_1, \dots, η_q be arbitrary non-negative integers satisfying

$$n - q \leq m, \quad \eta_1 + \dots + \eta_q \leq n - q.$$

Then there exists a pencil $\lambda E - Q \in \mathbb{F}^{m,n}[\lambda]$ satisfying $Q^*E = E^*Q \geq 0$ with precisely q left minimal indices η_1, \dots, η_q .

Proof. First observe that for each $k \geq 1$ there exists a positive definite Hankel matrix H with $\tilde{H} > 0$, where \tilde{H} is defined as in (20), when H is as in (19). Indeed, let $\xi_1 > \dots > \xi_k > 0$ and let

$$V = \begin{bmatrix} \xi_1 & \dots & \xi_1^n \\ \vdots & & \vdots \\ \xi_n & \dots & \xi_n^n \end{bmatrix} \in \mathbb{R}^{k,k}$$

be a Vandermonde matrix. Observe that

$$H = V^\top V = \left[\sum_{l=1}^n \xi_l^{i+j} \right]_{i,j=1}^k \quad (22)$$

is a positive definite Hankel matrix. Furthermore, $\tilde{H} > 0$ due to

$$x^* \tilde{H} x = \sum_{i,j=1}^{k-1} x_i \bar{x}_j \sum_{l=1}^k \xi_l^{i+j+1} = \sum_{l=1}^k \xi_l \left| \sum_{i=1}^{k-1} x_i \xi_l^i \right|^2 > 0, \quad \text{for } x \in \mathbb{F}^{k-1} \setminus \{0\}.$$

Taking $S = H^{1/2}$ (or S to be the Cholesky factor of H) we get by Proposition 16 (ii) a pencil $\lambda E_1 - Q_1$ satisfying $E_1^* Q_1 \geq 0$ with left minimal index $k - 1$. To obtain a pencil with the desired minimal indices it is enough to take the direct sum of pencils constructed as above for the minimal indices η_1, \dots, η_q and if necessary adding to the direct sum an appropriate pencil having no left minimal indices. (It is possible to find such a pencil, because its size is $(n - \eta - q) \times (m - \eta)$ with $\eta = \eta_1 + \dots + \eta_q$ which means that the number of rows does not exceed the number of columns by the assumption $n - q \leq m$.) \square

The natural equivalence relation for the set of pencils $\lambda E - Q$ satisfying $E^* Q = Q^* E$ is $(\lambda E_1 - Q_1) \sim (\lambda E_2 - Q_2)$ if and only if there exist unitary U and invertible X such that $U(\lambda E_1 - Q_1)X = \lambda E_2 - Q_2$. So far we have established the canonical form and the complete set of invariants in the regular case (Proposition 9) and in the singular case if $EQ^* = QE^*$ is additionally assumed (Proposition 11). We have also established all the possible Kronecker canonical forms in the general case (Theorem 15 and Corollary 17). However, in establishing the complete set of invariants of the equivalence relation \sim in the general case, we have met the following difficulty.

Remark 18 For any $k > 1$ there is a continuum of nonequivalent (in the sense of the relation \sim defined above) linear pencils $\lambda E - Q \in \mathbb{F}^{k,k-1}[\lambda]$ with $E^* Q = Q^* E \geq 0$ having precisely one left minimal index $k - 1$. This can be seen in the following way. Suppose that two pencils $\lambda S_1 E_0 - Q_0$ and $\lambda S_2 E_0 - Q_0$ as in Proposition 16 (ii) are equivalent, i.e., we have $U^*(\lambda S_1 E_0 - Q_0)X = \lambda S_2 E_0 - Q_0$ with unitary U and invertible X . Then by Proposition 16 (ii) the matrix $H_i := S_i^* S_i \in \mathbb{R}^{k,k}$ ($i = 1, 2$) is a Hankel matrix. The relation \sim implies that

$$X^* E_0^* H_1 E_0 X = E_0^* H_2 E_0, \quad X^* Q_0^* H_1 Q_0 X = Q_0^* H_2 Q_0$$

and thus the Hermitian pencils $\lambda E_0^* H_1 E_0 - Q_0^* H_1 Q_0$ and $\lambda E_0^* H_2 E_0 - Q_0^* H_2 Q_0$ are congruent.

Now let $\xi_1 > \dots > \xi_n > 0$ and let H be the positive definite Hankel matrix defined as in (22). Observe that there clearly is a continuum of mutually non-congruent pencils of the form

$$\lambda E_0^* H E_0 - Q_0^* H Q_0 = \lambda E_0^* \left[\sum_{l=1}^n \xi_l^{i+j} \right]_{i,j=1}^{k-1} E_0 - Q_0^* \left[\sum_{l=1}^n \xi_l^{i+j} \right]_{i,j=2}^k Q_0.$$

One can see this easily, e.g., by investigating the ξ_1 dependence of eigenvalues, keeping ξ_2, \dots, ξ_n fixed. But this immediately implies that under the equivalence relation \sim there is a continuum of nonequivalent pencils $\lambda E - Q$ with canonical form $\lambda E_0 - Q_0$ that satisfy $E^*Q = Q^*E \geq 0$.

In this section, we have considered the possible canonical forms for pencils $\lambda E - Q$ satisfying one or more of the conditions (3)–(12). In the next section, we will discuss the influence of these conditions on the pencil $P(\lambda)$ from (7).

4 Properties of the pencil $P(\lambda) = \lambda E - (J - R)Q$

In this section, we investigate the properties of the pencil $P(\lambda) = \lambda E - (J - R)Q$ from (7). For brevity, we will sometimes use the notation $L := J - R$ and thus write our pencil in the form

$$P(\lambda) = \lambda E - LQ, \quad E, Q \in \mathbb{F}^{n,m}, \quad L \in \mathbb{F}^{m,m}, \quad L + L^* \leq 0. \quad (23)$$

Note that the decomposition $L = J - R$, with $J = -J^*$, $R = R^*$ is unique. Hence, the condition $L + L^* \leq 0$ is equivalent to the condition $R \geq 0$.

Proposition 19 *Suppose that the pencil $P(\lambda)$ in (23) is regular with $E^*Q = Q^*E$. Then the pencil $\lambda E - Q$ is regular as well.*

Proof. Suppose that $\lambda E - Q$ is singular. Then by Corollary 14 the matrices E and Q have a common right nullspace which implies that E and LQ have a common right nullspace as well, contradicting the hypothesis that $P(\lambda)$ is regular. \square

In the regular case and with $Q > 0$, it is well-known, see e.g., [12, 21], that all eigenvalues of the pencil $P(\lambda)$ in (23) are in the closed left half complex plane and those on the imaginary axis are semisimple. We will now investigate if these properties remain valid under the weaker condition $E^*Q = Q^*E \geq 0$. Clearly, without this condition, we may have eigenvalues in the right half complex plane as is shown by the scalar example

$$E = \begin{bmatrix} -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 \end{bmatrix}, \quad L = J - R = \begin{bmatrix} 1 \end{bmatrix},$$

where the corresponding pencil $\lambda E - LQ = 1 - \lambda$ has the eigenvalue $\lambda_0 = 1$. But even if the condition $E^*Q = Q^*E \geq 0$ is satisfied we may have eigenvalues in the right half plane if the pencil $\lambda E - Q$ happens to be singular.

Example 20 Consider the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $a > 0$. Then we have $E^*Q = Q^*E = 0 \geq 0$ and $L + L^* = 0 \leq 0$, but the pencil

$$P(\lambda) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

has the eigenvalue $a > 0$ in the right half plane.

We highlight that in the previous example, the pencil $\lambda E - Q$ had a left minimal index larger than zero. (We recall that all right minimal indices are automatically zero by Theorem 15, but the left minimal indices may be arbitrary up to dimensional restrictions by Corollary 17.) In fact, it turns out that under the additional hypothesis that all left minimal indices of $\lambda E - Q$ are zero, the pencil $P(\lambda)$ has all eigenvalues in the closed left half complex plane, and those on the imaginary axis are semisimple with the possible exception of non semi-simplicity of the eigenvalues zero and infinity. With these preparations we are now able to formulate and prove our main result that does not only give information on the finite eigenvalues, but also restricts the possible index and minimal indices of a pencil as in (23).

Theorem 21 *Let $E, Q \in \mathbb{F}^{n,m}$ satisfy $E^*Q = Q^*E \geq 0$ and let all left minimal indices of $\lambda E - Q$ be equal to zero (if there are any). Furthermore, let $L \in \mathbb{F}^{m,m}$ be such that $R := -\frac{1}{2}(L + L^*) \geq 0$. Then the following statements hold for the pencil $P(\lambda) = \lambda E - LQ$.*

- (i) *If $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$ then $\operatorname{Re}(\lambda_0) \leq 0$.*
- (ii) *If $\omega \in \mathbb{R} \setminus \{0\}$ and $\lambda_0 = i\omega$ is an eigenvalue of $P(\lambda)$, then λ_0 is semisimple. Moreover, if the columns of $V \in \mathbb{C}^{m,k}$ form a basis of a regular deflating subspace of $P(\lambda)$ associated with λ_0 , then $RQV = 0$.*
- (iii) *The index of $P(\lambda)$ is at most two.*
- (iv) *All right minimal indices of $P(\lambda)$ are at most one (if there are any).*
- (v) *If in addition $\lambda E - Q$ is regular, then all left minimal indices of $P(\lambda)$ are zero (if there are any).*

Before we give the proof of Theorem 21 in the next section, we discuss a few consequences and examples.

Remark 22 Part (i) and (ii) of Theorem 21 generalize [21, Lemma 3.1] in which these results were proved for the case $E = I$ and $Q > 0$, and they also generalize [12, Theorem 2] where the regular case was covered under the additional assumption that for any eigenvector x associated with a nonzero eigenvalue the condition $Q^*Ex \neq 0$ is satisfied, but no necessary or sufficient conditions were given when this is the case. Theorem 21 now states that this extra assumption is not needed in the regular case (in fact, the proof of Theorem 21 shows that this condition is automatically satisfied), and it extends the assertion to the singular case under the additional assumption that the left minimal indices of the pencil $\lambda E - Q$ are all zero.

Example 23 We highlight that Theorem 21 gives no information about the semi-simplicity of the eigenvalue zero of the pencil $P(\lambda)$. Indeed, consider the matrices

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then with $E = C$ and $Q = D$ the pencil

$$P(\lambda) = \lambda E - (J - R)Q = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies $Q^*E = D \geq 0$ and has the eigenvalue zero with algebraic multiplicity two, but geometric multiplicity one. (We will see in Section 6 that sizes of Jordan blocks associated with the eigenvalue zero are also not restricted to be at most two, even under the additional hypothesis that $\lambda E - Q$ is regular.)

Example 24 With the matrices from Example 23, and $E = D$ and $Q = C$ the pencil

$$P(\lambda) = \lambda E - (J - R)Q = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

satisfies again $Q^*E = D \geq 0$ and has the eigenvalue ∞ with algebraic multiplicity two, but geometric multiplicity one. This shows that the case that the index of the pencil is two may indeed occur even when $\lambda E - Q$ is regular.

Example 25 The case that the pencil $P(\lambda)$ as in Theorem 21 has a right minimal index equal to one may indeed occur even in the case when $\lambda E - Q$ is regular. For an example, consider the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad LQ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $\lambda E - Q$ is regular, $E^*Q = Q^*E = 0$, and $L + L^* \leq 0$. The pencil $P(\lambda) = \lambda E - LQ$ has one right minimal index equal to one and one left minimal index equal to zero.

Example 26 If the pencil $\lambda E - Q$ in Theorem 21 is not regular, then not much can be said about the left minimal indices of the pencil $P(\lambda)$. In fact, they can be arbitrarily large as the following example shows. Let

$$E = Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n,n}, \quad R = 0 \in \mathbb{R}^{n,n}$$

and $L = J - R = J$. Then $Q^*E = E^*Q = Q^2 \geq 0$ and the pencil $\lambda E - Q$ has one left and right minimal index both being equal to zero. Furthermore, we obtain

$$P(\lambda) = \lambda E - LQ = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ -1 & \lambda & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \lambda & 0 \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix}.$$

Obviously this pencil has one right minimal index which is zero. Since its normal rank is $n - 1$, it has exactly one left minimal index. Observing that $\text{rank } E = \text{rank } P(\lambda_0) = n - 1$ for all $\lambda_0 \in \mathbb{C}$, it follows that $P(\lambda)$ does not have finite or infinite eigenvalues, and hence the left minimal index must be $n - 1$ which is the largest possible size for a left minimal index of an $n \times n$ singular matrix pencil.

As an application of Theorem 21, we obtain the following statement on the Kronecker structure of quadratic matrix polynomials that correspond to damped mechanical systems.

Corollary 27 *Let $S(\lambda) := \lambda^2 M + \lambda D + K \in \mathbb{F}^{n,n}[\lambda]$ be a quadratic matrix polynomial such that M, D, K are all Hermitian and positive semidefinite matrices. Then the following statements hold.*

- (i) *All eigenvalues of $S(\lambda)$ are in the closed left half complex plane and all finite nonzero eigenvalues on the imaginary axis are semisimple.*
- (ii) *The possible length of Jordan chains of $S(\lambda)$ associated with either with the eigenvalue ∞ or with the eigenvalue zero is at most two.*
- (iii) *All left and all right minimal indices of $S(\lambda)$ are zero (if there are any).*

Proof. The proof of (i) and the statement in (ii) on the length of the Jordan chains associated with the eigenvalue ∞ follows immediately from the fact that the companion linearization

$$P(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix} = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} - \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}$$

satisfies the hypothesis of Theorem 21 with

$$E = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad R = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}.$$

The remaining statement of (ii) then follows by applying the already proved part of (ii) to the reversal polynomial $\lambda^2 K + \lambda D + M$ of $S(\lambda)$.

To see (iii) observe that by Theorem 21 the left minimal indices of $P(\lambda)$ are all zero and the right minimal indices of $P(\lambda)$ are at most one. By [5, Theorem 5.10] the left minimal indices of $S(\lambda)$ coincide with those of $P(\lambda)$, and if $\varepsilon_1, \dots, \varepsilon_k$ are the right minimal indices of $S(\lambda)$, then $\varepsilon_1 + 1, \dots, \varepsilon_k + 1$ are the right minimal indices of $P(\lambda)$. This implies that all minimal indices of $S(\lambda)$ are zero (if there are any). \square

As another application of Theorem 21 we obtain a result for rectangular matrix pencils which in the case $E = I$, $Q > 0$ coincides with the well known Lyapunov stability condition [19] and can thus be seen as a generalization of this stability criterion. In the following Q^\dagger denotes the Moore-Penrose generalized inverse of a matrix Q , see, e.g., [14].

Corollary 28 *Let $A, E \in \mathbb{F}^{n,m}$ and assume that there exists $Q \in \mathbb{F}^{n,m}$ such that all minimal indices of the pencil $\lambda E - Q$ are zero (if there are any) and*

$$E^* Q \geq 0, \quad A Q^\dagger + Q^{\dagger*} A^* \leq 0, \quad \ker Q \subseteq \ker A. \quad (24)$$

Then, the statements (i)–(v) of Theorem 21 hold for the pencil $P(\lambda) = \lambda E - A \in \mathbb{F}^{n,m}[\lambda]$.

Proof. We set $L := A Q^\dagger$. The first and second inequality of (24) ensure that the assumptions of Theorem 21 are satisfied for this E, L, Q . Furthermore, as $\ker Q \subseteq \ker A$ and $Q^\dagger Q$ is the orthogonal projection onto $\text{im } Q^* \supset \text{im } A^*$, we have $L Q = A Q^\dagger Q = A$. Hence, the statements (i)–(v) of Theorem 21 hold for the pencil $P(\lambda)$. \square

In the case that $n = m$ and Q is invertible, the condition $A Q^\dagger + Q^{\dagger*} A^* \leq 0$ can be reformulated as $Q^* A + A^* Q \leq 0$, and thus we obtain the following simplified statement.

Corollary 29 *Let $A, E \in \mathbb{F}^{n,n}$ and assume that there exist an invertible $Q \in \mathbb{F}^{n,n}$ such that $E^* Q \geq 0$ and $Q^* A + A^* Q \leq 0$. Then, the statements (i)–(v) of Theorem 21 hold for the pencil $P(\lambda) = \lambda E - A \in \mathbb{F}^{n,n}[\lambda]$ with the assumption of regularity of $\lambda E - Q$ in (v) being automatically satisfied.*

Other generalizations of Lyapunov's theorem with focus on asymptotic stability have been obtained in [17, 28, 29] for the case of square pencils and in [16] for the rectangular case. Corollaries 28 and 29 give a generalization to the concept of *stability* in the sense of the conditions (i)–(v) in Theorem 21.

5 Proof of Theorem 21

In this section we will present a proof of our main result. To show the statements (iii)–(v) we will need the following lemma.

Lemma 30 *Let $E, A \in \mathbb{F}^{n,m}$ and $k \geq 2$. Assume that one of the following conditions holds*

- (I) *the pencil $\lambda E - A$ has index k ;*
- (II) *the pencil $\lambda E - A$ has a right minimal index $k - 1$.*

Then there exists a vector $x_1 \in \mathbb{F}^m$ orthogonal to the common nullspace of E and A such that the following two statements hold:

- (a) *There exist vectors $x_2, \dots, x_k \in \mathbb{F}^m$ such that $Ex_1 = 0, Ex_2 = Ax_1, \dots, Ex_k = Ax_{k-1}$, where in addition we have $Ax_k \neq 0$ in case (I) and $Ax_k = 0$ in case (II).*
- (b) *For any choice of vectors $x_2, \dots, x_k \in \mathbb{F}^m$ with $Ex_1 = 0, Ex_2 = Ax_1, \dots, Ex_k = Ax_{k-1}$ we have $Ax_1, \dots, Ax_{k-1} \neq 0$.*

Proof. First, assume that $\lambda E - A$ is in Kronecker canonical form, where without loss of generality the block \mathcal{N}_k associated with a block of size k of the eigenvalue ∞ , or the block associated with the right minimal index $k - 1$, respectively, comes first, i.e., we have

$$\lambda E - A = \begin{bmatrix} \lambda E_{11} - A_{11} & 0 \\ 0 & \lambda E_{22} - A_{22} \end{bmatrix},$$

where $\lambda E_{11} - A_{11} = \mathcal{N}_k$, or

$$\lambda E_{11} - A_{11} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}. \quad (25)$$

Here, in (25) we did on purpose not use the block \mathcal{L}_{k-1} as the representation of a block with right minimal index $k - 1$, but its reversal which is well-known to be equivalent to \mathcal{L}_{k-1} . In this way, we can treat the proofs for both cases simultaneously, because the block in (25) consists exactly of the first $k - 1$ rows of the block \mathcal{N}_k .

In the following, let $e_i^{(p)}$ denote the i -th standard basis vector of \mathbb{F}^p . Then $x_1 := e_1^{(m)}$ is orthogonal to the common nullspace of E and A , and it is straightforward to check that the vectors $x_2 = e_2^{(m)}, \dots, x_k = e_k^{(m)}$ satisfy (a). On the other hand, if $x_2, \dots, x_k \in \mathbb{F}^m$ are chosen such that they satisfy $Ex_1 = 0, Ex_2 = Ax_1, \dots, Ex_k = Ax_{k-1}$ then a straightforward computation shows that x_2, \dots, x_k must have the form

$$x_2 = \begin{bmatrix} e_2^{(k)} + \alpha_1 e_1^{(k)} \\ x_{2,2} \end{bmatrix}, \dots, x_k = \begin{bmatrix} e_k^{(k)} + \alpha_1 e_{k-1}^{(k)} + \dots + \alpha_{k-1} e_1^{(k)} \\ x_{k,2} \end{bmatrix}$$

for appropriate $\alpha_1, \dots, \alpha_{k-1} \in \mathbb{F}$ and $x_{22}, \dots, x_{k,2} \in \mathbb{F}^{m-k}$. Then we obtain

$$Ex_j = Ax_{j-1} = \begin{bmatrix} e_{j-1}^{(k-1)} + \alpha_1 e_{j-2}^{(k-1)} + \dots + \alpha_{j-2} e_1^{(k-1)} \\ A_{22} x_{j-1,2} \end{bmatrix} \neq 0$$

for $j = 2, \dots, k$ which shows (b). This also proves (a) and (b) for the general case with the possible exception of the orthogonality condition on x_1 . However, note that any vector $y \in \mathbb{F}^m$ satisfying $Ey = 0 = Ay$ can be added to x_1, x_2, \dots, x_k without changing any of the identities in (a) and (b). This shows that x_1 satisfying (a) and (b) can be chosen to be orthogonal to the common nullspace of E and A . \square

Proof (of Theorem 21): Since $Q^*E \geq 0$, we may assume without loss of generality by Theorem 15 that E and Q have the form

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad (26)$$

respectively, where E_{11}, Q_{11} are real and diagonal (which implies, in particular, that $Q^* = Q$) and satisfy $E_{11}^2 + Q_{11}^2 = I_{n_1}$. The fact that E_{22} and Q_{22} (and thus also E_{12} and Q_{12}) in (14) are zero follows from the assumption that $\lambda E - Q$ has only left minimal indices equal to zero (if any). To show the items (i)–(v) for the pencil $P(\lambda) = \lambda E - LQ$ we will again frequently use the decomposition $L = J - R$, where $R := \frac{1}{2}(L + L^*) \geq 0$ and $J := \frac{1}{2}(L - L^*)$.

(i) Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $P(\lambda)$ and let $v \neq 0$ be a regular eigenvector associated with λ_0 . Then we have $\lambda_0 Ev = LQv = (J - R)Qv$ and thus

$$\lambda_0 v^* Q^* E v = v^* Q^* J Q v - v^* Q^* R Q v.$$

Considering the real parts of both sides of this equation, we obtain

$$\operatorname{Re}(\lambda_0) \cdot v^* Q^* E v = -v^* Q^* R Q v,$$

where we used the fact that Q^*E and R are Hermitian and J is skew-Hermitian. If $Q^*Ev = 0$, then the special structure of E and Q implies that $v = y_1 + y_2$, where $Ey_1 = 0$ and $Qy_2 = 0$ and thus v can be expressed as a linear combination of vectors from the kernels of E and LQ , respectively which by Lemma 8 leads to a contradiction. Hence, we have $Q^*Ev \neq 0$, and since Q^*E is positive semidefinite, we obtain $v^*Q^*Ev > 0$ which finally implies

$$\operatorname{Re}(\lambda_0) = -\frac{v^* Q^* R Q v}{v^* Q^* E v} \leq 0.$$

For (ii) we first prove the 'moreover' part. For this, let the columns of $V \in \mathbb{C}^{m,k}$ form a basis of a regular deflating subspace of $P(\lambda)$ associated with the eigenvalue $\lambda_0 = i\omega$. Then we have to show $RQV = 0$.

By the definition of regular deflating subspaces and from the Kronecker canonical form it follows that there exists a matrix $W \in \mathbb{C}^{m,k}$ with full column rank such that

$$EV = W \quad \text{and} \quad (J - R)QV = WT, \quad (27)$$

where $T \in \mathbb{C}^{k,k}$ only has the eigenvalue $i\omega$. Without loss of generality we may assume that $T = i\omega I_k + N$ is in Jordan canonical form (JCF), where N is strictly upper triangular. Otherwise we choose P such that $P^{-1}TP$ is in JCF and set $\widetilde{W} = WP$ and $\widetilde{V} = VP$.

By the second identity in (27) we have $V^*Q^*(J - R)QV = V^*Q^*WT$, and taking the Hermitian part of both sides we obtain

$$0 \geq -2V^*Q^*RQV = V^*Q^*WT + T^*W^*Q^*V. \quad (28)$$

Since R is positive semidefinite, it remains to show that $V^*Q^*RQV = 0$, because then we also have $RQV = 0$. For this, we first note that we have

$$V^*Q^*W = W^*Q^*V > 0. \quad (29)$$

Indeed, it follows from the first identity in (27) that $V^*Q^*W = V^*Q^*EV \geq 0$. If there exists $x \neq 0$ such that $Q^*EVx = 0$, then with $y = Vx$ one has $QEy = 0$. Due to the specific form of Q and E this implies that $y = y_1 + y_2$ with $Ey_1 = 0$ and $Qy_2 = 0$. Hence, y is a vector belonging to a linear span of vectors from the kernels of E and LQ which, by Lemma 8, contradicts the fact that the columns of V span a regular deflating subspace associated with $\lambda_0 \neq 0$ and finishes the proof of (29).

Now let M be the inverse of the Hermitian positive definite square root of the Hermitian positive definite matrix $V^*Q^*W = W^*Q^*V$. Then it follows from (28) that the matrix

$$M(V^*Q^*WT + T^*V^*QW)M = M^{-1}TM + MT^*M^{-1}$$

is Hermitian negative semidefinite, because it is congruent to the right hand side of (28). Moreover, we obtain

$$\begin{aligned} \text{trace}(M^{-1}TM + MT^*M^{-1}) &= \text{trace}(M^{-1}TM) + \text{trace}(MT^*M^{-1}) \\ &= \text{trace}(T + T^*) = \text{trace}(N + N^*) = 0, \end{aligned}$$

because N has a zero diagonal. But this implies that $M^{-1}TM + MT^*M^{-1} = 0$ and hence by (28) also $-2V^*QRQV = 0$, which finishes the proof of the 'moreover' part.

Next, we will show that $i\omega \neq 0$ is a semisimple eigenvalue. For this, it remains to show that the matrix $T = i\omega I_k + N$ in (27) is diagonal, i.e., $N = 0$. Observe that with $RQV = 0$ the identities in (27) simplify to $EV = W$ and $JQV = WT$ which implies that

$$V^*Q^*JQV = V^*Q^*WT.$$

Using again the inverse M of the Hermitian positive definite square root of V^*Q^*WT , we obtain that

$$M^{-1}TM = MV^*Q^*WTM = MV^*Q^*JQVM,$$

which implies that T is similar to a matrix which is congruent to J , i.e., T is similar to a skew-Hermitian matrix. This, however, immediately implies that we must have $N = 0$. Thus, $i\omega$ is a semisimple eigenvalue of $\lambda E - (J - R)Q$ and assertion (ii) is proved.

For the remaining part of the proof, we will use a slight variation of (5). By scaling the nonzero parts of E and Q in (5) if necessary, we may assume without loss of generality that E and Q take the forms

$$E = \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_\ell & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (30)$$

where $Q_1 \in \mathbb{F}^{k,k}$ is positive definite and both E and Q are partitioned conformably with the partition (k, ℓ, q, p) of n , where $p, q \geq 0$.

To prove (iii), assume that the index ν of $P(\lambda)$ exceeds two. Then by Lemma 30 there exists a vector x_1 orthogonal to the common kernel of E and LQ that satisfies the conditions (a) and (b) of Lemma 30. In particular, by (a) there exists vectors x_2, \dots, x_ν such that

$$Ex_1 = 0, Ex_2 = LQx_1 \neq 0, Ex_3 = LQx_2 \neq 0, \dots, Ex_\nu = LQx_{\nu-1} \neq 0, LQx_\nu \neq 0. \quad (31)$$

Since x_1 is in the kernel of E , it must be of the form $x_1^\top = [0 \ 0 \ x_{13}^\top \ 0]^\top$, when partitioned conformably with respect to the partition (k, ℓ, q, p) of n . Then, we can apply a simultaneous similarity transformation to E, L , and Q with a unitary transformation matrix of the form $U = I_k \oplus I_\ell \oplus U_q \oplus I_p$, where $U_m x_{13} = \alpha e_q$ for some $\alpha \in \mathbb{R}$, e.g., we can take U_q to be an appropriate Householder matrix. Note that the simultaneous similarity transformation with U preserves all the properties of E, Q , and L . Finally, x_1 can be appropriately scaled such that $\alpha = 1$.

Next, let us extend the partitioning of E, Q, L, J , and R by splitting the third block row and column in two, i.e., the new partitioning is according to the partition $(k, \ell, q-1, 1, p)$ of n . Then we have

$$E = \text{diag}(I_k, I_\ell, 0, 0, 0), \quad Q = \text{diag}(Q_1, 0, I_{q-1}, 1, 0)$$

and we can express $L = J - R$ as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & J_{14} - R_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & J_{24} - R_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & J_{34} - R_{34} & L_{35} \\ -J_{14}^* - R_{14}^* & L_{42} & L_{43} & J_{44} - R_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & -J_{45}^* - R_{45}^* & L_{55} \end{bmatrix}.$$

Observe that

$$LQx_1 = \begin{bmatrix} J_{14} - R_{14} \\ J_{24} - R_{24} \\ J_{34} - R_{34} \\ J_{44} - R_{44} \\ -J_{45}^* - R_{45}^* \end{bmatrix} \in \mathbb{F}^n$$

has to be in the range of E in order to satisfy $Ex_2 = LQx_1$ for some x_2 . This implies the identities

$$J_{34} = R_{34}, \quad J_{44} = R_{44}, \quad -J_{45}^* = R_{45}^*. \quad (32)$$

From this we obtain $J_{44} = R_{44} = 0$, because R_{44} is a real scalar and J_{44} is purely imaginary. But then it follows from the positive-semidefiniteness of R that $R_{14} = 0, R_{24} = 0, R_{34} = 0$, and $R_{45} = 0$ which in turn implies that $J_{34} = 0$ and $J_{45} = 0$ by (32). Thus, using $L_{43} = -J_{34}^* + R_{34}^* = 0$ we find that LQ and LQx_1 take the simplified form

$$LQ = \begin{bmatrix} L_{11}Q_1 & 0 & L_{13} & J_{14} & 0 \\ L_{21}Q_1 & 0 & L_{23} & J_{24} & 0 \\ L_{31}Q_1 & 0 & L_{33} & 0 & 0 \\ -J_{14}^*Q_1 & 0 & 0 & 0 & 0 \\ L_{51}Q_1 & 0 & L_{53} & 0 & 0 \end{bmatrix} \quad \text{and} \quad LQx_1 = \begin{bmatrix} J_{14} \\ J_{24} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where we used the fact that the second diagonal block in Q is zero. At this point, we see that x_2 has to be of the form $x_2^\top = [J_{14}^\top \ J_{24}^\top \ x_{23}^\top \ x_{24}^\top \ x_{25}^\top]^\top$ in order to satisfy

$$\begin{bmatrix} J_{14} \\ J_{24} \\ 0 \\ 0 \\ 0 \end{bmatrix} = LQx_1 = Ex_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_2.$$

Since $Ex_3 = LQx_2$, the vector

$$LQx_2 = \begin{bmatrix} * \\ * \\ * \\ -J_{14}^* Q_1 J_{14} \\ * \end{bmatrix}$$

has to be in the range of E , which implies that $J_{14}^* Q_1 J_{14} = 0$, and hence $J_{14} = 0$ as Q_1 is positive definite. But then the vector $\tilde{x}_2 := [0 \ J_{24}^\top \ 0 \ 0 \ 0]^\top$ satisfies the identities

$$Ex_1 = 0, \quad E\tilde{x}_2 = LQx_1, \quad LQ\tilde{x}_2 = 0,$$

which is a contradiction to part (b) of Lemma 30

To prove (iv) assume that $P(\lambda)$ has a right minimal index $\varepsilon \geq 2$. Then, by Lemma 30, there exists a vector x_1 , orthogonal to the common nullspace of E and LQ , that satisfies the conditions (a) and (b) of Lemma 30. In particular, by (a) there exists vectors satisfying

$$Ex_1 = 0, \quad Ex_2 = LQx_1 \neq 0, \quad Ex_3 = LQx_2 \neq 0, \quad \dots, \quad Ex_{\varepsilon+1} = LQx_\varepsilon \neq 0, \quad LQx_{\varepsilon+1} = 0.$$

Since only x_1, x_2, x_3 have been used in (iii) to obtain a contradiction, we can follow exactly the lines of (iii) to obtain again a contradiction.

(v) Since $\lambda E - Q$ is regular, we have $p = 0$ in the partitioning (30). Assume that $P(\lambda)$ has a left minimal index of size $\eta > 0$. Then by applying Lemma 30 to the pencil $P(\lambda)^* = \lambda E^* - (LQ)^*$, there exists a vector x_1 , orthogonal to the common nullspace of E and LQ , such that the conditions (a) and (b) of Lemma 30 are satisfied. In particular, (a) implies the existence of vectors $x_2, \dots, x_{\eta+1}$ such that

$$x_1^* E = 0, \quad x_2^* E = x_1^* LQ \neq 0, \quad \dots, \quad x_{\eta+1}^* E = x_\eta^* LQ \neq 0, \quad x_{\eta+1}^* LQ = 0.$$

By an argument analogous to the one in the proof of (iii), we may assume without loss of generality that $x_1 = e_n$ is the last standard basis vector of \mathbb{F}^n . Next, partitioning E , Q , J , and R conformably with respect to the partition $(k, \ell, q-1, 1)$ of n we get

$$E = \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_\ell & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ -J_{12}^* & J_{22} & J_{23} & J_{24} \\ -J_{13}^* & -J_{23}^* & J_{33} & J_{34} \\ -J_{14}^* & -J_{24}^* & -J_{34}^* & J_{44} \end{bmatrix}, \quad R = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{12}^* & R_{22} & R_{23} & R_{24} \\ R_{13}^* & R_{23}^* & R_{33} & R_{34} \\ R_{14}^* & R_{24}^* & R_{34}^* & R_{44} \end{bmatrix}.$$

Then, similarly as in the proof of (iii), we obtain the identity

$$x_2^*E = x_1^*LQ = \begin{bmatrix} (-J_{14}^* - R_{14}^*)Q_1 & 0 & -J_{34}^* - R_{34}^* & J_{44} - R_{44} \end{bmatrix}$$

which implies that $J_{44} = R_{44} = 0$, $J_{34} = R_{34} = 0$, $R_{24} = 0$, and $R_{14} = 0$. Hence, the matrix LQ takes the simplified form

$$LQ = (J - R)Q = \begin{bmatrix} (J_{11} - R_{11})Q_1 & 0 & J_{13} - R_{13} & J_{14} \\ (-J_{12}^* - R_{12}^*)Q_1 & 0 & J_{23} - R_{23} & J_{24} \\ (-J_{13}^* - R_{13}^*)Q_1 & 0 & J_{33} - R_{33} & 0 \\ -J_{14}^*Q_1 & 0 & 0 & 0 \end{bmatrix}.$$

This implies that the vector x_2 must be of the form $x_2^* = \begin{bmatrix} -J_{14}^*Q_1 & 0 & x_{23}^* & x_{24}^* \end{bmatrix}$ for some $x_{23} \in \mathbb{F}^{q-1}$, $x_{24} \in \mathbb{F}$ in order to satisfy the identity

$$x_2^*E = x_1^*LQ = \begin{bmatrix} -J_{14}^*Q_1 & 0 & 0 & 0 \end{bmatrix}. \quad (33)$$

Let us assume first that $\eta > 1$. Then the vector

$$x_2^*LQ = \begin{bmatrix} * & * & * & -J_{14}^*Q_1J_{14} \end{bmatrix}$$

has to satisfy the identity $x_3^*E = x_2^*LQ$, which implies $J_{14}^*Q_1J_{14} = 0$ and thus $J_{14} = 0$ as Q_1 is positive definite. But this together with (33) implies that $x_1^*LQ = 0$ which contradicts the fact that x_1 is orthogonal to the common left nullspace of E and LQ .

In the second case that $\eta = 1$ we have a chain $x_1^*E = 0$, $x_2^*E = x_1^*LQ \neq 0$ and $x_2^*LQ = 0$, which again leads to $J_{14} = 0$ and $x_1^*LQ = 0$, a contradiction. Thus, $P(\lambda)$ cannot have a left minimal index of size larger than zero which finishes the proof. \square

6 Removing higher order Jordan blocks of the eigenvalue 0

In this section we will concentrate on the case that the pencil

$$P(\lambda) = \lambda E - LQ, \quad E, Q, L \in \mathbb{F}^{n,n} \quad (34)$$

with $Q^*E = E^*Q \geq 0$ and $L + L^* \leq 0$ is regular. As we have seen in the previous section, it may happen that the index of the pencil is two. In [1] it was shown that in the case of port-Hamiltonian DAEs with variable coefficients it is possible to perform an index reduction of the given port-Hamiltonian DAE such that the resulting DAE of index one is again a port-Hamiltonian DAE. Since the case with constant coefficients is a special case of the port-Hamiltonian DAEs considered in [1], the index reduction performed in Section 6 of that paper can also be applied in our case, so that we can assume that our pencil $P(\lambda)$ is of index at most one. However, as it was shown in the previous section, it is still possible that the eigenvalue 0 of $P(\lambda)$ is not semisimple. In this section, we will therefore provide a method for checking this property this and we also present a method of perturbing L to make 0 a semisimple eigenvalue, while keeping the structure assumptions on the pencil and all other eigenvalues and their Kronecker structures intact. The main ingredient of the method is a condensed form that provides particular information on the Kronecker structure of the eigenvalue zero.

Theorem 31 *Assume that $P(\lambda)$ from (34) with $E^*Q = Q^*E \geq 0$ and $L + L^* \leq 0$ is regular and of index at most one. Then there exist a partition (n_1, n_2, n_3, n_4) of n , a unitary matrix \tilde{U} and an invertible matrix \tilde{X} such that partitioning conformably with respect to (n_1, n_2, n_3, n_4) we have*

$$\begin{aligned} \tilde{Q} &:= \tilde{U}Q\tilde{X} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Q_{41} & Q_{42} & 0 & I \end{bmatrix}, \quad \tilde{E} := \tilde{U}E\tilde{X} = \begin{bmatrix} E_{11} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{L} &:= \tilde{U}L\tilde{U}^* = [L_{ij}]_{ij=1}^4, \quad \tilde{A} := \tilde{U}LQ\tilde{X} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ A_{21} & 0 & 0 & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \end{aligned} \quad (35)$$

where E_{11}, E_{22}, A_{11} are lower-triangular and invertible, and $A_{44}, \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ are invertible.

Proof. Note that the regularity of $P(\lambda)$ implies the regularity of $\lambda E - Q$ by Proposition 19. Hence, by Proposition 9, there exist a unitary matrix U and invertible matrix X such that

$$\hat{Q} := UQX = \begin{bmatrix} Q_1 & & \\ & 0 & \\ & & I \end{bmatrix}, \quad \hat{E} := UEX = \begin{bmatrix} E_1 & & \\ & I & \\ & & 0 \end{bmatrix},$$

with E_1 and Q_1 being diagonal and positive. The sizes of the block columns and rows are $n_1 + n_2, n_3, n_4$ in both matrices, later on the first block row and column will be split into blocks of sizes n_1, n_2 , respectively. Next, let

$$\hat{L} := ULU^* = [\hat{L}_{ij}]_{ij=1,\dots,3}, \quad ULQX = \begin{bmatrix} \hat{L}_{11}Q_1 & 0 & \hat{L}_{13} \\ \hat{L}_{21}Q_1 & 0 & \hat{L}_{22} \\ \hat{L}_{31}Q_1 & 0 & \hat{L}_{33} \end{bmatrix}.$$

be partitioned conformably with \hat{Q} and \hat{E} . Since $P(\lambda)$ is of index one, it follows that \hat{L}_{33} is invertible, see, e.g., [3]. Setting

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\hat{L}_{33}^{-1}\hat{L}_{31}Q_1 & 0 & I \end{bmatrix} \quad (36)$$

we get that

$$UEXT = \hat{E}, \quad U\hat{L}QXT = \begin{bmatrix} (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1 & 0 & \hat{L}_{13} \\ (\hat{L}_{21} - \hat{L}_{23}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1 & 0 & \hat{L}_{23} \\ 0 & 0 & \hat{L}_{33} \end{bmatrix}. \quad (37)$$

Observe that

$$Q_1^*E_1^{-1} > 0, \quad (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31}) + (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})^* \leq 0. \quad (38)$$

Indeed, the first inequality results directly from the form of Q_1 and E_1 and the second follows from $\hat{L} + \hat{L}^* \leq 0$ as

$$\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31} = S^*\hat{L}\hat{S}, \quad S := \begin{bmatrix} I & \\ & 0 \\ & & -\hat{L}_{33}^{-1}\hat{L}_{31} \end{bmatrix}.$$

Hence, we can apply [21, Lemma 3.1] to see that zero is a semisimple eigenvalue of the pencil $\lambda I - (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1E_1^{-1}$ and thus also a semisimple eigenvalue of the pencil $\lambda E_1 - (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1$, which has a generalized Schur form, see, e.g., [14],

$$W_1(\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1W_2 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}, \quad W_1E_1W_2 = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix} \quad (39)$$

with $A_{11}, E_{11} \in \mathbb{F}^{n_1, n_1}$, $E_{22} \in \mathbb{F}^{n_2, n_2}$ being lower triangular and invertible and W_1, W_2 being unitary. Setting

$$\tilde{U} = \begin{bmatrix} W_1 & & \\ & I & \\ & & I \end{bmatrix} U, \quad \tilde{X} = XT \begin{bmatrix} W_2 & & \\ & I & \\ & & I \end{bmatrix},$$

and splitting the first column and row conformably with (n_1, n_2) , where $n_1 = \dim \ker(\hat{L}_{21} - \hat{L}_{23}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1$, finishes the definition of \tilde{Q}, \tilde{X} and (n_1, n_2, n_3, n_4) .

The particular form of \tilde{A} and \tilde{E} follows now from (36), (37) and (39). Furthermore,

$$\tilde{Q} = \begin{bmatrix} W_1 & & \\ & I & \\ & & I \end{bmatrix} \hat{Q}T \begin{bmatrix} W_2 & & \\ & I & \\ & & I \end{bmatrix}$$

and hence $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = W_1Q_1W_2$ is invertible and the form of \tilde{Q} follows. \square

Let us discuss now some properties of the condensed form (35).

Corollary 32 *With the same notation and assumptions as in Theorem 31, the following conditions are equivalent:*

- (a) Zero is a semisimple eigenvalue of $P(\lambda)$,
- (b) $A_{32} = 0$,

$$(c) \operatorname{im} \begin{bmatrix} Q_{21} \\ Q_{22} \\ 0 \\ Q_{24} \end{bmatrix} \subseteq \ker \begin{bmatrix} L_{31} & L_{32} & 0 & L_{34} \end{bmatrix}.$$

Proof. (a) \Leftrightarrow (b): As A_{44} is invertible, the pencil $\lambda\tilde{E} - \tilde{A}$ is equivalent to the pencil

$$\lambda \begin{bmatrix} E_{11} & 0 & 0 & 0 \\ E_{12} & E_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & 0 & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix}.$$

which has a semisimple eigenvalue at 0 if and only if the matrix

$$C := \begin{bmatrix} E_{11} & 0 & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \\ A_{31} & A_{32} & 0 \end{bmatrix} = \begin{bmatrix} E_{11}^{-1}A_{11} & 0 & 0 \\ * & 0 & 0 \\ A_{31} & A_{32} & 0 \end{bmatrix},$$

has a semisimple eigenvalue at 0. Since $E_{11}^{-1}A_{11}$ is invertible, we immediately see that semi-simplicity of the eigenvalue zero is equivalent to $A_{32} = 0$.

(b) \Leftrightarrow (c): Let L_j (respectively, Q_j) denote for $j = 1, 2, 3, 4$ the j -th block row (block column) of the matrix \tilde{L} (matrix \tilde{Q}). Then $A_{32} = L_3Q_2$ and (c) is clearly equivalent to $L_3Q_2 = 0$. \square

Following the notation of (35) we also obtain the following result.

Corollary 33 *With the same notation and assumptions as in Theorem 31, any perturbation $L + M$ of L , where $\tilde{M} := \tilde{U}M\tilde{U}^*$ is of the form*

$$\tilde{M} = \begin{bmatrix} 0 & 0 & M_{13} & 0 \\ 0 & 0 & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & M_{43} & 0 \end{bmatrix},$$

leaves the pencil $P(\lambda)$ regular and the Kronecker form invariant, except for possible changes in the number and sizes of the blocks corresponding to the eigenvalue 0.

Proof. The assertion follows directly by computing

$$\tilde{A} + \tilde{M}\tilde{Q} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ A_{21} & 0 & 0 & A_{24} \\ A_{31} + W_{31} & A_{32} + W_{32} & 0 & A_{34} + W_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \quad W = [W_{ij}]_{i,j=1}^4 := \tilde{M}\tilde{Q}.$$

Due to the form of \tilde{E} in (35), the entries W_{31} and W_{34} do not have any influence on the Kronecker form and regularity of $\lambda\tilde{E} - \tilde{A}$ (and thus also $\lambda E - A$) and the entry W_{32} may cause changes to the Kronecker structure of the eigenvalue zero only. \square

Hence, given a pencil $P(\lambda)$ as in (34) with an eigenvalue 0 that is not semisimple, it is desirable to construct a stabilizing perturbation of the form above with $W_{32} = -A_{32}$ as this procedure will make the eigenvalue zero semisimple, but leaves all other eigenvalues and their Kronecker structure invariant. To express such a perturbation in terms of L and Q we need some preparations. In the following, for a linear subspace $\mathcal{X} \subseteq \mathbb{F}^n$ the symbol $P_{\mathcal{X}} \in \mathbb{F}^{n,n}$ denotes the orthogonal projection onto \mathcal{X} and $\mathcal{X} \ominus \mathcal{Y}$ stands for $P_{\mathcal{Y}^\perp}\mathcal{X}$.

Lemma 34 *Let $C \in \mathbb{F}^{m,l}$, $B \in \mathbb{F}^{n,m}$ and let $Z_0 \in \mathbb{F}^{m,l}$ be such that $B(C + Z_0) = 0$. Then $\text{im } C \ominus \ker B \subseteq \text{im } Z_0 \subseteq \text{im } C \oplus \ker B$ and the matrix Z with the smallest Frobenius norm satisfying $B(C + Z) = 0$ has the property that $\text{im } C \ominus \ker B = \text{im } Z$ and its Frobenius norm equals $\|Z\|_F = \|P_{\text{im } B^*}C\|_F$.*

Proof. The proof follows directly from the decomposition $\mathbb{F}^m = \ker B \oplus \text{im } B^*$. \square

Theorem 1.1 of [8] gives a characterization of positive semidefinite 2×2 block operators. We present here a matrix version for the reader's convenience.

Theorem 35 Let $B \in \mathbb{F}^{k,k}$ and $F \in \mathbb{F}^{m,m}$ be Hermitian positive semidefinite, let $E \in \mathbb{F}^{m,k}$ and let

$$T := \begin{bmatrix} B & E^* \\ E & F \end{bmatrix}.$$

Then $T \geq 0$ if and only if there exist a linear mapping $\Gamma : \text{im } F \rightarrow \text{im } B$ with $\|\Gamma\| \leq 1$ such that $E^* = B^{1/2}\Gamma F^{1/2}$. In particular, if $T \geq 0$ then $\text{im } E^* \subseteq \text{im } B^{1/2} = \text{im } B$.

The following lemma will serve as a prerequisite for constructing perturbations of R that preserve semi-definiteness.

Lemma 36 Let $B \in \mathbb{F}^{k,k}$ and $D \in \mathbb{F}^{m,m}$ be Hermitian, positive semidefinite, and let $C \in \mathbb{F}^{m,k}$ be such that

$$\begin{bmatrix} B & C^* \\ C & D \end{bmatrix} \geq 0.$$

Then, for any $Y \in \mathbb{F}^{m,k}$ with $\text{im } Y^* \subseteq \text{im } B$ we have that

$$T(Y, W) := \begin{bmatrix} B & (C + Y)^* \\ C + Y & D + W \end{bmatrix} \geq 0,$$

for $W := (\alpha^2 + 2\alpha)(D + \|Y\|I) + \|Y\|I$ with $\alpha = \|(B|_{\text{im } B})^{-\frac{1}{2}}Y^*(D + \|Y\|I)^{-\frac{1}{2}}\|$.

Proof. Theorem 35 adapted to the current situation states that $T(Y, W)$ is positive semidefinite if and only if there exists a linear map $\Gamma : \text{im}(D + W) \rightarrow \text{im } B$ with $\|\Gamma\| \leq 1$ such that

$$(C + Y)^* = B^{1/2} \Gamma (D + W)^{1/2}. \quad (40)$$

Let now W be as in the assertion and assume that $Y \neq 0$. The case $Y = 0$ is trivial, because then $W = 0$ as well. Observe that $D \geq 0$ implies that $D + W = (\alpha + 1)^2(D + \|Y\|I) > 0$. Setting

$$\Gamma := (B|_{\text{im } B})^{-\frac{1}{2}}(C + Y)^*(D + W)^{-\frac{1}{2}} = (1 + \alpha)^{-1}(B|_{\text{im } B})^{-\frac{1}{2}}(C + Y)^*(D + \|Y\|I)^{-\frac{1}{2}},$$

by Lemma 35 we have that $\text{im } C^* \subseteq \text{im } B$ and by the assumption that $\text{im } Y^* \subseteq \text{im } B$ we have that (40) is satisfied. Furthermore, by definition of α , we have that

$$\|\Gamma\| \leq (1 + \alpha)^{-1} \left(\|(B|_{\text{im } B})^{-\frac{1}{2}}C^*(D + \|Y\|I)^{-\frac{1}{2}}\| + \alpha \right).$$

Note that by Theorem 35, applied to $T(0, \|Y\|I) \geq 0$, there exist $\tilde{\Gamma} : \mathbb{F}^m \rightarrow \text{im } B$ with $\|\tilde{\Gamma}\| \leq 1$ such that $C^* = B^{1/2} \tilde{\Gamma} (D + \|Y\|I)^{1/2}$. Hence,

$$\|(B|_{\text{im } B})^{-\frac{1}{2}}C^*(D + \|Y\|I)^{-\frac{1}{2}}\| \leq \|\tilde{\Gamma}\| \leq 1,$$

which finishes the proof. \square

We have now prepared the tools for constructing a stabilizing perturbation of a pencil as in (34) by perturbing the matrix L in a structure preserving manner. Below $\|X\|_F$ stands for the Frobenius norm of the matrix X .

Theorem 37 Assume that $P(\lambda)$ from (34), with $E^*Q = Q^*E \geq 0$ and $L + L^* \leq 0$, is a regular pencil of index at most one, and let the decomposition (35) be given. Then, for every $Y := [Y_1 \ Y_2 \ Y_4]$ with

$$\text{im } Y^* \subseteq \text{im } R_0, \quad R_0 := \begin{bmatrix} R_{11} & R_{12} & R_{14} \\ R_{21} & R_{22} & R_{24} \\ R_{41} & R_{42} & R_{44} \end{bmatrix},$$

there exist $\Delta_R = \Delta_R^*$ such that $R + \Delta_R \geq 0$ and

$$\|\Delta_R\| \leq 2\|Y\| + \left(\|(R_0|_{\text{im } R_0})^{-1}\| \|Y\| + 2\|(R_0|_{\text{im } R_0})^{-\frac{1}{2}}\| \|Y\|^{\frac{1}{2}} \right) (\|R_{33}\| + \|Y\|) \quad (41)$$

and there exist $\Delta_J = -\Delta_J^*$ with the Frobenius norm bounded by

$$\|\Delta_J\|_F \leq 2\|P_{\text{im } Q_2}(L_3^* + Y^*)\|_F, \quad (42)$$

where

$$L_3 = [L_{31} \ L_{32} \ L_{33} \ L_{34}], \quad Q_2 = \begin{bmatrix} Q_{21} \\ Q_{22} \\ 0 \\ Q_{24} \end{bmatrix},$$

such that for every $s, t \in \mathbb{C}$ the pencil $\lambda E + (J + s\Delta_J - R - t\Delta_R)Q$ is regular and has the same Kronecker structure for the nonzero eigenvalues (including infinity) as $P(\lambda)$ and zero is a semisimple eigenvalue of $\lambda E - (J + \Delta_J - R - \Delta_R)Q$.

Proof. Without loss of generality we may assume that $\tilde{U} = I_n$ in the decomposition (35). We interchange the last two block rows and last two block columns, so that

$$R = \begin{bmatrix} R_0 & C^* \\ C & R_{33} \end{bmatrix}, \quad R_3 := [R_{31} \ R_{32} \ R_{34}].$$

Let $Y = [Y_1 \ Y_2 \ Y_4]$ with $\text{im } Y^* \subseteq \text{im } R_0$ and let $B := R_0$, $D = R_{33}$. In this setting we apply Lemma 36 getting a matrix W such that $R + \Delta_R \geq 0$, where

$$\Delta_R := \begin{bmatrix} 0 & Y^* \\ Y & W \end{bmatrix}.$$

The bound (41) then follows, since

$$\|(R_{33} + \|Y\|I)^{-\frac{1}{2}}\| \leq \|(\|Y\|I)^{-\frac{1}{2}}\| = \|Y\|^{-\frac{1}{2}}$$

and hence $\alpha \leq \|(R_0|_{\text{im } R_0})^{-\frac{1}{2}}\| \|Y\|^{\frac{1}{2}}$, with α as in Lemma 36. Furthermore, by Corollary 33 for any $t \in \mathbb{C}$ the pencil $\lambda E - (J - R - t\Delta_R)Q$ is regular and has the same Kronecker structure associated with the nonzero eigenvalues (including infinity).

Now let us define Δ_J to make the eigenvalue zero of the perturbed pencil semisimple. We reverse the interchange of the last two block rows and block columns, so that we are in the original setting of (35), and we define $Z = [Z_1 \ Z_2 \ 0 \ Z_4]$ as in Lemma 34 applied to $B =$

Q_2^* , $C = L_3^* + \hat{Y}^*$, where L_3 and Q_2 are defined as in the assertion and $\hat{Y} = [Y_1 \ Y_2 \ 0 \ Y_4]$. Then we get $Q_2^*(L_3^* + \hat{Y}^* + Z^*) = 0$, and defining

$$\Delta_J := \begin{bmatrix} 0 & 0 & -Z_1^* & 0 \\ 0 & 0 & -Z_2^* & 0 \\ Z_1 & Z_2 & 0 & Z_4 \\ 0 & 0 & -Z_4^* & 0 \end{bmatrix},$$

in the perturbed matrix $A + \Delta_R + \Delta_J$ the $(3, 2)$ block entry equals zero, which by Corollary 32 is equivalent to semi-simplicity of the zero eigenvalue. The estimate (42) follows by Lemma 34 and again by Corollary 33 for any $s \in \mathbb{C}$ the pencil $\lambda E - (J + s\Delta_J - R - t\Delta_R)Q$ is regular and has the same Kronecker structure at nonzero eigenvalues (including infinity). \square

Let us conclude this section with some comments.

- Theorem 37 can be applied with $Y = 0$. In this way we get a skew-symmetric perturbation of L .
- Note that condition (c) of Corollary 32 is implied by

$$\ker \begin{bmatrix} L_{11} & L_{12} & 0 & L_{14} \\ L_{21} & L_{22} & 0 & L_{24} \end{bmatrix} \subseteq \ker [L_{31} \ L_{32} \ 0 \ L_{34}],$$

and this leads to a bound $\|\Delta_J\|_F \leq 2 \|P_{\text{im}[L_1, L_2]}(L_3^* + Y^*)\|_F$, which, in general, is weaker than (42), however, it does not use the matrix Q .

- By Theorem 35, if $\text{im } Y^*$ is not contained in R_0 then the matrix $R + \Delta_R$ is not positive semidefinite.
- For sufficiently small Y with $\text{im}(Y + Y^*) \subseteq \text{im } R$ the perturbation $\tilde{\Delta}R := Y + Y^*$ preserves positive definiteness as well. Note that $\|\tilde{\Delta}R\| = \|Y\|$, while $\|\Delta_R\| = \mathcal{O}(\|Y\|^{1/2})$ as $\|Y\|$ tends to zero. However, note that the condition $\text{im}(Y + Y^*) \subseteq \text{im } R$ is stronger than $\text{im } Y^* \subseteq \text{im } R_0$ as it involves R_{33} as well. Furthermore, in applications, Y does not need to have a small norm.
- The matrix Δ_R constructed in the proof of Theorem 37 has the same form as the matrix M from Corollary 33. Hence, Δ_R is never positive semidefinite (unless it is zero). Therefore, strengthening the statement from $R + \Delta_R$ positive semidefinite to Δ_R positive semi-definite is in general not possible.
- The case $\Delta_J = 0$ can be considered as well. A sufficient condition for the existence of a symmetric perturbation Δ_L (of the form presented in the theorem) is that $P_{\text{im } Q_2} L_3^* \subseteq \text{im } R_0$. In this case the matrix $Y = -L_3 P_{\text{im } Q_2}$ satisfies $\text{im } Y^* \subseteq \text{im } R_0$ and for this particular choice of Y we get $\Delta_J = 0$.
- The matrices \tilde{U} and \tilde{X} , as well as the decomposition (35) can be computed efficiently, e.g., using the CS-decomposition and the generalized Schur decomposition of `matlab`.

Conclusions

We have studied matrix pencils associated with dissipative Hamiltonian descriptor systems. We have derived canonical and condensed forms and characterized spectral properties of such pencils. In particular, we have studied the size of Jordan blocks associated with the eigenvalues 0 and ∞ and the restrictions for the left and right minimal indices. When Kronecker blocks of size larger than one arise for the eigenvalue 0 (i.e. the associated system is not Lyapunov stable) then we have constructed minimal norm structure-preserving perturbations that make the system Lyapunov stable.

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