SOLVING SINGULAR GENERALIZED EIGENVALUE PROBLEMS BY A RANK-COMPLETING PERTURBATION

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Abstract. Generalized eigenvalue problems involving a singular pencil are very challenging to solve, both with respect to accuracy and efficiency. The existing package Guptri is very elegant but may sometimes be time-demanding, even for small and medium-sized matrices. We propose a simple method to compute the eigenvalues of singular pencils, based on one perturbation of the original problem of a certain specific rank. For many problems, the method is both fast and robust. This approach may be seen as a welcome alternative to staircase methods.

Key words. Singular pencil, singular generalized eigenvalue problem, rank-completing perturbation, Guptri, model updating, double eigenvalues, two-parameter eigenvalue problem, differential algebraic equations, quadratic two-parameter eigenvalue problem.

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1. Introduction.

We study the computation of eigenvalues of small to medium-sized matrix pencils $A-\lambda B$, where A and B are (real or complex) $n\times m$ matrices and where, in addition, the matrix pencil $A-\lambda B$ is singular, which means that $m\neq n$ or if $m=n$ then

$$\det(A-\lambda B)\equiv 0.$$

In these cases, the common definition of eigenvalues as roots of $\det(A-\lambda B)$ obviously does not make sense. Therefore, *finite eigenvalues* of a singular matrix pencil $A-\lambda B$ are typically defined as values $\lambda_0\in\mathbb{C}$ satisfying $\mathrm{rank}(A-\lambda_0 B)<\mathrm{nrank}(A,B)$, where

$$\mathrm{nrank}(A,B):=\max_{\zeta\in\mathbb{C}}\mathrm{rank}(A-\zeta B)$$

denotes the *normal rank* of the pencil $A-\lambda B$; see [10]. Similarly, we say that ∞ is an eigenvalue of the singular pencil $A-\lambda B$ if $\mathrm{rank}(B)<\mathrm{nrank}(A,B)$. In the following we will restrict ourselves to the case $m=n$ as the case $m\neq n$ can easily be reduced to the square case by adding an appropriate number of zero rows or columns.

The singular generalized eigenvalue problem (singular GEP) is well known to be ill-conditioned as arbitrarily small perturbation may cause drastic changes in the eigenvalues. A classical example is given by the pencils

$$A-\lambda B=\begin{bmatrix}1&0\\0&0\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&0\end{bmatrix}\quad\text{and}\quad \widetilde{A}-\lambda\widetilde{B}=\begin{bmatrix}1&\varepsilon_1\\\varepsilon_2&0\end{bmatrix}-\lambda\begin{bmatrix}1&\varepsilon_3\\\varepsilon_4&0\end{bmatrix},$$

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where $\varepsilon_1, \ldots, \varepsilon_4 \in \mathbb{C} \setminus \{0\}$; see [22]. While $A - \lambda B$ is singular and has only the eigenvalue 1, the perturbed pencil $\widetilde{A} - \lambda \widetilde{B}$ is regular and has the eigenvalues $\frac{\varepsilon_1}{\varepsilon_3}$ and $\frac{\varepsilon_2}{\varepsilon_4}$ that can be anywhere in the complex plane even for tiny absolute values of $\varepsilon_1, \ldots, \varepsilon_4$.

On the other hand, it was observed in [43] that a situation as above is exceptional and that generically small perturbations of a singular square pencil make the pencil regular and some of the eigenvalues of the perturbed pencil are very close to the original eigenvalues of the singular pencil. The following example illustrates this. The Matlab commands

```
A = diag([1 2 3 0 0 0]);
B = diag([2 3 4 0 0 0]);
eig(U'*A*V, U'*B*V)
```

where U and V are random 6×6 orthogonal matrices, give the eigenvalues

```
0.5000  0.6667  0.7500  0.1595  0.6756  0.6543
```

We see that the three (finite) eigenvalues of the regular part are correct. Following the terminology of [40], the other three values are "fake eigenvalues" and correspond to the singular part of the pencil. (Explicit error analysis for the regular eigenvalues of singular pencils has been undertaken in [9, 11].) Despite this observation, Van Dooren suggests in [40] to solve the singular generalized eigenvalue problem by first extracting the regular part and then use the QZ algorithm on that part. Wilkinson strongly supports that recommendation in [43].

A robust software package which follows Van Dooren's recommendation is Guptri [13, 17]. If the pencil is singular, then first a "staircase" algorithm is applied to deflate the singular part of the pencil, and then the QZ algorithm is used to compute the eigenvalues of the remaining regular part. While the results of Guptri are usually excellent, this method may turn out to be time-consuming; for instance, applying Guptri on a singular 300×300 pencil on our machine took over 20 seconds, while Matlab's `eig` on a random pencil of the same size took less than a second.[1] Another issue is the fact that the rank decisions, needed in staircase type methods, are involved. If the pencil has a minimal index of size η, then at least $\eta + 1$ such decisions have to be taken. Typically, these decisions tend to become more and more critical during the run of the staircase algorithm. See, e.g., [14] or [33, Ex. 18], where a variant of the staircase algorithm for the singular two-parameter eigenvalue problem, introduced in [31], fails in double precision but gets the right result in higher precision.

Another way of extracting the regular part that needs fewer rank decisions was suggested in [30]. One may interpret the singular pencil as a constant coefficient differential-algebraic equation and perform a regularization procedure with the help of an derivative array as described in [3]. In this way, the regular part of the pencil can be extracted by performing only three nullspace computations. However, the derivative array approach leads to an inflation of the system by a factor of at least $\eta + 1$, where η is the largest minimal index of the given pencil, and may therefore result in high computational costs.

In this paper, we propose a new method to compute the regular eigenvalues of a singular pencil. The method is based on considering perturbations of rank

$$k = n - \mathrm{nrank}(A, B)$$

[1] We note that this experiment has been performed some years ago. A current practical issue is that there is no publicly available 64-bit Guptri code.

that we will call *rank-completing perturbations* as the rank is exactly large enough to generically turn the pencil into a pencil of full normal rank. As we will show, the canonical form of the original regular part of the given pencil stays invariant under generic rank-completing perturbations.

The idea of computing eigenvalues of singular pencils with rank-completing perturbations is not completely new, and the following specific type has been used in system theory as early as in the 70s (without the use of the terminology "rank-completing perturbation"). If a system of the form

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$$

is given, where $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{r,n}$, and $D \in \mathbb{R}^{r,m}$ are the system matrices, x stands for the state of the system, u is the input, and y is the output, then the eigenvalues of the system pencil

$$S(\lambda) = \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix}$$

are of particular interest in control theory; see [15] and the references therein. (If the system is minimal, then these eigenvalues are also referred to as *transmission zeros* of the system.) Clearly, if $m \neq r$, then the pencil $S(\lambda)$ is rectangular and thus singular. For that case and under the additional assumptions $r < m$ and $\operatorname{nrank} S(\lambda) = n + r$, the following algorithm which is based on ideas of [6] has been proposed in [24] for the computation of the transmission zeros:

1: Select two random matrices $[E_1 \;\; F_1], [E_2 \;\; F_2] \in \mathbb{R}^{m-r,n+m}$ so that
$$S_1(\lambda) := \begin{bmatrix} \lambda I - A & B \\ -C & D \\ E_1 & F_1 \end{bmatrix} \text{ and } S_2(\lambda) := \begin{bmatrix} \lambda I - A & B \\ -C & D \\ E_2 & F_2 \end{bmatrix}$$
are regular.

2: Compute the eigenvalues $\mathcal{E}_i$ of $S_i(\lambda)$ for $i = 1, 2$.

3: Compute the intersection $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$.

Since for each regular eigenvalue λ_0 of $S(\lambda)$ we obviously have $\operatorname{nrank} S_i(\lambda_0) < n+m$ for $i = 1, 2$, it immediately follows that the eigenvalues of $S(\lambda)$ are contained both in the spectrum of $S_1(\lambda)$ and in the spectrum of $S_2(\lambda)$. Both extended matrix pencils will also give rise to two sets of fake eigenvalues. Since generically these sets will be disjoint if the applied perturbations are generated randomly, it follows that the set $\mathcal{E}$ will generically coincide with the set of regular eigenvalues of $S(\lambda)$.

As pointed out in [15], this method may encounter difficulties in distinguishing the finite zeros from the infinite ones, in particular if the latter occur with a high multiplicity. Another problem may occur in identifying which values belong to the intersection $\mathcal{E}$ and which do not. Although the original eigenvalues of the pencil theoretically coincide with a subset of each of the sets of eigenvalues of the two perturbed pencils, they may still differ slightly in practice due to finite precision arithmetic. Therefore, a tolerance has to be prescribed that decides when two values are considered to be equal. If this tolerance is chosen to be too small, then some of the regular eigenvalues may be missed. If, on the other hand the tolerance is set too high, then two fake eigenvalues of $S_1(\lambda)$ and $S_2(\lambda)$ that happen to be close are likely to be falsely identified as a regular eigenvalue.

In this paper, we show that the regular eigenvalues of a singular pencil can be efficiently computed with the help of *just one* rank-completing perturbation of the form

$$A - \lambda B + \tau\, U(D_A - \lambda D_B)V^*,$$

where U, V are $n \times k$ matrices with orthonormal columns, D_A, D_B are diagonal $k \times k$ matrices, and τ a nonzero real scalar. The orthonormality of the columns is not strictly necessary, but convenient, for instance since the norm of the perturbation can easily be controlled in this case with the help of the parameter τ. The problem of identifying the subset of regular eigenvalues from the computed eigenvalues of the perturbed pencil is then taken care of by the *key observation* that the left and right eigenvectors that correspond to the regular eigenvalues satisfy orthogonality relations with respect to the matrices U and V. Thus, instead of comparing two spectra of two different pencils the regular eigenvalues can be separated from the fake eigenvalues by using information from the corresponding left and right eigenvectors. We note that perturbations of singular matrix pencils have already been considered in [4, 9, 27, 38, 39], but it seems that a detailed investigation of rank-completing perturbations is new, except for [28], where the case of singular Hermitian pencils of normal rank $n-1$ was considered.

The rest of this paper is organized as follows. In Section 3 we review some motivating applications where one is interested in computing eigenvalues of a singular matrix pencil. The main theoretical results are presented in Section 4, while the method, which is based on these results, is introduced in Section 5, followed by some numerical experiments in Section 6. In Section 7 we discuss singular two-parameter eigenvalue problems and present a new numerical method for such problems. We summarize some conclusions in Section 8.

2. Preliminaries. Throughout the paper, we will interpret matrix pencils both as pairs of matrices $(A, B) \in \mathbb{C}^{n,m} \times \mathbb{C}^{n,m}$ or as $n \times m$ matrix polynomials $A - \lambda B$ of degree at most one and we will switch between these notation whenever it is useful.

An important tool in the theory of singular pencils is the Kronecker canonical form (KCF) of a pencil $A - \lambda B$; see, e.g., [16].

THEOREM 2.1 (Kronecker canonical form). *Let $A-\lambda B$ be a complex $n\times m$ matrix pencil. Then there exist nonsingular matrices $P \in \mathbb{C}^{n,n}$ and $Q \in \mathbb{C}^{m,m}$ such that*

$$P(A-\lambda B)Q = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ 0 & \mathcal{S}(\lambda) \end{bmatrix}, \qquad \mathcal{R}(\lambda) = \begin{bmatrix} J - \lambda I_r & 0 \\ 0 & I_s - \lambda N \end{bmatrix} \tag{2.1}$$

with J, N in Jordan normal form and in addition N being nilpotent, and

$$\mathcal{S}(\lambda) = diag\big(L_{m_1}(\lambda), \ldots, L_{m_k}(\lambda), L_{n_1}(\lambda)^\top, \ldots, L_{n_\ell}(\lambda)^\top\big),$$

where $L_j(\lambda) = [0\ \ I_j] - \lambda\,[I_j\ \ 0]$ is of size $j \times (j+1)$ and $m_i \geq 0$ for $i = 1, \ldots, k$ and $n_i \geq 0$ for $i = 1, \ldots, \ell$.

The pencils $\mathcal{R}(\lambda)$ and $\mathcal{S}(\lambda)$ in Theorem 2.1 are called the *regular* and the *singular part* of $A-\lambda B$, respectively. The eigenvalues of J are exactly the finite eigenvalues of $A-\lambda B$, while the eigenvalue 0 of N corresponds to the infinite eigenvalue of $A-\lambda B$. The parameters $m_1, \ldots, m_k$ are called the *right minimal indices* and the parameters $n_1, \ldots, n_\ell$ are called the *left minimal indices* of $A - \lambda B$. One can easily check that the normal rank $\mathrm{nrank}(A, B)$ of $A - \lambda B$ is then given by $\min(n-\ell, m-k)$. In the remainder of the paper, we will consider square pencils $A - \lambda B$, i.e., we have $n = m$. Note that this implies $k = \ell$, i.e., we have must have the same number of right and left minimal indices. However, the particular values of the left and right minimal indices may be distinct.

In contrast to the eigenvalues of singular pencils, the corresponding eigenvectors and deflating subspaces are not well defined. To understand why, we consider the following example borrowed from [29] (and slightly adapted). The pencil

$$A - \lambda B = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.2}$$

obviously has the regular part $\mathcal{R}(\lambda) = [1-\lambda]$ and thus the pencil $A-\lambda B$ has the single eigenvalue $\lambda_0 = 1$ with algebraic multiplicity one. Nevertheless, any vector of the form $x(\alpha, \beta) := [\alpha \;\; \beta \;\; \beta]^\top$ with $\alpha, \beta \in \mathbb{C}$ satisfies $\lambda_0 Bx = Ax$ and thus could be interpreted as an eigenvector of the pencil. One may argue that in this example the choice $\alpha \neq 0$ and $\beta = 0$ seems to be canonical and gives rise to a unique one-dimensional deflating subspace "corresponding" to the regular part of the pencil. But on the other hand it follows from the equality

$$\begin{bmatrix} 1/\alpha & 0 & 0 \\ -\beta/\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

that for any choice of α, β with $\alpha \neq 0$ also the vector $x(\alpha, \beta)$ can be used to extract the regular part of the pencil (and thus could also be considered as "corresponding" to the regular part). For this reason, we restrict ourselves to the computation of eigenvalues of singular pencils, but do not consider corresponding eigenvectors.

Instead of eigenvectors and deflating subspaces, the concept of *reducing subspaces* that was introduced in [41] is more adequate in the case of singular pencils. We say that a subspace $\mathcal{M}$ is a *reducing subspace* for the pencil $A - \lambda B$ if $\dim(A\mathcal{M} + B\mathcal{M}) = \dim(\mathcal{M}) - k$, where k is the number of right singular blocks. In the example above, the reducing subspace associated with the eigenvalue $\lambda_0 = 1$ is exactly given by all vectors $x(\alpha, \beta)$ with $\alpha, \beta \in \mathbb{C}$. The *minimal reducing subspace* $\mathcal{M}_{\rm RS}(A, B)$ is the intersection of all reducing subspaces. It is unique and can be numerically computed in a stable way from the generalized upper triangular form (Guptri), see, e.g., [13]. Guptri exists for every pencil $A - \lambda B$ and has the form

$$P^*(A - \lambda B)Q = \begin{bmatrix} A_r - \lambda B_r & \times & \times \\ 0 & A_{\rm reg} - \lambda B_{\rm reg} & \times \\ 0 & 0 & A_l - \lambda B_l \end{bmatrix},$$

where matrices P and Q are unitary, $A_r - \lambda B_r$ has only right singular blocks in its KCF, $A_l - \lambda B_l$ has only left singular blocks in its KCF, and $A_{\rm reg} - \lambda B_{\rm reg}$ has only regular blocks in its KCF.

3. Motivation and applications. Before proposing our new method, we first review some motivating applications besides the computation of transmission zeros mentioned in the introduction where one wants to compute the eigenvalues of a singular pencil.

3.1. Differential algebraic equations and descriptor systems. Linear differential algebraic equations (DAEs) with constant coefficients have the general form

$$E\dot{x} = Ax + f(t), \quad x(t_0) = x_0,$$

where $E, A \in \mathbb{R}^{k,n}$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $f : [t_0, \infty) \to \mathbb{R}^k$ is a given inhomogeneity; see [23]. Linear time-invariant descriptor systems consist of a DAE combined with a system input and output and take form

$$\begin{aligned} E\dot{x} =& Ax + Bu, \quad x(t_0) = x_0, \\ y =& Cx + Du. \end{aligned}$$

where, in addition, $B \in \mathbb{R}^{k,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$. Here, $x : [t_0, \infty) \to \mathbb{R}^n$ stands for the state of the descriptor system, $u : [t_0, \infty) \to \mathbb{R}^m$ is the input and $y : [t_0, \infty) \to \mathbb{R}^p$ the output. As is highlighted in [3], the problem may be well-posed even if the underlying pencil $A - \lambda E$ is singular. Indeed, even in the singular case the corresponding DAE may have a (even unique) solution for particular inhomogeneities f and for special initial conditions (t_0, x_0).

3.2. Double eigenvalue problem. Given two $n \times n$ matrices A and B, we are interested in all values λ such that $A + \lambda B$ has a double eigenvalue. For the generic case of a double eigenvalue, we look for independent vectors x and y such that

$$\begin{aligned} (A + \lambda B - \mu I)\, x &= 0, \\ (A + \lambda B - \mu I)^2\, y &= 0. \end{aligned}$$

This application is discussed in [33], together with a staircase algorithm to solve it; see also [21]. In the generic case, the problem has $n(n-1)$ solutions.

In [33], the problem is solved by linearizing the second equation first and then solving the obtained singular two-parameter eigenvalue problem. The values λ that we are looking for, are eigenvalues of the singular pencil $\Delta_1 - \lambda\Delta_0$ of size $3n^2 \times 3n^2$, where

$$\Delta_1 = A \otimes R - I \otimes P, \quad \Delta_0 = B \otimes R - I \otimes Q,$$

for

$$P = \begin{bmatrix} A^2 & AB + BA & -2A \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \; Q = \begin{bmatrix} 0 & B^2 & -B \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 & -B & I \\ 0 & 0 & 0 \\ -I & 0 & 0 \end{bmatrix}.$$

For more details, see [33] as well as [19] for other possible linearizations.

3.3. Singular two-parameter eigenvalue problems. In the case of a singular two-parameter eigenvalue problem (2EP) one has a pair of singular pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ (see also (7.2) and (7.3)), and the goal is to find finite regular eigenvalues (λ_0, μ_0), where (see, e.g., [31] for more details):

- λ_0 is a finite regular eigenvalue of $\Delta_1 - \lambda\Delta_0$,
- μ_0 is a finite regular eigenvalue of $\Delta_2 - \mu\Delta_0$,
- there exists a common regular eigenvector $z \neq 0$ such that $(\Delta_1 - \lambda_0\Delta_0)z = 0$, $(\Delta_2 - \mu_0\Delta_0)z = 0$, and $z \notin \mathcal{M}_{\mathrm{RS}}(\Delta_i, \Delta_0)$ for $i = 1, 2$.

This problem requires more than just solving one singular GEP. We discuss it in more details in Section 7 and present a new numerical method for its solution.

4. Rank-completing perturbations of singular pencils. Let $A - \lambda B$ be a singular pencil, where $A, B \in \mathbb{C}^{n,n}$ and $\operatorname{nrank}(A, B) = n - k$. In this section we investigate the effect of *rank-completing perturbations*, i.e., rank-k generic perturbations of the form

$$\widetilde{A} - \lambda \widetilde{B} := A - \lambda B + \tau\,(U D_A V^* - \lambda\, U D_B V^*), \tag{4.1}$$

where $D_A, D_B \in \mathbb{C}^{k,k}$ are diagonal matrices such that $D_A - \lambda D_B$ is a regular pencil, $U, V \in C^{n,k}$ have full column rank, and $\tau \in \mathbb{R}$ is nonzero. We investigate the dependence of eigenvalues and eigenvectors of the perturbed pencils on τ.

Above and in the following, the term generic is understood in the following sense: a set $\mathcal{A} \subseteq \mathbb{C}^m$ is called *algebraic*, if it is the set of common zeros of finitely many complex polynomials in m variables, and $\mathcal{A}$ is called *proper* if $\mathcal{A} \neq \mathbb{C}^m$. A set $\Omega \subseteq \mathbb{C}^m$ is called *generic* if its complement is contained in a proper algebraic set. In this sense we say that a property $\mathcal{P}$ holds generically with respect to the entries of U and V, if there exists a generic set $\Omega \subseteq (\mathbb{C}^{n,k})^2$ (whereby we interpret $(\mathbb{C}^{n,k})^2$ as $\mathbb{C}^{2nk}$) such that $\mathcal{P}$ holds for all pencils of the form (4.1) with $(U, V) \in \Omega$.

REMARK 4.1. Perturbations of the form (4.1) are not the most general perturbations of rank k. Indeed, completing U and V to nonsingular matrices $P = [U \ \ \widetilde{U}]$ and $Q = [V \ \ \widetilde{V}]$, we obtain that

$$P^{-1}(U D_A V^* - \lambda U D_B V^*)(Q^*)^{-1} = \begin{bmatrix} D_A - \lambda D_B & 0 \\ 0 & 0 \end{bmatrix},$$

which means that the perturbation pencil $U D_A V^* - \lambda U D_B V^*$ has the regular part $D_A - \lambda D_B$ of size $k \times k$, while, generically, a matrix pencil of rank $k < n$ would have no regular part [8]. Thus, more generally one could consider perturbations of the form

$$(A + U_1 V_1^* + V_2 U_2^*, B + U_1 W_1^* + W_2 U_2^*), \tag{4.2}$$

where $\ell \in \{0, 1, \ldots, k\}$ and where $U_1, V_1, W_1 \in \mathbb{C}^{n,\ell}$ and $U_2, V_2, W_2 \in \mathbb{C}^{n,k-\ell}$ have full column rank. However, we will restrict ourselves to perturbations of the form (4.1), because of their favorable properties.

Generically, a rank completing perturbation of $A - \lambda B$ will result in a pencil of full normal rank, i.e., the perturbed pencil is regular. We will show in the following, that if rank-completing perturbations of the form $\widetilde{A} - \lambda \widetilde{B} := A - \lambda B + \tau\,(U D_A V^* - \lambda U D_B V^*)$ as in (4.1) are considered, then generically the KCF of $\widetilde{A} - \lambda \widetilde{B}$ is for all $\tau \in \mathbb{R} \setminus \{0\}$ given by

$$\begin{bmatrix} \mathcal{R}_{\mathrm{reg}}(\lambda) & 0 & 0 \\ 0 & \mathcal{R}_{\mathrm{pre}}(\lambda) & 0 \\ 0 & 0 & \mathcal{R}_{\mathrm{ran}}(\lambda) \end{bmatrix},$$

where $\mathcal{R}_{\mathrm{reg}}(\lambda)$ is the regular part of the original pencil $A - \lambda B$, $\mathcal{R}_{\mathrm{pre}}(\lambda) = D_A - \lambda D_B$ and $\mathcal{R}_{\mathrm{ran}}(\lambda)$ only has simple eigenvalues that are different from the eigenvalues of $\mathcal{R}_{\mathrm{reg}}(\lambda)$ and $\mathcal{R}_{\mathrm{pre}}(\lambda)$. Thus, the eigenvalues of $\widetilde{A} - \lambda \widetilde{B}$ are exactly the p regular eigenvalues of $A - \lambda B$ (counted with multiplicities) and $n - p$ newly generated eigenvalues which consist of k "*prescribed*" eigenvalues which are the eigenvalues of the perturbation pencil $U(D_A - \lambda D_B)V^*$, and $n - p - k$ "*random*" eigenvalues that are gathered in the part $\mathcal{R}_{\mathrm{ran}}(\lambda)$.

We start by showing that under a rank-completing perturbation, the regular part of $A - \lambda B$ will stay invariant in the above sense. Although our main focus are square pencils, we will state some results in more generality covering also the case of rectangular pencils. The following proposition is a generalization of Theorem 4.2 in [28] (which dealt with Hermitian pencils) to a block case without any specific structure in matrices A and B.

PROPOSITION 4.2. *Let $A - \lambda B$ be an $n \times m$ singular matrix pencil having at least k left minimal indices. Furthermore, let $U \in \mathbb{C}^{n,k}$. Then generically (with respect to the entries of U) there exist nonsingular matrices P, Q such that*

$$P(A - \lambda B)Q = \begin{bmatrix} R(\lambda) & 0 \\ 0 & S(\lambda) \end{bmatrix} \quad \textit{and} \quad PU = \begin{bmatrix} 0 \\ \widetilde{U} \end{bmatrix},$$

where $R(\lambda)$ and $S(\lambda)$ are the regular and singular parts of $A - \lambda B$, respectively, and PU is partitioned conformably with $P(A - \lambda B)Q$.

Proof. Without loss of generality we may assume that $A - \lambda B$ is already in the KCF

$$A - \lambda B = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ 0 & \mathcal{S}(\lambda) \end{bmatrix}, \quad \mathcal{R}(\lambda) = \begin{bmatrix} J - \lambda I & 0 \\ 0 & I - \lambda N \end{bmatrix},$$

where $\mathcal{S}(\lambda)$ is the singular part and $\mathcal{R}(\lambda)$ is the regular part of $A - \lambda B$ with J, N being in Jordan normal canonical form and N being in addition nilpotent. Our main strategy is to use the part of U that corresponds to k arbitrarily chosen left minimal indices to introduce zeros in the components of U that correspond to the regular part of $A - \lambda B$. Here, we can treat the components of U corresponding to the "finite eigenvalue part" $J - \lambda I$ and the "infinite eigenvalue part" $I - \lambda N$ separately and we will give the proof only for the first case as the proof for the "infinite eigenvalue part" is completely analogously. Since we will only transform parts of the singular part $\mathcal{S}(\lambda)$ that correspond to k arbitrarily chosen left minimal indices and leave all other parts of $\mathcal{S}(\lambda)$ unchanged, it is sufficient to assume that $\mathcal{S}(\lambda)$ consists of only k singular blocks corresponding to k left minimal indices and that $\mathcal{R}(\lambda)$ does not have a part corresponding to infinite eigenvalues. This assumption will simplify the notation considerably.

Thus, we assume that U and $A - \lambda B$ have the forms

$$u = \begin{array}{c} \scriptstyle r \\ \scriptstyle n_1+1 \\ \vdots \\ \scriptstyle n_k+1 \end{array} \overset{\scriptstyle k}{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_k \end{bmatrix}} \quad \text{and} \quad A - \lambda B = \begin{array}{c} \scriptstyle r \\ \scriptstyle n_1+1 \\ \vdots \\ \scriptstyle n_k+1 \end{array} \overset{\begin{array}{cccc} \scriptstyle r & \scriptstyle n_1 & \scriptstyle \cdots & \scriptstyle n_k \end{array}}{\begin{bmatrix} J - \lambda I & & & \\ & L_{n_1}(\lambda)^\top & & \\ & & \ddots & \\ & & & L_{n_k}(\lambda)^\top \end{bmatrix}}.$$

In the following, we will use the notation $G_\ell^{(d)} = [0 \ \ I_\ell]^\top$ and $G_\ell^{(u)} = [I_\ell \ \ 0]^\top$ such that we can write $L_\ell(\lambda)^\top = G_\ell^{(d)} - \lambda G_\ell^{(u)}$ for a singular block corresponding to a left minimal index ℓ. For the transformation matrices P, Q we make the ansatz

$$P = \begin{array}{c} \scriptstyle r \\ \scriptstyle n_1+1 \\ \vdots \\ \scriptstyle n_k+1 \end{array} \overset{\begin{array}{cccc} \scriptstyle r & \scriptstyle n_1+1 & \scriptstyle \cdots & \scriptstyle n_k+1 \end{array}}{\begin{bmatrix} I & P_1 & \cdots & P_k \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}} \quad \text{and} \quad Q = \begin{array}{c} \scriptstyle r \\ \scriptstyle n_1 \\ \vdots \\ \scriptstyle n_k \end{array} \overset{\begin{array}{cccc} \scriptstyle r & \scriptstyle n_1 & \scriptstyle \cdots & \scriptstyle n_k \end{array}}{\begin{bmatrix} I & -P_1 G_{n_1}^{(u)} & \cdots & -P_k G_{n_k}^{(u)} \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}}$$

Since P and Q are designed in such a way that B remains unchanged, we obtain

$$P(A-\lambda B)Q = \begin{bmatrix} J-\lambda I & P_1 G_{n_1}^{(d)} - JP_1 G_{n_1}^{(u)} & \dots & P_k G_{n_k}^{(d)} - JP_k G_{n_k}^{(u)} \\ & L_{n_1}(\lambda)^\top & & \\ & & \ddots & \\ & & & L_{n_k}(\lambda)^\top \end{bmatrix}.$$

It remains to find solutions P_i to the equations $P_i G_{n_i}^{(d)} - JP_i G_{n_i}^{(u)} = 0$ for $i = 1, \dots, k$ to obtain $P(A-\lambda B)Q = A - \lambda B$. Setting $P_i = [p_{i,0} \;\; p_{i,1} \;\; \dots \;\; p_{i,n_i}]$ for $i = 1, \dots, k$, the equations to be solved take the form

$$[p_{i,1} \;\; \dots \;\; p_{i,n_i}] = P_i G_{n_i}^{(d)} = JP_i G_{n_i}^{(u)} = [Jp_{i,0} \;\; \dots \;\; Jp_{i,n_i-1}], \quad i = 1, \dots, k.$$

Thus, we may choose $P_i = [p_{i,0} \;\; Jp_{i,0} \;\; \dots \;\; J^{n_i} p_{i,0}]$, where $p_{i,0} \in \mathbb{C}^r$ is arbitrary. We will now use the freedom in the choice of $p_{i,0}$ to guarantee that PU has the desired form. To this end, let $u_{i,0}, \dots, u_{i,n_i} \in \mathbb{C}^k$ be the rows of U_i, $i = 1, \dots, k$. Then the first block component of PU is given by $U_0 + P_1 U_1 + \dots + P_k U_k$ and to set this component to zero, we have to solve the equation

$$-U_0 = \sum_{i=1}^{k} P_i U_i = \sum_{i=1}^{k} \sum_{j=0}^{n_i} J^j p_{j,0} u_{i,j}^\top \tag{4.3}$$

for $p_{1,0}, \dots, p_{k,0}$. Using the vec-operation that "vectorizes" a matrix by stacking the columns on top of another and recalling the well-known identity $\operatorname{vec}(XYZ) = (Z^\top \otimes X)\operatorname{vec}(Y)$ for matrices X, Y, Z, where $\otimes$ denotes the Kronecker product, we obtain that

$$-\operatorname{vec}(U_0) = \sum_{i=1}^{k} \sum_{j=0}^{n_i} \big(u_{i,j} \otimes J^j\big) p_{i,0} = M \cdot \begin{bmatrix} p_{1,0} \\ \vdots \\ p_{k,0} \end{bmatrix}, \tag{4.4}$$

where

$$M = \Big[\sum_{j=0}^{n_1} u_{1,j} \otimes J^j \;\; \dots \;\; \sum_{j=0}^{n_k} u_{k,j} \otimes J^j \Big] \in \mathbb{C}^{rk,rk}.$$

Clearly, the determinant of M is a polynomial in the nk entries of U (in fact, it only depends on the entries of $U_1, \dots, U_k$) which is nonzero for the particular choice $u_{1,0} = e_1, \dots, u_{k,0} = e_k$ and $u_{i,j} = 0$ for $j > 0$, where $e_1, \dots, e_k$ denote the standard basis vectors of $\mathbb{C}^k$. (Indeed, in this case M is just the identity of size $rk \times rk$.) Thus, generically (with respect to the entries of U), the matrix M is invertible, so the equation (4.4) and thus also (4.3) can be uniquely solved for $p_{1,0}, \dots, p_{k,0}$ which finishes the proof. □

The next result shows that the canonical form of the regular part of the original pencil stays invariant under a generic rank-completing perturbation of the form (4.1). Concerning the eigenvalues of the perturbed pencil that are also eigenvalues of the original singular pencil, the result also states that the corresponding left and right eigenvectors satisfy a particular orthogonality relation.

THEOREM 4.3. *Let $A - \lambda B$ be an $n \times n$ singular pencil of normal rank $n - k$, let $U, V \in \mathbb{C}^{n,k}$ have full column rank and let $D_A, D_B \in \mathbb{C}^{k,k}$ be such that $D_A - \lambda D_B$ is* *regular and all eigenvalues of $D_A - \lambda D_B$ are distinct from the eigenvalues of $A - \lambda B$. Then, generically with respect to the entries of U and V^*, the following statements hold for the pencil* (4.1)*:*

1. *For each $\tau \neq 0$, there exists nonsingular matrices $\widetilde{P}$ and $\widetilde{Q}$ such that*

$$\widetilde{P}(\widetilde{A} - \lambda\widetilde{B})\widetilde{Q} = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ 0 & \mathcal{R}_{\text{new}}(\lambda) \end{bmatrix}, \tag{4.5}$$

   where $\mathcal{R}(\lambda)$ is the regular part of the original pencil $A - \lambda B$, and $\mathcal{R}_{\text{new}}(\lambda)$ is regular and all its eigenvalues are distinct from the eigenvalues of $\mathcal{R}(\lambda)$.

2. *If λ_0 is a finite regular eigenvalue of $A - \lambda B$, i.e., rank$(A - \lambda_0 B) < n - k$, then λ_0 is an eigenvalue of* (4.1) *for each $\tau \neq 0$. Furthermore, the right null space*

$$\mathcal{N}_r(\lambda_0) := \ker\left(A + \tau\, UD_AV^* - \lambda\,(B + \tau\, UD_BV^*)\right)$$

   and the left null space

$$\mathcal{N}_l(\lambda) := \ker\left(\left(A + \tau\, UD_AV^* - \lambda\,(B + \tau\, UD_BV^*)\right)^*\right)$$

   are both constant in $\tau \neq 0$. In addition:
   (a) $\mathcal{N}_r(\lambda_0) \perp \operatorname{span}(V)$, i.e., if x is a right eigenvector of $\widetilde{A} - \lambda\widetilde{B}$ associated with λ_0, then $V^*x = 0$.
   (b) $\mathcal{N}_l(\lambda_0) \perp \operatorname{span}(U)$, i.e., if y is a left eigenvector of $\widetilde{A} - \lambda\widetilde{B}$ associated with λ_0, then $U^*y = 0$.

3. *If ∞ is a regular eigenvalue of $A - \lambda B$, i.e., rank$(B) < n - k$, then ∞ is an eigenvalue of* (4.1) *for each $\tau \neq 0$. The right and left null spaces*

$$\mathcal{N}_r(\infty) := \ker(B + \tau\, UD_BV^*) \quad \textit{and} \quad \mathcal{N}_l(\infty) := \ker\left((B + \tau\, UD_BV^*)^*\right)$$

   are both constant in $\tau \neq 0$. In addition:
   (a) $\mathcal{N}_r(\infty) \perp \operatorname{span}(V)$, i.e., if x is a right eigenvector of $\widetilde{A} - \lambda\widetilde{B}$ associated with ∞, then $V^*x = 0$.
   (b) $\mathcal{N}_l(\infty) \perp \operatorname{span}(U)$, i.e., if y is a left eigenvector of $\widetilde{A} - \lambda\widetilde{B}$ associated with ∞, then $U^*y = 0$.

Proof. First, we will assume that $A - \lambda B$ does not have one of the eigenvalues 0 or ∞ and we will show 1) and 2) for this particular case.

Applying Proposition 4.2, there exist nonsingular matrices P, Q such that

$$P(A - \lambda B)Q = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ 0 & \mathcal{S}(\lambda) \end{bmatrix}, \quad PU = \begin{bmatrix} 0 \\ U_2 \end{bmatrix}, \quad Q^*V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \tag{4.6}$$

where $\mathcal{R}(\lambda)$ and $\mathcal{S}(\lambda)$ are the regular and singular parts of $A - \lambda B$, respectively, both being in KCF, and where U and V are partitioned conformably with $A - \lambda B$. We will now show 1) and 2):

1) Since $A - \lambda B$ is square and of normal rank $n - k$, it has exactly k left minimal indices, say $n_1, \ldots, n_k$ and exactly k right minimal indices, say $m_1, \ldots, m_k$. We may assume without loss of generality that they are paired up to form square blocks of one left and right minimal index each, i.e., we may assume that $\mathcal{S}(\lambda)$ has the block diagonal form

$$\mathcal{S}(\lambda) = \operatorname{diag}\left(\begin{bmatrix} L_{n_1}(\lambda)^\top & 0 \\ 0 & L_{m_1}(\lambda) \end{bmatrix}, \ldots, \begin{bmatrix} L_{n_k}(\lambda)^\top & 0 \\ 0 & L_{m_k}(\lambda) \end{bmatrix}\right).$$

Then the perturbed pencil takes the form

$$P\big(A - \lambda B + \tau\,(UD_AV^* - \lambda UD_BV^*)\big)Q = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ \tau\,U_2(D_A - \lambda D_B)V_1^* & \mathcal{R}_{\rm new}(\lambda) \end{bmatrix},$$

where

$$\mathcal{R}_{\rm new}(\lambda) := \mathcal{S}(\lambda) + \tau\,U_2(D_A - \lambda D_B)V_2^*.$$

Clearly, the characteristic polynomial of $P\big(A + \tau\,UD_AV^* - \lambda\,(B + \tau\,UD_BV^*)\big)Q$ is given by $\det \mathcal{R}(\lambda) \cdot \det \mathcal{R}_{\rm new}(\lambda)$ and from the definition of $\mathcal{R}_{\rm new}(\lambda)$ it is clear that the coefficients of $\det \mathcal{R}_{\rm new}(\lambda)$ are polynomials in the entries of U_2 and V_2^* and thus also of U and V^*.

Now let λ_0 be an eigenvalue of $\mathcal{R}(\lambda)$, i.e., $\det \mathcal{R}(\lambda_0) = 0$. Note that if $e_{i,j}$ denotes the ith standard basis vector of $\mathbb{C}^j$, and $F = (\alpha_i - \lambda_0\beta_i)e_{n_i+1,n_i+m_i+1}e_{n_i+1,n_i+m_i+1}^*$, then

$$\det\left(\begin{bmatrix} L_{n_i}(\lambda_0)^\top & 0 \\ 0 & L_{m_i}(\lambda_0) \end{bmatrix} + F\right) = (-\lambda_0)^{n_i}(\alpha_i - \lambda_0\beta_i).$$

Thus, with e_i the ith standard basis vector in $\mathbb{C}^n$, for the particular choice

$$U_2 = V_2 = [e_{n_1+1} \;\; e_{n_1+m_1+1+n_2+1} \;\; \cdots \;\; e_{n_1+m_1+1+\cdots+n_{k_1}+m_{k-1}+1+n_k+1}]$$

we obtain that

$$\det \mathcal{R}_{\rm new}(\lambda_0) = (-\lambda_0)^{n_1+\cdots+n_k}(\alpha_1 - \lambda_0\beta_1)\cdots(\alpha_k - \lambda_0\beta_k)$$

which is nonzero as the eigenvalues of $D_A - \lambda D_B$ are by hypothesis distinct from λ_0. But then $\det \mathcal{R}_{\rm new}(\lambda_0)$ is generically nonzero (the set of all (U, V^*) for which $\det \mathcal{R}_{\rm new}(\lambda_0) = 0$ is by definition an algebraic set, because $\det \mathcal{R}_{\rm new}(\lambda_0)$ is a polynomial in the entries of U and V^*) which shows that $R_{\rm new}(\lambda_0)$ is generically regular and does not have λ_0 as an eigenvalue. Since intersections of finitely many generic sets are still generic we can conclude that the spectra of $\mathcal{R}(\lambda)$ and $\mathcal{R}_{\rm new}(\lambda)$ are disjoint. But then, it immediately follows from [25, Lemma 6.11] and [16, XII.2, Thm. 2] that the perturbed pencil $\widetilde{A} - \lambda\widetilde{B}$ has the KCF as given in (4.5).

2) Let λ_0 be a regular eigenvalue of $A - \lambda B$ and thus of $\mathcal{R}(\lambda)$. It then follows immediately from 1) that λ_0 is also an eigenvalue of $\widetilde{A} - \lambda\widetilde{B}$ for each $\tau \neq 0$. For the moment, let $\tau \neq 0$ be fixed and let the columns of Y form a basis of the left null space $\mathcal{N}_l(\lambda_0)$ of $\widetilde{A} - \lambda\widetilde{B}$. Partition

$$Y^*P^{-1} = [Y_1^* \;\; Y_2^*]$$

conformably with the partition in (4.6). Since λ_0 is not an eigenvalue of $\mathcal{R}_{\rm new}(\lambda)$ we obtain from $Y^*(\widetilde{A} - \lambda_0\widetilde{B}) = 0$ that $Y_2 = 0$. But this implies that the columns of Y form a basis for the left null space $\mathcal{N}_l(\lambda_0)$ for all values of $\tau \in \mathbb{R} \setminus \{0\}$ as the construction of the transformation matrices P and Q only depends on A, B, and U, but not on τ. Furthermore, we obtain

$$Y^*U = Y^*P^{-1}PU = \begin{bmatrix} Y_1^* & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ U_2 \end{bmatrix} = 0,$$

i.e., $N_l(\lambda_0)$ is orthogonal to the space spanned by the columns of U.

Observe that the statement on the right null space $\mathcal{N}_r(\lambda_0)$ does not follow immediately from the partitioning in (4.6) as in general we have $V_1 \neq 0$. But we can apply the already proved part of the theorem to the pencil $A^* - \lambda B^*$ and the perturbation $V(D_A^* - \lambda D_B^*)U^*$ to obtain the corresponding statements for the right null space $N_r(\lambda_0)$. This finishes the proof of 2).

Finally, assume that $A - \lambda B$ does have one of the eigenvalues 0 or ∞. Then apply a Möbius transformation of the form

$$\mathrm{M}_{\alpha,\beta}(A - \lambda B) := \alpha A + \beta B - \lambda\,(\alpha B - \beta A)$$

where $\alpha, \beta \in \mathbb{R}$ are such that $\alpha^2 + \beta^2 = 1$ and such that $\mathrm{M}_{\alpha,\beta}(A - \lambda B)$ does neither have the eigenvalues 0 nor ∞. Note that this Möbius transformation just has the effect of "rotating" eigenvalues on the extended real line $\mathbb{R} \cup \{\infty\}$, but it leaves eigenvectors and the Jordan structure invariant, see, e.g., [26]. The result then follows by applying the already proved parts of the theorem on $\mathrm{M}_{\alpha,\beta}(A - \lambda B)$ followed by applying the inverse Möbius transformation

$$\mathrm{M}_{\alpha,-\beta}(C - \lambda D) := \alpha C - \beta D - \lambda\,(\alpha D + \beta C).$$

to give the corresponding statements for $\widehat{A} - \lambda \widehat{B}$. In particular, this shows 3). □

Remark 4.4. We mention that part 1) in Theorem 4.3 is in line with one of the main results of [7], where it was shown that generically the regular part of a singular pencil stays invariant under generic perturbations that do not make the pencil regular. Part 1) of Theorem 4.3 extends this results (in the sense of the theorem) to the case of rank completing perturbation. Clearly, the regular part of the pencil will be completely changed if generic perturbations of a rank larger than the difference of the size and the normal rank of the pencil is applied.

Theorem 4.3 characterizes the properties of the regular eigenvalues of the perturbed pencil $\widetilde{A} - \lambda \widetilde{B}$ as in (4.1). We will next investigate the properties of the eigenvalues from the newly created block $\mathcal{R}_{\text{new}}$. We start with the following lemma that will be needed for the main results. The values $\gamma_1, \ldots, \gamma_k$ in the lemma are the eigenvalues that we will prescribe later in Theorem 4.6 using the matrices D_A and D_B.

Lemma 4.5. *Let $A - \lambda B$ be an $n \times n$ singular pencil of normal rank $n - k$ with left minimal indices $n_1, \ldots, n_k$ and right minimal indices $m_1, \ldots, m_k$. Furthermore, let $U, V \in \mathbb{C}^{n,k}$ have full column rank, $N := n_1 + \cdots + n_k$, $M := m_1 + \cdots + m_k$, and let $\gamma_1, \ldots, \gamma_k \in \mathbb{C}$ be given values that are distinct from the regular eigenvalues of $A - \lambda B$. Then, generically with respect to the entries of U and V^*, the following statements hold:*

1. *There exist exactly M pairwise distinct values $\alpha_1, \ldots, \alpha_M$ different from the eigenvalues of $A - \lambda B$ and different from $\gamma_1, \ldots, \gamma_k$ such that for each α_i there exists a nonzero vector z_i with $(A - \alpha_i B)z_i = 0$ and $V^* z_i = 0$.*
2. *There exist exactly N pairwise distinct values $\beta_1, \ldots, \beta_N$ different from the eigenvalues of $A - \lambda B$ and different from $\gamma_1, \ldots, \gamma_k$ and $\alpha_1, \ldots, \alpha_M$ such that for each β_i there exists a nonzero vector w_i with $w_i^*(A - \beta_i B) = 0$ and $w_i^* U_i = 0$.*
3. *For any given set of k linearly independent vectors $t_1, \ldots, t_k \in \mathbb{C}^k$ there exist nonzero vectors $s_1, \ldots, s_k$ with $(A - \gamma_i B)s_i = 0$ and $t_i = V^* s_i$ for $i = 1, \ldots, k$.*

Proof. 1) Without loss of generality we may assume that $A - \lambda B$ is in KCF such that the blocks $L_{m_1}(\lambda), \ldots, L_{m_k}(\lambda)$ associated with the right minimal indices appear first in the form. Then for each $\alpha \in \mathbb{C}$ different from the eigenvalues of $A - \lambda B$ the columns of

$$\begin{bmatrix} q_1(\alpha) & \ldots & q_k(\alpha) \end{bmatrix} = \left[\begin{array}{ccc} q_{11}(\alpha) & & 0 \\ & \ddots & \\ 0 & & q_{kk}(\alpha) \\ \hline 0 & \ldots & 0 \end{array}\right] \quad \text{with} \quad q_{jj}(\alpha) = \begin{bmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{m_j} \end{bmatrix}$$

form a basis for $\ker(A - \alpha B)$ for each $\alpha \in \mathbb{C}$. (When α is a regular eigenvalue, there are additional vectors in $\ker(A-\alpha B)$ since the rank of $A-\alpha B$ drops below the normal rank $n - k$.)

We are looking for $z \neq 0$ and α such that $V^*z = 0$ and $(A - \alpha B)z = 0$. Since we want α to be distinct from the regular eigenvalues of $A - \lambda B$, the vector z has to be of the form

$$z = c_1\, q_1(\alpha) + \cdots + c_k\, q_k(\alpha),$$

where $c = [c_1 \ \ldots \ c_k]^T \neq 0$. From $V^*z = 0$ we get the equation

$$G(\alpha)\, c = 0, \tag{4.7}$$

where $G(\alpha)$ is a $k \times k$ matrix whose element $g_{ij}(\alpha) = v_i^* q_j(\alpha)$ is a polynomial in α which generically with respect to the entries of v_i^* will have degree m_j for $i, j = 1, \ldots, k$. Equation (4.7) has a nontrivial solution if and only if $\det G(\alpha) = 0$, where $\det G(\alpha)$ is a polynomial in α which generically with respect to the entries of V^* is of degree M. Thus $\det G(\alpha)$ will have M roots $\alpha_1, \ldots, \alpha_M$ (counted with multiplicities).

On the other hand, for each fixed $\mu \in \mathbb{C}$, we have that $\det G(\mu)$ is also a polynomial in the entries of $V^* = [v_1 \ \cdots \ v_k]^*$. For the particular choice

$$v_1 = e_1, \quad v_2 = e_{m_1+2}, \quad \ldots, \quad v_k = e_{m_1+\cdots+m_{k-1}+k}$$

we obtain that $v_i^* q_j(\mu) = \delta_{ij}$ so that $G(\mu) = I_k$ shows that $\det G(\mu)$ is a nonzero polynomial in the entries of V^*. It thus follows that generically with respect to the entries of V^* we will have $\det G(\mu) \neq 0$, and consequently the fixed value μ will generically not be among the roots of $G(\alpha)$ as a polynomial in α. Since the intersection of finitely many generic sets is still generic, it follows that we can generically exclude finitely many values from the zeros $\alpha_1, \ldots, \alpha_M$ of $G(\alpha)$. This shows that generically with respect to the entries of V^*, the values $\alpha_1, \ldots, \alpha_M$ are different from the eigenvalues of $A - \lambda B$ and also from the given values $\gamma_1, \ldots, \gamma_k$.

Next we show that the roots $\alpha_1, \ldots, \alpha_M$ of $p(\alpha) := \det G(\alpha)$ generically are pairwise distinct which is exactly the case if none of these values is a root of $q(\alpha) = p'(\alpha)$, where q is the formal derivative of p with respect to α. Also recall that the polynomials p and q have a common root if and only if their resultant is zero, which is given by the determinant of the Sylvester matrix $S(p, q)$ associated with p and q. Since the entries of the Sylvester matrix are coefficients of the polynomials p and q, it follows that $\det S(p, q)$ is a polynomial with respect to the entries of V^*. It remains to show that $\det S(p, q)$ is a nonzero polynomial (because then we will have that $\det S(p, q) \neq 0$ is a generic property with respect to the entries of V^*), and for this it is enough to show that for a particular choice of the entries of V we have that the values $\alpha_1,\ldots,\alpha_M$ are pairwise distinct. Now taking $v_1 = e_{m_1+1} - \varepsilon_1 e_1$, $v_2 = e_{m_1+m_2+2} - \varepsilon_2 e_{m_1+2}$, $\ldots$, $v_k = e_{m_1+\cdots+m_k+k} - \varepsilon_k e_{m_1+\cdots+m_{k-1}+k}$, with $\varepsilon_1,\ldots,\varepsilon_k > 0$, we obtain that $v_i^* q_j(\alpha) = \delta_{ij}\alpha^{m_j} - \varepsilon_j$ and thus $G(\alpha)$ is diagonal and

$$\det G(\alpha) = (\alpha^{m_1} - \varepsilon_1)\cdots(\alpha^{m_k} - \varepsilon_k).$$

Since the roots of each factor $(\alpha^{m_j} - \varepsilon_j)$ are m_j pairwise distinct complex numbers lying on a circle centered at zero with radius ε_j^{1/m_j}, it remains to choose the values $\varepsilon_1,\ldots,\varepsilon_k$ in such a way that the k radii are pairwise distinct to obtain the desired example.

2) In a way similar to the one in 1) we can consider the left null space for $A-\alpha B$ and show the existence of $\beta_1,\ldots,\beta_N$ and the corresponding nonzero vectors $w_1,\ldots,w_N$, where now the statements are generic with respect to the entries of U. In particular, by interpreting V as already fixed, this shows that generically with respect to the entries of U, the values $\beta_1,\ldots,\beta_N$ are not only different from the eigenvalues of $A-\lambda B$ and $\gamma_1,\ldots,\gamma_k$, but also from the values $\alpha_1,\ldots,\alpha_M$ constructed in 1).

3) With the same notation as in 1) we now aim to solve the equations

$$s_i = c_1 q_1(\gamma_i) + \cdots + c_k q_k(\gamma_i) \quad \text{and} \quad V^* s_i = t_i,$$

or, equivalently, $G(\gamma_i)c = t_i$ for $i = 1,\ldots,k$. Since γ_i is different from the values $\alpha_1,\ldots,\alpha_M$, we have $\det G(\gamma_i) \neq 0$ and hence $G(\gamma_i)\,c = t_i$ is uniquely solvable for c for $i = 1,\ldots,k$. ◻

The following theorem encapsulates the main result on the new eigenvalues of our perturbed pencil.

THEOREM 4.6. *Let $A-\lambda B$ be an $n\times n$ singular pencil of normal rank $n-k$ with left minimal indices $n_1,\ldots,n_k$ and right minimal indices $m_1,\ldots,m_k$. Furthermore, let $U,V \in \mathbb{C}^{n,k}$ have full column rank and let $D_A = \mathrm{diag}(a_1,\ldots,a_k), D_B = \mathrm{diag}(b_1,\ldots,b_k) \in \mathbb{C}^{k,k}$ be such that $D_A - \lambda D_B$ is regular and such that all (not necessarily pairwise distinct) values $\gamma_i := \frac{a_i}{b_i}$, $i = 1,\ldots,k$, are different from the eigenvalues of $A-\lambda B$. (Here, $\frac{a_i}{b_i}$ is interpreted as the infinite eigenvalue, if $b_i = 0$.) Finally, let $N := n_1 + \cdots + n_k$ and $M := m_1 + \cdots + m_k$. Then generically with respect to the entries of U and V^*, the following statements hold:*

1. *The pencil (4.1) has M simple eigenvalues $\alpha_1,\ldots,\alpha_M$ which are independent of $\tau \neq 0$, so that for each of these eigenvalues its right eigenvector x_i is constant in $\tau \neq 0$ (up to scaling) and satisfies $V^*x_i = 0$, while the left eigenvector y_i is a linear function of τ (up to scaling) and satisfies $U^*y_i \neq 0$ for all $\tau \neq 0$.*
2. *The pencil (4.1) has N simple eigenvalues $\beta_1,\ldots,\beta_N$ which are independent of $\tau \neq 0$, so that for each of these eigenvalues its left eigenvector y_i is constant for $\tau \neq 0$ (up to scaling) and satisfies $U^*y_i = 0$, while the right eigenvector x_i is a linear function of τ (up to scaling) and satisfies $V^*x_i \neq 0$ for all $\tau \neq 0$.*
3. *For each $\tau \neq 0$ each γ_i is an eigenvalue of (4.1) with the same algebraic multiplicity as for the pencil $D_A - \lambda D_B$. Furthermore, the left and right null spaces $\mathcal{N}_l(\gamma_i)$ and $\mathcal{N}_r(\gamma_i)$ of (4.1) associated with γ_i are constant in τ. In addition, we have:*
   (a) *$\mathcal{N}_r(\gamma_i) \cap \ker(V^*) = \{0\}$, i.e., for each right eigenvector x of (4.1) associated with γ_i we have $V^*x \neq 0$.*

(b) $\mathcal{N}_l(\gamma_i) \cap \ker(U^*) = \{0\}$, *i.e., for each left eigenvector* y *of* (4.1) *associated with* γ_i *we have* $U^* y \neq 0$.

(Note that the simplicity of the eigenvalues $\alpha_1, \dots, \alpha_M, \beta_1, \dots, \beta_N$ *implies in particular that they are all different from the eigenvalues of* $A - \lambda B$ *and from the eigenvalues* $\gamma_1, \dots, \gamma_k$*.)*

Proof. Without loss of generality, we may assume that the infinite eigenvalue is not among the eigenvalues of $D_A - \lambda D_B$. Otherwise, we may as in the proof of Theorem 4.3 apply a Möbius transformation to both $A - \lambda B$ and $D_A - \lambda D_B$ such that $D_A - \lambda D_B$ does not have the eigenvalue ∞, apply the statement that was proved for this special situation, and finally transform back with the inverse Möbius transformation to obtain the desired result.

Observe that generically with respect to the entries of U and V^*, the statements of Lemma 4.5 will hold, where for the values $\gamma_1, \dots, \gamma_k$ we will take the eigenvalues of the pencil $D_A - \lambda D_B$ and for the vectors $t_1, \dots, t_k$ we will take the standard basis vectors $e_1, \dots, e_k$ from $\mathbb{C}^k$. We now show 1)–3).

1) By Lemma 4.5 there exist exactly M pairwise distinct values $\alpha_1, \dots, \alpha_M$ different from the eigenvalues of $A - \lambda B$ and from $\gamma_1, \dots, \gamma_k$, and nonzero vectors $z_1, \dots, z_M$ such that $(A - \alpha_i B) z_i = 0$ and $V^* z_i = 0$ for $i = 1, \dots, M$. From this we obtain that $(A - \alpha_i B + \tau\, U D_A V^* - \alpha_i \tau\, U D_B V^*)\, z_i = 0$ which means that α_i is an eigenvalue of (4.1) for $\tau \neq 0$ with a right eigenvector z_i that is invariant under τ.

Considering now τ as a variable, it follows that the pencil

$$G_i + \tau H_i := (A - \alpha_i B) + \tau\, U (D_A - \alpha_i D_B) V^* \tag{4.8}$$

is singular. Suppose that $\operatorname{nrank}(G_i, H_i) = n - j$ for $j \geq 1$, which means that (4.8) has j right and j left minimal indices. Then we know from $G_i z_i = 0$ and $H_i z_i = 0$ that one of these right minimal indices is equal to zero. The remaining $j - 1$ right minimal indices are all larger than zero, because otherwise there would exist $y_i \in \ker(G_i) \cap \ker(H_i)$ linearly independent of z_i which implies that α_i would be a multiple eigenvalue of (4.1) in contradiction to Lemma 4.5.

Now suppose that (4.8) has a left minimal index being zero. Then there exists a vector $w_i \neq 0$ such that $w_i^* G_i = 0$ and $w_i^* H_i = 0$, which implies $w_i^* U = 0$, because V was assumed to have full rank and $D_A - \alpha_i D_B$ is nonsingular since the values α_i are different from the eigenvalues of $D_A - \lambda D_B$. But then by Lemma 4.5 α_i is equal to one of the values $\beta_1, \dots, \beta_N$ which is a contradiction. Thus, all left minimal indices of (4.8) are larger than or equal to one.

Furthermore, we know that $\operatorname{rank}(G_i) = n - k$ and $\operatorname{rank}(H_i) = k$ since α_i differs from all finite regular eigenvalues of $A - \lambda B$ and all eigenvalues of $D_A - \lambda D_B$. It follows that in the KCF of the pencil (4.8) there are at least $n - k - j$ blocks associated with the eigenvalue infinity and at least $k - j$ blocks associated with the eigenvalue zero.

By a simple computation we obtain that the dimension of the KCF of (4.8) is at least $(n + j - 1) \times (n + j - 1)$; therefore the only option is $j = 1$ and hence (4.8) has exactly one right minimal index (being zero) and exactly one left minimal index, say p. Then another simple computation shows that the dimension of the KCF of (4.8) is at least $(n + p - 1) \times (n + p - 1)$ showing that the left minimal index p must be equal to one. Consequently, there exist linearly independent vectors s_i and t_i such that

$$\begin{aligned} s_i^*(A-\alpha_i B) &= 0,\\ t_i^*(A-\alpha_i B) + s_i^* U(D_A-\alpha_i D_B)V^* &= 0,\\ t_i^* U(D_A-\alpha_i D_B)V^* &= 0 \end{aligned}$$

and $s_i^* U \neq 0$. Up to scaling, the left eigenvector y_i of (4.1) associated with α_i then has the form $y_i(\tau) = s_i + \tau t_i$ and is a linear function of τ.

2) This follows completely analogously to 1).

3) Clearly, the standard basis vectors $e_1, \ldots, e_k$ are eigenvectors of the pencil $D_A - \lambda D_B$ associated with the eigenvalues $\gamma_1, \ldots, \gamma_k$. By Lemma 4.5, there exists k (necessarily linearly independent) vectors $s_1, \ldots, s_k \in \mathbb{C}^n$ such that $(A - \gamma_i B)s_i = 0$ and $e_i = V^* s_i$ for $i = 1, \ldots, k$. Then for each $i = 1, \ldots, k$ and each $\tau \neq 0$ we have

$$(\widetilde{A} - \gamma_i \widetilde{B})s_i = (A - \gamma_i B)s_i + \tau\, U(D_A - \gamma_i D_B)V^* s_i = 0.$$

This implies that the values $\gamma_1, \ldots, \gamma_k$ are eigenvalues of $\widetilde{A} - \lambda\widetilde{B}$ with the same algebraic multiplicities as for $D_A - \lambda D_B$. Furthermore, it follows that the null space $\mathcal{N}_r(\gamma_i)$ does not depend on τ and by construction we have $\mathcal{N}_r(\gamma_i) \cap \ker(V^*) = \{0\}$.

By applying Lemma 4.5 to the pencil $A^* - \lambda B^*$ we obtain the analogous statements for the left null spaces $\mathcal{N}_l(\gamma_i)$. ◻

Summary 4.7. Summarizing the results from Theorem 4.3 and Theorem 4.6, let $A - \lambda B$ be an $n \times n$ singular pencil of normal rank $n - k$, and with left minimal indices $n_1, \ldots, n_k$ and right minimal indices $m_1, \ldots, m_k$, and let U, V, D_A, D_B, N, and M be as in Theorem 4.6. Since the regular part of $A - \lambda B$ then has the size $r := n - N - M - k$ and we have found $N + M + k$ new eigenvalues in Theorem 4.6, we have classified all eigenvalues of the perturbed pencil

$$\widetilde{A} - \lambda\widetilde{B} := A - \lambda B + \tau\,(U D_A V^* - \lambda\, U D_B V^*)$$

into the following three groups:

1. *Regular eigenvalues*: There are r such eigenvalues that are exactly the eigenvalues of $A - \lambda B$. The corresponding right eigenvectors x and left eigenvectors y satisfy $V^* x = 0$ and $U^* y = 0$.
2. *Prescribed eigenvalues*: There are k such eigenvalues that coincide with the k eigenvalues of $D_A - \lambda D_B$. The corresponding right eigenvectors x and left eigenvectors y satisfy both $V^* x \neq 0$ and $U^* y \neq 0$.
3. *Random eigenvalues*: These are the remaining $N + M$ eigenvalues. They are simple and if μ is such an eigenvalue, x is a corresponding right eigenvector and y is a corresponding left eigenvector, then we either have $V^* x = 0$ and $U^* y \neq 0$, or $V^* x \neq 0$ and $U^* y = 0$.

Thus, the eigenvalues of $A - \lambda B$ can be identified from the eigenvalues of $\widetilde{A} - \lambda\widetilde{B}$ by investigating orthogonality properties of the corresponding left and right eigenvectors. We will use this observation in the following section for the development of an algorithm for computing the eigenvalues of a singular square pencil.

Remark 4.8. If A and B are symmetric, then it seems that for our current approach it is necessary to use nonsymmetric rank completing perturbations. Namely, when a symmetric perturbation of the form $\tau\, U(D_A - \lambda D_B)U^*$ is used, there is an issue with the third group in Summary 4.7 as random eigenvalues appear either as double real eigenvalues or in complex conjugate pairs, and in the former case the orthogonality constraints cannot be satisfied. We leave the study of structured singular pencils for a future research.

5. A perturbation method for singular generalized eigenvalue problems. In this section we explain how it is possible in the generic case to extract the finite regular eigenvalues by solving only one perturbed eigenvalue problem. The key are the orthogonality properties that are existent or non-existent for the eigenvectors associated with regular, prescribed, and random eigenvalues, respectively. We can exploit these properties and obtain reliable results already in double precision.

Let $A-\lambda B$ be a singular $n\times n$ pencil with a normal rank $n-k$, where $k>0$. We determine $\mathrm{nrank}(A,B)$ by computing $\mathrm{rank}(A-\zeta B)$ for a random ζ. As we have shown in the previous section, if we take two random $n\times k$ matrices U and V with orthonormal columns, a regular $k\times k$ diagonal pencil $D_A-\lambda D_B$, and $\tau\neq 0$, then the perturbed pencil (4.1) is regular. Regular eigenvalues of $A-\lambda B$ (theoretically) remain constant under this perturbation. In contrast, eigenvalues that originate from the singular part of the pencil (the so called "random" eigenvalues) may be "anywhere in the complex plane". In addition, (4.1) also has k "prescribed" eigenvalues that coincide with the eigenvalues of $D_A-\lambda D_B$.

Theoretically, if we compute all eigenvalues λ_i together with the left and right eigenvectors x_i and y_i for $i=1,\dots,n$ of (4.1), then $\max(\|V^*x_i\|,\|U^*y_i\|)=0$ for a regular eigenvalue and $\max(\|V^*x_i\|,\|U^*y_i\|)>0$ for a prescribed or a random eigenvalue, so we could apply this criterion to extract the regular eigenvalues. In the following we will discuss how the above criterion is affected by computation in finite precision and how does it depend on τ. We will also introduce other criteria that may be used for the same purpose or to further separate regular eigenvalues into finite and infinite ones.

If x_i and y_i are normalized left and right eigenvectors of the perturbed problem (4.1) for an eigenvalue λ_i, we can compute the number

$$s(\lambda_i)=y_i^*\widetilde{B}x_i=y_i^*Bx_i+\tau\, y_i^*UD_BV^*x_i. \tag{5.1}$$

It is easy to see that $s(\lambda_i)\neq 0$ for a simple finite eigenvalue λ_i. As explained in the following lemma, which is a straightforward generalization of the standard result for a pencil $A-\lambda I$, see, e.g., [42, Sec. 2.9], $1/|s(\lambda_i)|$ occurs in the expression for a standard condition number of a simple finite eigenvalue.

LEMMA 5.1. *Let λ_i be a simple finite eigenvalue of a regular matrix pencil $\widetilde{A}-\lambda\widetilde{B}$ and let x_i and y_i be its normalized left and right eigenvectors. If we perturb the pencil into $(\widetilde{A}+\theta E)-\lambda(\widetilde{B}+\theta F)$ for a small $\theta>0$, then λ_i perturbs into*

$$\lambda_i+\theta\,\frac{y_i^*Ex_i-\lambda_i y_i^*Fx_i}{s(\lambda_i)}+\mathcal{O}(\theta^2). \tag{5.2}$$

For a simple finite regular eigenvalue λ_i we have $V^*x_i=0$ and $U^*y_i=0$. This implies $s(\lambda_i)=y_i^*Bx_i$ and consequently $s(\lambda_i)$ does not change with $\tau\neq 0$. For a regular infinite eigenvalue we have $y_i^*B=0$, $Bx_i=0$, $V^*x_i=0$, and $U^*y_i=0$, therefore $s(\infty)=0$, again independent of $\tau\neq 0$. On the other hand, we can show that values $s(\lambda)$ of prescribed and random eigenvalues depend on τ and go to 0 as τ goes to 0. For this, we need the following lemma.

LEMMA 5.2. *Let $A - \lambda B$ be a singular pencil and let α be different from all eigenvalues of $A - \lambda B$, i.e., rank*$(A - \alpha B) = \mathrm{nrank}(A, B)$. *If $y^*(A - \alpha B) = 0$ and $(A - \alpha B)x = 0$ then $y^*Ax = y^*Bx = 0$.*

Proof. We know from the structure of the left and right singular blocks that $x \in \mathcal{N}_r(\alpha)$ can be written as $x = q_1 + \alpha q_2 + \cdots + \alpha^p q_{p+1}$, where the vectors $q_1, \ldots, q_{p+1}$ form the chain

$$Aq_1 = 0, \; Aq_2 = Bq_1, \; \ldots, \; Aq_{p+1} = Bq_p, \; Bq_{p+1} = 0$$

for certain $p \geq 0$. Similarly, $y \in \mathcal{N}_l(\alpha)$ can be written as $y = w_1 + \overline{\alpha} w_2 + \cdots + \overline{\alpha}^r w_{r+1}$, where the vectors $w_1, \ldots, w_{r+1}$ form the chain

$$A^*w_1 = 0, \; A^*w_2 = B^*w_1, \; \ldots, \; A^*w_{r+1} = B^*w_r, \; B^*w_{r+1} = 0$$

for certain $r \geq 0$. To show $y^*Bx = 0$ it is enough to show that $w_i^*Bq_j = 0$ for all $i = 1, \ldots, r+1$ and $j = 1, \ldots, p+1$. For $j = p+1$ this follows from $Bq_{p+1} = 0$, so we can assume that $j \leq p$. It follows that $w_i^*Bq_j = w_i^*Aq_{j+1}$. If $i = 1$, then $w_1^*A = 0$, if not, we can continue to $w_i^*Bq_j = w_i^*Aq_{j+1} = w_{i-1}^*Bq_{j+1}$. As we continue in this manner, we eventually reach either $w_1^*A = 0$ or $Bq_{p+1} = 0$. It follows that $y^*Bx = 0$ and from $y^*Ax = \alpha y^*Bx$ we get $y^*Ax = 0$ as well. □

LEMMA 5.3. *Let λ_i be a prescribed or random eigenvalue of* (4.1) *under the assumptions of Theorem* 4.6. *Then there exists a nonzero constant c_i such that $s(\gamma_i) = c_i\tau$.*

Proof. First, let λ_i be a prescribed eigenvalue. Then by the proof of Theorem 4.6 the corresponding left and right eigenvectors satisfy $y_i^*(A - \lambda_i B) = 0$ and $(A - \lambda_i B)x_i = 0$ which by Lemma 5.2 implies $y_i^*Bx_i = 0$. But then we have $s(\lambda_i) = c_i\tau$ with $c_i = y_i^*UD_BV^*x_i$ and c_i must be nonzero, because γ_i is a simple eigenvalue of $\widetilde{A} - \lambda\widetilde{B}$ for $\tau \neq 0$.

Next, let λ_i be a random eigenvalue, such that $V^*x_i = 0$ and $U^*y_i \neq 0$. We know (see the proof of Theorem 4.6) that y_i is a linear function of τ as $y_i(\tau) = s_i + \tau t_i$, where $s_i^*(A - \lambda_i B) = 0$. Since $(A - \lambda_i B)x_i = 0$, it follows from Lemma 5.2 that $s_i^*Bx_i = 0$ and $y_i^*\widetilde{B}x_i = y_i^*Bx_i = \tau\, t_i^*Bx_i$. The case $V^*x_i \neq 0$ and $U^*y_i = 0$ can be shown analogously. □

So, if we take a small τ and if all finite regular eigenvalues are simple and none of them is too ill-conditioned, then we can separate the finite regular eigenvalues from the remaining ones with the help of the values $s(\lambda)$.

Let ε be the machine precision and let the matrices A and B be scaled in such way that $\|A\| = \|B\| = 1$. If all finite regular eigenvalues are simple and not too ill-conditioned, then we expect to observe the situation in Table 5.1, where $c > 0$ is a constant, independent of τ, and possibly different for each eigenvalue and each entry in the table.

We now explain the values in the Table 5.1. We will start with column $|s(\lambda)|$ and a finite regular eigenvalue, where we assume that all finite regular eigenvalues are simple and well conditioned. It follows that λ is a simple eigenvalue of $\widetilde{A} - \lambda\widetilde{B}$, therefore $y^*\widetilde{B}x \neq 0$ and, since this value is independent of τ and ε, we have $|s(\lambda)| = c$. For an infinite eigenvalue we should have $y^*\widetilde{B}x = 0$ in exact computation, instead, in finite precision, we get $|y^*\widetilde{B}x| < c\varepsilon$. Finally, in the generic case, if λ_i is a prescribed or

TABLE 5.1
Characteristics of the eigenvalues of the perturbed pencil as in (4.1).

Eigenvalue λ	$\lvert s(\lambda)\rvert$	$\lVert V^*x\rVert$	$\lVert U^*y\rVert$
Finite regular eigenvalue of $A-\lambda B$	c	$< c\varepsilon/\tau$	$< c\varepsilon/\tau$
Infinite regular eigenvalue of $A-\lambda B$	$< c\varepsilon$	$< c\varepsilon/\tau$	$< c\varepsilon/\tau$
Prescribed eigenvalue of $D_A-\lambda D_B$	$c\tau$	c	c
Random eigenvalue from an L_p block	$c\tau$	$< c\varepsilon/\tau$	c
Random eigenvalue from an L_p^T block	$c\tau$	c	$< c\varepsilon/\tau$

random eigenvalue then it follows from Lemma 5.3 that $|s(\lambda_i)| = c_i\tau$ for a positive constant c_i.

Finally, we turn to the values in the columns $\|V^*x\|$ and $\|U^*x\|$ that are marked by $< c\varepsilon/\tau$. In exact arithmetic, these values should be zero. In finite precision however, due to the supposed backward stability of the applied eigenproblem solver, the computed eigenvalues and eigenvectors of the matrix pencil $\widetilde{A}-\lambda\widetilde{B}$ are exact eigenpairs of a perturbed pencil $\widetilde{A}+E-\lambda\,(\widetilde{B}+F)$, where $\|E\| \le c_1\,\|\widetilde{A}\|\,\varepsilon$ and $\|F\| \le c_2\,\|\widetilde{B}\|\,\varepsilon$. If we assume that all finite eigenvalues of $\widetilde{A}-\lambda\widetilde{B}$ are simple, then we can say something about the eigenvector perturbations. This is done in the following lemma whose proof is omitted, because it is a straightforward generalization of the result for the pencil $A-\lambda I$ from [42, Sec. 2.10].

LEMMA 5.4. *Let all finite eigenvalues λ_i of $\widetilde{A}-\lambda\widetilde{B}$ be simple and let x_i and y_i be corresponding left and right normalized eigenvectors. If the pencil is perturbed into $\widetilde{A}+\theta E-\lambda\,(\widetilde{B}+\theta F)$, then the eigenvector x_i is perturbed into*

$$\widetilde{x}_i = x_i + \theta \sum_{k=1,k\neq i}^{n} \frac{y_k^*(E-\lambda_i F)x_i}{(\lambda_i-\lambda_k)\,s(\lambda_k)}\,x_k + \mathcal{O}(\theta^2).$$

Suppose that λ_i is a finite regular eigenvalue of $A-\lambda B$. Then λ_i is also an eigenvalue of $\widetilde{A}-\lambda\widetilde{B}$ and $V^*x_i = 0$, where x_i is an exact normalized right eigenvector. In finite precision, x_i becomes perturbed in directions of other eigenvectors and it follows from Lemma 5.4 that a contribution in direction of another eigenvector depends on the condition number of the corresponding eigenvalue. The only contributions that affect the value of $\|V^*\widetilde{x}_i\|$ are those related to prescribed eigenvalues or random eigenvalues from left singular blocks, as right eigenvectors of other eigenvalues are orthogonal to V. As condition numbers of these eigenvalues are equal to $1/(c\tau)$ and $\|V^*x_j\| = c$ for the corresponding right eigenvectors, it follows from Lemma 5.4 and the backward stability of the computed eigenpairs that $\|V^*\widetilde{x}_i\| < c\varepsilon/\tau$.

Next, we now turn to the question what choices for the value τ are most beneficial. If we use a very small τ, i.e., τ close to ε, then prescribed and random eigenvalues are very ill-conditioned, and perturbations of eigenvectors may affect the values of $\|V^*x_i\|$ and $\|U^*y_i\|$ that much that none of these values may be close to zero, when they should be. So, if τ is too small, we may not be able to use the values of $\|V^*x_i\|$ and $\|U^*y_i\|$ to extract the regular eigenvalues. However, if all finite regular eigenvalues are simple, then we may still use the values $|s(\lambda_i)|$ to extract the finite regular eigenvalues.

On the other hand, if we use a large τ, then all eigenvalues (except the infinite one) are expected to be well-conditioned which means that the eigenvectors will not change much and the computed left and right eigenvectors will be orthogonal to

V or U in finite precision, when they should be. So, for large τ, we can first use $\max(\|V^*x_i\|, \|U^*y_i\|)$ to extract the regular eigenvalues and then use $|s(\lambda_i)|$ to distinguish the finite regular eigenvalues from the infinite one. In practice, we see this as a better option, because it does not depend on finite regular eigenvalues being simple. However, we should not choose τ too large as this may decrease the precision of the computed finite regular eigenvalues. Since the computed eigenvalues are, due to assumed backward stability, exact eigenvalues of a slightly perturbed pencil $\widetilde{A} - \lambda\widetilde{B}$, it is safe to use τ up to the limit $\|\widetilde{A}\| \approx \|A\|$ and $\|\widetilde{B}\| \approx \|B\|$.

Based on the above discussion, we summarize our method in Algorithm 1, where we assume that the pencil is scaled in such way that $\|A\| = \|B\| = 1$. We note that this scaling is mainly for convenience, to determine an appropriate default value for τ in Algorithm 1. For efficiency, we can also use the 1-norm scaling $\|A\|_1 = \|B\|_1 = 1$.

Algorithm 1: Computing finite regular eigenvalues of a singular pencil (A, B) by a rank-completing perturbation.

Input: A and B such that $\|A\| = \|B\| = 1$, perturbation constant τ (default 10^{-2}), thresholds δ_1 (default $\varepsilon^{1/2}$) and δ_2 (default $10^2\,\varepsilon$).
Output: Eigenvalues of the finite regular part.

Compute the normal rank of pencil $A - \lambda B$: $k = \operatorname{rank}(A - \zeta B)$ for random ζ
Select random $n \times k$ matrices U and V with orthonormal columns.
Select diagonal $k \times k$ matrices D_A and D_B such that the eigenvalues of (D_A, D_B) are (likely) different from those of (A, B) (default: choose diagonal elements of D_A and D_B uniformly random from the interval $[1, 2]$).
Compute the eigenvalues λ_i, $i = 1, \ldots, n$, and right and left eigenvectors x_i and y_i of $(\widetilde{A}, \widetilde{B}) = (A + \tau\, U D_A V^*,\ B + \tau\, U D_B V^*)$.
Compute $s_i = y_i^* \widetilde{B} x_i$ for $i = 1, \ldots, n$.
Compute $\zeta_i = \max(\|V^*x_i\|,\ \|U^*y_i\|)$ for $i = 1, \ldots, n$.
Return all eigenvalues λ_i for $i = 1, \ldots, n$, where $\zeta_i < \delta_1$ and $|s_i| > \delta_2$.

As we will illustrate by experiments in the next section, the above approach seems to work very well in double precision for small or moderate singular pencils. Of course, if some of the eigenvalues are very ill-conditioned (for instance when some of the eigenvalues are multiple), then the method can fail in extracting some of the finite regular eigenvalues. However, its advantage over staircase-based methods may be the following observation: if we make a wrong rank decision in a staircase algorithm, then the method usually fails completely and returns no eigenvalues at all; see Example 6.4 in the next section. In contrast, the method proposed here is able to detect (if not all) at least the well-conditioned finite regular eigenvalues of the pencil under consideration.

6. Numerical examples.

In this section we demonstrate the method with several numerical examples computed in Matlab 2015b.

EXAMPLE 6.1. For the first example we take a 7×7 pencil $A - \lambda B$, where

$$A = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 & 5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ 2 & -1 & -1 & -1 & -1 & -1 & -1 \\ 2 & 5 & 5 & 5 & 5 & 5 & 5 \\ 2 & 5 & 5 & 4 & 4 & 4 & 4 \\ 2 & 5 & 5 & 6 & 5 & 5 & 5 \\ 2 & 5 & 5 & 6 & 7 & 7 & 7 \\ 2 & 5 & 5 & 6 & 7 & 6 & 6 \end{bmatrix}.$$

The matrices are built in such way that the KCF of the pencil contains blocks of all four possible types, $\mathrm{nrank}(A,B)=6$ and the pencil is singular. Its KCF has blocks $J_1(1/2)$, $J_2(1/3)$, N_1, L_1, and L_2^T. If we apply Algorithm 1, we get the values in the following table:

k	λ_k	$\lvert s_k\rvert$	$\lVert V^*x_k\rVert$	$\lVert U^*y_k\rVert$
1	0.333333	$1.5\cdot10^{-2}$	$1.3\cdot10^{-15}$	$1.3\cdot10^{-14}$
2	0.500000	$9.5\cdot10^{-4}$	$1.3\cdot10^{-14}$	$1.9\cdot10^{-14}$
3	∞	$3.8\cdot10^{-19}$	$2.8\cdot10^{-15}$	$1.3\cdot10^{-14}$
4	$-0.244794+0.421723i$	$7.8\cdot10^{-3}$	$5.8\cdot10^{-2}$	$5.6\cdot10^{-15}$
5	$-0.244794-0.421723i$	$7.8\cdot10^{-3}$	$5.8\cdot10^{-2}$	$5.6\cdot10^{-15}$
6	0.383682	$2.1\cdot10^{-4}$	$2.6\cdot10^{-2}$	$4.2\cdot10^{-1}$
7	0.478292	$2.6\cdot10^{-4}$	$9.2\cdot10^{-15}$	$5.2\cdot10^{-1}$

The values in the table follow the pattern from the previous section and it is easy to detect that λ_1 and λ_2 are finite regular eigenvalues, λ_3 is a regular infinite eigenvalue, λ_4,λ_5, and λ_7 are random eigenvalues, and λ_6 is the prescribed eigenvalue.

EXAMPLE 6.2. For the next example we take example C3 from [12]. This example comes from control theory and belongs to a set of examples C1, C2, and C3, where each has successively more ill-conditioned eigenvalues. The pencil has the form

$$A-\lambda B=\begin{bmatrix}1&-2&100&0&0\\1&0&-1&0&0\\0&0&0&1&-75\\0&0&0&0&2\end{bmatrix}-\lambda\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}.$$

Its KCF contains blocks L_2, $J_1(1)$, and $J_1(2)$. As the pencil is rectangular, we add a zero line to make it square. This adds an L_0^T block to the KCF. Algorithm 1 returns the following table for $A-\lambda B$, from which the finite regular eigenvalues 1 and 2 can be extracted.

k	λ_k	$\lvert s_k\rvert$	$\lVert V^*x_k\rVert$	$\lVert U^*y_k\rVert$
1	1.000000	$1.2\cdot10^{-2}$	$1.7\cdot10^{-15}$	$1.9\cdot10^{-15}$
2	2.000000	$1.2\cdot10^{-2}$	$2.3\cdot10^{-15}$	$1.7\cdot10^{-15}$
3	$-0.693767+1.563033i$	$2.1\cdot10^{-2}$	$2.3\cdot10^{-15}$	$5.0\cdot10^{-1}$
4	$-0.693767-1.563033i$	$2.1\cdot10^{-2}$	$2.3\cdot10^{-15}$	$5.0\cdot10^{-1}$
5	78.673901	$2.8\cdot10^{-3}$	$3.3\cdot10^{-1}$	$6.4\cdot10^{-1}$

As in [12] we add some noise and perturb initial $A-\lambda B$ into $\widehat{A}-\lambda\widehat{B}$ by adding $10^{-6}\mathtt{rand}(4,5)$ to A and B. Regular eigenvalues of $\widehat{A}-\lambda\widehat{B}$ can still be extracted by Algorithm 1 if we adjust the parameter δ_1. The values we get are in the following table:

k	λ_k	$\lvert s_k\rvert$	$\lVert V^*x_k\rVert$	$\lVert U^*y_k\rVert$
1	0.999990	$7.6\cdot10^{-3}$	$2.6\cdot10^{-15}$	$5.2\cdot10^{-7}$
2	2.000058	$7.6\cdot10^{-3}$	$2.8\cdot10^{-15}$	$5.6\cdot10^{-7}$
3	101.850555	$8.2\cdot10^{-4}$	$1.3\cdot10^{-1}$	$3.7\cdot10^{-1}$
4	-14.308508	$1.9\cdot10^{-2}$	$5.8\cdot10^{-16}$	$4.5\cdot10^{-1}$
5	15.734162	$9.3\cdot10^{-3}$	$1.0\cdot10^{-17}$	$4.7\cdot10^{-1}$

EXAMPLE 6.3. This is an example from [14, Sec. 5], where the staircase algorithm fails to find a regular subspace of proper size under a small random perturbation. We take

$$A_1-\lambda B_1=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}-\lambda\begin{bmatrix}\delta&0&0&0\\0&\delta&0&0\\0&1&0&0\end{bmatrix},$$

where $\delta=1.5\cdot10^{-8}$. The KCF structure of the pencil is $J_2(0)$ and L_1 which means that 0 is a double regular eigenvalue. It is reported in [14] that if we add a random perturbation of size 10^{-14} to the pencil, then Guptri reports the regular part $J_1(0)$ and we could confirm this using a Matlab implementation of Guptri in [34]. If we enlarge the perturbation to 10^{-11}, Guptri returns no regular part at all, while Algorithm 1 returns two finite regular eigenvalues λ_1 and λ_2 from the following table.

k	λ_k	$\lvert s_k\rvert$	$\lVert V^*x_k\rVert$	$\lVert U^*y_k\rVert$
1	$-1.4306543 \cdot 10^{-3}$	$1.6 \cdot 10^{-11}$	$5.5 \cdot 10^{-17}$	$6.7 \cdot 10^{-10}$
2	$9.9599790 \cdot 10^{-4}$	$1.6 \cdot 10^{-11}$	0.0	$6.7 \cdot 10^{-10}$
3	$-2.2641370 \cdot 10^{7}$	$5.2 \cdot 10^{-9}$	$2.9 \cdot 10^{-18}$	$2.6 \cdot 10^{-6}$
4	$1.1878888 \cdot 10^{0}$	$1.6 \cdot 10^{-3}$	$2.0 \cdot 10^{-1}$	$7.8 \cdot 10^{-1}$

EXAMPLE 6.4. For this example we take the singular pencil $\Delta_1 - \lambda\Delta_0$ of size 300×300 from [33, Ex. 18]. This example is related to two random matrices A and B of size 10×10 in a way that the finite regular eigenvalues of $\Delta_1 - \lambda\Delta_0$ are exactly the values λ such that $A + \lambda B$ has a multiple eigenvalue (see Section 3.2). We know from the properties of the problem, that there are 90 such values λ and that the KCF of $\Delta_1 - \lambda\Delta_0$ contains 100 N_1 and 10 left and 10 right singular blocks. The conjecture from [33] is that the singular blocks are 5 L_4^T, 5 L_5^T, 5 L_5, and 5 L_6 blocks.

This example is also available as `demo_double_eig_mp` in toolbox MultiParEig [35]. The staircase algorithm in MultiParEig fails to extract the finite regular part of size 90 in double precision, but manages to extract all 90 finite regular eigenvalues using quadruple precision and the Multiprecision Computing Toolbox [36]. If we apply Algorithm 1 on $\Delta_1 - \lambda\Delta_0$ in double precision, we get the following values:

k	λ_k	$\lvert s_k\rvert$	$\lVert V^*x_k\rVert$	$\lVert U^*y_k\rVert$
1	$0.508999 + 2.016378i$	$3.0 \cdot 10^{-3}$	$2.3 \cdot 10^{-14}$	$1.6 \cdot 10^{-14}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
89	$4.266290 - 0.925962i$	$1.4 \cdot 10^{-6}$	$2.6 \cdot 10^{-13}$	$7.2 \cdot 10^{-14}$
90	-0.628208	$3.2 \cdot 10^{-7}$	$2.8 \cdot 10^{-14}$	$1.3 \cdot 10^{-11}$
91	∞	$1.1 \cdot 10^{-17}$	$7.1 \cdot 10^{-15}$	$7.1 \cdot 10^{-15}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
190	∞	$2.8 \cdot 10^{-21}$	$5.9 \cdot 10^{-15}$	$7.9 \cdot 10^{-15}$
191	-6.276934	$3.2 \cdot 10^{-7}$	$2.7 \cdot 10^{-14}$	$4.5 \cdot 10^{-5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
300	7.125982	$2.3 \cdot 10^{-5}$	$1.7 \cdot 10^{-1}$	$1.0 \cdot 10^{-2}$

From the columns $\|V^*x_k\|$ and $\|U^*y_k\|$ we get $\max_{k=1,\ldots,190}(\max(\|V^*x_k\|, \|U^*y_k\|)) = 1.3 \cdot 10^{-11}$ and $\min_{k=191,\ldots,300} \max(\|V^*x_k\|, \|U^*y_k\|) = 4.5 \cdot 10^{-5}$, which shows a clear gap that separates regular eigenvalues from the prescribed and random ones. Next, in the set of regular eigenvalues there is also a clear gap between s_{90} and s_{91} that separates finite regular eigenvalues from infinite ones, since $\min_{k=1,\ldots,90} |s_k| = 3.2 \cdot 10^{-7}$ and $\max_{k=91,\ldots,190} |s_k| = 1.1 \cdot 10^{-17}$.

7. The singular two-parameter eigenvalue problem. In a two-parameter eigenvalue problem (2EP) [1] we have the equations

$$
\begin{aligned}
(A_1 + \lambda B_1 + \mu C_1)\, x_1 &= 0, \\
(A_2 + \lambda B_2 + \mu C_2)\, x_2 &= 0,
\end{aligned}
\tag{7.1}
$$

where A_1, B_1, and C_1 are of size $n_1 \times n_1$, and A_2, B_2, and C_2 are of size $n_2 \times n_2$. Sought are scalars λ, μ and nonzero vectors x_1 and x_2 such that (7.1) is satisfied. We say that (λ, μ) is an eigenvalue of the 2EP and the tensor product $x_1 \otimes x_2$ is the corresponding eigenvector. Define the operator determinants

$$\begin{aligned}\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2,\\ \Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2,\\ \Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2.\end{aligned} \tag{7.2}$$

Then problem (7.1) is related to a coupled pair of GEPs

$$\begin{aligned}\Delta_1 \, z &= \lambda \, \Delta_0 \, z,\\ \Delta_2 \, z &= \mu \, \Delta_0 \, z\end{aligned} \tag{7.3}$$

for a decomposable tensor $z = x_1 \otimes x_2$. If Δ_0 is nonsingular, then Atkinson [1] shows that the solutions of (7.1) and (7.3) agree and the matrices $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute. In the nonsingular case the two-parameter problem (7.1) has $n_1 n_2$ eigenvalues and it can be solved with a variant of the QZ algorithm on (7.3); see [18].

It turns out that for many problems occurring in practice both pencils (Δ_1, Δ_0) and (Δ_2, Δ_0) are singular and we have a singular two-parameter eigenvalue problem [31]. Applications involve delay-differential equations [20], quadratic two-parameter eigenvalue problems [32, 19], model updating [5], roots of systems of bivariate polynomials [37, 2], and others.

There exists a staircase type algorithm that works on both singular pencils (7.3) simultaneously and extracts regular finite eigenvalues; see [32] and an implementation in [35]. However, as we have shown in Examples 6.3 and 6.4, a staircase algorithm can fail. In this section we propose another method that can be applied to a singular 2EP, which in some cases finds regular finite eigenvalues when the staircase algorithm fails, while in some other cases the situation is exactly the opposite.

We can apply Algorithm 1 to $\Delta_1 z = \lambda \Delta_0 z$, one out of two singular pencils in (7.3) to compute λ_i components of eigenvalues (λ_i, μ_i). This is, however, only half of the required information and for each λ_i we have to find the corresponding μ_i. Subsequently, we insert $\lambda = \lambda_i$ in (7.1) and search for common eigenvalues μ of a pair of pencils $(A_1 - \lambda_i B_1) - \mu C_1$ and $(A_2 - \lambda_i B_2) - \mu C_2$ that could be singular as well. We detect the common eigenvalues by comparing the sets of computed eigenvalues for the first and the second pencil, for which we use Algorithm 1 again. The overall method is given in Algorithm 2.

Algorithm 2: Computing finite regular eigenvalues of a singular 2EP

Input: Matrices $A_1, B_1, C_1, A_2, B_2, C_2$ from (7.1) which also provide Δ_1 and Δ_0 from (7.2); threshold δ (default $\delta = \varepsilon^{1/2}$), and parameters for Algorithm 1.
Output: Finite regular eigenvalues and eigenvectors of (7.1).

```
1: Compute finite regular eigenvalues λ_1, ..., λ_r of Δ_1 − λΔ_0 using Alg. 1.
2: for j = 1, ..., r
3:    Compute eigenvalues μ_1^(1), ..., μ_{m_1}^(1) of (A_1 − λ_j B_1) − μC_1 using Alg. 1.
4:    Compute eigenvalues μ_1^(2), ..., μ_{m_2}^(2) of (A_2 − λ_j B_2) − μC_2 using Alg. 1.
5:    Reorder eigenpairs: |μ_1^(1) − μ_1^(2)| ≤ ··· ≤ |μ_m^(1) − μ_m^(2)| for m = min(m_1, m_2).
6:    for k = 1, ..., m
7:       if |μ_k^(1) − μ_k^(2)| < δ then add (λ_j, ½(μ_k^(1) + μ_k^(2))) to list of eigenvalues.
```

Some remarks:

- If we know that each eigenvalue has a unique λ component, then we can replace Lines 6 and 7 by selecting $(\lambda_j, \frac{1}{2}(\mu_1^{(1)} + \mu_1^{(2)})),\ x_1^{(1)} \otimes x_1^{(2)})$ regardless of the difference $|\mu_1^{(1)} - \mu_1^{(2)}|$.
- If $n_1 = n_2 = n$ then the complexity of Line 1 is $\mathcal{O}(n^6)$ while the complexity of Lines 2 to 7 is at most $\mathcal{O}(n^5)$ in case $r = \mathcal{O}(n^2)$.

EXAMPLE 7.1. Consider a system of bivariate polynomials

$$\begin{aligned} p_1(\lambda,\mu) &= 1 + 2\lambda + 3\lambda + 4\lambda^2 + 5\lambda\mu + 6\mu^2 + 7\lambda^3 + 8\lambda^2\mu + 9\lambda\mu^2 + 10\mu^3 = 0,\\ p_2(\lambda,\mu) &= 10 + 9\lambda + 8\mu + 7\lambda^2 + 6\lambda\mu + 5\mu^2 + 4\lambda^3 + 3\lambda^2\mu + 2\lambda\mu^2 + \mu^3 = 0. \end{aligned}$$

Using a uniform determinantal representation from [2], we write the above system as a two parameter eigenvalue problem of the form

$$A_1 + \lambda B_1 + \mu C_1 = \begin{bmatrix} 0 & 0 & 4+7\lambda & 1 & 0\\ 0 & 5+8\lambda & 2 & -\lambda & 1\\ 6+9\lambda+10\mu & 3 & 1 & 0 & -\lambda\\ 1 & -\mu & 0 & 0 & 0\\ 0 & 1 & -\mu & 0 & 0 \end{bmatrix},$$

$$A_2 + \lambda B_2 + \mu C_2 = \begin{bmatrix} 0 & 0 & 7+4\lambda & 1 & 0\\ 0 & 6+3\lambda & 9 & -\lambda & 1\\ 5+2\lambda+\mu & 8 & 10 & 0 & -\lambda\\ 1 & -\mu & 0 & 0 & 0\\ 0 & 1 & -\mu & 0 & 0 \end{bmatrix},$$

where $p_i(\lambda,\mu) = \det(A_i + \lambda B_i + \mu C_i)$ for $i = 1, 2$. The obtained two-parameter eigenvalue problem is singular and has 9 regular eigenvalues (λ_j, μ_j) which are exactly the 9 solutions of the initial polynomial system.

If we apply Algorithm 2 to the above problem, we get all 9 solutions. In Line 2 we compute first components $\lambda_1, \dots, \lambda_9$ as finite regular eigenvalues of the corresponding singular pencil $\Delta_1 - \lambda\Delta_0$ from (7.3), whose KCF contains 4 L_0, 4 L_0^T, 2 N_4, 1 N_2, 2 N_1, and 9 J_1 blocks. For each λ_j we compute the candidates for μ_j in Lines 4 and 5, where the KCF of singular pencils $(A_i - \lambda_j B_i) - \mu C_i$ contains 1 N_2 and 3 J_1 blocks for $i = 1, 2$ and $j = 1, \dots, 9$.

We remark that the above approach might also fail, in particular if we apply it to systems of bivariate polynomials of high degree. It can happen that some of the eigenvalues of $\Delta_1 - \lambda\Delta_0$ are so ill-conditioned that the algorithm cannot separate them from the infinite eigenvalues. In such a case a possible solution would be to apply computation in higher precision, using, e.g., the Multiprecision Computing Toolbox [36].

8. Conclusions. We have proposed a method to approximate the finite eigenvalues of a singular pencil by means of a so-called *rank-completing perturbation*, i.e., a random perturbation of the pencil of rank k, where $n - k$ is the normal rank of the pencil of dimension $n \times n$. The use of such perturbations ensures that, generically, the regular finite and infinite eigenvalues remain fixed, while there appear newly generated eigenvalues. For most problems we can well distinguish the original regular eigenvalues from the newly created ones by looking at the angles of the eigenvectors with respect to the perturbation spaces, and at the condition numbers of the eigenvalues. Therefore, this method may be useful for a wide range of applications.

When the eigenvalues of the original problem are very ill-conditioned, it may become difficult to distinguish between original eigenvalues and newly created ones, and it is an outcome to have another alternative type of methods available: the class of staircase algorithms, such as e.g., Guptri [17] or a staircase type algorithm for singular two-parameter eigenvalue problems [32] in [35]. These methods can be rapid and accurate for many problems. However, the key part of staircase techniques are a number of rank decisions, which can be difficult and ill-posed, see e.g., [14] and Examples 6.3 and 6.4. In such cases, these methods may not return even a single eigenvalue.

A code for the approach developed in this paper can be obtained via [35].

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REFERENCES

[1] F. V. Atkinson, *Multiparameter Eigenvalue Problems*, Academic Press, New York, 1972.
[2] A. Boralevi, J. van Doornmalen, J. Draisma, M. E. Hochstenbach, and B. Plestenjak, *Uniform determinantal representations*, SIAM J. Appl. Algebra Geometry, 1 (2017), pp. 415–441.
[3] P. Benner, P. Losse, V. Mehrmann, and M. Voigt, *Numerical Linear Algebra Methods for Linear Differential-Algebraic Equations.* In: A. Ilchmann, T. Reis (eds), Surveys in Differential-Elgebraic Equations III. Differ.-Algebr. Equa. Forum, pp. 117-175, Springer, Cham, 2015.
[4] R. Byers, C. He, and V. Mehrmann, *Where is the nearest non-regular pencil?*, Linear Algebra Appl., 285 (1998), pp. 81–105.
[5] N. Cottin, *Dynamic model updating—a multiparameter eigenvalue problem*, Mechanical systems and signal processing, 15 (2001), pp. 649–665.
[6] E.J. Davison and S.H. Wang, *Properties and calculation of transmission zeros of linear multivariable systems*, Automatica, 10 (1974), pp. 643-658.
[7] F. De Terán and F. M. Dopico, *Low rank perturbation of Kronecker structures without full rank*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 496–529.
[8] ———, *A note on generic Kronecker orbits of matrix pencils with fixed rank*, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 491–496.
[9] F. De Terán, F. M. Dopico, and J. Moro, *First order spectral perturbation theory of square singular matrix pencils*, Linear Algebra Appl., 429 (2008), pp. 548–576.
[10] J. Demmel, *Generalized Non-Hermitian Eigenproblems.* Section 2.6 in: Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst (eds), *Templates for the Solution of Algebraic Eigenvalue Problems: a practical guide*, pp. 28–36, SIAM, Philadelphia, 2000.
[11] J. Demmel and B. Kågström, *Computing stable eigendecompositions of matrix pencils*, Linear Algebra Appl., 88/89 (1987), pp. 139–186.
[12] ———, *Accurate solutions of ill-posed problems in control theory*, SIAM J. Matrix. Anal. Appl., 9 (1988), pp. 126–145.
[13] ———, *The generalized Schur decomposition of an arbitrary pencil $A-\lambda B$–robust software with error bounds and applications. Part I: theory and algorithms*, ACM Trans. Math. Software, 19 (1993), pp. 160–174.
[14] A. Edelman and Y. Ma, *Staircase failures explained by orthogonal versal forms*, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 1004–1025.
[15] A. Emami-Naeini and P. Van Dooren, *Computation of zeros of linear multivariable systems,* Automatica 18 (1982), pp. 415-430.
[16] F.R. Gantmacher, *Theory of Matrices. Volume 1 and 2*, Chelsea, New York, 1959.
[17] *Guptri*, software for singular pencils, `www8.cs.umu.se/research/nla/singular_pairs/guptri/`.
[18] M. E. Hochstenbach, T. Košir, and B. Plestenjak, *A Jacobi–Davidson type method for*

the nonsingular two-parameter eigenvalue problem, SIAM J. Matrix Anal. Appl., 26 (2005), pp. 477–497.

[19] M. E. Hochstenbach, A. Muhič, and B. Plestenjak, *On linearizations of the quadratic two-parameter eigenvalue problem*, Linear Algebra Appl., 436 (2012), pp. 2725–2743.

[20] E. Jarlebring and M. E. Hochstenbach, *Polynomial two-parameter eigenvalue problems and matrix pencil methods for stability of delay-differential equations*, Linear Algebra Appl., 431 (2009), pp. 369–380.

[21] E. Jarlebring, S. Kvaal, and W. Michiels, *Computing all pairs* (λ,μ) *such that* λ *is a double eigenvalue of* $A+\mu B$, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 902–927.

[22] B. Kågström, *Singular matrix pencils.* Section 8.7 in: Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst (eds), *Templates for the Solution of Algebraic Eigenvalue Problems: a practical guide*, pp. 260–277, SIAM, Philadelphia, 2000.

[23] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations: Analysis and Numerical Solution*, EMS Publishing House, Zürich, Switzerland, 2006.

[24] A.J. Laub and B.C. Moore, *Calculation of transmission zeros using QZ techniques*, Automatica, 14 (1978), pp. 557–566.

[25] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Skew-symmetric matrix polynomials and their Smith forms*, Linear Algebra Appl., 438 (2013), pp. 4625–4653.

[26] ———, *Möbius transformations of matrix polynomials*, Linear Algebra Appl., 470 (2015), pp. 120–184.

[27] C. Mehl, V. Mehrmann, and M. Wojtylak, *On the distance to singularity via low rank perturbations*, Operators and Matrices, 9 (2015), pp. 733–772.

[28] ———, *Parameter-dependent rank-one perturbations of singular Hermitian or symmetric pencils*, SIAM J. Matrix Anal. Appl., 38 (2017), pp. 72–95.

[29] ———, *Linear algebra properties of dissipative Hamiltonian descriptor systems*, Preprint 1-2018, Institut für Mathematik, TU Berlin (2018), submitted.

[30] V. Mehrmann, private communication, 2018.

[31] A. Muhič and B. Plestenjak, *On the singular two-parameter eigenvalue problem*, Electron. J. Linear Algebra, 18 (2009), pp. 420–437.

[32] ———, *On the quadratic two-parameter eigenvalue problem and its linearization*, Linear Algebra Appl., 432 (2010), pp. 2529–2542.

[33] ———, *A method for computing all values* λ *such that* $A+\lambda B$ *has a multiple eigenvalue*, Linear Algebra Appl., 440 (2014), pp. 345–359.

[34] *MCS Toolbox.* The Matrix Canonical Structure Toolbox for Matlab, `www.cs.umu.se/english/research/groups/matrix-computations/stratigraph`.

[35] *MultiParEig.* Toolbox for multiparameter eigenvalue problems, `www.mathworks.com/matlabcentral/fileexchange/47844-multipareig`.

[36] *Multiprecision Computing Toolbox.* Advanpix, Tokyo. `www.advanpix.com`.

[37] B. Plestenjak and M. E. Hochstenbach, *Roots of bivariate polynomial systems via determinantal representations*, SIAM J. Sci. Comput., 38 (2016), pp. A765–A788.

[38] N. Valeev, *On a spectral property of irregular pencils*, Ufa Mathematical Journal, 4 (2012), pp. 44–52.

[39] ———, *On quasiregular spectrum of matrix pencils*, Doklady Mathematics, vol. 88, Springer, 2013, pp. 545–547.

[40] P. Van Dooren, *The computation of Kronecker's canonical form of a singular pencil*, Linear Algebra Appl., 27 (1979), pp. 103–140.

[41] ———, *Reducing subspaces: Definitions, properties and algorithms.* In: B. Kågström, A. Ruhe (eds), Matrix Pencils. Lecture Notes in Mathematics, vol 973. Springer, Berlin, Heidelberg, 1983, pp. 58–73.

[42] J.H. Wilkinson, *Algebraic Eigenvalue Problem*, Oxford University Press, Oxford, 1965.

[43] ———, *Kronecker's canonical form and the QZ algorithm*, Linear Algebra Appl., 28 (1979), pp. 285–303.