Generic rank-one perturbations of structured regular matrix pencils

Leonhard Batzke*

January 29, 2014

Abstract

Classes of regular, structured matrix pencils are examined with respect to their spectral behavior under a certain type of structure-preserving rank-1 perturbations. The observed effects are as follows: On the one hand, generically the largest Jordan block at each eigenvalue gets destroyed or becomes size one under a rank-1 perturbation, depending on that eigenvalue occuring in the perturbating pencil or not. On the other hand, certain Jordan blocks of T-alternating matrix pencils occur in pairs, so that in some cases, the largest block cannot just be destroyed or shrinked to size one without violating the pairing. Thus, the largest remaining Jordan block will typically increase in size by one in these cases. Finally, these results are shown to carry over to the classes of palindromic and symmetric matrix pencils.

Key words. Matrix pencil, alternating matrix pencil, palindromic matrix pencil, symmetric matrix pencil, perturbation theory, rank one perturbation, generic perturbation.

AMS subject classification. 15A18, 15A21, 15A22, 15B57, 47A55.

1 Introduction

Low-rank perturbations of matrices have been studied by different authors in [9, 19, 20, 21, 22]. It is well-understood that under a rank-1 perturbation generically (i.e., under a typical perturbation) the largest Jordan block of a matrix corresponding to each eigenvalue is destroyed. Generalizations to matrix pencils have been made in in recent years; it was shown in [3] that if a regular pencil is subjected to a low-rank perturbation, at each eigenvalue generically not only certain Jordan blocks will disappear but also certain Jordan blocks will become size one. On the other hand, the behavior of singular pencils under low-rank perturbations was examined in [2] and shown to be very different: generically, existing regular blocks are preserved and certain singular blocks become regular ones.

---

*Institut für Mathematik, MA 4-5, Technische Universität Berlin, 10623 Berlin, Germany, Email: `batzke@math.tu-berlin.de`.

Structure-preserving rank-1 perturbations were investigated in [16, 17, 18] for different types of structured matrices. The focus of [16] lay on J-Hamiltonian matrices, which are known to show a certain pairing of blocks in Jordan canonical form, leading to restrictions on the Jordan form of the perturbed matrix since it is required to be J-Hamiltonian as well. Resulting from this, in some cases the generic behavior was observed to include one block growing in size by one so that the Jordan structure of the perturbed matrix has the pairing characteristic for J-Hamiltonian matrices; this effect is substantially different from the unstructured case. A similar pattern could be identified for real H-skew-symmetric matrices under real H-nonnegative rank-1 perturbations investigated in [4].

In this work, we want to examine structure-preserving rank-1 perturbations of the following classes of structured matrix pencils.

Definition 1.1 *A matrix pencil $\lambda E - A$ with $E, A \in \mathbb{C}^{n,n}$ is called:*

- T-even *if E is skew-symmetric and A is symmetric.*
- T-odd *if E is symmetric and A is skew-symmetric.*
- T-alternating *if it is either T-even or T-odd.*
- T-palindromic *if $E = -A^T$.*
- T-anti-palindromic *if $E = A^T$.*
- palindromic *if it is either T-palindromic or T-anti-palindromic.*
- symmetric *if E and A are both symmetric.*

Our motivation for considering these classes of structured matrix pencils is that they frequently occur in various applications. A T-palindromic matrix pencil is, e.g., obtained from the vibration analysis of rail tracks under periodic excitation. As described in [8], this problem is modeled by an eigenvalue problem of the form

$$\frac{1}{\kappa}(A_0^T + \kappa A_1 + \kappa^2 A_0)y = 0,$$

where $A_1, A_0 \in \mathbb{C}^{n,n}$ and $A_1 = A_1^T$. Now, a matrix polynomial, i.e., an expression of the form $P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j$, where $A_j \in \mathbb{C}^{n,n}$ for $j = 0, \ldots, n$, is called T-palindromic if $P(\lambda)^T = \lambda^k P(1/\lambda)$ holds. Observe that this definition is consistent with the above definition of T-palindromic matrix pencils and that $A_0^T + \lambda A_1 + \lambda^2 A_0$ is indeed T-palindromic. Such polynomial eigenvalue problems are ususally treated by linearization, i.e., by solving a generalized eigenvalue problem with equivalent spectral information. If the matrix polynomial is structured, extensive research has been conducted in [12, 13, 14] to determine structure-preserving linearizations (or a lack thereof), since they are preferable when applying numerical algorithms. E.g., the T-palindromic pencil

$$\lambda \begin{bmatrix} A_0^T & A_1 - A_0 \\ A_0^T & A_0^T \end{bmatrix} + \begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix}$$

is a linearization of the above T-palindromic matrix polynomial that can instead be investigated.

Also, structured matrix pencils occur in applications in control, e.g., the T-even pencil

$$\left(\begin{bmatrix} 0 & E & 0 \\ -E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A & B \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \right)$$

has to be studied to solve the optimality system of the linear-quadratic optimal control problem. Although in most these applications the occuring matrix pencils will be real, in the real case the sign characteristic (see [10, 25]) plays an important role and has to be considered under perturbation. Thus, we will focus on the complex case in this paper referring the real case to future research.

It is well-known that T-alternating and palindromic matrix pencils show a certain symmetry in their eigenstructure, i.e., their eigenvalues occur in pairs of the form $(\lambda, -\lambda)$ and (λ, λ^{-1}), respectively. Hence, considering structure-preserving rank-1 perturbations of those, there will be restrictions on the Jordan structure of the perturbed pencil, especially at certain 'critical points' in the spectrum where the eigenvalue pairing degenerates, namely at 0 and ∞ for T-alternating pencils and at ± 1 for palindromic pencils. This leads to the question of how the generic Jordan structure of the perturbed pencil will be remedied with that. We consider the following example that was already examined in [24]:

Example 1.2 The T-even matrix pencil

$$(E, A) = \left(\begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & 0 & & \\ 0 & -1 & & \end{bmatrix}, \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \right) \in \mathbb{C}^{4,4} \times \mathbb{C}^{4,4}$$

only has the eigenvalue ∞ with the partial multiplicities $(2, 2)$. Since there is no rank-1 perturbation of the skew-symmetric matrix E, let us consider the T-even rank-1 perturbation $(0, bb^T)$. We will show in Section 3 that the perturbed pencil $(E, A + bb^T)$ generically only has the eigenvalue ∞ with the partial multiplicities $(3, 1)$.

This example has interesting characteristics: Generically, one block of (E, A) grows in size from 2 to 3, while the other one decreases in size from 2 to 1. Thus, the pair of blocks of size 2 is separated under perturbation although the algebraic multiplicity at ∞ remains constant. This is a feature that is not shared by the J-Hamiltonian matrices from [16] and in particular, there is no intuitive explanation for the fact that the Jordan structure is generically being altered under this rank-1 perturbation even though no instance of the eigenvalue ∞ is destroyed. Thus, it will be our incentive to clarify the principles governing these generic, structure-preserving rank-1 perturbations in this work.

In Section 2, we will specify the considered perturbations and then prove a version of the partial Brunovsky form that contains valuable information about the Jordan structure of a perturbed pencil. This allows us to characterize rank-1 perturbations of unstructured matrix pencils in terms of the typical Jordan structure of the perturbed pencil. Then, we will introduce structured Kronecker forms that are helpful for dealing with structured matrix pencils. In Section 3, generic, T-alternating perturbations are shown to consist of rank-1 and rank-2 perturbations, followed by a characterization of the Jordan structure of T-alternating matrix pencils under generic, structure-preserving rank-1 perturbations (rank-2 perturbations are postponed to a later paper). In Section 4, these results are applied to structure-preserving rank-1 perturbations of palindromic matrix pencils using Cayley transformations. In Section 5, generic, structure-preserving rank-1 perturbations of symmetric matrix pencils are examined, followed by a conclusion in the final section.

As suggested above, the behavior of singular matrix pencils turned out to be greatly different, which is why we will restrict ourselves to regular pencils $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ in this work, i.e., $\det(\lambda E - A)$ shall not be the constant zero polynomial. We denote by $e_{k,n}$ the kth unit vector of length n (omitting the second index n if clear from the context) and denote by $J_n(\lambda)$ an $n \times n$ Jordan block corresponding to the eigenvalue λ. Further, for square matrices X and Y (not necessarily of the same dimension), define $X \oplus Y := \operatorname{diag}(X, Y)$ and let $X^{\oplus p} := X \oplus \cdots \oplus X$ (p times). Finally, let R_n and Σ_n denote the $n \times n$ matrices

$$R_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \quad \text{and} \quad \Sigma_n = \begin{bmatrix} (-1)^n & & & \\ & \ddots & & \\ & & (-1)^2 & \\ & & & (-1)^1 \end{bmatrix}.$$

2 Preliminaries

Before starting to investigate low-rank perturbations, let us consider the following two definitions in order to clarify under what conditions we regard a property as typical.

Definition 2.1 *A set $\mathcal{A} \subseteq \mathbb{C}^n$ is called* algebraic *if there exist finitely many polynomials $p_1(x_1, \dots, x_n), \dots, p_k(x_1, \dots, x_n)$, such that $(a_1, \dots, a_n) \in \mathcal{A}$ if and only if*

$$p_j(a_1, \dots, a_n) = 0 \quad \textit{for} \quad j = 1, \dots, k.$$

An algebraic set $\mathcal{A} \subseteq \mathbb{C}^n$ is called proper *if $\mathcal{A} \neq \mathbb{C}^n$ holds.*

Definition 2.2 *A set $\Omega \subseteq \mathbb{C}^n$ is called* generic *if its complement $\mathbb{C}^n \setminus \Omega$ is contained in a proper algebraic set.*

Note that the intersection of finitely many generic sets is again generic. Subsets of $\mathbb{C}^{n,m}$ or $\mathbb{C}^{n,m} \times \mathbb{C}^{n,m}$ are called generic if they can be canonically identified with generic subsets of $\mathbb{C}^{nm}$ or $\mathbb{C}^{2nm}$, respectively. Further, for any invertible matrix $X \in \mathbb{C}^{n,n}$ and a generic set $\Omega \subseteq \mathbb{C}^n$, the set $X\Omega$ is generic as well. Finally, a property is called generic if there exists a generic set Ω such that this property is satisfied for all $x \in \Omega$.

2.1 Properties of generic perturbations

Let $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular matrix pencil. Subject of our investigation will be rank-k perturbations of the form

$$\left[\ \widetilde{u}_1 \ \ \ldots \ \ \widetilde{u}_k\ \right] \big(\delta E, \delta A\big) \left[\ \widetilde{v}_1 \ \ \ldots \ \ \widetilde{v}_k\ \right]^T, \tag{2.1}$$

assuming that $(\delta E, \delta A)$ is an element of some generic subset of $\mathbb{C}^{k,k} \times \mathbb{C}^{k,k}$ and that $[\widetilde{u}_1, \ldots, \widetilde{u}_k, \widetilde{v}_1, \ldots, \widetilde{v}_k]$ is an element of some generic subset of $\mathbb{C}^{n,2k}$. Then, we can characterize certain properties of $(\delta E, \delta A)$ in the following theorem.

Theorem 2.3 *There exists a generic set* $\Gamma \subseteq \mathbb{C}^{k,k} \times \mathbb{C}^{k,k}$ *such that any* $(E,A) \in \Gamma$ *is regular and has distinct eigenvalues.*

Proof. Let $(E,A) \in \mathbb{C}^{k,k} \times \mathbb{C}^{k,k}$ be an arbitrary matrix pencil. Consider its characteristic polynomial $\sum_{j=0}^{k} c_j \lambda^j := \det(\lambda E - A)$ and observe that the coefficients $c_j = c_j(E,A)$ for $j = 0, \ldots, n$ depend polynomially on the entries of E and A and that $c_j(E,A) \neq 0$ for at least one j if (E,A) is regular. Hence, the set of regular pencils

$$\big\{(E,A) \in \mathbb{C}^{k,k} \times \mathbb{C}^{k,k} \ \mid \ \exists j \in \{0,1,\ldots,k\} \ \text{ with } \ c_j(E,A) \neq 0\,\big\}$$

is a generic subset of $\mathbb{C}^{k,k} \times \mathbb{C}^{k,k}$.

Recall that the Sylvester resultant matrix of two polynomials $s(\lambda)$ and $t(\lambda)$, denoted by $S\big(s(\lambda), t(\lambda)\big)$, is a square matrix of dimension $\deg(s) + \deg(t)$. It is well-known that it consists of the coefficients of $s(\lambda)$ and $t(\lambda)$ (but not the variable λ) and that the rank defect of $S\big(s(\lambda), t(\lambda)\big)$ is exactly the degree of the greatest common divisor of $s(\lambda)$ and $t(\lambda)$ (see, e.g., [11]). We therefore define

$$p(E,A) := \det S\Big(\det(\lambda E - A), \frac{\partial}{\partial \lambda} \det(\lambda E - A)\Big) \tag{2.2}$$

and note that $p(E,A)$ depends polynomially on the on the entries of E and A. We observe that $p(E,A) = 0$ if and only if the pencil (E,A) has a multiple finite eigenvalue. To account for the infinite eigenvalue of (E,A), define

$$q(E,A) := \det S\Big(\det(\lambda A - E), \frac{\partial}{\partial \lambda} \det(\lambda A - E)\Big) \tag{2.3}$$

and observe that the pencil (E,A) does not have multiple eigenvalues (neither finite nor infinite) if and only if $p(E,A)q(E,A) \neq 0$. Clearly, $p(E,A)q(E,A)$ depends polynomially on the entries of E and A and it is not constantly zero since there exist pencils with distinct eigenvalues. We conclude the assertion by defining

$$\begin{aligned} \Gamma := \big\{(E,A) \in \mathbb{C}^{k,k} \times \mathbb{C}^{k,k} \ \mid \ & p(E,A)q(E,A) \neq 0 \ \text{ and} \\ & \exists j \in \{0,1,\ldots,k\} \ \text{ with } \ c_j(E,A) \neq 0\,\big\}. \quad \Box \end{aligned} \tag{2.4}$$

Since any regular pencil with distinct eigenvalues is diagonalizable, diagonalizability is a generic property in $\mathbb{C}^{k,k} \times \mathbb{C}^{k,k}$ as well. Hence, if $(\delta E, \delta A)$ from (2.1) is an element of Γ as in (2.4), there exist invertible $V, W \in \mathbb{C}^{k,k}$ such that $V(\delta E, \delta A)W$ is diagonal; thus (2.1) can be transformed to

$$\begin{bmatrix} u_1 & \dots & u_k \end{bmatrix} V\big(\delta E, \delta A\big)W \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix}^T \tag{2.5}$$

setting $u_j = \widetilde{u}_j V^{-1}$ and $v_j = \widetilde{v}_j W^{-T}$ for $j = 1, \dots, k$. Then, considering all $[\widetilde{u}_1, \dots, \widetilde{u}_k, \widetilde{v}_1, \dots, \widetilde{v}_k]$ that are elements of some generic subset of $\mathbb{C}^{n,2k}$ is equivalent to considering all $[u_1, \dots, u_k, v_1, \dots, v_k]$ that are elements of some generic subset of $\mathbb{C}^{n,2k}$, since the respective generic sets can be transformed into one another by multiplication with invertible matrices. Therefore, it comes with no loss of generality to assume that (2.1) has the shape (2.5) already. As (2.5) is the sum of rank-1 pencils of the form $(\beta uv^T, \alpha uv^T)$, a perturbation of this form will in the following be considered a generic rank-1 perturbation if (u, v) is an element of some generic subset of $\mathbb{C}^n \times \mathbb{C}^n$.

Note that other unstructured rank-1 perturbations have the forms

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix},$$

that do not stem from generic pencils of the form (2.1), but since they do not have structured rank-1 analogues, we will exclude them from our study of unstructured rank-1 perturbations.

Further, for all $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$; $u, v \in \mathbb{C}^n$; and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ the following inequalities hold (see, e.g., [3, Section 1]):

$$\begin{aligned} \operatorname{rank}(\delta E - \gamma A) - \operatorname{rank}(\delta\beta uv^T - \gamma\alpha uv^T) &\leq \operatorname{rank}\big(\delta(E + \beta uv^T) - \gamma(A + \alpha uv^T)\big) \qquad (2.6) \\ &\leq \operatorname{rank}(\delta E - \gamma A) + \operatorname{rank}(\delta\beta uv^T - \gamma\alpha uv^T). \end{aligned}$$

Therefore, if $\widehat{\lambda} = \gamma/\delta$ (defining $\widehat{\lambda} = \infty$ if $\delta = 0$) is an eigenvalue of (E, A), its geometric multiplicity cannot decrease by more than one if $\widehat{\lambda}\beta \neq \alpha$ and its geometric multiplicity cannot change at all if $\widehat{\lambda}\beta = \alpha$. Hence, for perturbations of the form $(\beta uv^T, \alpha uv^T)$ we expect different effects depending on α being equal to $\widehat{\lambda}\beta$ or not.

2.2 The partial Brunovsky form

Before analyzing rank-1 perturbations of unstructured matrix pencils in closer detail, let us cover a few more prerequisites. Jordan chains of regular matrix pencils are defined as follows, see, e.g., [7].

Definition 2.4 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular pencil. The ordered set of vectors $\{x_1, \dots, x_p\} \subseteq \mathbb{C}^n$ is called a Jordan chain of length p corresponding to an eigenvalue $\widehat{\lambda} \in \mathbb{C}$ of (E, A) if $x_1 \neq 0$ and:*

$$(\widehat{\lambda}E - A)x_1 = 0 \quad \textit{and} \quad (\widehat{\lambda}E - A)x_j = -Ex_{j-1}, \qquad j = 2, \dots, p.$$

Similarly, $\{x_1,\ldots,x_p\}$ *is called a Jordan chain of length* p *corresponding to* ∞ *if* $x_1 \neq 0$ *and:*

$$Ex_1 = 0 \quad \textit{and} \quad Ex_j = Ax_{j-1}, \quad j = 2,\ldots,p.$$

The following notion of the sum of two Jordan chains was initially used in [16].

Definition 2.5 *Let* $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ *be regular and let* $X = \{x_1,\ldots,x_p\}$ *and* $Y = \{y_1,\ldots,y_q\}$ *be two Jordan chains of* (E,A) *of lengths* p *and* q*, respectively, associated with the same eigenvalue* $\widehat{\lambda}$*. Then the sum* $X+Y$ *is defined to be* $Z = \{z_1,\ldots,z_{\max\{p,q\}}\}$*, where*

$$z_j = \begin{cases} x_j & \textit{if } p \geq q \\ y_j & \textit{if } p < q \end{cases}, \quad j = 1,\ldots,|p-q|$$

and

$$z_j = \begin{cases} x_j + y_{j-p+q} & \textit{if } p \geq q \\ y_j + x_{j-q+p} & \textit{if } p < q \end{cases}, \quad j = |p-q|+1,\ldots,\max\{p,q\}.$$

It is straightforward to check that the sum $Z = X + Y$ of two Jordan chains associated with an eigenvalue $\widehat{\lambda}$ is again a Jordan chain associated with $\widehat{\lambda}$.

Let us now consider the following notation from [18]: for $v = [v_1,\ldots,v_n]^T \in \mathbb{C}^n$ and $k,p \in \mathbb{N}$ with $p \leq n$, we define $\operatorname{Toep}(v,k,p)$ to be the upper triangular $p \times p$ Toeplitz matrix, whose first row consists of the first p entries of the series $(v_{k+1},\ldots,v_n,0,0,\ldots)$. Also, we define $\operatorname{Toep}(v) := \operatorname{Toep}(v,0,n)$.

The following theorem is a key ingredient for characterizing the Jordan structure of perturbed pencils. Its proof uses elements of [18, Theorem 4.4] and [24, Theorem 4.9] and goes back to the *Brunovsky canonical form* of linear, time-invariant control systems introduced in [1].

Theorem 2.6 (partial Brunovsky form) *Let* $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ *be regular and* $\widehat{\lambda} \in \mathbb{C}$ *an eigenvalue with*

$$\begin{aligned} E &= I_{n_1} \oplus \cdots \oplus I_{n_m} \oplus \widetilde{E} \in \mathbb{C}^{n,n} \\ A &= J_{n_1}(\widehat{\lambda}) \oplus \cdots \oplus J_{n_m}(\widehat{\lambda}) \oplus \widetilde{A} \in \mathbb{C}^{n,n} \end{aligned} \tag{2.7}$$

where $n_1 \geq \cdots \geq n_m > 0$ *such that* $\widehat{\lambda}$ *is not an eigenvalue of* $(\widetilde{E},\widetilde{A})$*. Further, set* $a := n_1 + \cdots + n_m$ *and let* $\alpha,\beta \in \mathbb{C}$ *and* $u,v \in \mathbb{C}^n$ *with*

$$v^T = \left[\ (v^{(1)})^T,\ \ldots,\ (v^{(m)})^T,\ \widetilde{v}^T\ \right], \quad (v^{(j)})^T = \left[\ v_1^{(j)},\ \ldots,\ v_{n_j}^{(j)}\ \right] \in \mathbb{C}^{1,n_j}, \quad j = 1,\ldots,m.$$

If we define

$$k_j := \max\left\{k \mid v_1^{(j)} = v_2^{(j)} = \cdots = v_k^{(j)} = 0\right\}, \quad j = 1,\ldots,m,$$

then the following statements hold:

1) *There is an invertible matrix* $S \in \mathbb{C}^{n,n}$ *such that*

$$S(E + \beta uv^T, A + \alpha uv^T)S^{-1} = (E + \beta we^T, A + \alpha we^T) \tag{2.8}$$

holds, where $w = Su$ *and*

$$e^T = \begin{bmatrix} e^T_{k_1+1,n_1}, & \dots, & e^T_{k_m+1,n_m}, & \widetilde{e}^T \end{bmatrix}$$

for a suitable $\widetilde{e} \in \mathbb{C}^{n-a}$ *defining* $e_{k_j+1,n_j} = 0$ *if* $k_j = n_j$.

2) *If the pencil* (2.8) *is regular, then it has at least* $m-1$ *linearly independent chains of lengths at least* $n_2, \dots, n_m$ *corresponding to* $\widehat{\lambda}$. *If the (generic) condition that the first component of* $v^{(j)}$ *is nonzero for* $j = 1, \dots, m$ *holds, then they are given by:*

$$\begin{array}{cccc} e_1 - e_{n_1+1}, & e_2 - e_{n_1+2}, & \dots, & e_{n_2} - e_{n_1+n_2}; \\ e_1 - e_{n_1+n_2+1}, & e_2 - e_{n_1+n_2+2}, & \dots, & e_{n_3} - e_{n_1+n_2+n_3}; \\ \vdots & \vdots & \ddots & \vdots \\ e_1 - e_{n_1+\cdots+n_{m-1}+1}, & e_2 - e_{n_1+\cdots+n_{m-1}+2}, & \dots, & e_{n_m} - e_{n_1+\cdots+n_m}. \end{array} \tag{2.9}$$

Proof. The matrix $S := S_1 \oplus \cdots \oplus S_m \oplus I_{n-a}$, where

$$S_j = \begin{cases} \mathrm{Toep}(v^{(j)}, k_j, n_j) & \text{if } k_j < n_j, \\ I_{n_j} & \text{if } k_j = n_j, \end{cases}$$

is invertible by our definition of k_j for $j = 1, \dots, m$. It is straightforward to check that S commutes with both E and A (see also [5, Chapter 8]), and we easily confirm that $v^T S^{-1} = e^T$ because of

$$e^T_{k_j+1,n_j} S_j = (v^{(j)})^T, \quad j = 1, \dots, m.$$

Thus, it remains to prove 2). If the pencil (2.8) is regular, for $1 \le i < \ell \le m$ we define the *chain corresponding to the* i*th and* ℓ*th block* to be $\{x_1, \dots, x_{n_\ell}\}$, where

$$x_j = \begin{cases} e_{n_1+\cdots+n_{i-1}+j} & \text{if } k_i \ge k_\ell \\ -e_{n_1+\cdots+n_{\ell-1}+j} & \text{if } k_i < k_\ell \end{cases}, \quad j = 1, \dots, \min\{n_\ell, |k_i - k_\ell|\}$$

and

$$x_j = \begin{cases} e_{n_1+\cdots+n_{i-1}+j} - e_{n_1+\cdots+n_{\ell-1}+j-k_i+k_\ell} & \text{if } k_i \ge k_\ell \\ e_{n_1+\cdots+n_{i-1}+j-k_\ell+k_i} - e_{n_1+\cdots+n_{\ell-1}+j} & \text{if } k_i < k_\ell \end{cases}, \quad j = |k_i - k_\ell| + 1, \dots, n_\ell.$$

Now, we will show that for each $1 \le i < \ell \le m$, the chain corresponding to the ith and ℓth block is indeed a chain corresponding to the eigenvalue $\widehat{\lambda}$ of the pencil (2.8) of length n_ℓ

according to Definition 2.4. Thus, consider the case $n_\ell > k_\ell - k_i > 0$ (otherwise the proof is analogous). We obtain

$$\big(\widehat{\lambda}(E + \beta w e^T) - A - \alpha w e^T\big)x_1 = \big(\widehat{\lambda}E - A + (\widehat{\lambda}\beta - \alpha)we^T\big)(-e_{n_1+\cdots+n_{\ell-1}+1}) = 0$$

because of $k_\ell \geq 1$ and (2.7). Further, using (2.7) we obtain for $j = 2, \ldots, k_\ell - k_i$:

$$\begin{aligned}&\big(\widehat{\lambda}(E + \beta w e^T) - A - \alpha w e^T\big)x_j = \big(\widehat{\lambda}E - A + (\widehat{\lambda}\beta - \alpha)we^T\big)(-e_{n_1+\cdots+n_{\ell-1}+j})\\ &= e_{n_1+\cdots+n_{\ell-1}+j-1} = -(E + \beta we^T)x_{j-1}\end{aligned}$$

and also

$$\begin{aligned}&\big(\widehat{\lambda}(E + \beta w e^T) - A - \alpha w e^T\big)x_{k_\ell-k_i+1}\\ &= \big(\widehat{\lambda}E - A + (\widehat{\lambda}\beta - \alpha)we^T\big)(e_{n_1+\cdots+n_{i-1}+1} - e_{n_1+\cdots+n_{\ell-1}+k_\ell-k_i+1})\\ &= e_{n_1+\cdots+n_{\ell-1}+k_\ell-k_i} = -(E + \beta we^T)x_{k_\ell-k_i}.\end{aligned} \tag{2.10}$$

Finally, again using (2.7) it is for $j = k_\ell - k_i + 2, \ldots, n_\ell$:

$$\begin{aligned}&\big(\widehat{\lambda}(E + \beta w e^T) - A - \alpha w e^T\big)x_j\\ &= \big(\widehat{\lambda}E - A + (\widehat{\lambda}\beta - \alpha)we^T\big)(e_{n_1+\cdots+n_{i-1}+j-k_\ell+k_i} - e_{n_1+\cdots+n_{\ell-1}+j})\\ &= -e_{n_1+\cdots+n_{i-1}+j-k_\ell+k_i-1} + e_{n_1+\cdots+n_{\ell-1}+j-1} = -(E + \beta we^T)x_{j-1}.\end{aligned} \tag{2.11}$$

We note that (2.10) also holds if $k_i = 0$ and that (2.11) also holds if $j = k_\ell + 1 \geq k_\ell - k_i + 2$ or $j = k_\ell + 2$ (these are the special cases where we^T contributes to the computation) since the $(n_1 + \cdots + n_{i-1} + k_i + 1)$th column and the $(n_1 + \cdots + n_{\ell-1} + k_\ell + 1)$th column of we^T are identical. Thus, the above defined chain corresponding to the ith and ℓth block is indeed a Jordan chain of (2.8) corresponding to $\widehat{\lambda}$ of length n_ℓ.

Now, it remains to select $m - 1$ linearly independent ones of the lengths at least $n_2, \ldots, n_m$ to conclude the proof. To achieve this, define $i := 1$. Then, for $\ell = 2, \ldots, m$:

- denote by C_ℓ the chain corresponding to the ith and ℓth block.
- if $k_i > k_\ell$, set $i := \ell$.

Thus, we obtain the chains $C_2, \ldots, C_m$ of the desired lengths, which are easily checked to be linearly independent. Also, they are equal to the chains in (2.9) if $k_1 = \cdots = k_m$; this includes the generic case that $k_1 = \cdots = k_m = 1$, i.e., the first component of each $v^{(j)}$ is nonzero. □

Remark 2.7 An analogous result for the infinite eigenvalue of (E, A) can be obtained by applying Theorem 2.6 to the eigenvalue $\widehat{\lambda} = 0$ of the reverse pencil (A, E).

Example 2.8 Consider the matrix pencil $(E, A) = \big(I_{10}, J_4(0) \oplus J_3(0) \oplus J_3(0)\big)$, i.e., we have $\widehat{\lambda} = 0$ and $(n_1, n_2, n_3) = (4, 3, 3)$, and the perturbation (uv^T, uv^T) where $v^T = [e_{2,4}^T, e_{3,3}^T, e_{1,3}^T]$ and $u \in \mathbb{C}^{10}$ is arbitrary. Then, the perturbed pencil

$$\left(\left[\begin{array}{cccc|ccc|ccc}
1 & u_1 & & & & & u_1 & u_1 & & \\
 & u_2{+}1 & & & & & u_2 & u_2 & & \\
 & u_3 & 1 & & & & u_3 & u_3 & & \\
 & u_4 & & 1 & & & u_4 & u_4 & & \\
\hline
 & u_5 & & & 1 & & u_5 & u_5 & & \\
 & u_6 & & & & 1 & u_6 & u_6 & & \\
 & u_7 & & & & & u_7{+}1 & u_7 & & \\
\hline
 & u_8 & & & & & u_8 & u_8{+}1 & & \\
 & u_9 & & & & & u_9 & u_9 & 1 & \\
 & u_{10} & & & & & u_{10} & u_{10} & & 1
\end{array}\right], \left[\begin{array}{cccc|ccc|ccc}
0 & u_1{+}1 & & & & & u_1 & u_1 & & \\
 & u_2 & 1 & & & & u_2 & u_2 & & \\
 & u_3 & 0 & 1 & & & u_3 & u_3 & & \\
 & u_4 & & 0 & & & u_4 & u_4 & & \\
\hline
 & u_5 & & & 0 & 1 & u_5 & u_5 & & \\
 & u_6 & & & & 0 & u_6{+}1 & u_6 & & \\
 & u_7 & & & & & u_7 & u_7 & & \\
\hline
 & u_8 & & & & & u_8 & u_8 & 1 & \\
 & u_9 & & & & & u_9 & u_9 & 0 & 1 \\
 & u_{10} & & & & & u_{10} & u_{10} & & 0
\end{array}\right] \right)$$

is in partial Brunovsky form and we read off $(k_1, k_2, k_3) = (1, 2, 0)$. The linearly independent chains of lengths n_2 and n_3 constructed in the proof of Theorem 2.6 are given by $-e_5, e_1 - e_6, e_2 - e_7$ and $e_1, e_2 - e_8, e_3 - e_9$. Note that the latter chain can be extended to length 4 by adding the vector $e_4 - e_{10}$. ☐

2.3 Unstructured rank-1 perturbations

Low-rank perturbations of regular, unstructured matrix pencils have been investigated in [3]; in the main theorem [3, Theorem 3.3] the generic behavior of the Jordan structure of regular matrix pencils under perturbations $\lambda B_1 + B_0$ is described, where $\operatorname{rank}(\widehat{\lambda} B_1 + B_0)$ and $\operatorname{rank}(B_1)$ are prescribed. The generic behavior is then given by the largest $\operatorname{rank}(\widehat{\lambda} B_1 + B_0)$ blocks corresponding to $\widehat{\lambda}$ being destroyed and the second largest $\operatorname{rank}(B_1)$ blocks corresponding to $\widehat{\lambda}$ being turned into blocks of size one.

In the following theorem, we will consider perturbations of the form $(\lambda\beta - \alpha)uv^T$, thereby prescribing their *normal rank* (i.e., the maximum rank for all $\lambda \in \mathbb{C}$) to be equal to one.

Theorem 2.9 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular matrix pencil with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda} \in \mathbb{C}$. Then, for all $(\alpha, \beta) \in (\mathbb{C} \times \mathbb{C}) \setminus \{0\}$ there is a generic set $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that for all $(u, v) \in \Omega$ the perturbed pencil $(E + \beta uv^T, A + \alpha uv^T)$ is regular and has the partial multiplicities*

$$(n_2, \ldots, n_m) \quad \text{or} \quad (n_2, \ldots, n_m, 1)$$

associated with $\widehat{\lambda}$ if $\alpha \neq \widehat{\lambda}\beta$ or $\alpha = \widehat{\lambda}\beta$, respectively.

Proof. Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and let $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}$ be fixed. Consider the polynomial

$$\sum_{j=0}^{n} c_j \lambda^j = \det\big(\lambda(E + \beta uv^T) - A - \alpha uv^T\big),$$

whose coefficients $c_j = c_j(u,v)$ depend polynomially on the entries of $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$. Therefore, since $c_j(0,0) \neq 0$ for at least one j (recall that (E,A) is regular), we have that $c_j(u,v) \neq 0$ for at least one j on a set $\Lambda \subseteq \mathbb{C}^n \times \mathbb{C}^n$ that is by definition generic, i.e., $(E + \beta uv^T, A + \alpha uv^T)$ is regular for all $(u,v) \in \Lambda$.

Hence, for all $(u,v) \in \Lambda$ the perturbed pencil $(E + \beta uv^T, A + \alpha uv^T)$ has partial multiplicities *greater than or equal to* $(n_2, \ldots, n_m)$ associated with $\widehat{\lambda}$ (i.e., the perturbed pencil has at least $m-1$ Jordan chains of lengths at least $n_2, \ldots, n_m$ associated with $\widehat{\lambda}$) by Theorem 2.6. If $\alpha = \widehat{\lambda}\beta$, it has one additional chain of length at least one because of inequality (2.6), then the resulting partial multiplicities associated with $\widehat{\lambda}$ are greater than or equal to $(n_2, \ldots, n_m, 1)$. Thus, for all $(u,v) \in \Lambda$ we factorize

$$\det\big((\lambda + \widehat{\lambda})(E + \beta uv^T) - (A + \alpha uv^T)\big) = \begin{cases} \lambda^{n_2 + \cdots + n_m}\, q_1(\lambda) & \text{if } \alpha \neq \widehat{\lambda}\beta, \\ \lambda^{n_2 + \cdots + n_m + 1}\, q_2(\lambda) & \text{if } \alpha = \widehat{\lambda}\beta \end{cases}$$

for suitable polynomials $q_1(\lambda)$ and $q_2(\lambda)$. For continuity reasons this factorization must even hold for all $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$. Also, the coefficients $q_1(0)$ and $q_2(0)$ depend polynomially on the entries of (u,v). Let us now assume that (E,A) is in *Weierstraß canonical form* (see, e.g., [6, Chapter 12]) and that the blocks are arranged such that the blocks corresponding to $\widehat{\lambda}$ come first and are ordered decreasingly with respect to their size. Consider the perturbation $(u,v) = (e_{n_1}, e_1)$, then the perturbed first block is given by

$$-J_{n_1}(\widehat{\lambda} - \lambda) + (\beta\lambda - \alpha)e_{n_1}e_1^T,$$

whose algebraic multiplicity at $\widehat{\lambda}$ is clearly given by 0 if $\alpha \neq \widehat{\lambda}\beta$ and 1 if $\alpha = \widehat{\lambda}\beta$. As the other blocks are unchanged, $q_1(0)$ and $q_2(0)$ are indeed nonzero for this particular choice of (u,v), and hence they are nonzero for all $(u,v) \in \widetilde{\Lambda}$, where $\widetilde{\Lambda} \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is a suitable generic set. Then, the assertion is obtained as the intersection of two generic sets is again generic. □

Remark 2.10 Analogous results for the infinite eigenvalue of regular pencils (E,A) can be obtained by applying Theorem 2.9 to the eigenvalue $\widehat{\lambda} = 0$ of the reverse pencil (A,E).

2.4 Structured Kronecker canonical forms

Before turning to perturbations of structured matrix pencils, we will present structured Kronecker forms for different types of structured matrix pencils.

Recall that T-alternating matrix pencils exhibit the eigenvalue pairing $(\widehat{\lambda}, -\widehat{\lambda})$, which degenerates for the eigenvalues 0 and ∞. This leads to a pairing of these blocks in Kronecker form, which is reflected in the following T-even Kronecker form that was deduced in [25]. While there is a version of the following theorem for arbitrary T-even matrix pencils, we restrict ourselves to the regular case here.

Theorem 2.11 (T-even Kronecker form) *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular, T-even matrix pencil. Then, there is a nonsingular matrix $X \in \mathbb{C}^{n,n}$, such that*

$$X(\delta E - \gamma A)X^T = \mathcal{K}_{\mathcal{I}} \oplus \mathcal{K}_{\mathcal{Z}} \oplus \mathcal{K}_{\mathcal{F}},$$

where

$$\begin{aligned}
\mathcal{K}_{\mathcal{I}} &= \mathcal{I}_{2\delta_1+1} \oplus \cdots \oplus \mathcal{I}_{2\delta_\ell+1} \oplus \mathcal{I}_{2\epsilon_1} \oplus \cdots \oplus \mathcal{I}_{2\epsilon_m}, \\
\mathcal{K}_{\mathcal{Z}} &= \mathcal{Z}_{2\rho_1+1} \oplus \cdots \oplus \mathcal{Z}_{2\rho_r+1} \oplus \mathcal{Z}_{2\sigma_1} \oplus \cdots \oplus \mathcal{Z}_{2\sigma_s}, \\
\mathcal{K}_{\mathcal{F}} &= \mathcal{F}_{\phi_1} \oplus \cdots \oplus \mathcal{F}_{\phi_t},
\end{aligned}$$

and the blocks are given as follows:

1) *$\mathcal{I}_{2\delta_j+1}$ is one odd-sized block that corresponds to the eigenvalue ∞:*

$$\delta \begin{bmatrix}
 & & & & & & 0 \\
 & & & & & \iddots & 1 \\
 & & & & \iddots & \iddots & \\
 & & & 0 & 1 & & \\
 & & \iddots & -1 & & & \\
 & \iddots & \iddots & & & & \\
0 & -1 & & & & &
\end{bmatrix} - \gamma \begin{bmatrix}
 & & 1 \\
 & \iddots & \\
1 & &
\end{bmatrix} \in \mathbb{C}^{(2\delta_j+1)\times(2\delta_j+1)}.$$

2) *$\mathcal{I}_{2\epsilon_j}$ contains two even-sized blocks that correspond to the eigenvalue ∞:*

$$\delta \left[\begin{array}{cccc|cccc}
 & & & & & & & 0 \\
 & & & & & & \iddots & 1 \\
 & & & & & \iddots & \iddots & \\
 & & & & 0 & 1 & & \\
\hline
 & & & 0 & & & & \\
 & & \iddots & -1 & & & & \\
 & \iddots & \iddots & & & & & \\
0 & -1 & & & & & &
\end{array}\right] - \gamma \left[\begin{array}{ccc|ccc}
 & & & & & 1 \\
 & & & & \iddots & \\
 & & & 1 & & \\
\hline
 & & 1 & & & \\
 & \iddots & & & & \\
1 & & & & &
\end{array}\right] \in \mathbb{C}^{(4\epsilon_j)\times(4\epsilon_j)}.$$

3) *$\mathcal{Z}_{2\rho_j+1}$ contains two odd-sized blocks that correspond to the eigenvalue 0:*

$$\delta \left[\begin{array}{ccc|ccc}
 & & & & & 1 \\
 & & & & \iddots & \\
 & & & 1 & & \\
\hline
 & & -1 & & & \\
 & \iddots & & & & \\
-1 & & & & &
\end{array}\right] - \gamma \left[\begin{array}{cccc|cccc}
 & & & & & & & 0 \\
 & & & & & & \iddots & 1 \\
 & & & & & \iddots & \iddots & \\
 & & & & 0 & 1 & & \\
\hline
 & & & 0 & & & & \\
 & & \iddots & 1 & & & & \\
 & \iddots & \iddots & & & & & \\
0 & 1 & & & & & &
\end{array}\right] \in \mathbb{C}^{(4\rho_j+2)\times(4\rho_j+2)}.$$

4) *$\mathcal{Z}_{2\sigma_j}$ is one even-sized block that corresponds to the eigenvalue 0:*

$$\delta \left[\begin{array}{cccccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array}\right] - \gamma \left[\begin{array}{cccc} & & & 0 \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{array}\right] \in \mathbb{C}^{(2\sigma_j)\times(2\sigma_j)}.$$

5) *$\mathcal{F}_{\phi_j}$ contains two $\phi_j \times \phi_j$ blocks that correspond to the eigenvalues $\lambda_j, -\lambda_j \in \mathbb{C}\setminus\{0\}$:*

$$\delta \left[\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array}\right] - \gamma \left[\begin{array}{ccc|ccc} & & & & & \lambda_j \\ & & & & \iddots & 1 \\ & & & \iddots & \iddots & \\ & & & \lambda_j & 1 & \\ \hline & & \lambda_j & & & \\ & \iddots & 1 & & & \\ \iddots & \iddots & & & & \\ \lambda_j & 1 & & & & \end{array}\right] \in \mathbb{C}^{(2\phi_j)\times(2\phi_j)}.$$

The following structured Kronecker form was deduced for arbitrary symmetric pencils in [25], we restrict ourselves to regular pencils here. Note that symmetric pencils do not show any specific eigenvalue pairing.

Theorem 2.12 (Symmetric Kronecker form) *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular, symmetric matrix pencil. Then, there is a nonsingular matrix $X \in \mathbb{C}^{n,n}$, such that*

$$X(\delta E - \gamma A)X^T = \mathcal{K}_\mathcal{I} \oplus \mathcal{K}_\mathcal{F},$$

where

$$\begin{aligned} \mathcal{K}_\mathcal{I} &= \mathcal{I}_{\delta_1} \oplus \cdots \oplus \mathcal{I}_{\delta_\ell}, \\ \mathcal{K}_\mathcal{F} &= \mathcal{F}_{\epsilon_1} \oplus \cdots \oplus \mathcal{F}_{\epsilon_m}, \end{aligned}$$

and the blocks are given as follows:

1) *$\mathcal{I}_{\delta_j}$ is one $\delta_j \times \delta_j$ block that corresponds to the eigenvalue ∞:*

$$\delta \left[\begin{array}{cccc} & & & 0 \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{array}\right] - \gamma \left[\begin{array}{ccc} & & 1 \\ & \iddots & \\ 1 & & \end{array}\right] \in \mathbb{C}^{\delta_j \times \delta_j}.$$

2) *$\mathcal{F}_{\epsilon_j}$ is one $\epsilon_j \times \epsilon_j$ block that corresponds to the eigenvalue $\lambda_j \in \mathbb{C}$:*

$$\delta \left[\begin{array}{ccc} & & 1 \\ & \iddots & \\ 1 & & \end{array}\right] - \gamma \left[\begin{array}{cccc} & & & \lambda_j \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ \lambda_j & 1 & & \end{array}\right] \in \mathbb{C}^{\epsilon_j \times \epsilon_j}.$$

3 T-alternating rank-1 perturbations

We will now turn to T-alternating matrix pencils. Since results on T-even pencils will easily generalize to the T-odd case, let us focus on these first. Subject of our investigation are T-even rank-k perturbations of the form

$$\begin{bmatrix} u_1 & \dots & u_k \end{bmatrix} \left(\delta E, \delta A\right) \begin{bmatrix} u_1 & \dots & u_k \end{bmatrix}^T, \tag{3.1}$$

where $(\delta E, \delta A)$ is a generic T-even $k \times k$ pencil and $u_1, \dots, u_k \in \mathbb{C}^n$ are certain generic vectors. Here, we consider a subset of the set of T-even $k \times k$ pencils

$$\mathcal{E}_k := \{(E, A) \in \mathbb{C}^{k,k} \times \mathbb{C}^{k,k} \mid (E, A) \text{ is } T\text{-even}\}$$

to be generic, if it can canonically be identified with a generic subset of $\mathbb{C}^{k^2}$. Hence, we denote with $[(E, A)]_{\mathcal{B}} \in \mathbb{C}^{k^2}$ the coordinates of the T-even matrix pencil (E, A) with respect to a basis $\mathcal{B}$ of $\mathcal{E}_k$. In order to characterize the pencil $(\delta E, \delta A)$ from (3.1), consider the following lemma.

Lemma 3.1 *There exists a generic set* $\Gamma \subseteq \mathcal{E}_k$ *such that any* $(E, A) \in \Gamma$ *is regular and has distinct eigenvalues.*

Proof. We follow the procedure laid out in the proof of Theorem 2.3. Given $(E, A) \in \mathcal{E}_k$, we define $\sum_{j=0}^{k} c_j \lambda^j := \det(\lambda E - A)$, then the coefficients $c_j = c_j(E, A)$ depend polynomially on the coordinates $[(E, A)]_{\mathcal{B}}$ and not all $c_j(E, A)$ are constantly zero as there exist regular, T-even pencils.

Further, define $p(E, A)$ as in (2.2) and $q(E, A)$ as in (2.3), then $p(E, A)$ and $q(E, A)$ both depend polynomially on the coordinates $[(E, A)]_{\mathcal{B}}$ and $p(E, A)q(E, A)$ is not constantly zero since there exist T-even pencils with distinct eigenvalues. Hence, the set

$$\Gamma := \left\{(E, A) \in \mathcal{E}_k \mid p(E, A)q(E, A) \neq 0 \text{ and } \exists j \in \{0, 1, \dots, k\} \text{ with } c_j(E, A) \neq 0 \right\} \tag{3.2}$$

of regular, T-even pencils with distinct eigenvalues is a generic subset of $\mathcal{E}_k$. □

Let us now go back to the perturbation (3.1): if $(\delta E, \delta A)$ is an element of Γ from (3.2), it is regular and has distinct eigenvalues. Similar to the treatment of unstructured matrix pencils as in (2.1), we assume without loss of generality that $(\delta E, \delta A)$ is in T-even Kronecker form as in Theorem 2.11. Then, $(\delta E, \delta A)$ can only consist of the following types of blocks:

- $\mathcal{I}_1$ – a single 1×1 block corresponding to the eigenvalue ∞,
- $\mathcal{F}_1$ – two paired 1×1 blocks corresponding to the eigenvalues $\widehat{\lambda}, -\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$,

as all other regular blocks in T-even Kronecker form have multiple eigenvalues. If k is even, $(\delta E, \delta A)$ consists of $k/2$ blocks of the form $\mathcal{F}_1$, which all have distinct eigenvalues. If k is odd, δE is singular and thus $(\delta E, \delta A)$ has one block $\mathcal{I}_1$; since all other eigenvalues have to be distinct (and distinct from ∞), there occur $(k-1)/2$ blocks of the type $\mathcal{F}_1$. Therefore, the perturbation (3.1) is the sum of, on the one hand, rank-1 perturbations of the form

$$(0, uu^T) \quad \text{where} \quad u \in \Omega \subseteq \mathbb{C}^n \quad \text{is a generic set,}$$

that we study in this paper, and on the other hand of rank-2 perturbations of the form

$$\begin{bmatrix} u & v \end{bmatrix} \Big(\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \Big) \begin{bmatrix} u^T \\ v^T \end{bmatrix} \quad \text{where} \quad (u,v) \in \Psi \subseteq \mathbb{C}^n \times \mathbb{C}^n \quad \text{is generic,}$$

that will be treated in future work. Let us now examine the T-even perturbation result from the introduction more closely.

Example 1.2 continued. Consider the T-even matrix pencil

$$(E,A) = \left(\begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & 0 & & \\ 0 & -1 & & \end{bmatrix}, \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \right) \in \mathbb{C}^{4,4} \times \mathbb{C}^{4,4}.$$

Then, defining

$$\Omega := \left\{ [b_1, b_2, b_3, b_4]^T \in \mathbb{C}^4 \;\middle|\; b_1, b_3 \neq 0 \;\text{ and }\; 1 + 2b_1b_4 + 2b_2b_3 \neq 0 \right\}, \tag{3.3}$$

which is clearly a generic subset of $\mathbb{C}^4$, we aim to show that the perturbed pencil $(E, A+bb^T)$ has the only eigenvalue ∞ with partial multiplicities $(3,1)$ for all $b \in \Omega$.

First, observe that the pencil $(\widehat{E}, \widehat{A}) := \Sigma R_4\,(E,A)\,\Sigma$, where $\Sigma = \operatorname{diag}(1,-1,1,1)$, is in Weierstraß canonical form. Analogously, the transformed perturbation is given by $(0, uv^T)$, where $u = \Sigma R_4 b = [b_4, -b_3, b_2, b_1]^T$ and $v^T = b^T\Sigma = [b_1, -b_2, b_3, b_4]$. Then, we transform the pencil $(\widehat{E}, \widehat{A} + uv^T)$ to partial Brunovsky form as in Theorem 2.6. Therefore, consider that the matrix $S := \operatorname{Toep}(b_1, -b_2) \oplus \operatorname{Toep}(b_3, b_4)$ is invertible for all $b \in \Omega$ and that

$$S(\widehat{E}, \widehat{A} + uv^T)S^{-1} = \left(\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 1 + b_1b_4 + b_2b_3 & 0 & b_1b_4 + b_2b_3 & 0 \\ -b_1b_3 & 1 & -b_1b_3 & 0 \\ b_1b_4 + b_2b_3 & 0 & 1 + b_1b_4 + b_2b_3 & 0 \\ b_1b_3 & 0 & b_1b_3 & 1 \end{bmatrix} \right)$$

holds. Now, it is easy to verify (see Definition 2.4) that the above pencil has the following chains associated with ∞:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{b_1b_3} \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1+b_1b_4+b_2b_3}{b_1b_3} \\ 0 \\ \frac{b_1b_4+b_2b_3}{b_1b_3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which are linearly independent due to the last condition in (3.3). Hence, the perturbed pencil has the partial multiplicities $(3,1)$ associated with ∞ for all $b \in \Omega$. $\square$

Observe that in this example, a generic, structure-preserving perturbation of a T-even matrix pencil is different from a generic, unstructured perturbation, which would produce the partial multiplicities $(2, 1)$ (see Theorem 2.9). Further, it is not obvious why the partial multiplicities are generically altered at all in this example, as no instance of the eigenvalue ∞ is destroyed under rank-1 perturbation. Using the same approach as in Example 1.2, we will investigate T-alternating matrix pencils under generic, structure-preserving rank-1 perturbations in the following theorem.

Theorem 3.2 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular matrix pencil that is T-even or T-odd and has the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda} \in \mathbb{C} \cup \{\infty\}$. Then, there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$ the perturbed pencil $(E, A + uu^T)$ or $(E + uu^T, A)$, respectively, is regular and its partial multiplicities at $\widehat{\lambda}$ are given by Table* 3.1.

Table 3.1: Rank-1 perturbations of T-alternating matrix pencils.

parity	eigenvalue $\widehat{\lambda}$	case	multiplicities
T-even	$\widehat{\lambda} = 0$	n_1 even	$(n_2, n_3, \ldots, n_m)$
		n_1 odd	$(n_2 + 1, n_3, \ldots, n_m)$
	$\widehat{\lambda} = \infty$	n_1 even	$(n_2 + 1, n_3, \ldots, n_m, 1)$
		n_1 odd	$(n_2, n_3, \ldots, n_m, 1)$
	$\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$		$(n_2, n_3, \ldots, n_m)$
T-odd	$\widehat{\lambda} = 0$	n_1 even	$(n_2 + 1, n_3, \ldots, n_m, 1)$
		n_1 odd	$(n_2, n_3, \ldots, n_m, 1)$
	$\widehat{\lambda} = \infty$	n_1 even	$(n_2, n_3, \ldots, n_m)$
		n_1 odd	$(n_2 + 1, n_3, \ldots, n_m)$
	$\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$		$(n_2, n_3, \ldots, n_m)$

First, we note that it suffices to prove the assertion for T-even pencils, as the analogous result for T-odd pencils (E, A) then follows by considering the T-even reverse pencil (A, E). Hence, let (E, A) be T-even and consider the polynomial

$$\sum_{j=0}^{n} c_j \lambda^j = \det\left(\lambda E - A - uu^T\right),$$

whose coefficients $c_j = c_j(u)$ depend polynomially on the entries of $u \in \mathbb{C}^n$. Therefore, since $c_j(0) \neq 0$ for at least one j, the set

$$\Lambda := \left\{\, u \in \mathbb{C}^n \mid \exists j \in \{0, 1, \ldots, n\} \text{ with } c_j(u) \neq 0 \,\right\} \tag{3.4}$$

is by definition a generic subset of $\mathbb{C}^n$ and $(E, A + uu^T)$ is regular for all $u \in \Lambda$.

Also, the following lemma will be essential in our considerations.

Lemma 3.3 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and T-even. Also, let there be a generic $\Psi \subseteq \mathbb{C}^n$ such that for all $u \in \Psi$ the perturbed pencil $(E, A + uu^T)$ is regular and has partial multiplicities greater than or equal to $(\widetilde{n}_1, \ldots, \widetilde{n}_{\widetilde{m}})$ associated with some $\widehat{\lambda} \in \mathbb{C}$. If $(E, A + uu^T)$ has the algebraic multiplicity $\widetilde{n}_1 + \cdots + \widetilde{n}_{\widetilde{m}}$ at $\widehat{\lambda}$ for one $u \in \mathbb{C}^n$, it is regular and has the partial multiplicities $(\widetilde{n}_1, \ldots, \widetilde{n}_{\widetilde{m}})$ at $\widehat{\lambda}$ on some generic subset of $\mathbb{C}^n$.*

Proof. By hypothesis, for all $u \in \Psi$ we have

$$\det\left((\lambda + \widehat{\lambda})E - A - uu^T\right) = \lambda^{\widetilde{n}_1 + \cdots + \widetilde{n}_{\widetilde{m}}} q(\lambda)$$

for a suitable polynomial $q(\lambda)$, noting that the coefficient $q(0)$ depends polynomially on the entries of u. For continuity reasons, this factorization even holds for all $u \in \mathbb{C}^n$. Since there is one particular $u \in \mathbb{C}^n$ such that $q(0) \neq 0$, by definition $q(0) \neq 0$ is satisfied on some generic set $\widetilde{\Psi} \subseteq \mathbb{C}^n$. Then, $\Psi \cap \widetilde{\Psi}$ is clearly generic and for all $u \in \Psi \cap \widetilde{\Psi}$, the perturbed pencil $(E, A + uu^T)$ is regular and has the partial multiplicities $(\widetilde{n}_1, \ldots, \widetilde{n}_{\widetilde{m}})$ associated with $\widehat{\lambda}$. □

We continue by verifying the partial multiplicities given in the top half of Table 3.1 by proving the Lemmas 3.4, 3.5, and 3.6. If not stated otherwise, we will in the following assume (E, A) to be in T-even Kronecker form as in Theorem 2.11, where the blocks corresponding to $\widehat{\lambda}$ come first and are ordered decreasingly with respect to their size.

3.1 Case $\widehat{\lambda} = 0$

Lemma 3.4 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and T-even with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with the eigenvalue 0. Then, there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$ the perturbed pencil $(E, A + uu^T)$ is regular and has the partial multiplicities at 0 given by*

$$(n_2, n_3, \ldots, n_m) \quad \text{or} \quad (n_2 + 1, n_3, \ldots, n_m)$$

if n_1 is even or odd, respectively.

Proof. **Case n_1 even**. For all $u \in \Lambda$ as in (3.4), Theorem 2.6 implies that $(E, A + uu^T)$ has partial multiplicities greater than or equal to $(n_2, \ldots, n_m)$ associated with 0. Regard the largest block of (E, A) corresponding to 0 given by

$$\left(\begin{bmatrix} 0 & R_{n_1/2} \\ -R_{n_1/2} & 0 \end{bmatrix}, R_{n_1} J_{n_1}(0) \right)$$

and consider the perturbation $(0, uu^T) = (0, e_1 e_1^T)$. As the perturbed pencil $(E, A + uu^T)$ has the algebraic multiplicity $n_2 + \cdots + n_m$ at 0 for this particular u, by Lemma 3.3 it is regular and has the partial multiplicities $(n_2, \ldots, n_m)$ at 0 for all u that are elements of a suitable generic subset of $\mathbb{C}^n$.

Case n_1 odd. As above, $(E, A + uu^T)$ has partial multiplicities greater than or equal to

$(n_2, \ldots, n_m)$ at 0 for all $u \in \Lambda$. Now, we claim that these partial multiplicities are even greater than or equal to $(n_2+1, n_3, \ldots, n_m)$ on some generic subset of $\mathbb{C}^n$. To prove this, we will show in the following that $(E, A+uu^T)$ generically has a chain of length at least n_2+1 associated with 0.

First, let us partition the pencil $(E, A) = (P \oplus \widetilde{E}, J \oplus \widetilde{A})$ such that (P, J) has only the eigenvalue 0 and $(\widetilde{E}, \widetilde{A})$ does not have the eigenvalue 0. To show that there is a chain of the desired length associated with 0, we will transform (J, P) into Weierstraß canonical form, hence grouping together Jordan blocks of the same size, i.e.,

$$(n_1, n_2, \ldots, n_m) = (\underbrace{s_1, \ldots, s_1}_{t_1}, \ldots, \underbrace{s_\nu, \ldots, s_\nu}_{t_\nu}),$$

such that $s_1 > s_2 > \cdots > s_\nu > 0$. Observe that the pencil $T^{-1}P^T(P, J)T$ is in Weierstraß canonical form, where $T := \bigoplus_{i=1}^{\nu} T_i$ with

$$T_i := \begin{bmatrix} \Sigma_{s_i} & 0 \\ 0 & I_{s_i} \end{bmatrix}^{\oplus t_i/2} \qquad \text{or} \qquad T_i := \begin{bmatrix} \Sigma_{s_i/2} & 0 \\ 0 & I_{s_i/2} \end{bmatrix}^{\oplus t_i}$$

if s_i is odd or even, respectively. We go on to transform the whole perturbed pencil

$$(T^{-1} \oplus I_{n-a})(P^T \oplus I_{n-a})(E, A+bb^T)(T \oplus I_{n-a}) =: (\widehat{E}, \widehat{A} + uv^T), \tag{3.5}$$

where the vector characterizing our symmetric rank-1 matrix shall be denoted by $b \in \mathbb{C}^n$ in this argument. Then, the transformed perturbation uv^T is given by:

$$u = (T^{-1} \oplus I_{n-a})(P^T \oplus I_{n-a})b \quad \text{and} \quad v = (T \oplus I_{n-a})^T b. \tag{3.6}$$

Let now the vectors b, u, v have the partitioning

$$* = \begin{bmatrix} *^{(1)} \\ \vdots \\ *^{(\nu)} \\ \widetilde{*} \end{bmatrix}, \quad *^{(i)} = \begin{bmatrix} *^{(i,1)} \\ \vdots \\ *^{(i,t_i)} \end{bmatrix}, \quad *^{(i,j)} = \begin{bmatrix} *_1^{(i,j)} \\ \vdots \\ *_{s_i}^{(i,j)} \end{bmatrix} \in \mathbb{C}^{s_i}, \quad j = 1, ..., t_i, \quad i = 1, ..., \nu, \tag{3.7}$$

i.e., let $* \in \{b, u, v\}$. In order to apply Theorem 2.6 to the perturbed pencil (3.5) and its eigenvalue 0, let us from now on assume that the first component of each $v^{(i,j)}$ is different from zero and that $(E, A+bb^T)$ is regular. Both are generic conditions on b since v is obtained from b by multiplication with an invertible matrix; hence they are satisfied for all $b \in \Gamma$, where $\Gamma \subseteq \mathbb{C}^n$ is a certain generic set. Denoting by a the dimension of P (and J), as in the proof of Theorem 2.6 we define

$$S := \left(\bigoplus_{i=1}^{\nu} \bigoplus_{j=1}^{t_i} \operatorname{Toep}(v^{(i,j)}) \right) \oplus I_{n-a}, \tag{3.8}$$

observing that S is invertible for all $b \in \Gamma$. Then,

$$S(\widehat{E}, \widehat{A} + uv^T)S^{-1} = (\widehat{E}, \widehat{A} + we^T) \tag{3.9}$$

is in partial Brunovsky form with $e^T = [e_{1,n_1}^T, \ldots, e_{1,n_m}^T, \widetilde{e}^T] = v^T S^{-1}$ and $w = Su$ that we assume to be partitioned as in (3.7). We continue by computing certain entries of w noting that we study the case that $n_1 = s_1$ is odd and hence t_1 is even. By (3.6), $u^{(1)}$ and $v^{(1)}$ are given by

$$u^{(1)} = \begin{bmatrix} 0 & -\Sigma_{s_1} R_{s_1} & & & \\ R_{s_1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\Sigma_{s_1} R_{s_1} \\ & & & R_{s_1} & 0 \end{bmatrix} b^{(1)} \quad \text{and} \quad v^{(1)} = \begin{bmatrix} \Sigma_{s_1} & & & & \\ & I_{s_1} & & & \\ & & \ddots & & \\ & & & \Sigma_{s_1} & \\ & & & & I_{s_1} \end{bmatrix} b^{(1)},$$

from which we extract

$$w_{s_1}^{(1,2s-1)} = v_1^{(1,2s-1)} u_{s_1}^{(1,2s-1)} = -b_1^{(1,2s-1)} b_1^{(1,2s)} = -u_{s_1}^{(1,2s)} v_1^{(1,2s)} = -w_{s_1}^{(1,2s)} \tag{3.10}$$

for all $s = 1, \ldots, t_1/2$. Under this condition, we will show that the perturbed pencil has a Jordan chain of length $s_1 + 1 = n_2 + 1$ by following the steps of the proof of [16, Theorem 4.2]. First, (3.9) has for all $b \in \Gamma$ the following chains associated with its zero eigenvalue:

$$\begin{aligned} C_{1,s} &:= \left\{ e_{2(s-1)s_1+1} - e_{(2s-1)s_1+1}, \ldots, e_{(2s-1)s_1} - e_{2ss_1} \right\}, & s &= 1, ..., t_1/2, \\ C_{\ell,s} &:= \left\{ -e_1 + e_{\sum_{k=1}^{\ell-1} t_k s_k + (s-1)s_\ell + 1}, \ldots, -e_{s_\ell} + e_{\sum_{k=1}^{\ell-1} t_k s_k + s s_\ell} \right\}, & s &= 1, ..., t_\ell, \ell = 2, ..., \nu, \end{aligned} \tag{3.11}$$

as they are easily obtained from the ones in (2.9). We consider the following linear combination of them

$$\sum_{s=1}^{t_1/2} \alpha_{1,s} C_{1,s} + \sum_{\ell=2}^{\nu} \sum_{s=1}^{t_\ell} \alpha_{\ell,s} C_{\ell,s}, \tag{3.12}$$

where the sums are to be interpreted according to Definition 2.5. If $b \in \Gamma$, then (3.12) is a Jordan chain of (3.9) associated with 0, where the parameters $\alpha_{1,1}, \ldots, \alpha_{1,t_1/2}, \alpha_{2,1}, \ldots, \alpha_{\nu,t_\nu} \in \mathbb{C}$ remain to be specified. Also, (3.12) has length s_1 if at least one $\alpha_{1,s}$ for $s = 1, \ldots, t_1/2$ is nonzero. We will denote by y the last vector of (3.12) that we partition as in (3.7) and observe that

$$\begin{aligned} & y_{s_1}^{(1,2s-1)} = \alpha_{1,s}, y_{s_1}^{(1,2s)} = -\alpha_{1,s}, && s = 1, \ldots, t_1/2, \\ & y_{s_\ell}^{(\ell,s)} = \alpha_{\ell,s}, && s = 1, \ldots, t_\ell,\ \ell = 2, \ldots, \nu. \end{aligned} \tag{3.13}$$

To show that (3.12) can be extended to length $s_1 + 1$, we have to find an $x \in \mathbb{C}^n$ such that

$$(\widehat{A} + we^T)x = \widehat{E}y$$

for a specific choice of the α's from (3.13) with at least one $\alpha_{1,s}$ being nonzero. Partitioning x as in (3.7) and making the ansatz $\widetilde{x} = 0$, the system reduces to

$$\left(\bigoplus_{\ell=1}^{\nu}\bigoplus_{s=1}^{t_\ell} J_{s_\ell}(0) + we^T\right)\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(\nu)} \end{bmatrix} = I_a \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(\nu)} \end{bmatrix}.$$

We write down this system component-wise already inserting (3.13):

$$w_k^{(\ell,s)}\left(\sum_{i=1}^{\nu}\sum_{j=1}^{t_i} x_1^{(i,j)}\right) + x_{k+1}^{(\ell,s)} = y_k^{(\ell,s)}, \qquad k = 1,...,s_\ell - 1,\ s = 1,...,t_\ell,\ \ell = 1,...,\nu, \tag{3.14}$$

$$w_{s_\ell}^{(\ell,s)}\left(\sum_{i=1}^{\nu}\sum_{j=1}^{t_i} x_1^{(i,j)}\right) = \alpha_{\ell,s}, \qquad s = 1,...,t_\ell,\ \ell = 2,...,\nu, \tag{3.15}$$

$$w_{s_1}^{(1,2s-1)}\left(\sum_{i=1}^{\nu}\sum_{j=1}^{t_i} x_1^{(i,j)}\right) = \alpha_{1,s}, \qquad s = 1,...,t_1/2, \tag{3.16}$$

$$w_{s_1}^{(1,2s)}\left(\sum_{i=1}^{\nu}\sum_{j=1}^{t_i} x_1^{(i,j)}\right) = -\alpha_{1,s}, \qquad s = 1,...,t_1/2. \tag{3.17}$$

First, set $x_1^{(1,1)} = 1$ and $x_1^{(i,j)} = 0$ for all $(i,j) \neq (1,1)$. Then, (3.14) can be solved by choosing $x_{k+1}^{(\ell,s)} := y_k^{(\ell,s)} - w_k^{(\ell,s)}$ for $k = 1,\ldots,s_\ell - 1$, $s = 1,\ldots,t_\ell$, and $\ell = 1,\ldots,\nu$. To solve (3.15), define $\alpha_{\ell,s} := w_{s_\ell}^{(\ell,s)}$ for $s = 1,\ldots,t_\ell$ and $\ell = 2,\ldots,\nu$. Finally, (3.16) is solved by setting $\alpha_{1,s} := w_{s_1}^{(\ell,2s-1)}$ for $s = 1,\ldots,t_1/2$ noting that (3.17) is also satisfied because of equation (3.10). Also, at least one $\alpha_{1,s}$ being nonzero is a generic condition on b, so that there is a chain of the pencil $(E, A + bb^T)$ of length at least $s_1 + 1 = n_2 + 1$ for all $b \in \Psi$, where $\Psi \subseteq \mathbb{C}^n$ is a suitable generic set.

We remain to prove that the partial multiplicities of $(E, A+bb^T)$ at 0 cannot generically exceed $(n_2 + 1, n_3, \ldots, n_m)$. Hence, consider (E, A) to be perturbed by $(0, bb^T)$ with $b = e_1 + e_{n_1+2}$, then the first blocks of $(E, A + bb^T)$ are given by

$$\left(\begin{bmatrix} 0 & R_{n_1} \\ -R_{n_1} & 0 \end{bmatrix}, \begin{bmatrix} e_1e_1^T & R_{n_1}J_{n_1}(0) + e_1e_2^T \\ R_{n_1}J_{n_1}(0) + e_2e_1^T & e_2e_2^T \end{bmatrix}\right)$$

of which we aim to compute the determinant using the result from the appendix. Thus, we left-multiply the above pencil by $I_{n_1} \oplus (-I_{n_1})$ (thereby changing the determinant by a factor of -1) and set the dummy elements to $x = -1$ and $y = 1$, then the determinant of the obtained pencil is equal to $-\lambda^{2n_1} + 2\lambda^{n_1+1}$. Hence, by Lemma 3.3, $(E, A + bb^T)$ is regular and its partial multiplicities at 0 are given by $(n_2 + 1, n_3 \ldots, n_m)$ for all b that are elements of a suitable generic subset of $\mathbb{C}^n$. □

3.2 Case $\widehat{\lambda} = \infty$

Lemma 3.5 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and T-even with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with the eigenvalue ∞. Then, there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$ the perturbed pencil $(E, A + uu^T)$ is regular and has the partial multiplicities at ∞ given by*

$$(n_2 + 1, n_3, \ldots, n_m, 1) \quad \text{or} \quad (n_2, n_3, \ldots, n_m, 1)$$

if n_1 is even or odd, respectively.

Proof. **Case n_1 even**. First, by (2.6) the geometric multiplicity of (E, A) at ∞ cannot change under perturbations $(0, uu^T)$. Also applying Theorem 2.6, the perturbed pencil $(E, A + uu^T)$ has partial multiplicities greater than or equal to $(n_2, \ldots, n_m, 1)$ associated with ∞ for all $u \in \Lambda$ as in (3.4). As the eigenvalue ∞ of the T-even pencil (E, A) is the eigenvalue 0 of the T-odd reverse pencil (A, E), we will in the following arguments consider (A, E) and its zero eigenvalue under perturbations $(uu^T, 0)$.

Now, we claim that the partial multiplicities of $(A + uu^T, E)$ at 0 are greater than or equal to $(n_2 + 1, n_3, \ldots, n_m, 1)$ for all u that are elements of some generic subset of $\mathbb{C}^n$, i.e., that $(A + uu^T, E)$ generically has a chain of length at least $n_2 + 1$ associated with 0. Hence, we follow the same steps as in the proof of Lemma 3.4, partitioning the pencil $(A, E) = (P \oplus \widetilde{A}, J \oplus \widetilde{E})$ such that (P, J) only has the eigenvalue 0 and $(\widetilde{A}, \widetilde{E})$ does not have the eigenvalue 0. Then, we transform (P, J) into Weierstraß canonical form again grouping together blocks of the same size, i.e.,

$$(n_1, n_2, \ldots, n_m) = (\underbrace{s_1, \ldots, s_1}_{t_1}, \ldots, \underbrace{s_\nu, \ldots, s_\nu}_{t_\nu}),$$

such that $s_1 > s_2 > \cdots > s_\nu > 0$. Now, the pencil $T^{-1}P(P, J)T$ is in Weierstraß canonical form, where $T := \bigoplus_{i=1}^{\nu} T_i$ with

$$T_i := \begin{bmatrix} \Sigma_{s_i} & 0 \\ 0 & I_{s_i} \end{bmatrix}^{\oplus t_i/2} \quad \text{or} \quad T_i := \begin{bmatrix} \Sigma_{(s_i-1)/2} & 0 \\ 0 & I_{(s_i+1)/2} \end{bmatrix}^{\oplus t_i}$$

if s_i is even or odd, respectively. We transform the whole perturbed pencil

$$(T^{-1} \oplus I_{n-a})(P \oplus I_{n-a})(A + bb^T, E)(T \oplus I_{n-a}) =: (\widehat{A} + uv^T, \widehat{E}), \tag{3.18}$$

where the vector characterizing our symmetric rank-1 matrix is again denoted by $b \in \mathbb{C}^n$. We obtain for the transformed perturbation

$$u = (T^{-1} \oplus I_{n-a})(P \oplus I_{n-a})b \quad \text{and} \quad v = (T \oplus I_{n-a})^T b,$$

assuming b, u, v to have the partitioning (3.7). As in the proof of Lemma 3.4, we now aim to apply Theorem 2.6 to (3.18); hence we choose a generic $\Gamma \subseteq \mathbb{C}^n$ such that for all $b \in \Gamma$, the first component of each $v^{(i,j)}$ is different from zero and $(A+bb^T, E)$ is regular. For $b \in \Gamma$, we define S as in (3.8), since then

$$S(\widehat{A}+uv^T, \widehat{E})S^{-1} = (\widehat{A}+we^T, \widehat{E}) \tag{3.19}$$

is in partial Brunovsky form with $e^T = [e^T_{1,n_1}, \ldots, e^T_{1,n_m}, \widetilde{e}^T] = v^T S^{-1}$ and $w = Su$. Now, as $n_1 = s_1$ is even and hence t_1 is also even, we have

$$u^{(1)} = \begin{bmatrix} 0 & \Sigma_{s_1} R_{s_1} & & & \\ R_{s_1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \Sigma_{s_1} R_{s_1} \\ & & & R_{s_1} & 0 \end{bmatrix} b^{(1)} \quad \text{and} \quad v^{(1)} = \begin{bmatrix} \Sigma_{s_1} & & & & \\ & I_{s_1} & & & \\ & & \ddots & & \\ & & & \Sigma_{s_1} & \\ & & & & I_{s_1} \end{bmatrix} b^{(1)},$$

thus we obtain, partitioning w as in (3.7):

$$w^{(1,2s-1)}_{s_1} = v^{(1,2s-1)}_1 u^{(1,2s-1)}_{s_1} = -b^{(1,2s-1)}_1 b^{(1,2s)}_1 = -u^{(1,2s)}_{s_1} v^{(1,2s)}_1 = -w^{(1,2s)}_{s_1} \tag{3.20}$$

for all $s = 1, \ldots, t_1/2$. Now, for all $b \in \Gamma$ the pencil (3.19) has the Jordan chains from (3.11) associated with its zero eigenvalue. Additionally, it has the eigenvector e_1, which is linearly independent from all vectors in (3.11). Thus, let us now consider the following linear combination of these chains

$$\widehat{\alpha}\,\{e_1\} + \sum_{s=1}^{t_1/2} \alpha_{1,s} C_{1,s} + \sum_{\ell=2}^{\nu} \sum_{s=1}^{t_\ell} \alpha_{\ell,s} C_{\ell,s}, \tag{3.21}$$

where the sums are to be interpreted according to Definition 2.5. If $b \in \Gamma$, then (3.21) is a Jordan chain of (3.19) associated with 0, where $\widehat{\alpha}, \alpha_{1,1}, \ldots, \alpha_{1,t_1/2}, \alpha_{2,1}, \ldots, \alpha_{\nu,t_\nu} \in \mathbb{C}$ remain to be specified. Also, (3.21) has length s_1 if at least one $\alpha_{1,s}$ is nonzero. Now, let y be the last vector of (3.21) that is partitioned as in (3.7), then we observe that (3.13) holds. To show that the chain (3.21) can be extended to length s_1+1, we have to find an $x \in \mathbb{C}^n$ with

$$\widehat{E}x = (\widehat{A}+we^T)y$$

for a particular choice of the α's such that at least one $\alpha_{1,s}$ is nonzero. Thus, let x have the partitioning (3.7); then as $\widetilde{y} = 0$ holds we make the ansatz $\widetilde{x} = 0$, which yields the system

$$\bigoplus_{\ell=1}^{\nu} \bigoplus_{s=1}^{t_\ell} J_{s_\ell}(0) \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(\nu)} \end{bmatrix} = I_a \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(\nu)} \end{bmatrix} + e^T y \begin{bmatrix} w^{(1)} \\ \vdots \\ w^{(\nu)} \end{bmatrix}. \tag{3.22}$$

Now, because of the form of e, we obtain

$$e^T y = \sum_{i=1}^{\nu} \sum_{j=1}^{t_i} y^{(i,j)}_1. \tag{3.23}$$

If $s_\nu > 1$, then all chains $C_{1,s}$ and $C_{\ell,s}$ defined in (3.11) have lengths greater than one. Thus, their last vectors' 1st entries, (n_1+1)th entries, …, $(n_1+\cdots+n_{m-1}+1)$th entries, will be zero. Therefore, in (3.23) these chains do not contribute to the summation and we obtain $e^T y = y_1^{(1,1)} = \widehat{\alpha}$ in this case.

On the other hand, if $s_\nu = 1$, then of $\{e_1\}$ and the chains from (3.11), only $\{e_1\}$ and $C_{\nu,s}$ for $s = 1,\dots,t_\nu$ have length one. Thus, as in the above case, only those will contribute to the summmation in (3.23). Then, from (3.11) we observe that the first entry of each $C_{\nu,s}$ (only consisting of one vector) is -1. Hence, by (3.21) we have

$$y_1^{(1,1)} = \widehat{\alpha} - \alpha_{\nu,1} - \cdots - \alpha_{\nu,t_\nu}$$

and from (3.13) we obtain

$$y_1^{(\nu,s)} = y_{s_\nu}^{(\nu,s)} = \alpha_{\nu,s}, \quad s = 1,\dots,t_\nu.$$

Therefore, by (3.23) we follow that $e^T y = \widehat{\alpha}$ in this case as well. Now, (3.22) is rewritten as follows: we define $\widehat{\alpha} := 1$, then inserting (3.13) yields

$$\begin{aligned} x_{k+1}^{(\ell,s)} &= y_k^{(\ell,s)} + w_k^{(\ell,s)}, && k = 1,...,s_\ell - 1,\ s = 1,...,t_\ell,\ \ell = 1,...,\nu, && (3.24)\\ 0 &= \alpha_{\ell,s} + w_{s_\ell}^{(\ell,s)}, && s = 1,...,t_\ell,\ \ell = 2,...,\nu, && (3.25)\\ 0 &= \alpha_{1,s} + w_{s_1}^{(1,2s-1)}, && s = 1,...,t_1/2, && (3.26)\\ 0 &= -\alpha_{1,s} + w_{s_1}^{(1,2s)}, && s = 1,...,t_1/2. && (3.27) \end{aligned}$$

This system is solvable by defining $x_{k+1}^{(\ell,s)}$ according to (3.24) for $k = 1,\dots,s_\ell - 1$, $s = 1,\dots,t_\ell$, and $\ell = 1,\dots,\nu$; defining $\alpha_{\ell,s}$ according to (3.25) for $s = 1,\dots,t_\ell$ and $\ell = 2,\dots,\nu$; and finally setting $\alpha_{1,s} := -w_{s_1}^{(1,2s-1)}$ for $s = 1,\dots,t_1/2$, which solves (3.26) and (3.27) because of (3.20).

Hence, we have solved (3.22), where at least one $\alpha_{1,s}$ being nonzero is a generic condition on b, so that there is a chain of length at least $s_1 + 1 = n_2 + 1$ of the perturbed pencil $(A + bb^T, E)$ for all $b \in \Psi$, where $\Psi \subseteq \mathbb{C}^n$ is a suitable generic set.

Finally, we claim that the partial multiplicities of $(A + bb^T, E)$ at 0 cannot generically exceed $(n_2+1, n_3, \dots, n_m, 1)$. Hence, consider the first blocks of (A, E) at 0 to be perturbed by $(bb^T, 0)$ with $b = e_1 + e_{n_1+2}$, given by

$$\left(\begin{bmatrix} e_1e_1^T & R_{n_1} + e_1e_2^T \\ R_{n_1} + e_2e_1^T & e_2e_2^T \end{bmatrix}, \begin{bmatrix} 0 & R_{n_1}J_{n_1}(0) \\ -R_{n_1}J_{n_1}(0) & 0 \end{bmatrix} \right).$$

By the computation in the appendix (with $x = y = \lambda$), this pencil has the determinant $\lambda^{2n_1} + 2\lambda^{n_1+2}$ for this particular b. By Lemma 3.3, the perturbed pencil $(A + bb^T, E)$ is therefore regular and has the partial multiplicities $(n_2+1, n_3, \dots, n_m, 1)$ at 0 for all b that are elements of a suitable generic subset of $\mathbb{C}^n$.

Case n_1 odd. As in the above case, the perturbed pencil $(E, A + uu^T)$ has the partial multiplicities greater than or equal to $(n_2, \ldots, n_m, 1)$ associated with ∞ for all $u \in \Lambda$ as in (3.4); thus the same holds for its reversal $(A + uu^T, E)$ at 0. The first block of (A, E) corresponding to 0 is given by

$$\left(R_{n_1}, \begin{bmatrix} 0 & R_{(n_1+1)/2} \\ -R_{(n_1-1)/2} & 0 \end{bmatrix} J_{n_1}(0)\right),$$

hence it is straightforward that the pencil $(A + uu^T, E)$ with $u = e_1$ has the algebraic multiplicity $n_2 + \cdots + n_m + 1$ at 0. Then, by Lemma 3.3, $(A + uu^T, E)$ is regular and has the partial multiplicities $(n_2, \ldots, n_m, 1)$ at 0 for all u that are elements of a suitable generic subset of $\mathbb{C}^n$; the same is true for $(E, A + uu^T)$ at the eigenvalue ∞. $\square$

3.3 Case $\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$

Lemma 3.6 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and T-even with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$. Then, there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$ the perturbed pencil $(E, A + uu^T)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ given by $(n_2, \ldots, n_m)$.*

Proof. From Theorem 2.6 we obtain that $(E, A + uu^T)$ has partial multiplicities greater than or equal to $(n_2, \ldots, n_m)$ at $\widehat{\lambda}$ for all $u \in \Lambda$ as in (3.4). The largest Jordan block of (E, A) corresponding to $\widehat{\lambda}$ is paired to one corresponding to $-\widehat{\lambda}$, which we perturb by $(0, uu^T)$ with $u = e_1 + e_{n_1+1}$. Then, the perturbed first blocks

$$\begin{bmatrix} -e_1 e_1^T & -R_{n_1} J_{n_1}(\widehat{\lambda} - \lambda) - e_1 e_1^T \\ -R_{n_1} J_{n_1}(\widehat{\lambda} + \lambda) - e_1 e_1^T & -e_1 e_1^T \end{bmatrix}$$

are observed to have full rank at $\lambda = \pm\widehat{\lambda}$. As no other blocks of (E, A) are affected by this perturbation, $(E, A + uu^T)$ has the algebraic multiplicity $n_2 + \cdots + n_m$ at $\widehat{\lambda}$ and $-\widehat{\lambda}$ for this particular u and therefore, by Lemma 3.3, the assertion follows. $\square$

4 Palindromic rank-1 perturbations

Now, we aim to generalize the results on T-alternating rank-1 perturbations to the palindromic case. In order to relate T-alternating and palindromic matrix pencils, we introduce the Cayley transformations with pole at $+1$ and -1 given by

$$\mathcal{C}_{+1}(P)(\mu) = (1 - \mu) P\left(\frac{1 + \mu}{1 - \mu}\right) \quad \text{and} \quad \mathcal{C}_{-1}(P)(\mu) = (1 + \mu) P\left(\frac{\mu - 1}{1 + \mu}\right),$$

respectively, which are defined for any $P(\lambda) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ (and can be generalized to matrix polynomials). It is well-known that the structure of $P(\lambda)$ corresponds to the structure of

Table 4.1: Cayley transforms of structured matrix pencils.

$P(\lambda)$	$\mathcal{C}_{-1}(P)(\mu)$	$\mathcal{C}_{+1}(P)(\mu)$
T-palindromic	T-odd	T-even
T-anti-palindromic	T-even	T-odd
T-even	T-palindromic	T-anti-palindromic
T-odd	T-anti-palindromic	T-palindromic

its Cayley transforms as described in Table 4.1, which is extracted from [12] and reduced to the case of matrix pencils (instead of matrix polynomials).

Subject of our study in this section will be T-palindromic rank-k perturbations of the form

$$\left[\begin{array}{ccc} u_1 & \dots & u_k \end{array}\right] \delta P(\lambda) \left[\begin{array}{ccc} u_1 & \dots & u_k \end{array}\right]^T, \tag{4.1}$$

where $\delta P(\lambda)$ is a generic, T-palindromic $k \times k$ pencil and $u_1, \dots, u_k \in \mathbb{C}^n$ are certain generic vectors. In analogy to the previous section, a subset of the set of T-palindromic pencils $\mathcal{P}_k = \{\lambda B + B^T \mid B \in \mathbb{C}^{k,k}\}$ is considered generic, if it can canonically be identified with a generic subset of $\mathbb{C}^{k,k}$. In order to further characterize $\delta P(\lambda)$, let us consider the following lemma.

Lemma 4.1 *Let $\Gamma \subseteq \mathcal{E}_k$ be a generic set of T-even pencils. Then the set $\mathcal{C}_{-1}(\Gamma)$ is a generic subset of $\mathcal{P}_k$.*

Proof. Let $\mathcal{B}$ be a basis of $\mathcal{E}_k$ and let again $[\lambda E - A]_{\mathcal{B}} \in \mathbb{C}^{k^2}$ denote the coordinates of $\lambda E - A \in \mathcal{E}_k$ with respect to $\mathcal{B}$. Since Γ is generic, there is a nonzero polynomial $p : \mathbb{C}^{k^2} \to \mathbb{C}$ such that $\lambda E - A \notin \Gamma$ implies $p\big([\lambda E - A]_{\mathcal{B}}\big) = 0$.

Assume $\lambda B + B^T \notin \mathcal{C}_{-1}(\Gamma)$, then $\mathcal{C}_{+1}(P)(\lambda) = \lambda(B - B^T) + B + B^T \notin \Gamma$, which implies $p\big([\lambda(B - B^T) + B + B^T]_{\mathcal{B}}\big) = 0$. Now, $B \mapsto p\big([\lambda(B - B^T) + B + B^T]_{\mathcal{B}}\big)$ is a polynomial and not constantly zero; hence $\mathcal{C}_{-1}(\Gamma)$ is a generic subset of $\mathcal{P}_k$. □

Now, let $\Gamma \subseteq \mathcal{E}_k$ be the generic set from Lemma 3.1, then the set $\mathcal{C}_{-1}(\Gamma)$ is by Lemma 4.1 a generic subset of $\mathcal{P}_k$ and it contains T-palindromic $k \times k$ pencils that are regular and have distinct eigenvalues by [26, Theorem 3.4] (see also [15]). Assuming $\delta P(\lambda) \in \mathcal{C}_{-1}(\Gamma)$, we apply $\mathcal{C}_{+1}$ to the pencil (4.1) and obtain, as $u_1, \dots, u_k$ do not depend on λ:

$$\left[\begin{array}{ccc} u_1 & \dots & u_k \end{array}\right] \mathcal{C}_{+1}(\delta P)(\mu) \left[\begin{array}{ccc} u_1 & \dots & u_k \end{array}\right]^T,$$

which is T-even with $\mathcal{C}_{+1}(\delta P)(\mu) \in \Gamma$. Again, without loss of generality let $\mathcal{C}_{+1}(\delta P)(\mu)$ be in T-even Kronecker form as in Theorem 2.11; then it consists of $k/2$ blocks of the type $\mathcal{F}_1$ if k is even and of $(k-1)/2$ blocks of the type $\mathcal{F}_1$ and one block $\mathcal{I}_1$ if k is odd, as all other blocks are singular or involve multiple eigenvalues. Since blocks $\mathcal{F}_1$ correspond to rank-2 perturbations of $P(\lambda)$, we will only study perturbations corresponding to $\mathcal{I}_1$, i.e., ones of the form $\mathcal{C}_{+1}(\Delta P)(\mu) = u\,\mathcal{I}_1\,u^T$, thus

$$\Delta P(\lambda) = u\,\mathcal{C}_{-1}(\mathcal{I}_1)(\lambda)\,u^T = (\lambda + 1)uu^T.$$

Note that it is also possible to characterize the Jordan structure of $\delta P(\lambda)$ from (4.1) by employing a T-palindromic Kronecker form that was deduced in [23], which also yields the above form of generic, structure-preserving rank-1 perturbations. To characterize the change in Jordan structure of T-palindromic matrix pencils under perturbations $\Delta P(\lambda)$, where u is an element of some generic subset of $\mathbb{C}^n$, the following theorem is obtained parallel to Theorem 3.2.

Theorem 4.2 *Let $P(\lambda) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a T-palindromic or T-anti-palindromic, regular matrix pencil with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda} \in \mathbb{C} \cup \{\infty\}$. Then, there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$, the perturbed pencil*

$$P(\lambda) + (\lambda + 1)uu^T \quad or \quad P(\lambda) + (\lambda - 1)uu^T,$$

respectively, is regular and has the partial multiplicities associated with $\widehat{\lambda}$ given by Table 4.2.

Table 4.2: Rank-1 perturbations of palindromic matrix pencils.

parity	eigenvalue $\widehat{\lambda}$	case	multiplicities
T-palindromic	$\widehat{\lambda} = 1$	n_1 even	$(n_2, n_3, \ldots, n_m)$
		n_1 odd	$(n_2 + 1, n_3, \ldots, n_m)$
	$\widehat{\lambda} = -1$	n_1 even	$(n_2 + 1, n_3, \ldots, n_m, 1)$
		n_1 odd	$(n_2, n_3, \ldots, n_m, 1)$
	$\widehat{\lambda} \neq \pm 1$		$(n_2, n_3, \ldots, n_m)$
T-anti-palindromic	$\widehat{\lambda} = 1$	n_1 even	$(n_2 + 1, n_3, \ldots, n_m, 1)$
		n_1 odd	$(n_2, n_3, \ldots, n_m, 1)$
	$\widehat{\lambda} = -1$	n_1 even	$(n_2, n_3, \ldots, n_m)$
		n_1 odd	$(n_2 + 1, n_3, \ldots, n_m)$
	$\widehat{\lambda} \neq \pm 1$		$(n_2, n_3, \ldots, n_m)$

Proof. We will only prove the assertion if $P(\lambda)$ is T-palindromic, the treatment of the other case is analogous. We define $\Delta P(\lambda) := (\lambda + 1)uu^T$; since $\mathcal{C}_{+1}$ and $\mathcal{C}_{-1}$ are linear transformations on the vector space $\mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$, the transformed pencil is given by

$$\mathcal{C}_{+1}(P + \Delta P)(\mu) = \mathcal{C}_{+1}(P)(\mu) + \mathcal{C}_{+1}(\Delta P)(\mu),$$

where both pencils on the right hand side are T-even. Observe that $\mathcal{C}_{+1}(P)(\mu)$ has the partial multiplicities $(n_1, \ldots, n_m)$ associated with the transformed eigenvalue $\widehat{\mu} = (\widehat{\lambda} - 1)/(\widehat{\lambda} + 1)$ if $\widehat{\lambda} \neq -1$ and $\widehat{\mu} = \infty$ if $\widehat{\lambda} = -1$ by [26, Theorem 3.4]. In particular,

$$\mathcal{C}_{+1}(\Delta P)(\mu) = (1 - \mu)\left[\left(\frac{1 + \mu}{1 - \mu}\right) + 1\right] uu^T = 2uu^T$$

is a structure-preserving rank-1 perturbation of $\mathcal{C}_{+1}(P)(\mu)$. Hence, by Theorem 3.2 there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$, the perturbed pencil $\mathcal{C}_{+1}(P)(\mu)+\mathcal{C}_{+1}(\Delta P)(\mu)$ is regular and its partial multiplicities at $\widehat{\mu}$ are given by the upper half of Table 3.1, where $\widehat{\lambda}$ has to be replaced by the transformed eigenvalue $\widehat{\mu}$.

We can now apply the inverse transformation $\mathcal{C}_{-1}$ to $\mathcal{C}_{+1}(P+\Delta P)(\mu)$ to obtain that the perturbed pencil $P(\lambda) + \Delta P(\lambda)$ is regular and has the partial multiplicities in the upper half of Table 4.2 at $\widehat{\lambda} = (1+\widehat{\mu})/(1-\widehat{\mu})$ if $\widehat{\mu} \neq 1$ and at $\widehat{\lambda} = \infty$ if $\widehat{\mu} = 1$ for all $u \in \Omega$ by [26, Theorem 3.4]. □

5 Symmetric rank-1 perturbations

Subject of this section will be symmetric rank-k perturbations of the form

$$\begin{bmatrix} u_1 & \dots & u_k \end{bmatrix} (\delta E, \delta A) \begin{bmatrix} u_1 & \dots & u_k \end{bmatrix}^T, \tag{5.1}$$

where $(\delta E, \delta A)$ is a generic, symmetric $k \times k$ pencil and $u_1, \dots, u_k \in \mathbb{C}^n$ are certain generic vectors. Analogously to Sections 3 and 4, we consider a subset of the set of symmetric $k \times k$ pencils

$$\mathcal{S}_k := \{(E, A) \in \mathbb{C}^{k,k} \times \mathbb{C}^{k,k} \mid (E, A) \text{ is symmetric } \}$$

to be generic, if it can canonically be identified with a generic subset of $\mathbb{C}^{k(k+1)}$; also we denote by $[(E, A)]_{\mathcal{B}} \in \mathbb{C}^{k(k+1)}$ the coordinates of $(E, A) \in \mathcal{S}_k$ with respect to a basis $\mathcal{B}$ of $\mathcal{S}_k$.

Lemma 5.1 *There exists a generic set $\Gamma \subseteq \mathcal{S}_k$ such that any $(E, A) \in \Gamma$ is regular and has distinct eigenvalues.*

Proof. We follow the procedure of the proofs of Theorem 2.3 and Lemma 3.1. Given $(E, A) \in \mathcal{S}_k$ and $\sum_{j=0}^{k} c_j \lambda^j := \det(\lambda E - A)$, the coefficients $c_j = c_j(E, A)$ depend polynomially on the coordinates $[(E, A)]_{\mathcal{B}}$ and not all $c_j(E, A)$ are constantly zero.

Further, define $p(E, A)$ as in (2.2) and $q(E, A)$ as in (2.3), which both depend polynomially on the coordinates $[(E, A)]_{\mathcal{B}}$ and observe that $p(E, A)q(E, A)$ is not constantly zero. Hence, the set

$$\Gamma := \left\{(E, A) \in \mathcal{S}_k \mid p(E, A)q(E, A) \neq 0 \text{ and } \exists j \in \{0, 1, \dots, k\} \text{ with } c_j(E, A) \neq 0\right\} \tag{5.2}$$

of regular, symmetric pencils with distinct eigenvalues is a generic subset of $\mathcal{S}_k$. □

If $(\delta E, \delta A)$ is an element of Γ from (5.2), it is regular and has distinct eigenvalues; without loss of generality let $(\delta E, \delta A)$ be in symmetric Kronecker form as in Theorem 2.12. Then, its diagonal blocks will all have size 1×1 and thus (5.1) is the sum of symmetric rank-1 pencils of the form $(\beta uu^T, \alpha uu^T)$. Hence, in the following theorem, we will examine this type of symmetric rank-1 perturbations if u is an element of a suitable generic subset of $\mathbb{C}^n$.

Theorem 5.2 *Let $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be symmetric and regular with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda} \in \mathbb{C}$. Then, for all $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}$ there is a generic set $\Omega \subseteq \mathbb{C}^n$ such that for all $u \in \Omega$, the perturbed pencil $(E + \beta uu^T, A + \alpha uu^T)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ given by*

$$(n_2, \ldots, n_m) \quad \text{or} \quad (n_2, \ldots, n_m, 1) \tag{5.3}$$

if $\alpha \neq \widehat{\lambda}\beta$ or $\alpha = \widehat{\lambda}\beta$, respectively.

Proof. Let $(\alpha,\beta) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}$ be fixed and consider the polynomial

$$\sum_{j=0}^{n} c_j \lambda^j = \det\big(\lambda(E + \beta uu^T) - A - \alpha uu^T\big),$$

whose coefficients $c_j = c_j(u)$ depend polynomially on the entries of $u \in \mathbb{C}^n$. Therefore, since $c_j(0) \neq 0$ holds for at least one j, we have that $c_j(u) \neq 0$ for at least one j on a set $\Lambda \subseteq \mathbb{C}^n$ that is by definition generic, i.e., $(E + \beta uu^T, A + \alpha uu^T)$ is regular for all $u \in \Lambda$.

By (2.6) the geometric multiplicity of (E,A) at $\widehat{\lambda}$ at most decreases by $\operatorname{rank}\big((\widehat{\lambda}\beta - \alpha)uu^T\big)$ under the given perturbation; adding the chains from Theorem 2.6, the perturbed pencil $(E + \beta uu^T, A + \alpha uu^T)$ has partial multiplicities greater than or equal to the ones in (5.3) at $\widehat{\lambda}$ for all $u \in \Lambda$. Hence, for all $u \in \Lambda$ we can factorize

$$\det\big((\lambda + \widehat{\lambda})(E + \beta uu^T) - (A + \alpha uu^T)\big) = \begin{cases} \lambda^{n_2+\cdots+n_m}\, q_1(\lambda) & \text{if } \alpha \neq \widehat{\lambda}\beta, \\ \lambda^{n_2+\cdots+n_m+1}\, q_2(\lambda) & \text{if } \alpha = \widehat{\lambda}\beta \end{cases}$$

for suitable polynomials $q_1(\lambda)$ and $q_2(\lambda)$. Note that the coefficients $q_1(0)$ and $q_2(0)$ depend polynomially on the entries of u and that for continuity reasons the above factorization even holds for all $u \in \mathbb{C}^n$. Assuming that (E,A) is in symmetric Kronecker form as in Theorem 2.12, where the blocks corresponding to $\widehat{\lambda}$ come first and are ordered decreasingly with respect to their size, the largest block corresponding to $\widehat{\lambda}$ is given by $\big(R_{n_1}, R_{n_1}J_{n_1}(\widehat{\lambda})\big)$. Then, $q_1(0)$ and $q_2(0)$ are nonzero, e.g., for $u = e_1$, since the first block of the perturbed pencil

$$-R_{n_1}J_{n_1}(\widehat{\lambda} - \lambda) + (\lambda\beta - \alpha)e_1e_1^T$$

at $\widehat{\lambda}$ clearly has the algebraic multiplicity 0 if $\alpha \neq \widehat{\lambda}\beta$ and 1 if $\alpha = \widehat{\lambda}\beta$. Hence, we have $q_1(0) \neq 0$ and $q_2(0) \neq 0$ for all $u \in \widetilde{\Lambda}$, where $\widetilde{\Lambda} \subseteq \mathbb{C}^n$ is by definition generic. Thus, for all $u \in \Lambda \cap \widetilde{\Lambda}$ the partial multiplicities of $(E + \beta uu^T, A + \alpha uu^T)$ at $\widehat{\lambda}$ are given by (5.3), from which the assertion follows as $\Lambda \cap \widetilde{\Lambda}$ is generic. $\square$

Remark 5.3 An analogous result for the infinite eigenvalue of (E,A) is obtained as well since the reverse pencil (A,E) is also symmetric and hence its 0 eigenvalue shows the above behavior.

6 Conclusion

Structure-preserving rank-1 perturbations of classes of structured, regular matrix pencils were investigated and the generic Jordan structure of the perturbed pencil at the old eigenvalues was determined with respect to the parameters of the perturbation. In interplay of two effects was observed:

First, the largest block at each eigenvalue disappears under a generic rank-1 perturbation if this eigenvalue is not present in the perturbating pencil; otherwise the largest block shrinks to size one. Second, for T-alternating and palindromic matrix pencils, there are certain pairing conditions on the Jordan structure of the perturbed pencil so that in some cases, the largest remaining block generically increased in size by one. In extreme cases, e.g., in Example 1.2, the algebraic multiplicity is not affected by a generic, structure-preserving rank-1 perturbation, but we are still able to observe the above described principles.

Since symmetric matrix pencils do not have any restrictions on their eigenvalues, their behavior under generic, structure-preserving rank-1 perturbations does not deviate from the unstructured situation. We remain to examine generic, structure-preserving rank-2 perturbations of T-alternating and palindromic regular matrix pencils to completely characterize low-rank perturbations in these cases, which will be addressed in future work.

Acknowledgement

The author is very grateful to Christian Mehl for many useful suggestions for this manuscript and his continuous support.

References

[1] P. Brunovsky. A classification of linear controllable systems. *Kybernetika (Prague)*, 6:173–188, 1970.

[2] F. De Terán and F. Dopico. Low rank perturbation of Kronecker structures without full rank. *SIAM J. Matrix Anal. Appl.*, 29:496–529, 2007.

[3] F. De Terán, F. Dopico, and J. Moro. Low rank perturbation of Weierstrass structure. *SIAM J. Matrix Anal. Appl.*, 30:538–547, 2008.

[4] J.H. Fourie, G.J. Groenewald, D.B. Janse van Rensburg, and A.C.M. Ran. Rank one perturbations of H-positive real matrices. *Linear Algebra Appl.*, 439:653–674, 2013.

[5] F.R. Gantmacher. *Theory of Matrices*, volume 1. Chelsea, New York, 1959.

[6] F.R. Gantmacher. *Theory of Matrices*, volume 2. Chelsea, New York, 1959.

[7] I. Gohberg, P. Lancaster, and L. Rodman. *Matrices and Indefinite Scalar Products.* Birkhäuser, Basel, 1983.

[8] A. Hilliges, C. Mehl, and V. Mehrmann. On the solution of palindromic eigenvalue problems. In *Proceedings of the European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS)*, Jyväskylä, Finland, 2004. CD-ROM.

[9] L. Hörmander and A. Melin. A remark on perturbations of compact operators. *Math. Scand.*, 167:113–118, 1994.

[10] P. Lancaster and L. Rodman. Canonical forms for symmetric/skew-symmetric real matrix pairs under strict equivalence and congruence. *Linear Algebra Appl.*, 406:1–76, 2005.

[11] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, Orlando, 1985.

[12] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[13] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432(4):867–891, 2010.

[14] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms for palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[15] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. *Möbius transformations of matrix polynomials.* MIMS EPrint 2014.2, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2014.

[16] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Eigenvalue perturbation theory of classes of structured matrices under generic structured rank one perturbations. *Linear Algebra Appl.*, 435:687–716, 2011.

[17] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Perturbation theory of selfadjoint matrices and sign characteristics under generic structured rank one perturbations. *Linear Algebra Appl.*, 436:4027–4042, 2012.

[18] C. Mehl, V. Mehrmann, A.C.M. Ran, and L. Rodman. Jordan forms of real and complex matrices under rank one perturbations. *Oper. Matrices*, 7:381–398, 2013.

[19] J. Moro and F. Dopico. Low rank perturbation of Jordan structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.

[20] S.V. Savchenko. Typical changes in spectral properties under perturbations by a rank-one operator. *Mat. Zametki*, 74:590–602, 2003. (Russian). Translation in Mathematical Notes. 74:557-568, 2003.

[21] S.V. Savchenko. On the change in the spectral properties of a matrix under a perturbation of sufficiently low rank. *Funkts. Anal. Prilozh.*, 38:85–88, 2004. (Russian). Translation in Funct. Anal. Appl. 38:69-71, 2004.

[22] S.V. Savchenko. Laurent expansion for the determinant of the matrix of scalar resolvents. *Mathematicheskii Sbornik*, 196:121–144, 2005. (Russian). Translation in Sbornik: Mathematics. 196:743-764, 2005.

[23] C. Schröder. *Palindromic and Even Eigenvalue Problems - Analysis and Numerical Methods.* PhD thesis, Technical University Berlin, Germany, 2008.

[24] C. Schütt. *Störungstheorie strukturierter Matrizenbüschel.* Master's thesis, Technical University Berlin, Germany, 2013. (German).

[25] R.C. Thompson. Pencils of Complex and Real Symmetric and Skew Matrices. *Linear Algebra Appl.*, 147:323–371, 1991.

[26] F. Tisseur and I. Zaballa. *Finite and infinite elementary divisors of matrix polynomials: a global approach.* MIMS EPrint 2012.78, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2012.

Appendix

We aim to compute the determinant of the matrix pencil

$$T(\lambda) = \begin{array}{c} \left[\begin{array}{cccc|cccc} x & & & & 0 & x & \cdots & \lambda \\ & & & & & & \iddots & -1 \\ & & & & & \iddots & \iddots & \\ & & & & \lambda & -1 & & \\ \hline 0 & & & \lambda & 0 & 0 & & \\ y & & \iddots & 1 & 0 & y & & \\ \vdots & \iddots & \iddots & & & & & \\ \lambda & 1 & & & & & & \end{array}\right] \begin{array}{l} n_1 \\ \\ n_1 \end{array} \\ \quad n_1 \qquad\qquad\qquad n_1 \qquad\quad \end{array},$$

where x and y are dummy elements that remain to be specified. We make a Laplace expansion with respect to the first column and obtain

$$\det T(\lambda) = x \det T_1(\lambda) + (-1)^{n_1+1} y \det T_2(\lambda) - \lambda \det T_3(\lambda),$$

where $T_j(\lambda)$ for $j = 1, 2, 3$ are the respective submatrices of $T(\lambda)$. It is

$$\det T_1(\lambda) = \begin{array}{|ccc|ccc|c} & & & & \lambda & -1 & \\ & & & \iddots & \iddots & & n_1-1 \\ & & & \lambda & -1 & & \\ \hline & & \lambda & 0 & 0 & & \\ & \iddots & 1 & 0 & y & & n_1 \\ \lambda & \iddots & & & & & \\ 1 & & & & & & \\ & n_1-1 & & & n_1 & & \end{array} = (-1)^{n_1-1} \begin{array}{|ccc|ccc|c} -1 & \lambda & & & & & \\ \ddots & \ddots & & & & & n_1-1 \\ & -1 & \lambda & & & & \\ \hline & 0 & 0 & \lambda & & & \\ & y & 0 & 1 & \ddots & & n_1 \\ & & & & \ddots & \lambda & \\ & & & & & 1 & \\ & n_1 & & & n_1-1 & & \end{array}$$

$$= (-1) \begin{vmatrix} 1 & \lambda & 0 \\ 0 & 0 & \lambda \\ y & 0 & 1 \end{vmatrix} = -y\lambda^2.$$

We further compute $\det T_2(\lambda)$ to be equal to

$$\begin{array}{|ccc|cccc|c} & & & 0 & x & & \lambda & \\ & & & & & \iddots & -1 & n_1 \\ & & & & \iddots & \iddots & & \\ & & & \lambda & -1 & & & \\ \hline & & \lambda & & & & & \\ & \lambda & 1 & & & & & \\ \iddots & \iddots & & & & & & n_1-1 \\ \lambda & \iddots & & & & & & \\ 1 & & & & & & & \\ & n_1-1 & & & n_1 & & & \end{array} = (-1)^{n_1-1} \begin{array}{|cccc|cccc|c} 1 & & & & & & & & \\ \lambda & \ddots & & & & & & & \\ & \ddots & \ddots & & & & & & n_1-1 \\ & & \lambda & 1 & & & & & \\ & & & \lambda & & & & & \\ \hline & & & & \lambda & -1 & & & \\ & & & & & \ddots & \ddots & & n_1 \\ & & & & & & \ddots & -1 & \\ & & & & 0 & x & & \lambda & \\ & n_1-1 & & & & n_1 & & & \end{array}$$

$$= (-1)^{n_1-1}\lambda^2\left[\lambda^{n_1-1} + x\right]$$

and $\det T_3(\lambda)$ to be equal to

$$\begin{array}{|ccc|cccc|c} & & & 0 & x & \cdots & \lambda & \\ & & & & & \iddots & -1 & n_1 \\ & & & & \iddots & \iddots & & \\ & & & \lambda & -1 & & & \\ \hline & & \lambda & 0 & 0 & & & \\ & \iddots & 1 & 0 & y & & & n_1-1 \\ \iddots & \iddots & & & & & & \\ \lambda & 1 & & & & & & \\ & n_1-1 & & & n_1 & & & \end{array} = (-1)^{n_1-1} \begin{array}{|ccc|cccc|c} \lambda & 1 & & & & & & \\ \ddots & \ddots & & & & & & n_1-1 \\ & \ddots & 1 & 0 & y & & & \\ & & \lambda & 0 & 0 & & & \\ \hline & & & \lambda & -1 & & & \\ & & & & \ddots & \ddots & & n_1 \\ & & & & & \ddots & -1 & \\ & & & 0 & x & \cdots & \lambda & \\ & n_1-1 & & & n_1 & & & \end{array}$$

$$= (-1)^{n_1-1}\lambda^{n_1}\left[\lambda^{n_1-1} + x\right].$$

Altogether we obtain

$$\begin{aligned}\det T(\lambda) &= x \det T_1(\lambda) + (-1)^{n_1+1} y \det T_2(\lambda) - \lambda \det T_3(\lambda) \\ &= (-1)^{n_1} \lambda^{2n_1} + \big(y + (-1)^{n_1} x\big) \lambda^{n_1+1}.\end{aligned}$$