Arithmetical Foundations

Recursion.Evaluation.Consistency

Ω_1 1

für

ANGELA & *FRANCISCUS*

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Preface

Johannes Zawacki, my high school teacher, told us about Gödel's second theorem, on non-provability of consistency of mathematics within mathematics. Bonmot of André Weil: *Dieu existe parceque la Mathématique est consistente, et le diable existe parceque nous ne pouvons pas prouver cela – God exists since Mathematics is consistent, and the devil exists since we cannot prove that.*

The problem with 19th/20th century mathematical foundations, clearly stated in Skolem 1919, is unbound infinitistic (non-constructive) formal existential quantification.

In his 1973 Oberwolfach talk André Joyal sketched a categorical – map based – version of the Gödel theorems. A categorical version of the unrestricted non-constructive existential quantifier was still inherent.

The *consistency formula* of **set** theory (and of arbitrary quantified arithmetical theories), namely: *not exists a proof code for (the code of) false*, can be introduced as a (primitive) recursive – Gödel 1931 – free variable predicate:

"For all arithmetised *proofs* k : k does not *prove* (code of) false."

Language restriction to the constructive (categorical) free-variables theory **PR** of primitive recursion or appropriate extensions opens the possibility to circumvent the two Gödel's incompleteness issues:

We discuss iterative map code evaluation in direction of *(termination conditioned) soundness,* and based on this, decidability of primitive recursive predicates.

In combination with Gödel's classical theorems this leads to unexpected consequences, namely to *consistency provability* and logical *soundness* for *recursive descent* theory $\pi\mathbf{R}$: theory of primitive recursion strengthened by an axiom schema of *non-infinite descent,* descent in complexity of *complexity controlled iterations* like in particular (iterative) p. r.-map-code evaluation.

We show an antithesis to Weil's above: *Set theoretically God need not to exist, since his* – BOURBAKI *'s* – THÉORIE DES ENSEMBLES *is inconsistent. The devil does not need to exist, since we can prove inside free-variables recursive mathematics this mathematics consistency formula. By the same token God may exist.*

Berlin, December 2018 M. Pfender

Contents

Introduction

We fix *constructive foundations* for arithmetic on a map theoretical, algorithmic level. In contrast to elementhood and quantification based traditional foundations such as Principia Mathematica **PM**, Zermelo-Fraenkel **set** theory **ZF**, or v. Neumann-Gödel-Bernays **set** theory **NGB**, our *fundamental primitive recursive theory* **PR** has as its *basic* "undefined" (not further defined) terms just terms for objects and maps. On that language level it is *variable free,* and it is free from formal quantification on *individuals* like numbers or number pairs.

Theory **PR** is strongly finitistic with only *bound* existential quantification, in the sense of SKOLEM 1919/1970, p. 153.[1]

PR is a formal, *combinatorial category* with cartesian i. e. universal product and a natural numbers object (NNO) $\mathbb{N}$, a *p. r. cartesian category,* cf. ROMÀN 1989.

The NNO $\mathbb{N}$ admits *iteration of endo maps* and the *full schema*

[1] "Was ich nun in dieser Abhandlung zu zeigen wünsche ist folgendes: *Faßt man die allgemeinen Sätze der Arithmetik als Funktionalbehauptungen auf, und basiert man sich auf der rekurrierenden Denkweise, so läßt sich diese Wissenschaft in folgerichtiger Weise ohne Anwendung der Russel-Whitehead'schen Begriffe "always" und "sometimes" begründen.*"

of primitive recursion. Such NNO has been introduced in categorical terms by Freyd 1972, on the basis of the NNO of Lawvere 1964.

We remain on the purely syntactical level of this categorical theory and later extensions: no formal semantics necessary into an outside, non-combinatorial world, cf. Hilbert's formalistic program.

Fundamental (categorical) p. r. theory **PR** is developped from the endomap iteration scheme (§) of Eilenberg/Elgot 1970. We take as additional axiom Freyd's *uniqueness* of the initialised iterated endo map. This gives the full schema of primitive recursion including uniqueness of p. r. maps defined by that scheme.

Into our variable-free setting are introduced *free variables,* formally interpreted as names for identity and projection maps. As a consequence, we have in the present context 'free variable' as a *defined* notion. We have object and map constants such as terminal object, NNO, zero constant and successor map, and use free metavariables for objects and for maps.

Fundamental arithmetic is further developped along Goodstein's 1971 *Free Variables Arithmetic* whose *uniqueness rules* are derived as theorems of categorical theory **PR**, with its "eliminable" notion of *free variable.* This gives the expected structure theorem for algebra and order on NNO $\mathbb{N}$. "On the way", via Goodstein's *truncated subtraction* and his commutativity of the maximum function, we obtain the *equality definability theorem:* If *predicative equality* of two p. r. maps is derivably true, then map equality *between* these maps is derivable.

The game is enriched by an (embedding, hence conservative) extension of theory **PR** by *abstraction* of *predicates* into new *(sub)objects.* This enrichment makes emerging theory $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ more comfortable, in direction to **set** theory with its *sets* and *subsets,* called *sets* of emerging theory **PRa**, of *primitive recursion with predicate abstraction.*

PRa has a *universal object* $\mathbb{X}$, of all internal *numerals* and (nested) pairs of internal numerals as well as two-element set $\mathbb{2} = \{0, 1\} \subset \mathbb{N}$.

For the rôle of (2-valued) boolean *truth algebra* we add[2] formally an extra *truth object* **2** with basic *truth value* true : $\mathbb{1} \to \mathbf{2}$ and basic binary logic operator $\smallsetminus = \alpha \smallsetminus \beta : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ (α *but not* β).

Theory **PR** is extended this way into a boolean cartesian p. r. theory named **PR2**. It has object **2** as additional basic object, and has constants false, true : $\mathbb{1} \to \mathbf{2}$, and all (boolean) operators making **2** into a 2-valued boolean algebra.

Boolean p. r. theory **PR2** is extended into boolean p. r. *constructive set theory* $\mathbf{S} = \mathbf{PR2} + (\text{abstr})$ with *predicate-into-subobject abstraction* in the same way as fundamental p. r. theory **PR** has been extended into p. r. theory $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ with **PR**-predicate abstraction. Only – formal – difference: **PR2** predicates (to be abstracted) are **PR2** maps $\chi : A \to \mathbf{2}$, whereas **PR** predicates are special **PR** maps $\chi : A \to \mathbb{N}$.

Over (extended) theory **S** is constructed a theory $\widehat{\mathbf{S}}$ of *partial* p. r. maps[3] with half-terminal diagonal symmetric monoidal structure in the sense of Budach/Hoehncke 1975; μ-recursive maps and `while` loop `program`s turn out to be just partial p. r. maps; in particular map code evaluation will be such a (formally) partial map.

The crucial problem with these formally partial recursive maps is

[2] suggestion of J. Sablatnig
[3] specialising the *Korrespondenzen* of Brinkmann/Puppe 1969

termination. A special class of non-p. r. recursive maps whose non-termination is excluded by **axiom,** is given by *Complexity Controlled Iteration* ("CCI").

Extra **axiom** (π), of *non-infinite descent* of CCI's, constitutes iterative descent theory $\pi\mathbf{R}$ over p. r. theory $\widehat{\mathbf{S}}$ of partial p. r. maps. Descent theory $\pi\mathbf{R}$ is introduced mainly as a framework for *evaluation,* of $\mathbf{S}$ map codes on suitable arguments.

Evaluation is defined as a CCI with complexity values descending in linearily ordered set, *ordinal (semiring)* $\mathbb{N}[\omega]$ of polynomes in one indeterminate ω intended to take (arbitrarily) big values.

Since theories (**PR** and) **S** are formally free of variables and quantification, we code (*gödelise*) just *maps* $f : A \to B$, into natural numbers *map code sets,* with p. r. enumerated *internal, arithmetised notion of equality* '$\check{=}$'.

Map codes of theory **S** are *evaluated* on universal set $\mathbb{X}_{\mathbf{2}} \supset \mathbb{X}$ of **S**, of internal truth values, numerals as well as (nested) pairs of these. Evaluation is defined as a *complexity controlled iteration* of a p. r. *evaluation step* (on pairs of map codes and arguments) "until" *map complexity* 0 is reached in left component as well as, by this, *evaluation result* in right component.

Evaluation turns internally equal map codes $\ulcorner f \urcorner \check{=} \ulcorner g \urcorner$ of theory **S** into **S** *predicatively* equal maps. This *termination-conditioned soundness* is arithmetically central.

The strengthened frame $\pi\mathbf{R}$ – strengthened over theory $\widehat{\mathbf{S}}$ of partial p. r. maps – derives free-variable *consistency predicate* $\mathrm{Con}_{\mathbf{S}}$ for theory **S** and relative, S-to-$\pi\mathbf{R}$ *evaluation soundness,* from termination-conditioned soundness of **S**.

Logically central is **decision** by theory $\pi\mathbf{R}$ of each p. r. predicate χ, essentially via p. r. enumerative race for a (first) *counterexample* versus a (first) $\mathbf{S}$ *proof* index k, $\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\chi\urcorner)$. Well-definedness of predicate decision follows from relative soundness. Since consistency formulae Con of "all" theories can be expressed as (free variable) p. r. predicates, this leads to:

1. *Self-consistency* of iterative descent theory $\pi\mathbf{R}$: *Consistency* derivation $\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$ of theory $\pi\mathbf{R}$, genuine subsystem of **set** theory.

2. *Soundness* and ω-*completeness* of theory $\pi\mathbf{R}$.

In Appendix A we resolve Ackermann's double recursive function $\Psi = \Psi(m, n)$ into a complexity controlled `while` loop, not infinitely looping within theory $\pi\mathbf{R}$.

Ackermann has shown by this example that there are number theoretic "functions" which are recursive but not primitive recursive – not "recursive" in Gödel's original sense.

In Appendix B we show that (already a special instance of) axiom of Choice **AC** is inconsistent over (categorical) recursion theory, and hence in particular that classical **set** theory is (just) *inconsistent.*

Part I

RECURSION

Chapter 1

Cartesian language

We develop from scratch the free-variables "but" categorial language of cartesian products, possibly nested, cartesian products of fundamental object $\mathbb{1}$, *one-element set,* and *natural numbers object* "NNO" $\mathbb{N}$. NNO $\mathbb{N}$ comes with *zero* map $0 : \mathbb{1} \to \mathbb{N}$ and *successor* (endo) map $\mathrm{s} : \mathbb{N} \to \mathbb{N}$.

We ***define/interpret*** *free variables* as *identity maps* resp. left or right *projections* – possibly nested – out of cartesian products, onto their *factors.* Within the **axioms** for *cartesian theories* (bearing on objects and maps) we specify use and interpretation of these free variables which can be seen as *components* in terms of Linear Algebra.[1][2]

A special rôle is played by *terminal object* $\mathbb{1}$. It works as the *empty cartesian product* $\mathbb{N}^0$, comes with a (unique) "projection" map $\Pi : A \to \mathbb{1}$ for each object A, and is the *domain* object for *con-*

[1]K. Polthier

[2]in subsection 3 we show on the example of a distributive law how to transform a free-variables equation into a variable-free map equation.

crete "elements" $\boldsymbol{a} : \mathbb{1} \to A$ of A, in particular for *(concrete) numbers* $\boldsymbol{n} : \mathbb{1} \to \mathbb{N}$. We turn to the formal development of the *cartesian theory* **CA** generated over the NNO $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{\mathrm{s}} \mathbb{N}$.

1.1 Fundamental object language symbols

The set of fundamental symbols of cartesian language **CA** is

$$\{\mathbb{1}, \mathbb{N}, \times, 0, \mathrm{s}, \mathrm{id}, \circ, \Pi, \ell, \mathrm{r}\}, \text{and equality sign '='}$$

$\mathbb{1}$ is the *one-element object,* $\mathbb{N}$ the *Natural Numbers Object, NNO,* of theories **CA** and **PR** to come, $\times$ the cartesian product of objects and of maps. 0 is the *zero constant* $0 : \mathbb{1} \to \mathbb{N}$, s is the "fundamental" *successor function* $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ to formalise counting.

Identity is the family of *identity maps* to all objects, these *objects* obtained out of objects $\mathbb{1}$ and $\mathbb{N}$ by *cartesian product* $\times$;

$\circ$ is map *composition,* occasionally replaced by concatenation, Π symbolises the family of *terminal maps* into object $\mathbb{1}$, ℓ and r are left resp. right *projections* out of cartesian product $A \times B$ onto *factors* A and B respectively.

Theory **PR** below – of *primitive recursion* – will come with an additional symbol § for endomap *iteration.*[3]

1.2 Cartesian category axioms

We give here the axioms of cartesian categorical theory **CA** in a fully formal way using Gentzen bars for expression of metamathematical

[3] Eilenberg/Elgot 1970

inferences. The most characteristic such axioms are marked by a •

• **Ax** [$\mathbb{N}$]

(no antecedent for this inference)

{**Obj** $\mathbb{1}$, $\mathbb{N}$}

one-element object and *natural numbers object;*

map $0 : \mathbb{1} \to \mathbb{N}$ ***zero*** *constant*

map $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ ***successor*** *function*

Ax [id]

Obj A

---

map $\mathrm{id}_A = \mathrm{id} : A \to A$

identity *map*

Ax [*reflexivity*]

map f

$f = f$

Ax [*symmetry*]

map f, g;

$f = g$

$g = f$

$$\textbf{Ax}\ [\,\textit{transitivity}\,]\quad \frac{\textbf{map}\ f, g, h;\quad f = g;\ \ g = h}{f = h}$$

• $$\textbf{Ax}\ [\,\circ\,]\quad \frac{f : A \to B;\ \ g : B \to C}{\textbf{map}\ (g\,f) = (g \circ f) = g(f) : A \to C;\quad (g \circ f) : A \to B \to C}$$

map ***composition***

(outmost brackets may be omitted)

$$\textbf{Ax}\ [\,\circ\,\text{sub}\,]\quad \frac{f, \tilde{f} : A \to B;\ \ g : B \to C;\ f = \tilde{f}}{g \circ f = g \circ \tilde{f}}$$

Leibniz' ***substitutivity***

Substitution of equals into same gives equals.

$$\textbf{Ax}\ [\,\text{sub}\,\circ\,]\quad \frac{f : A \to B;\ \ g, \tilde{g} : B \to C;\ g = \tilde{g}}{g \circ f = \tilde{g} \circ f}$$

second Leibniz' ***substitutivity***

Substitution of same into equals gives equals.

$$\textbf{Ax}\ [\circ\ \mathrm{id}]\quad \frac{f : A \to B}{\begin{array}{l} f \circ \mathrm{id} = f \circ \mathrm{id}_A = f; \\ \mathrm{id} \circ f = \mathrm{id}_B \circ f = f \end{array}}$$

***neutrality** of identities to composition*

It follows a first statement on the use of free variables.

$$\textbf{Lemma}\ [\circ\ \textbf{var}]\quad \frac{\begin{array}{l} f : A \to B; \\ \textbf{var}\ a \in A,\ a := \mathrm{id}_A \end{array}}{f(a) = f(\mathrm{id}_A)\ =_{\text{by def}}\ f \circ \mathrm{id}_A = f}$$

free variable as identity,

$f(a) \in B$ *"dependent variable"* **q. e. d.**

Next axiom is ***associativity*** *of composition.*

$$\bullet\ \textbf{Ax}\ [\mathrm{ass}\ \circ]\quad \frac{\begin{array}{l} f : A \to B;\ g : B \to C;\ h : C \to D \\ \textbf{var}\ a \in A,\ a := \mathrm{id}_A \end{array}}{\begin{array}{l} (h \circ g) \circ f = h \circ (g \circ f) : A \to D \\ = h \circ g \circ f = h\,g\,f = h(g(f(a)))) \end{array}}$$

Counting Remark: Up to insertion of (composition-neutral) identities, the maps of *category theory* generated over $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ are just the iterated $\mathrm{s} \circ \ldots \circ \mathrm{s} : \mathbb{N} \xrightarrow{\mathrm{s}} \ldots \xrightarrow{\mathrm{s}} \mathbb{N}$ of the successor map, as well as the

numerals to be used in particular for metamathematical purpose.

$$\frac{\text{(empty antecedent)}}{0 : \mathbb{1} \to \mathbb{N}\ \textbf{numeral}}$$

$$\frac{\boldsymbol{n} : \mathbb{1} \to \mathbb{N}\ \textbf{numeral}}{(\mathrm{s} \circ \boldsymbol{n}) : \mathbb{1} \to \mathbb{N}\ \textbf{numeral}}$$

example: $3 = (\mathrm{s} \circ (\mathrm{s} \circ (\mathrm{s} \circ 0)))$

Cartesian structure

For each object is given a *terminal map* to object $\mathbb{1}$,

$$\textbf{Ax}\ [\,\Pi\,]\quad \frac{\textbf{Obj}\ A}{\textbf{map}\ \Pi = \Pi_A : A \to \mathbb{1}}$$

terminal *map*

• $$\textbf{Ax}\ [\,!\Pi\,]\quad \frac{f : A \to \mathbb{1}}{f = \Pi_A}$$

uniqueness

– equivalent to *naturality* of family Π given by (commutativity of) every DIAGRAM of form

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \Big\downarrow \Pi_A & = & \Big\downarrow \Pi_B \\ \mathbb{1} & \xrightarrow{\mathrm{id}} & \mathbb{1} \end{array}$$

Remark: This naturality **axiom** for family Π is not required for *half*-terminal *monoidal* categories, introduced in Budach & Hoehncke 1975. Theory $\widehat{\mathbf{S}}$ to come of *partially defined* (primitive) recursive maps is of that type.

Notation: Equality sign '=' inserted into (part of) a diagram means commutativity of (that part of) a diagram, equality of composition of arrows along both paths.

- **Ax** [**Obj** $\times$] $\quad \dfrac{\mathbf{Obj}\ A, B}{\mathbf{Obj}\ (A \times B)}$

  (binary) ***cartesian product*** *of objects.*

  Iteration gives *nested* products.

  Outmost brackets may be omitted.

We introduce use of pairs of free variables as pairs of left and right *projections:*

Obj A, B

var $a \in A$, **var** $b \in B$

• **Ax** $[\,\ell, \mathrm{r}\,]$ ————————————

map $\ell = \ell_{A,B} : A \times B \to A$

map $\mathrm{r} = \mathrm{r}_{A,B} : A \times B \to B$

left resp. right ***projection***

$a = \ell_{A,B}, \;\; b = \mathrm{r}_{A,B}$

variables as projections.

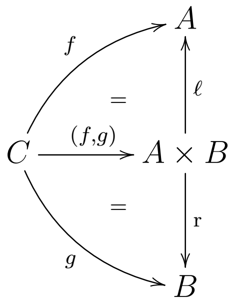
map $f : C \to A, \;\; g : C \to B$

• **Ax** $[\,\mathrm{indu}\,]$ ————————————

map $(f, g) : C \to A \times B$

induced *map into product*

$\ell \circ (f, g) = f, \;\; \mathrm{r} \circ (f, g) = g$



Godement's DIAGRAM

uniqueness of horizontal arrow see below. This is the very beginning of map-theoretic, element-free category theory.

Ax [sub(,)]

$$\frac{f, \tilde{f} : C \to A;\ g, \tilde{g} : C \to B; \quad f = \tilde{f};\ g = \tilde{g}}{(f,g) = (\tilde{f}, \tilde{g})}$$

compatibility *of inducing* with '='

Ax [distr]

$$\frac{h : D \to C,\ f : C \to A,\ g : C \to B}{(f,g) \circ h = (f \circ h, g \circ h) : D \to (A \times B)}$$

distributivity *of compositionn over forming the induced map into product.*

Use of **free variable** for induced map:

Lemma

$$\frac{\textbf{var } c \in C,\ c := \mathrm{id}_C}{}$$

$\ell \circ (f, g)(c) = \ell \circ (f(c), g(c)) = f(c),$

$\mathrm{r} \circ (f, g)(c) = \mathrm{r} \circ (f(c), g(c)) = g(c)$

q. e. d.

Ax [retr. pairing]

$$\frac{h : C \to (A \times B)}{(\ell_{A,B} \circ h, \mathrm{r}_{A,B} \circ h) = h}$$

pairing *is* ***retractive***

(*even* ***isomorphic***)

Lemma [!(,)]

$$\frac{f : C \to A;\ g : C \to B;\quad h : C \to (A \times B);\quad \ell_{A,B} \circ h = f;\ \mathrm{r}_{A,B} \circ h = g}{h = (f, g)}$$

uniqueness *of induced map*

Proof:

$$
\begin{aligned}
h &= \mathrm{id}_{A\times B} \circ h \\
&= (\ell_{A,B} \circ \mathrm{id}_{A\times B}, \mathrm{r}_{A,B} \circ \mathrm{id}_{A\times B}) \circ h \quad \text{[retr. pairing]} \\
&= (\ell_{A,B}, \mathrm{r}_{A,B}) \circ h \\
&= (\ell_{A,B} \circ h, \mathrm{r}_{A,B} \circ h) \quad \text{[distr]} \\
&= (f, g) : C \to A \times B \quad \text{[sub(,) antecedent]}
\end{aligned}
$$

q. e. d.

Lemma [(ℓ, r)]

$$\frac{\textbf{Obj } A, B}{(\ell_{A,B}, \mathrm{r}_{A,B}) = \mathrm{id}_{A\times B}}$$

Proof: uniqueness of induced into product $A \times B$ **q. e. d.**

Def [$\times$ maps]

$$\frac{f : A \to A',\ g : B \to B' \qquad \textbf{var } a := \ell_{A,B},\ b := \mathrm{r}_{A,B}}{\begin{gathered}(f \times g) = (f \circ \ell, g \circ \mathrm{r}) : (A \times B) \to (A' \times B') \\ f \times g = (f \times g)(a, b) = (f(a), g(b))\end{gathered}}$$

cartesian map product

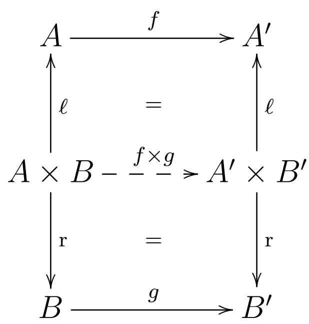
[**unary** $\times$]

$$\frac{f : A \to A',\ g : B \to B',}{\begin{gathered}(A \times g) \ =_{\mathrm{def}}\ (\mathrm{id}_A \times g) : A \times B \to A \times B' \\ (f \times B) \ =_{\mathrm{def}}\ (f \times \mathrm{id}_B) : A \times B \to A' \times B\end{gathered}}$$

Theorem [$\mathrm{nat}_{\ell,r}$] $\dfrac{\textbf{map } f : A \to A',\ g : B \to B'}{\ell \circ (f \times g) = f \circ \ell;\ \mathrm{r} \circ (f \times g) = g \circ \mathrm{r}}$

naturality *of projection families* ℓ *and* r

Proof: Uniqueness of induced map into product $A' \times B'$, consider



Cartesian map product DIAGRAM

q. e. d.

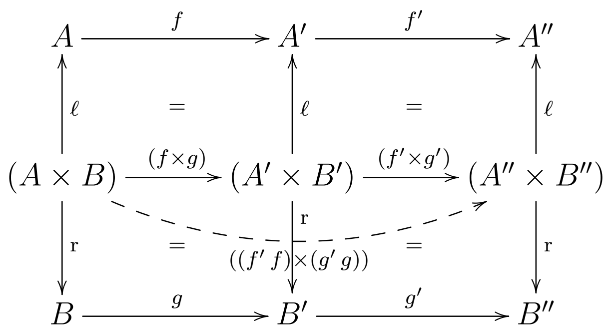
Theorem [$\times\ \circ$]

$$f : A \to A',\ f' : A' \to A'';$$
$$g : B \to B',\ g' : B' \to B'';$$

$$\mathrm{id}_A \times \mathrm{id}_B = \mathrm{id}_{A\times B} : A \times B \to A \times B$$
$$(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g) :$$
$$(A \times B) \to (A'' \times B'')$$

bifunctoriality *of cartesian product*

Proof: Uniqueness of induced map into product $A'' \times B''$ in

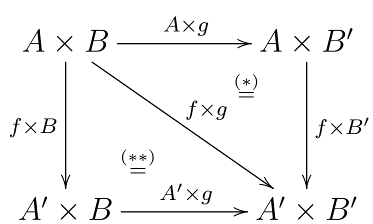


Cartesian bifunctoriality DIAGRAM

q. e. d.

Corollary [× id ∘]
$$\frac{f : A \to A',\ g : B \to B'}{f \times g = (f \times B') \circ (A \times g)}$$
$$= (A' \times g) \circ (f \times B)$$

map product ***decomposition***



map product decomposition DIAGRAM

Proof:

$$(f \times B') \circ (A \times g) = (f \times \mathrm{id}_{B'}) \circ (\mathrm{id}_A \times g)$$
$$= (f \circ \mathrm{id}_A) \times (\mathrm{id}_{B'} \circ g) \qquad \text{(by bifunctoriality)}$$
$$= f \times g \quad (*)$$

the latter by compatibility of (,) with equality, which entails compatibilityindcompatibility of × with equality.

Analogously

$$(A' \times g) \circ (f \times B) = (\mathrm{id}_{A'} \times g) \circ (f \times \mathrm{id}_B)$$
$$= (\mathrm{id}_{A'} \circ f) \times (f \circ \mathrm{id}_B) \qquad \text{(by bifunctoriality)}$$
$$= f \times g \quad (**)$$

q. e. d.

Distributivity Corollary [Distr × ∘ (,)]

$$\frac{f : C \to A,\ g : C \to B,\ f' : A \to A',\ g' : B \to B'}{(f' \times g') \circ (f, g) = (f' \circ f, g' \circ g) : C \to A' \times B'}$$

Proof:

$$(f' \times g') \circ (f, g)$$
$$= (f' \circ \ell_{A',B'}, g' \circ \mathrm{r}_{A',B'}) \circ (f, g)$$
$$= (f' \circ \ell_{A',B'} \circ (f, g), g' \circ \mathrm{r}_{A',B'} \circ (f, g)) \text{ by } \mathbf{Ax}\ [\,\mathrm{distr}\,]$$
$$= (f' \circ (\ell_{A',B'} \circ (f, g)), g' \circ (\mathrm{r}_{A',B'} \circ (f, g)))$$
$$= (f' \circ f, g' \circ g) \ \textbf{q. e. d.}$$

1.3 Interpretation of free variables

We start with a ("generic") example of *elimination* of free variables by their *interpretation into (possibly nested) projections* within a ring R.

A distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$ gets the map interpretation

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c) :$$
$$R^3 \ =_{\text{by def}} \ R^2 \times R \ =_{\text{by def}} \ (R \times R) \times R \to R$$

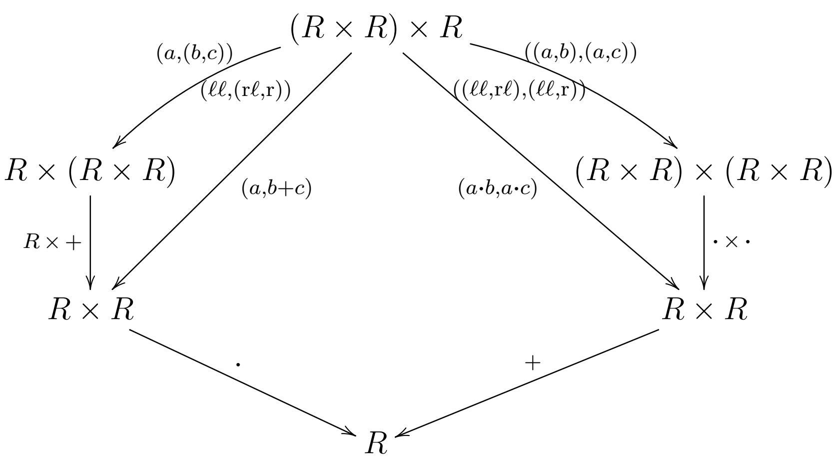
with *systematic* interpretation of variables:

$$a := \ell\ \ell,\ b := \mathrm{r}\ \ell,\ c := \mathrm{r} : R^3 = (R \times R) \times R \to R$$

and infix writing of operations $x\, op\, y : R \times R \to R$ prefix interpreted as $op \circ (x, y)$, here

$$\cdot \circ (a, + \circ (b, c)) = + \circ (\cdot \circ (a, b), \cdot \circ (a, c)) : R^3 \to R$$

In form of a commuting diagram:



An *iterated map*[4] $f^{\S} : A \times \mathbb{N}$ may be written in free-variables notation as

$$f^{\S} = f^{\S}(a, n) = f^n(a) : A \times \mathbb{N} \to A$$

with $a := \ell : A \times \mathbb{N} \to A$, and $n := \mathrm{r} : A \times \mathbb{N} \to \mathbb{N}$

Systematic map interpretation of free-variables equations:

1. Extract the common codomain (domain of values), say B, of both sides of the equation (this codomain may be implicit);

2. "Expand" operator priority into additional bracket pairs;

3. Transform infix into prefix notation on both sides of the equation;

4. Order the (finitely many) variables appearing in the equation, for example lexically;

5. If these variables $a_1, a_2, \ldots, a_{\boldsymbol{m}}$ range over the objects $A_1, A_2, \ldots, A_{\boldsymbol{m}}$, then fix as common *domain object* (source of commuting diagram), the object

$$A = A_1 \times A_2 \times \ldots \times A_{\boldsymbol{m}} \ =_{\mathrm{def}}\ (\ldots((A_1 \times A_2) \times \ldots) \times A_{\boldsymbol{m}});$$

6. *Interpret* the variables as *identities* or (possibly nested) *projections,* will say: *replace,* within the equation, all the occurences of a *variable* by the corresponding – in general *binary nested* – projection;

7. Replace each symbol " 0 " by " $0\, \Pi_D$ " where " D " is the (common) domain of (both sides) of the equation;

[4] see below

8. Insert composition symbol $\circ$ between terms which are not bound together by an *induced map operator* as in (f_1, f_2);

9. By the above, we have the following two-maps-cartesian-Product **rule,** forth and back: For

   $a := \ell_{A,B} : (A \times B) \to A$, $b := \mathrm{r}_{A,B} : (A \times B) \to B$ and $f : A \to A'$ as well as $g : B \to B'$, the following identity holds:

$$\begin{aligned}
&(f \times g)(a, b) = (f \times g) \circ (\ell_{A,B}, \mathrm{r}_{A,B}) \\
&= (f \times g) \circ \mathrm{id}_{(A \times B)} = (f \times g) \\
&= (f \circ \ell_{A,B}, g \circ \mathrm{r}_{A,B}) \\
&= (f \circ a, g \circ b) = (f(a), g(b)) : A \times B \to A' \times B'
\end{aligned}$$

10. For free variables $a \in A$, $n \in \mathbb{N}$ *interpret* the term $f^n(a)$ as the map $f^{\S}(a, n) : A \times \mathbb{N} \to A$, *iterated* of endomap $f : A \to A$, see next chapter.

These 10 *interpretation* steps transform a cartesian [a cartesian p. r.] free-variables equation into a variable-free, categorical equation of theory **CA** [and of **PR** to come]:

Elimination of (free) variables by their interpretation as *projections,* and vice versa: *Introduction of free variables* as *names* for identities resp. projections. We allow for mixed notation too. All this, for the time being, just in the context of cartesian theories.

All of our theories are free from classical, (axiomatic) formal unbound quantification.[5] Free-variables equations are understood intuitively as *universally quantified.* But a free variable $a \in A$ occurring

[5] critizised by SKOLEM 1919

only in the premise of an *implication* takes (in suitable context), the meaning

> *for any given* $a \in A$: premise $(\dots a) \implies$ conclusion, i. e.
> *if exists* $a \in A$ s. t. premise $(\dots a)$, *then* conclusion;
> provided that (free) variable $a \in A$ does not occur
> in conclusion.

Chapter 2

Primitive Recursion

We introduce Gödel's primitive recursion – called by him just *recursion*[1] –, beginning with the iteration schema in EILENBERG/ELGOT 1970. We show the *full schema of primitive recursion* and uniqueness of the NNO $\mathbb{N}$ within the categorical theory **PR** of primitive recursion to be described in this chapter.

[1]later Ackermann found a *recursive* function which is *n*ot primitive recursive. Cf. **Appendix A.** The same holds for *evaluation* of primitive recursive map codes below.

2.1 Iteration axioms added

• **Ax** [§] $\dfrac{f : A \to A \text{ (endomap)},\ \textbf{var}\ a \in A,\ \textbf{var}\ n \in \mathbb{N}}{}$

$f^{\S} = f^{\S}(a, n) : A \times \mathbb{N} \to A$ (*iterated*);

$f^{\S}(a, 0) := f^{\S}(\mathrm{id}_A, 0_A) = f^{\S}(\mathrm{id}_A, 0\,\Pi_A) = a = \mathrm{id}_A :$

$A \to A \times \mathbb{N}$ (*anchoring*);

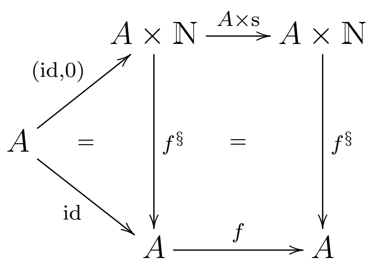
$f^{\S} \circ (A \times \mathrm{s}) = f^{\S}(a, \mathrm{s}\,n) = f \circ f^{\S} = f(f^{\S}(a, n)) :$

$A \times \mathbb{N} \to A \to A$ (*iteration step*);

$f^n(a) := f^{\S}(a, n)$

apply *iteratively endomap* f *to initial argument* a*,*

iterate n *times.*



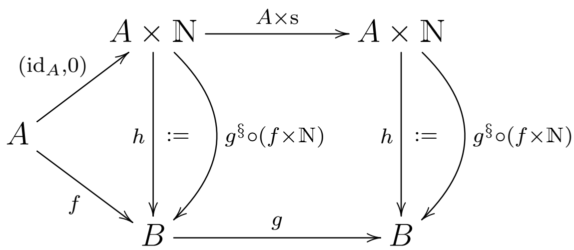
(§)

Iteration DIAGRAM

$$f : A \to B;\ g : B \to B;$$

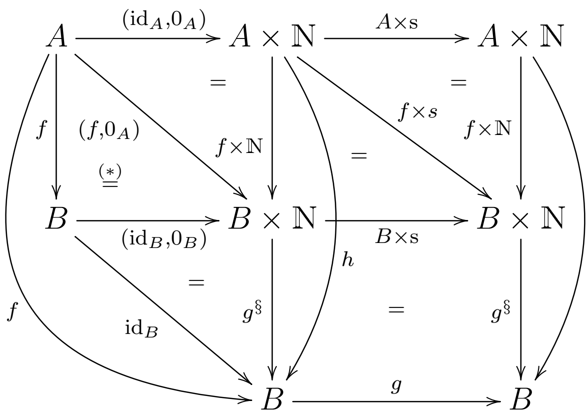
$$h := g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}}) : A \times \mathbb{N} \to B \times \mathbb{N} \to B$$

Lemma [FR]



commutes

Proof: Consider DIAGRAM



In particular equation $(*)$ holds by uniqueness of terminal map $A \to \mathbb{1} : 0_B\, f = 0\,\Pi_B\, f = 0\,\Pi_A = 0_A$ and "then" by distributivity of $\circ$ over $(\ ,\ )$ **q. e. d.**

$$\bullet\ \textbf{Ax}\ [\,\mathrm{FR!}\,]\quad \frac{\begin{array}{l} f : A \to B;\ g : B \to B;\ h : A \times \mathbb{N} \to B; \\ \textbf{var}\ a \in A,\ \textbf{var}\ n \in \mathbb{N}; \\ h(a,0) = f(a); \\ h(a, \mathrm{s}\, n) = g\, h(a,n) \end{array}}{\begin{array}{l} h = g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}})\ \text{i. e.} \\ h(a,n) = g^n(f(a)) : A \times \mathbb{N} \to B : \end{array}}$$

Freyd's ***uniqueness*** *of the iterated endomap* g ***initialised*** *by map* f

[“$g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}})$ does the job”, see [FR] above.]

$$\textbf{Corollary}\ [\,\S!\,]\quad \frac{\begin{array}{l} f : A \to A;\ h : A \times \mathbb{N} \to A \\ \textbf{var}\ a \in A,\ \textbf{var}\ n \in \mathbb{N}; \\ h(a,0) = a = \mathrm{id}_A(a); \\ h(a, \mathrm{s}\, n) = f\, h(a,n) \end{array}}{h = f^{\S}}$$

uniqueness *of “simply” iterated* $f^{\S}$

Lemma [§ =]
$$\frac{f, \tilde{f} : A \to A;\ f = \tilde{f}}{f^{\S} = \tilde{f}^{\S} : A \times \mathbb{N} \to A}$$

compatibility *of iteration* § *with equality*

Proof:

$\underline{\mathrm{Ax}}$ [§], [sub ∘]
$$\frac{\tilde{f} = f}{\begin{array}{l}\tilde{f}^{\S}(a, 0) = \mathrm{id}_A \\ \tilde{f}^{\S}(a, s\,n) = \tilde{f} \circ \tilde{f}^{\S}(a, n) = f \circ \tilde{f}^{\S}(a, n)\end{array}}$$

and – the latter postcedent –

[§!]
$$\frac{\begin{array}{l}\tilde{f}^{\S}(a, 0) = \mathrm{id}_A \\ \tilde{f}^{\S}(a, s\,n) = f \circ \tilde{f}^{\S}(a, n)\end{array}}{\tilde{f}^{\S} = f^{\S}}$$

q. e. d.

2.2 Full schema of primitive recursion

Already for definition and characterisation of *multiplication* and moreover for proof of the laws of Arithmetic, the following ***full schema*** (pr) of primitive recursion is needed:

Theorem (pr)

$$\frac{\begin{array}{l} g = g(a) : A \to B \\ h = h((a,n),b) : (A \times \mathbb{N}) \times B \to B \end{array}}{\begin{array}{l} f = f(a,n) : A \times \mathbb{N} \to B \ \mathbf{s.t.} \\ \text{(anchor)} \quad f(a,0) = g(a), \text{ and} \\ \text{(step)} \quad f(a, \mathrm{s}\, n) = h((a,n), f(a,n)). \\ f =: \mathrm{pr}[g,h] \\ + \\ \text{(pr!) } \textit{uniqueness of } f \text{ to satisfy} \\ \text{these (anchor) and (step) equations.} \end{array}}$$

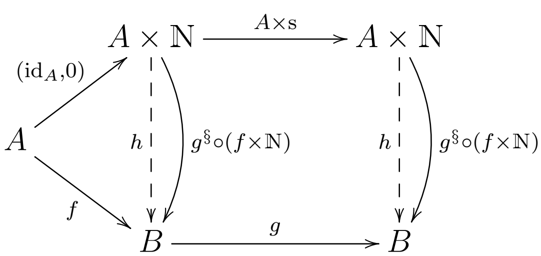
Interpretation: *General primitive recursive map $f = f(a,b)$* ***initialised*** *by a map $g = g(a)$ and iteratively extended using a* ***step*** *map $h = h((a,n),b)$ which depends on previous value b but (possibly) also from initial argument $a \in A$ as well as from running recursion parameter $n \in \mathbb{N}$.*

Schema (pr) without use of free variables:[2]

[2] see Freyd 1972 and (then) Pfender, Kröplin, and Pape 1994

$$
\text{(pr)}\quad \frac{\begin{array}{l} g : A \to B \\ h : (A \times \mathbb{N}) \times B \to B \end{array}}{\begin{array}{l} \mathrm{pr}[g, h] := f : A \times \mathbb{N} \to B \\ f(\mathrm{id}_A, 0) = g : A \to B \\ f\,(\mathrm{id}_A \times \mathrm{s}) = h\,(\mathrm{id}_{A\times\mathbb{N}}, f) : \\ \quad (A \times \mathbb{N}) \to (A \times \mathbb{N}) \times B \to B \\ (\mathrm{pr!}) : f \ \textit{unique.} \end{array}}
$$

Schema (pr) is a consequence of iteration schema **Ax** [§] and *uniqueness of the initialised iterated* h, this taken as **axiom** (FR!), commuting diagram[3]



Remarks:

- Full schema (pr) of primitive recursion is an **axiom** in the classical theory of primitive recursion, subsystem of any classical (gödelian) arithmetical theory **T**.

[3] Freyd 1972

- Free-Variables Arithmetics of the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$ can be based on the axioms of the cartesian theory **PR** of primitive recursion as defined in the above.

- Goodstein's[4] uniqueness axioms U_1 to U_4 – basic for his *Free-Variables Arithmetics* – are theorems of **PR**.

- In "Begründung der elementären Arithmetik durch die rekurrierende Denkweise ohne die Anwendung scheinbarer Veränderlichen mit unendlichem Ausdehnungsbereich",
  Skolem 1919 exhibits the strongly finitistic logical kernel of Principia Mathematica **PM,** and forshadows in particular Goodstein 1971.

2.3 Proof of full schema

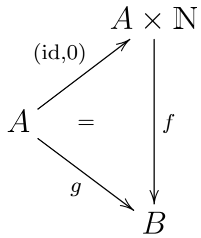
Proof of schema (pr) out of [§] and (FR!) : [5]

Construction of the map $f = \mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$ out of data $g : A \to B$ (*initialisation*) and $h : (A \times \mathbb{N}) \times B \to B$ (*iteration step*):

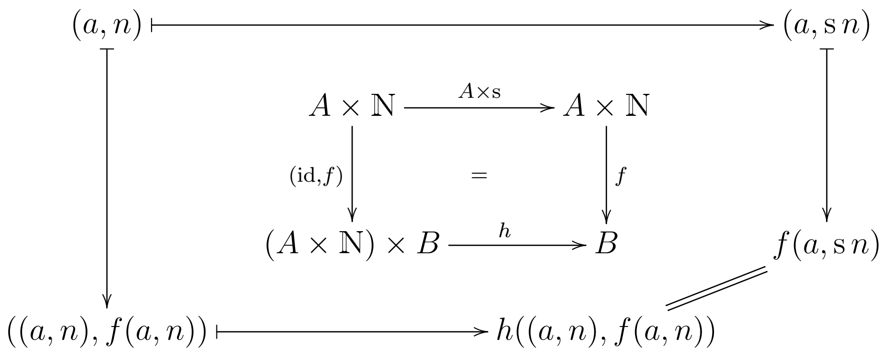
Wanted $f : A \times \mathbb{N} \to B$ is to satisfy (init) und (step) given as the two commuting diagrams

[4] Goodstein 1971

[5] this proof and everything before has been verified by A. Cloete and G. Myrach within the proof verification system *HOL light*



(init)

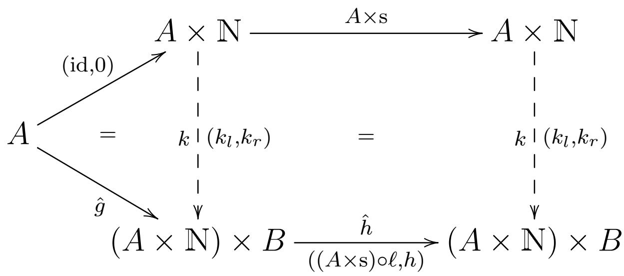


(step)

With $\hat{g} := ((\mathrm{id}_A, 0), g)$ and $\hat{h} := ((A \times \mathrm{s}) \circ \ell, h)$ we get by (FR!) a uniquely determined map

$$k = (k_l, k_r) : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$$

satisfying

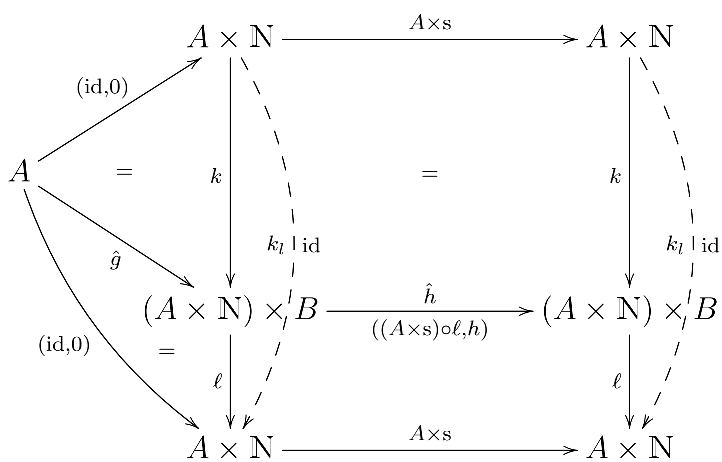


i. e.

$$k \circ (\mathrm{id}_A, 0) = \hat{g} \quad \text{and}$$
$$k \circ (A \times \mathrm{s}) = \hat{h} \circ k$$

[It will turn out that $k = (\mathrm{id}_{A\times\mathbb{N}}, f)$ for wanted map $f : A \times \mathbb{N} \to B$.]

For our unique k consider first its left component $k_l = \ell \circ k : A \times \mathbb{N} \to A \times \mathbb{N}$ unique – by (FR!) – in

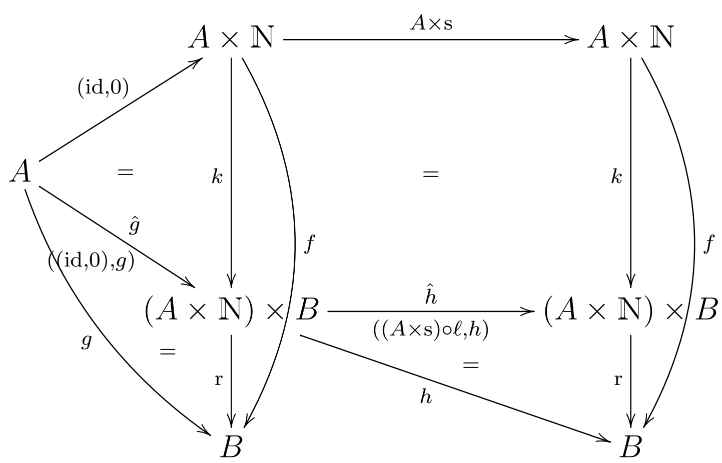


We have

$$\ell \circ k \circ (\mathrm{id}_A, 0) = \ell \circ \hat{g} = (\mathrm{id}_A, 0) \quad \text{and}$$
$$\ell \circ k \circ (A \times \mathrm{s}) = \ell \circ \hat{h} \circ k = (A \times \mathrm{s}) \circ \ell \circ k$$

Since these two equations hold likewise for $\mathrm{id}_{A\times\mathbb{N}}$ instead of $\ell \circ k$, equation $\ell \circ k = \mathrm{id}_{A\times\mathbb{N}}$ follows by uniqueness (FR!) of such a map.

Taking now $f := \mathrm{r} \circ k : A \times \mathbb{N} \to B$ we have the following diagram for this (unique) right component of $k : A \times \mathbb{N} \to (A \times \mathbb{N}) \times B$:



Obtain

$$k = (\ell \circ k, \mathrm{r} \circ k) = (\mathrm{id}_{A\times\mathbb{N}}, f)$$
$$f \circ (\mathrm{id}_A, 0) = \mathrm{r} \circ k \circ (\mathrm{id}_A, 0) = \mathrm{r} \circ \hat{g} = g \quad \text{and}$$
$$f \circ (A \times \mathrm{s}) = \mathrm{r} \circ k \circ (A \times \mathrm{s}) = \mathrm{r} \circ \hat{h} \circ k$$
$$= h \circ k = h \circ (\mathrm{id}_{A\times\mathbb{N}}, f)$$

So this map $f : A \times \mathbb{N} \to B$ is *available* to fullfill the requirements of $\mathrm{pr}\,[\,g, h\,] : A \times \mathbb{N} \to B$.

Uniqueness proof for such map f: Let f' be a map assumed likewise to satisfy equations (init) and (step).

Then take $k' := (\mathrm{id}_{A\times\mathbb{N}}, f') : A\times\mathbb{N} \to (A\times\mathbb{N}) \to B$ and calculate:

$$\begin{aligned}
k' \circ (\mathrm{id}_A, 0) &= (\mathrm{id}_{A\times\mathbb{N}}, f') \circ (\mathrm{id}_A, 0) \\
&= ((\mathrm{id}_A, 0), f' \circ (\mathrm{id}_A, 0)) \\
&= ((\mathrm{id}_A, 0), g) = \hat{g} \quad \text{as well as} \\
k' \circ (A \times \mathrm{s}) &= (\mathrm{id}_{A\times\mathbb{N}}, f') \circ (A \times \mathrm{s}) \\
&= ((A \times \mathrm{s}), f' \circ (A \times \mathrm{s})) \\
&= ((A \times \mathrm{s}), h) = \hat{h} \circ k'
\end{aligned}$$

Since by (FR!) k above is the *unique* map to satisfy the equations above, we have necessarily $k' = k$ and hence $f' = \mathrm{r} \circ k' = \mathrm{r} \circ k = f :$ $A \times \mathbb{N} \to B$ **q. e. d.**

2.4 Program version of full schema

$$g = g(a) : A \to B \quad \text{(init)}$$

$$h = h((a,n),b) : (A \times \mathbb{N}) \times B \to B \quad \text{(step)}$$

(pr prog)

```
function f = pr[g, h]
= pr[g, h](a, n) : A × ℕ → B :
var b ∈ B
b := g(a);
for j := 0 to n − 1 do
{b := h((a, j), b)
od
result  f(a, n) := b
```

Dangerous bound: Recursion parameter $j \in [0, n-1]$ in a `for` loop given by *full schema* may be *used* within this loop, but not *modified* in the loop body, as for example by a statement of form $j := j+2$. Same for the *passive* parameter $a \in A$.

Examples of use of the *full schema,* in particular of dependence of recursion step from *passive parameter* $a \in A$ and/or from *recursion parameter* $n \in \mathbb{N}$ will be given at several occasions in the below. Mentioned is here the recursive definition of the *faculty* function

$$\text{fac} = \text{fac}(n) = n! : \mathbb{N} \to \mathbb{N}.$$

2.5 Uniqueness of the NNO

Category theorists like constructions which are *uniquely* given by their defining properties, unique up to *natural isomorphisms,* or – functorial constructions – up to natural equivalence. For the (binary) cartesian product with its projection families as *natural map* families, this is true by considerations earlier above, same for terminal object $\mathbb{1}$ and the family $\Pi : A \to \mathbb{1}$ of terminal maps (projections).

Now what about the Natural Numbers Object

$\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$?

This DIAGRAM has the property wanted, property which should be called *categoricity:* by its LAWVERE *existence* and *uniqueness* properties below, it is just the *initial diagram*

$$\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$$

of form

$$\mathbb{1} \xrightarrow{a_0} A \xrightarrow{f} A$$

So *purely map theoretically* the notion of an NNO *is categoric:* Within a cartesian map theory NNO $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ is unique up to *natural isomorphism.*

Specialised, sequences definition of NNO: LAWVERE defines the NNO $\mathbb{N}$ as follows:

$a_0 : \mathbb{1} \to A$ a *point*

$f : A \to A$ an endo map *to be iterated*

(NNO$_{\text{FWL}}$) ―――――――――――――――――――――

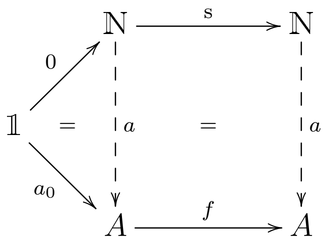
$a : \mathbb{N} \to A$ *resulting sequence*

$a \circ 0 = a_0 : \mathbb{1} \to A$ *start of sequence*

$a \circ \mathrm{s} = f \circ a : \mathbb{N} \to A$ *progress of sequence*

$+$ *uniqueness* of such sequence $a : \mathbb{N} \to A$

in DIAGRAM form:



LAWVERE NNO DIAGRAM

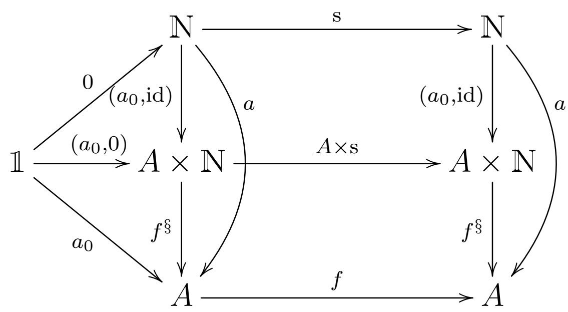
We show that this early NNO scheme is obtained from FREYD's scheme.

NNO Lemma: For $a_0 : \mathbb{1} \to A$ and $f : A \to A$ (antecedent in LAWVERE's NNO scheme) the map

$$a =_{\text{def}} f^{\S} \circ (a_0, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \to A \times \mathbb{N} \xrightarrow{f^{\S}} A$$

uniquely makes the above diagram commute.

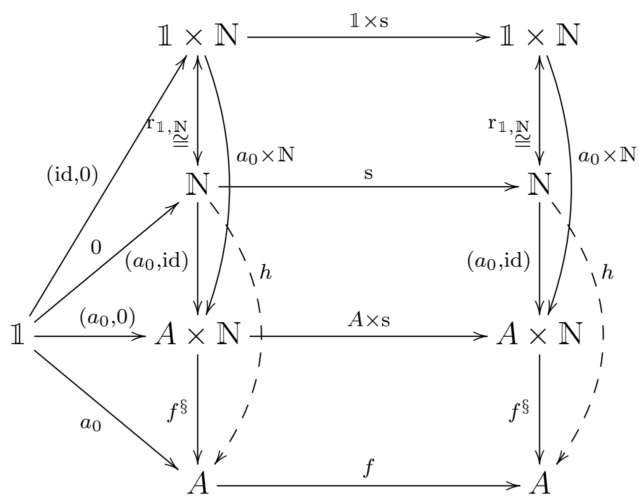
Proof: Consider the following DIAGRAM:



This diagram commutes with $a := f^{\S} \circ (a_0, \mathrm{id}_{\mathbb{N}})$, unique a as is seen by extending the diagram with **isomorphism**

$$\mathrm{r}_{\mathbb{1},\mathbb{N}} : \mathbb{1} \times \mathbb{N} \to \mathbb{N}, \ \text{inverse } (\Pi_{\mathbb{N}}, \mathrm{id}_{\mathbb{N}})$$

into commuting DIAGRAM



FREYD to LAWVERE NNO specialisation DIAGRAM

$h = h(n) : \mathbb{N} \to A$ is to be another *sequence* assumed to fullfill the postcedent above in place of $a : \mathbb{N} \to A$. By uniqueness of the initialised iterated $f^{\S} \circ (a_0 \times \mathrm{id}_{\mathbb{N}})$ it must equal

$$a = f^{\S} \circ (a_0, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \to A \ \ \textbf{q. e. d.}$$

Remark: Conversely LAWVERE's NNO is said to have the properties of an NNO in FREYD's version quoted above. But for his proof of this assertion FREYD relies on internal hom structure – *axiomatic* exponentiation B^A – coming with *axiomatic* internal evaluation $\epsilon_{A,B} : B^A \times A \to B$ which is available in his context of an Elementary (higher order) Topos, not available in present context.

In RCF 3 in the References it is shown that the initial *cartesian closed* theory with NNO admits *code self-evaluation* and hence is inconsistent. This is one motivation for not considering here higher order recursion theory. The other motivation is simplicity: the Gödelian case is built on first-order in SMORYNSKI 1977, no power sets needed.

2.6 Hilbert's infinite hotel

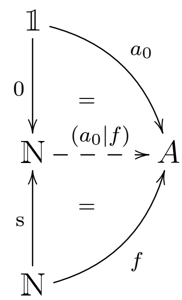
$\mathbb{N} \cong \mathbb{1} + \mathbb{N}$

$\mathbb{N}$ *is isomorphic to the coproduct of* $\mathbb{1}$ *and* $\mathbb{N}$

paradoxon on infinity

"But" maps $a_0 : \mathbb{1} \to A,\ f : \mathbb{N} \to A$ induce a unique map $(a_0|f) : \mathbb{N} \to A$ "out of the sum/coproduct"

$$\mathbb{1} \xrightarrow{0} \mathbb{N} \xleftarrow{\mathrm{s}} \mathbb{N}$$ such that

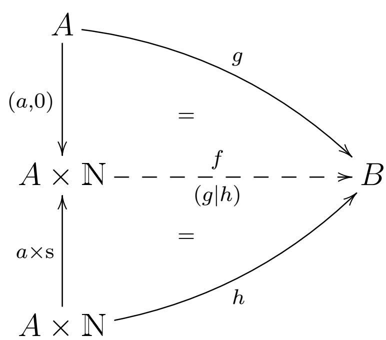


[Coproducts are *universal*, hence unique up to isomorphism.]

We **prove** a more general, *parametrised* version of **coproduct property** of $\mathbb{1} \xrightarrow{0} \mathbb{N} \xleftarrow{\mathrm{s}} \mathbb{N}$ namely: For A an arbitrary ("parameter") object A

$$A \times \mathbb{N} \cong A \times (\mathbb{1} + \mathbb{N}) \cong A + (A \times \mathbb{N})$$
$$[\cong (A \times \mathbb{1}) + (A \times \mathbb{N})]$$

Proof: We obtain, via full schema (pr) the following **coproduct** diagram where $a := \mathrm{id}_A : A \to A$, and "inducing" maps $g : A \to B$, $h : A \times \mathbb{N} \to B$ are given. They induce a unique map $f = (g|h) : A \times \mathbb{N}$ out of the *coproduct* $A \times \mathbb{N}$, what we have to show:



Map

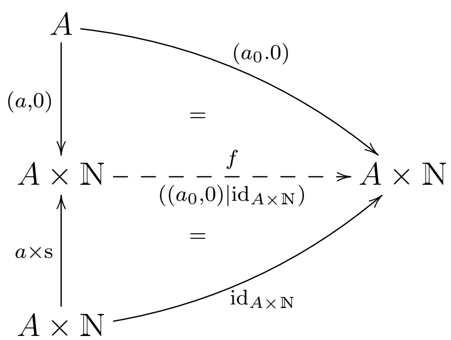
$$f = (g|h) \ =_{\mathrm{def}} \ \mathrm{pr}[g, h \circ \ell] : A \times \mathbb{N} \to B$$

is the *unique* commutative fill-in into this *coproduct diagram,* since by full scheme (pr) of primitive recursion

$$f(a, 0) = g(a) : A \to B$$
$$f(a, \mathrm{s}\, n) = h(a, n) = (h \circ \ell)((a, n), f(a, n)) : (A \times \mathbb{N}) \to B$$

Infinite-hotel interpretation:

Replace within the latter coproduct diagram object B by $A \times \mathbb{N}$, component map g by $(a_0, 0) : A \to A \times \mathbb{N}$ and $h = \mathrm{id}_{A \times \mathbb{N}}$, and get special "hotel" coproduct diagram



Hotel $\mathbb{N}$ has an infinite number $n \in \mathbb{N}$ of rooms. Each room n is occupied by a guest $(a, n) \in A \times \mathbb{N}$.

A new guest $a_0 \in A$ arrives at that fully occupied hotel. Since the hotel is infinite, the manager has (at least) 2 possibilies to host all present guests *and* the new one:

- the *actual*-infiniteness possibility: per simultaneous message he asks all present guests to change to respective next room:

  $(a, n) \mapsto (a, n + 1)$, and hosts simultaneously the new guest a_0 in room 0, $a_0 \mapsto (a_0, 0) \in A \times \mathbb{N}$.

- the *potential*-infiniteness possibility: The hotel has *potential* for an infinity of rooms (new rooms can be aquired in time or even constructed). All rooms the manager has at his disposal at present are occupied. A new guest arrives. The manager travels along all of these rooms and aquires at his disposal a next room. Then he travels backwards and asks subsequently the finitely many present guests to move "upwards", first the guest with highest room number, and eventually allocates room 0 to the arriving guest.

- the latter possibility is realised mathematically by interpretation of $A \times \mathbb{N}$ as the – (one-sided) potentially infinite – tape of a TURING machine, and the hotel manager as the (processing) head of a (very simple) such machine. A is the *tape alphabet* of the TURING machine. In computer science this simple TURING machine is – works as – a (potentially infinite) STACK.

Chapter 3

Algebra and order on the NNO

In "Development of Mathematical Logic" (Logos Press 1971) R. L. Goodstein gives four basic uniqueness-rules for free-variable Arithmetics. We show here these rules for theory **PR** and that these four rules are sufficient for proving the commutative and associative laws for multiplication and the distributive law, for addition as well as for truncated subtraction $a \setminus n$ noted $a \dot{-} n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by Goodstein.

For our *evaluation and consistency* considerations below we need from present chapter equality *predicate* $[a \doteq b] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and that this predicate **defines** map equality, see *equality definability scheme.* This scheme is a consequence of (Goodstein's) max commutativity which is difficult to show and which you may take on faith.

3.1 Free-variable NNO Algebra

Basic **GA**[1] operations are *addition* '$+$', *predecessor* 'pre', *truncated subtraction* '$\smallsetminus$' [in GOODSTEIN predecessor written $\mathrm{pre}\, n := n \dot{-} 1$], as well as *multiplication* '$\cdot$'.

We[2] include into Goodstein's uniqueness rules a "passive parameter" a. These extended rules are derivable by use of Freyd's uniqueness theorem (pr!), part of *full scheme* (pr) of primitive recursion which he deduces from his uniqueness (FR!) of the *initialised iterated.*

Goodstein's rules parametrised

Let $f, g : A \times \mathbb{N} \to \mathbb{N}$ be maps, $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ the successor map $n \mapsto n+1$ and $\mathrm{pre} : \mathbb{N} \to \mathbb{N}$ the predecessor map, here usually written as $n \mapsto n \smallsetminus 1$.

Then Goodstein's rules read:

$$\mathrm{U}_1 \quad \frac{f(a, sn) = f(a, n) : A \times \mathbb{N} \to B}{f(a, n) = f(a, 0) : A \times \mathbb{N} \to B}$$

no change by application of successor

infers equality *with value at zero* for f

[1] Goodstein Arithmetic

[2] Sandra Andrasek and the author

$$\mathrm{U}_2 \quad \frac{f(a, \mathrm{s}\, n) = \mathrm{s}\, f(a, n) : A \times \mathbb{N} \to \mathbb{N}}{f(a, n) = f(a, 0) + n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of successors into $+n$

$$\mathrm{U}_3 \quad \frac{f(a, sn) = \mathrm{pre}\ f(a, n) : A \times \mathbb{N} \to \mathbb{N}}{f(a, n) = f(a, 0) \smallsetminus n : A \times \mathbb{N} \to \mathbb{N}}$$

accumulation of predecessors into $\smallsetminus n$

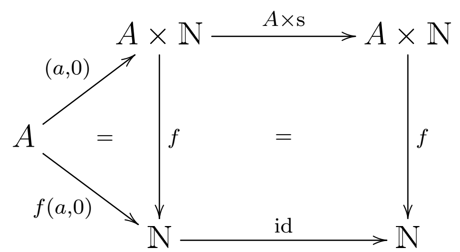
$$\mathrm{U}_4 \quad \frac{\begin{array}{l} f(a, 0) = g(a, 0) : A \to \mathbb{N} \\ f(a, sn) = g(a, sn) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a, n) = g(a, n) : A \times \mathbb{N} \to \mathbb{N}}$$

uniqueness of map definition by case-distinction

Rule U_4 is nothing else than *uniqueness* of the *induced map out of the sum* $A \times \mathbb{N} \cong (A \times \mathbb{1}) + (A \times \mathbb{N})$, this sum canonically realised via *injections* $\iota = (\mathrm{id}_A, 0) : A \to A \times \mathbb{N}$ as well as – right injection – $\kappa = \mathrm{id}_A \times \mathrm{s} : A \times \mathbb{N} \to A \times \mathbb{N}$.

Proof of these four rules is straight forward for theory **PR** using Freyd's uniqueness (FR!) and uniqueness clause (pr!) of the *full scheme of primitive recursion* respectively, as follows:

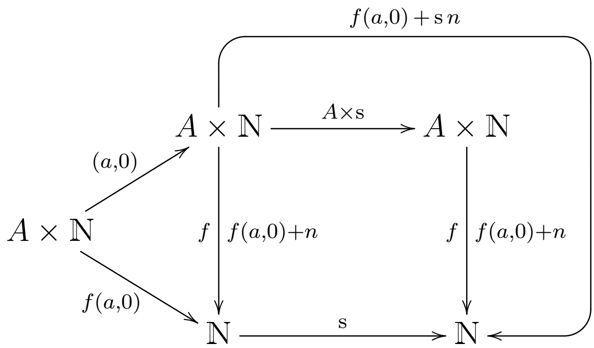
For scheme U_1 consider, with free variable $a := \ell : A \times \mathbb{N} \to A$,



(FR!)

$$f(a,n) = f = \mathrm{id}^{\S}_{\mathbb{N}} \circ (f(a,0) \times \mathbb{N})$$
$$= \ell_{\mathbb{N},\mathbb{N}} \circ (f(a,0) \times \mathbb{N}) : A \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$= f(a,0) \circ \ell_{A,\mathbb{N}} : A \times \mathbb{N} \to A \to \mathbb{N}$$
$$= f(a,0) : A \times \mathbb{N} \xrightarrow{A \times \Pi_{\mathbb{N}}} A \times \mathbb{1} \xrightarrow{A \times 0} A \times \mathbb{N} \xrightarrow{f} \mathbb{N}$$

Proof of U_2 of *"summing up successors":*



pentagon commutative for both f, $f(a,0)+n$

(FR!)

$$f(a,n) = f(a,0) + n$$

Proof of U_3 is exactly analogous to the above: Replace in statement of U_2 and its proof *stepwise augmentation* $f(a, sn) = \mathrm{s}\ f(a,n)$

by *stepwise descent*

$$f(a, \mathrm{s}\,n) = f(a,n) \smallsetminus 1 \ =_{\text{by def}} \ \mathrm{pre}\, f(a,n)$$

On right hand side replace *successor* $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ by *predecessor* $\mathrm{pre} : \mathbb{N} \to \mathbb{N}$ which in turn is defined by the full scheme (pr) of primitive recursion. In *postcedent* replace *iterated successor* $a + n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by *iterated predecessor* $a \smallsetminus n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

[In GOODSTEIN's *original,* $\mathrm{pre}(n) = n \smallsetminus 1 : \mathbb{N} \to \mathbb{N}$ is a *basic,* "undefined" map constant]

We give a **direct proof** of U_4 :

We tailor first this scheme for convenient use of "full" uniqueness scheme (pr!) as follows:

$$\frac{\begin{array}{l} f = f(a,n),\ f' = f'(a,n) : A \times \mathbb{N} \to B \\ f(a,0) = f'(a,0) : A \to B \\ f(a,\mathrm{s}\,n) = f'(a,\mathrm{s}\,n) : A \times \mathbb{N} \to A \times \mathbb{N} \to B \end{array}}{f = f' : A \times \mathbb{N} \to B.} \quad U_4$$

Choose the *anchor map*

$$g = g(a) := f(a,0) = f'(a,0) :$$
$$A \to A \times \mathbb{N} \to B$$

and the *step map*

$$h = h((a,n),b) := f(a,\mathrm{s}\,n) = f'(a,\mathrm{s}\,n) :$$
$$(A \times \mathbb{N}) \times B \xrightarrow{\ \ell\ } A \times \mathbb{N} \to B$$

We obtain via the *full* scheme (pr!) of primitive recursion:

$$\text{(pr!)}\quad \frac{\begin{array}{l} f(a,0) = g(a) = f'(a,0) \ \ \text{(anchor hypothesis)} \\ f(a,\mathrm{s}\,n) = h((a,n), f(a,n)) = f'(a,\mathrm{s}\,n) \ \text{(step hypothesis)} \end{array}}{f = \mathrm{pr}[\,g,h] = f' : A \times \mathbb{N} \to B \quad \textbf{q. e. d.}}$$

Combination of *reflexivity, symmetry,* and *transitivity* of *equality* $f = g : A \to B$ between maps with the defining *equations* for the fundamental *operations* and with *rules* U_1 to U_4 above, **defines** categorical Goodstein's **free-variables Arithmetic** which we name **Goodstein Arithmetic GA**.

Arithmetical equations

We **quote** here – with *passive parameters* made visible – GOODSTEIN's arithmetical equations together with his **proofs.**

The first equation is (Goodstein's statement numbers)

Lemma:

$$(a \smallsetminus n) \smallsetminus 1 =^{\mathbf{GA}} (a \smallsetminus 1) \smallsetminus n : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad (1.)$$

$a \in \mathbb{N}$ free, “passive” $a := \ell : A \times \mathbb{N} \to A$

$n \in \mathbb{N}$ free, *recursive,* $n := \mathrm{r} : A \times \mathbb{N} \to \mathbb{N}$

Proof:

$$\mathrm{U}_3 \quad \frac{(a \smallsetminus \mathrm{s}\,n) \smallsetminus 1 \ =_{\text{by def}} \ ((a \smallsetminus n) \smallsetminus 1) \smallsetminus 1}{(a \smallsetminus n) \smallsetminus 1 = ((a \smallsetminus 0) \smallsetminus 1) \smallsetminus n}$$

$$=_{\text{by def}} \ (a \smallsetminus 1) \smallsetminus n : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \quad \textbf{q. e. d.}$$

Next equation is

stepwise simplification rule for truncated subtraction:

$$\mathrm{s}\ a \smallsetminus \mathrm{s}\ b = a \smallsetminus b : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad (1.1)$$

Proof:

$$\mathrm{U}_3 \quad \frac{\mathrm{s}\ a \smallsetminus \mathrm{s}\ \mathrm{s}\ b \ =_{\text{by def}} \ (\mathrm{s}\ a \smallsetminus \mathrm{s}\ b) \smallsetminus 1}{\mathrm{s}\ a \smallsetminus \mathrm{s}\ b = (\mathrm{s}\ a \smallsetminus \mathrm{s}\ 0) \smallsetminus b}$$

$$=_{\text{by def}} \ a \smallsetminus b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

the latter by definition of the *predecessor* "$\smallsetminus 1$" **q. e. d.**

Lemma: $a \smallsetminus a = 0 : \mathbb{N} \to \mathbb{N}$ (1.2)

Proof:

$$\mathrm{U}_1 \quad \frac{\mathrm{s}\ a \smallsetminus \mathrm{s}\ a = a \smallsetminus a \quad \text{(by stepwise simplification 1.1 above)}}{a \smallsetminus a = 0 \smallsetminus 0 \ =_{\text{by def}} \ 0}$$

q. e. d.

Lemma: $0 \smallsetminus a = 0 : \mathbb{N} \to \mathbb{N}$ (1.3)

Proof:

$$0 \smallsetminus \mathrm{s}\,a \ =_{\text{by def}} \ (0 \smallsetminus a) \smallsetminus 1$$
$$= (0 \smallsetminus 1) \smallsetminus a \quad \text{(by (1.) above)}$$
$$= 0 \smallsetminus a : \mathbb{N} \to \mathbb{N}$$

$$\mathrm{U}_1 \ \overline{\qquad\qquad\qquad\qquad}$$

$$0 \smallsetminus a = 0 \smallsetminus 0 = 0 : \mathbb{N} \to \mathbb{N} \quad \textbf{q. e. d.}$$

Proposition:

$$a \smallsetminus (b + c) = (a \smallsetminus b) \smallsetminus c : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \qquad (1.31)$$

Proof:

$$a \smallsetminus (b + \mathrm{s}\,c) \ =_{\text{by def}} \ a \smallsetminus \mathrm{s}\,(b + c) \quad \text{(definition of +)}$$
$$=_{\text{by def}} \ (a \smallsetminus (b + c)) \smallsetminus 1 \quad \text{(definition of} \smallsetminus \text{)}$$

$$\mathrm{U}_3 \ \overline{\qquad\qquad\qquad\qquad}$$

$$a \smallsetminus (b + c) = (a \smallsetminus (b + 0)) \smallsetminus c \ =_{\text{by def}} \ (a \smallsetminus b) \smallsetminus c$$

q. e. d.

Full Simplification:

$$(a + n) \smallsetminus (b + n) = a \smallsetminus b : \mathbb{N}^2 \times \mathbb{N} \to \mathbb{N} \qquad (1.4)$$

Proof:

$$(a + \mathrm{s}\,n) \smallsetminus (b + \mathrm{s}\,n)$$

$$=_{\text{by def}}\ \mathrm{s}\,(a+n) \smallsetminus \mathrm{s}\,(b+n) = (a+n) \smallsetminus (b+n)$$

by *substitution* – realised essentially as composition – of $(a+n)$ into a and $(a+n)$ into b within *stepwise simplification equation* 1.1 above

$$\mathrm{U}_1 \quad \overline{}$$

$$(a+n) \smallsetminus (b+n) = (a+0) \smallsetminus (b+0) \ =_{\text{by def}}\ a \smallsetminus b.$$

Lemma: $0 + n = n\ [\ =_{\text{by def}}\ n + 0\ ]\ : \mathbb{N} \to \mathbb{N}$ (2)

Proof:

$$\frac{\mathrm{id}_{\mathbb{N}}\,\mathrm{s}\,a = \mathrm{s}\,a}{\mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a}\ \mathrm{U}_2$$

and hence

$$a = \mathrm{id}_{\mathbb{N}}(a) = \mathrm{id}_{\mathbb{N}}(0) + a = 0 + a : \mathbb{N} \to \mathbb{N} \quad \textbf{q. e. d.}$$

Lemma: $a + \mathrm{s}\,b = \mathrm{s}\,a + b : \mathbb{N} \times \mathbb{N} \to B$ (2.1)

Proof by U_2 as follows, with free variable $b := \mathrm{r} : \mathbb{N}^2 \to \mathbb{N}$ as *recursion variable:*

For $f = f(a,b)\ =_{\text{def}}\ a + \mathrm{s}\,b : \mathbb{N} \times \mathbb{N} \to N$:

$$U_2 \quad \frac{f(a, \mathrm{s}\, b) \ =_{\text{by def}} \ a + \mathrm{s}\,\mathrm{s}\, b = \mathrm{s}(a + \mathrm{s}\, b) = \mathrm{s}\, f(a, b) : \mathbb{N}^2 \to \mathbb{N}}{f(a, b) = a + \mathrm{s}\, b = f(a, 0) + b}$$

$=_{\text{by def}} \ (a + \mathrm{s}\, 0) + b \ =_{\text{by def}} \ \mathrm{s}\, a + b$ **q. e. d.**

Theorem:

$$a + b = b + a : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \tag{2.2}$$

$a := \ell : \mathbb{N}^2 \to \mathbb{N}$

$b := \mathrm{r} : \mathbb{N}^2 \to \mathbb{N}$

Proof:

$$U_4 \quad \frac{\begin{array}{l} a + 0 \ =_{\text{by def}} \ a = 0 + a \text{ by (2) above} \\ a + \mathrm{s}\, b = \mathrm{s}\, a + b \text{ by (2.1) above (and symmetry of equality)} \end{array}}{a + b \ =_{\text{by def}} \ f(a, b) = g(a, b)}$$

$=_{\text{by def}} \ \mathrm{s}\, a + b : \mathbb{N}^2 \to \mathbb{N}$ **q. e. d.**

This gives also sort of *permutability* for truncated subtraction:

$$(a \smallsetminus b) \smallsetminus c = (a \smallsetminus c) \smallsetminus b : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

Proof:

$(a \smallsetminus b) \smallsetminus c = a \smallsetminus (b + c)$ by (1.31) above

$= a \smallsetminus (c + b)$ by commutativity of addition

$= (a \smallsetminus c) \smallsetminus b$ again by (1.31) **q. e. d.**

From *full simplification* (1.4) and *left neutrality* of zero (2) above with respect to addition we get immediately *"one-term" simplification*

Lemma:

$$(a+n) \smallsetminus n = (a+n) \smallsetminus (0+n) = a : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad (2.3)$$

Associativity of Addition

$$(a+b)+c = a+(b+c) : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$$

Proof: for $f((a,b),c) \ =_{\text{def}} \ a+(b+c) : \mathbb{N}^2 \times \mathbb{N}$:

$$\frac{\begin{array}{l} f((a,b),\mathrm{s}\,c) = a+(b+\mathrm{s}\,c) = a+\mathrm{s}(b+c) \\ = \mathrm{s}(a+(b+c)) = \mathrm{s}\,f((a,b),c) \end{array}}{\begin{array}{l} a+(b+c) = f((a,b),c) = f((a,b),0)+c \\ =_{\text{by def}} \ (a+(b+0))+c = (a+b)+c \end{array}} \ U_2$$

q. e. d.

Recall p. r. **Definition** of *Multiplication:*

$$a \cdot 0 = 0 : \mathbb{N} \to \mathbb{N}$$
$$a \cdot (n+1) = (a \cdot n) + a$$

For this operation we have not only *annihilation by zero from the right* but also

Left zero-Annihilation $0 \cdot n = 0 : \mathbb{N} \to \mathbb{N}$.

Proof:

$$U_1 \quad \frac{0 \cdot \mathrm{s}\, n = (0 \cdot n) + 0 = 0 \cdot n}{0 \cdot n = 0 \cdot 0 = 0} \qquad \textbf{q. e. d.}$$

For proving the other equational laws making the natural numbers object $\mathbb{N}$ into a *unitary commutative semiring* with in addition truncated subtraction introduced above GOODSTEIN's derived scheme V_4 below is helpfull.

For proof of that scheme we rely on

Commutativity of maximum operation:[3]

$$\begin{aligned} &\max(a, b) \ =_{\mathrm{def}} \ a + (b \smallsetminus a) \\ &= b + (a \smallsetminus b) \ =_{\text{by def}} \ \max(b, a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \end{aligned}$$

Proof[4]: As a first step we show

Diagonal Reduction Lemma for maximum:

$$\max(a, b) = \max(a \smallsetminus 1, b \smallsetminus 1) + \mathrm{sgn}(a + b)$$

Proof of Lemma: first we show **equation**

$$\max(a, \mathrm{s}\, b) = \max(a \smallsetminus 1, \mathrm{s}\, b \smallsetminus 1) + \mathrm{sgn}(a + \mathrm{s}\, b) \tag{1}$$

[3]in GOODSTEIN 1964 this is taken as an axiom

[4]GOODSTEIN 1971 adapted by G. Myrach

[where $\mathrm{sgn}(0) = 0,\ \mathrm{sgn}(\mathrm{s}\,n) = 1$] as follows:

$$\max(0 \smallsetminus 1, \mathrm{s}\,b) = \mathrm{s}\,b$$
$$= \max(0 \smallsetminus 1, s\,b \smallsetminus 1) + \mathrm{sgn}(0 + s\,b) \qquad (2)$$

and

$$\max(\mathrm{s}\,a, \mathrm{s}\,b) = \mathrm{s}\,a + (\mathrm{s}\,b \smallsetminus \mathrm{s}\,a)$$
$$= \mathrm{s}\,a + (b \smallsetminus a) = \mathrm{s}(a + (b \smallsetminus a))$$
$$= \mathrm{s}\ \max(a, b) = \max(a, b) + 1$$
$$= \max(\mathrm{s}\,a \smallsetminus 1, \mathrm{s}\,b \smallsetminus 1) + \mathrm{sgn}(\mathrm{s}\,a + \mathrm{s}\,b) \qquad (3)$$

From (2) and (3) **follows equation** (1) by uniqueness rule U_4.

Furthermore

$$\max(a, 0) = a = (a \smallsetminus 1) + \mathrm{sgn}(a)$$
$$= \max(a \smallsetminus 1, 0 \smallsetminus 1) + \mathrm{sgn}(a + 0) \qquad (4)$$

Together with (1) above this gives again by U_4 the **Diagonal Reduction Lemma.**

From this we get immediately by substitution

Opposite Diagonal Reduction Lemma for maximum:

$$\max(b, a) = \max(b \smallsetminus 1, a \smallsetminus 1) + \mathrm{sgn}(b + a)$$
$$= \max(b \smallsetminus 1, a \smallsetminus 1) + \mathrm{sgn}(a + b) \quad \textbf{q. e. d.}$$

Let *increment* map

$\phi = \phi(n, (a, b)) : \mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}$ be defined by
$\phi(0, (a, b)) = 0 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and
$\phi(\mathrm{s}\, n, (a, b)) = \phi(n, (a, b)) + \mathrm{sgn}((a \setminus n) + (b \setminus n)) :$
$\mathbb{N} \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N}$

We show for this ϕ

$$\max(a \setminus n, b \setminus n) + \phi(n, (a, b))$$
$$= \max(a \setminus \mathrm{s}\, n, b \setminus \mathrm{s}\, n) + \phi(\mathrm{s}\, n, (a, b)) \qquad (5)$$

as well as

$$\max(b \setminus n, a \setminus n) + \phi(n, (a, b))$$
$$= \max(b \setminus \mathrm{s}\, n, a \setminus \mathrm{s}\, n) + \phi(\mathrm{s}\, n, (a, b)) \qquad (6)$$

(same increment)

First we show equation (5): Substitution of $(a \setminus n)$ for a and $(b \setminus n)$ for b within **Reduction Lemma** above gives

$$\max(a \setminus n, b \setminus n)$$
$$= \max((a \setminus n) \setminus 1, (b \setminus n) \setminus 1) + \mathrm{sgn}((a \setminus n) + (b \setminus n))$$

Adding $\phi(n, (a, b))$ to both sides of this equation gives

$$\max(a \setminus n, b \setminus n) + \phi(n, (a + b))$$
$$= \max((a \setminus n) \setminus 1, (b \setminus n) \setminus 1)$$
$$+ \mathrm{sgn}((a \setminus n) + (b \setminus n)) + \phi(n, (a + b))$$
$$=_{\text{by def}} \max(a \setminus \mathrm{s}\, n, b \setminus \mathrm{s}\, n) + \phi(\mathrm{s}\, n, (a, b))$$

i. e. equation (5)

We show equation (6): By substitution of $(b \smallsetminus n)$ for b and $(a \smallsetminus n)$ for a in **Opposite Reduction Lemma** and addition of $\phi(n,(a,b))$ on both sides we get

$$
\begin{aligned}
&\max(b \smallsetminus n, a \smallsetminus n) + \phi(n,(a,b)) \\
&= \max((b \smallsetminus n) \smallsetminus 1, (a \smallsetminus n) \smallsetminus 1) \\
&\quad + \operatorname{sgn}((b \smallsetminus n) + (a \smallsetminus n)) + \phi(n,(a,b)) \\
&= \max((b \smallsetminus n) \smallsetminus 1, (a \smallsetminus n) \smallsetminus 1) \\
&\quad + \operatorname{sgn}((a \smallsetminus n) + (b \smallsetminus n)) + \phi(n,(a,b)) \\
&=_{\text{by def}}\ \max((b \smallsetminus n) \smallsetminus 1, (a \smallsetminus n) \smallsetminus 1) + \phi(\mathrm{s}\,n,(a,b)) \\
&= \max(b \smallsetminus \mathrm{s}\,n, a \smallsetminus \mathrm{s}\,n) + \phi(\mathrm{s}\,n,(a,b))
\end{aligned}
$$

i. e. equation (6)

From the two Lemmata we get by uniqueness U_1

$$
\begin{aligned}
&\max(a \smallsetminus n, b \smallsetminus n) + \phi(n,(a,b)) \\
&= \max(a \smallsetminus 0, b \smallsetminus 0) + \phi(0,(a,b)) = \max(a,b) + 0 = \max(a,b) \\
&\quad \text{as well as} \\
&\max(b \smallsetminus n, a \smallsetminus n) + \phi(n,(a,b)) \\
&= \max(b \smallsetminus 0, a \smallsetminus 0) + \phi(0,(a,b)) = \max(b,a) + 0 = \max(b,a)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\max(a,b) = \max(a \smallsetminus n, b \smallsetminus n) + \phi(n,(a,b)) \ \text{ as well as} \\
&\max(b,a) = \max(b \smallsetminus n, a \smallsetminus n) + \phi(n,(a,b))
\end{aligned}
$$

and so, by substitution of b into n :

$$\begin{aligned}
&\max(a,b) = \max(a \smallsetminus b, b \smallsetminus b) + \phi(b,a,b) \\
&= (a \smallsetminus b) + \phi(b,(a,b)) \\
&= \max(b \smallsetminus b, a \smallsetminus b) + \phi(b,(a,b)) \\
&= \max(b,a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}
\end{aligned}$$

q. e. d. max commutativity.

This given we **show** for **GA** (and hence for **PR**) scheme

$$\mathrm{V}_4 \quad \frac{\begin{array}{l} f,\ g,\ h : A \times \mathbb{N} \to \mathbb{N} \\ f(a,0) = g(a,0) : A \to \mathbb{N} \\ f(a,sn) = f(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \\ g(a,sn) = g(a,n) + h(a,n) : A \times \mathbb{N} \to \mathbb{N} \end{array}}{f(a,n) = g(a,n).}$$

Rule V_4 can be derived by applying rule U_1 to the distance map

$$\begin{aligned}
&d(a,n) = |f(a,n), g(a,n)| = |f(a,n) - g(a,n)| \\
&\ =_{\text{by def}}\ (f(a,n) \smallsetminus g(a,n)) + (g(a,n) \smallsetminus f(a,n)) : \\
&A \times \mathbb{N} \to \mathbb{N}^2 \xrightarrow{+} \mathbb{N}
\end{aligned}$$

$$\begin{aligned} d(a,0) &= (f(a,0) \dot{-} g(a,0)) + (g(a,0) \dot{-} f(a,0)) = 0 \\ d(a,sn) &= (f(a,sn) \dot{-} g(a,sn)) + (g(a,sn) \dot{-} f(a,sn)) \\ &= (f(a,n) + h(a,n)) \dot{-} (g(a,n) + h(a,n)) \\ &\quad + (g(a,n) + h(a,n)) \dot{-} (f(a,n) + h(a,n)) \\ &= (f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) \\ &= d(a,n) : A \times \mathbb{N} \to \mathbb{N} \end{aligned}$$

whence by U_1:

$$d(a,n) = d(a,0) = 0 \text{ i.e.}$$
$$(f(a,n) \dot{-} g(a,n)) + (g(a,n) \dot{-} f(a,n)) = 0 \text{ whence}$$
$$f(a,n) \dot{-} g(a,n) = 0 = g(a,n) \dot{-} f(a,n) : A \times \mathbb{N} \to \mathbb{N}$$

and hence

$$\begin{aligned} f(a,n) &= f(a,n) + (g(a,n) \dot{-} f(a,n)) \\ &= \max(f(a,n), g(a,n)) \\ &= \max(g(a,n), f(a,n)) \\ &= g(a,n) + (f(a,n) \dot{-} g(a,n)) \\ &= g(a,n) \quad \textbf{q. e. d.} \end{aligned}$$

3.2 Equality definability

Individual equality is **defined** as equality *predicate*

$$[m \doteq n] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

via weak order as follows:

$$[m \le n] \ =_{\mathrm{def}} \ \neg\, [m \smallsetminus n] : \mathbb{N}^2 \to \mathbb{N} \to \mathbb{N} \text{ where}$$

protoboolean operation negation given as

$$\neg\, n \ =_{\mathrm{def}} \ 1 \smallsetminus n \text{ directly p. r. defined by}$$

$$\neg\, 0 \ =_{\mathrm{def}} \ 1 = \mathrm{s}\, 0 : \mathbb{1} \to \mathbb{N}$$

$$\neg\, \mathrm{s}\, n \ =_{\mathrm{def}} \ 0 : \mathbb{1} \to \mathbb{N}$$

This order on $\mathbb{N}$ is *reflexive* and *transitive.*

Individual equality – first on $\mathbb{N}$ – then is easily **defined** by

$$[m \doteq n] \ =_{\mathrm{def}} \ [m \le n \ \wedge \ n \le m]$$
$$=_{\mathrm{by\,def}} \ [m \le n] \cdot [n \le m] : \mathbb{N}^2 \to \mathbb{N}$$

[It is a *protopredicate.*]

We now have at our disposition all ingredients for

Equality definability theorem

$$f = f(a) : A \to B, \ g = g(a) : A \to B \text{ in } \mathbf{PR}$$

$$\mathbf{PR} \vdash \ \mathrm{true}_A \ =_{\mathrm{by\,def}} \ 1 \circ \Pi_A = [f(a) \doteq_B g(a)] :$$
$$A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \xrightarrow{\doteq_B} \mathbb{N}$$

(EqDef) $\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$

$$\mathbf{PR} \vdash \ f = g : A \to B \ \text{ i. e. } \ f =^{\mathbf{PR}} g : A \to B$$

A map equation which holds true predicatively for “all” arguments individually gives rise to an argument-free *categorical equation **between** the maps concerned.*

Proof: We begin with the special case $B = \mathbb{N}$: Let $f,\ g : A \to \mathbb{N}$ **PR** maps satisfying the *antecedent* of (EqDef). Then

$$\begin{aligned}\mathbf{PR} \vdash f(a) &= f(a) + 0 = f(a) + (g(a) \smallsetminus f(a)) \text{ by antecedent} \\ &= \max(f(a), g(a)) \text{ by definition of } \max(m, n) \\ &= \max(g(a), f(a)) \text{ by } \max \text{ commutativity} \\ &= g(a) + (f(a) \smallsetminus g(a)) \\ &= g(a) + 0 = g(a) : A \to B\end{aligned}$$

The general case for codomain object B follows since *individual equality* on (binary) cartesian products is canonically defined *componentwise* and B is a cartesian product of $\mathbb{N}$'s **q. e. d.**

Equality convention

Motivated by **equality definability** just proved, we write from now on $f(a) = g(a)$ or $[f(a) = g(a)]$ or $[f = g]$ instead of $f(a) \doteq g(a)$.

These *fundamentals* given we continue with properties of the algebraic structure on $\mathbb{N}$.

3.3 Further Algebra on the NNO

Theorem: In free–variables arithmetics the *commutative law* for *multiplication:* $n \cdot m = m \cdot n$ holds.

Proof: We need the following

Lemma:

(i) $0 \cdot n = 0$

(ii) $sa \cdot n = a \cdot n + n$

Proof:

(i) $0 \cdot 0 = 0$ and
$0 \cdot sn = 0 \cdot (n+1) = 0 \cdot n + 0 = 0 \cdot n = 0 \cdot 0 = 0.$

(ii) We show $f(a,n) := sa \cdot n = g(a,n) := a \cdot n + n$ using V_4: $f(a,0) = g(a,0)$ because for $n = 0$ we get $(sa) \cdot 0 = 0$ as well as $a \cdot 0 + 0 = a \cdot 0 = 0$.

$$\begin{aligned} f(a,sn) &= (sa) \cdot (sn) = (a+1) \cdot (n+1) \\ &= (a+1) \cdot n + (a+1) = (sa) \cdot n + sa \\ &= f(a,n) + h(a,n) \qquad \text{with} \qquad h(a,n) := sa \\ g(a,sn) &= a \cdot (sn) + sn = a \cdot (n+1) + (n+1) \\ &= a \cdot n + a + n + 1 = a \cdot n + n + a + 1 \\ &= a \cdot n + n + sa \\ &= g(a,n) + h(a,n). \end{aligned}$$

So V_4 gives $f(a,n) = g(a,n)$ i.e. $sa \cdot n = a \cdot n + n$.

q. e. d.

We continue with the proof of $a \cdot n = n \cdot a$:

From $a \cdot 0 = 0 = 0 \cdot a$ and $a \cdot sn = a \cdot n + n = sn \cdot a$ by the Lemma, we conclude $a \cdot n = n \cdot a$ by V_4.

q. e. d.

Theorem: In free–variable arithmetics multiplication distributes over addition: $a \cdot (m+n) = a \cdot m + a \cdot n$.

Proof: Case $n = 0$ is trivial by definition of $+$ and $\cdot$.

From the hypothesis $a \cdot (m+n) = a \cdot m + a \cdot n$ we infer the next step $a \cdot (m + sn) = a \cdot m + a \cdot sn$ by rule V_4 above – with passive parameter (a, m) – as follows:

$$\begin{aligned} \text{with} \quad & f((a,m),n) := a \cdot (m+n) \\ & g((a,m),n) := a \cdot m + a \cdot n \qquad \text{and} \\ & h((a,m),n) := a \end{aligned}$$

we have

$$\begin{aligned} f((a,m),sn) &= a \cdot (m+sn) = a \cdot (m + (n+1)) \\ &= a \cdot ((m+n)+1) = a \cdot (m+n) + a \\ &= f((a,m),n) + h((a,m),n) \\ g((a,m),sn) &= a \cdot m + a \cdot sn = a \cdot m + a \cdot (n+1) \\ &= a \cdot m + a \cdot n + a \\ &= g((a,m),n) + h((a,m),n). \end{aligned}$$

From this V_4 gives

$f((a,m),n) = g((a,m),n)$ i. e.

$a \cdot (m+n) = a \cdot m + a \cdot n$

q. e. d.

Theorem: In free–variable arithmetics the associative law holds:

$$a \cdot (m \cdot n) = (a \cdot m) \cdot n$$

Proof: We prove the law applying rule V_4 with "active" parameter n and passive parameter (a, m) to

$$f((a,m),n) := a \cdot (m \cdot n)$$
$$g((a,m),n) := (a \cdot m) \cdot n \quad \text{and}$$
$$h((a,m),n) := a \cdot m$$

For $n = 0$ we have: $a \cdot (m \cdot n) = a \cdot 0 = 0$ and on the other hand: $(a \cdot m) \cdot 0 = 0$.

For V_4–step we have:

$$\begin{aligned} f((a,m),sn) &= a \cdot (m \cdot sn) = a \cdot (m \cdot (n+1)) \\ &= a \cdot (m \cdot n + m) = a \cdot (m \cdot n) + a \cdot m \\ &= f((a,m),n) + h((a,m),n) \\ g((a,m),sn) &= (a \cdot m) \cdot (n+1) = (a \cdot m) \cdot n + a \cdot m \\ &= g((a,m),n) + h((a,m),n). \end{aligned}$$

By V_4 we get

$$f((a,m),n) = g((a,m),n) \text{ i. e.}$$
$$a \cdot (m \cdot n) = (a \cdot m) \cdot n$$

q. e. d.

Minus distributivity theorem: In free–variable arithmetics *multiplication distributes* over *truncated subtraction:*

$$a \cdot (m \smallsetminus n) = a \cdot m \smallsetminus a \cdot n$$

Proof by V_4 as follows.

$$f((a,m),n) := a \cdot (m \smallsetminus n)$$
$$g((a,m),n) := a \cdot m \smallsetminus a \cdot n$$

Anchoring

$$f((a,m),0) := a \cdot (m \smallsetminus 0) = a \cdot m$$
$$= a \cdot m \smallsetminus a \cdot 0 = g((a,m),0)$$

V_4 *progress* $h((a,m),n) := 0$

$$V_4 \quad \frac{\begin{array}{l} f((a,m),0) = g((a,m),0) : A \times \mathbb{N} \to \mathbb{N} \\ f((a,m),sn) = f((a,m),n) + 0 : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \\ g((a,m),sn) = g((a,m),n) + 0 : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \end{array}}{\begin{array}{l} f((a,m),n) = g((a,m),n) : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N} \\ \text{i. e. } a \cdot (m \smallsetminus n) = a \cdot m \smallsetminus a \cdot n \\ A \times (\mathbb{N} \times \mathbb{N}) \to \mathbb{N} \end{array}}$$

q. e. d.

Proposition: Addition and multiplication in free-variable arithmetics are weakly monotonous i. e.

$$\begin{aligned} m \le n &\implies m \smallsetminus n = 0 \\ &\implies (a+m) \smallsetminus (a+n) = 0 \text{ by absorption law for } \smallsetminus \\ &\implies a+m \ \le\ a+n \end{aligned}$$

$$\begin{aligned} m \le n &\implies m \smallsetminus n = 0 \\ &\implies (a \cdot m) \smallsetminus (a \cdot n) = a \cdot (m \smallsetminus n) = 0 \\ &\implies a \cdot m \ \le\ a \cdot n \end{aligned}$$

where *protoboolean* ***implication*** is defined as the p. r. predicate

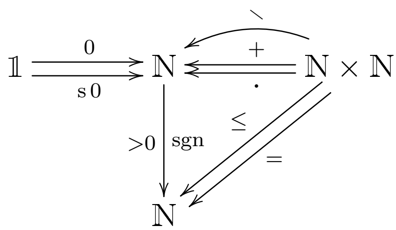
$$[a \implies b] \ =_{\mathrm{def}} \ [a \leq b] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

cf. chapter on *arithmetical logic* below **q. e. d.**

Putting things together, we obtain

3.4 Structure theorem for the NNO

- $\mathbb{N}$ admits the structure



  of a *unitary commutative semiring with zero,* combined with

- a foundational important additional algebraic structure namely *truncated subtraction* $m \smallsetminus n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with its *simplification properties,* and such that multiplication *distributes* over this kind of subtraction;

- linear *order* $[m \leq n] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as a reflexive and transitive *predicate* – this order is p. r. *decidable;*

- $\max(a, b) \ =_{\mathrm{def}} \ a + (b \smallsetminus a) = b + (a \smallsetminus b) = \max(b, a) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is in fact the *maximum* with respect to the order

  $[a \leq b] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$

Furthermore we have

- fundamental equality *predicate*

$$[m = n] \ =_{\text{by def}} \ [m \leq n] \wedge [m \geq n] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

which is an *equivalence predicate,* and which makes up a *trichotomy* with strict order

$$[m < n] \ =_{\text{def}} \ \text{sgn}(n \smallsetminus m) = [m \leq n] \wedge \neg[m = n] : \\ \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Proof of the latter assertion as **exercise.**

- **Algebra Combined with Order:** As expected, addition is strongly monotonic in both arguments, multiplication is strongly monotonic for both arguments strictly greater than zero, and truncated subtraction is weakly monotonic in its first and weakly antitonic in its second argument.

Proofs as **exercises.**

3.5 Exponentiation and faculty

- **exponentiation**

NNO exponentiation $\exp(a, n) = a^n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is **defined** (iteratively) p. r. as follows:

$$a^0 = \exp(a, 0) = 1 : A = \mathbb{N} \xrightarrow{\Pi} \mathbb{1} \xrightarrow{0} \mathbb{N} = B$$

$$a^{\text{s}\,n} = a^n \cdot a = \exp(a, n) \cdot a :$$

$$(A \times \mathbb{N}) \times B = (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{\exp \times \ell\,\ell} \mathbb{N} \times \mathbb{N} \xrightarrow{\cdot} \mathbb{N} = B$$

- **super exponentiation**

  super exponentiation $\mathrm{sexp}(a, n) = a^{\uparrow n} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is **defined** iteratively p. r. as follows:

$$a^{\uparrow 0} = \mathrm{sexp}(a, 0) = a^0 = 1 : A = \mathbb{N} \xrightarrow{\Pi} \mathbb{1} \xrightarrow{0} \mathbb{N} = B$$
$$a^{\uparrow \mathrm{s}\, n} = a^{a \uparrow n} = a^{(a \uparrow n)} = \exp(a, \mathrm{sexp(a, n)}) :$$
$$(A \times \mathbb{N}) \times B$$
$$= (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{\ell} \mathbb{N} \times \mathbb{N} \xrightarrow{(\ell, \mathrm{sexp})} \mathbb{N} \times \mathbb{N} \xrightarrow{\exp} \mathbb{N} = B$$

- **faculty** $\mathrm{fac} = \mathrm{fac}(n) : \mathbb{N} \to \mathbb{N}$ is **defined** by full schema as follows:

$$0! = \mathrm{fac}(0) = 1 : A = N \xrightarrow{\Pi} \mathbb{1} \xrightarrow{1} \mathbb{N} = B$$
$$(n + 1)! = \mathrm{fac}(\mathrm{s}\, n) = n! \cdot (n + 1) = h((a, n), \mathrm{fac}(a, n)) :$$
$$(A \times \mathbb{N}) \times B \to B \text{ with}$$
$$h = h((a, n), b) = (n + 1) \cdot b :$$
$$(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{(\mathrm{s\,r}) \times \mathbb{N}} \mathbb{N} \times \mathbb{N} \xrightarrow{\cdot} \mathbb{N}$$

  We have here an example where step function of full schema depends not only from previous value b but also from *recursion parameter* n. [5]

- **Binomial coefficients**

[5] an example asked for by K. Polthier

$$\text{(choose)}\quad \frac{\begin{array}{l} g(n) = 1\,\Pi_{\mathbb{N}}(n) : \mathbb{N} \to \mathbb{1} \to \mathbb{Q} \\ h = h((n,k),b) = b \cdot \frac{n-k}{1+k} : (\mathbb{N} \times \mathbb{N}) \times \mathbb{Q} \to \mathbb{Q} \quad \text{(step)} \end{array}}{\begin{array}{l} \texttt{function } \binom{n}{k} = \mathrm{pr}[g,h](n,k) : (\mathbb{N} \times \mathbb{N}) \to \mathbb{Q}, \\ \binom{n}{0} = 1 : \mathbb{N} \to \mathbb{Q} \quad \text{(init)} \\ \binom{n}{k+1} = \binom{n}{k} \cdot \frac{n-k}{1+k} : \mathbb{N} \times \mathbb{N} \to \mathbb{Q} \end{array}}$$

This is an example again where the recursion step depends not only on the actual value of the recursive function to be constructed, but also from the actual value of the *recursion parameter,* here $k \in \mathbb{N}$.

Exercise

- **show** $\binom{m}{n} \in \mathbb{N}$
- **show** $\binom{m}{n} = \frac{n!}{k!(n-k!)}$
- **show** the **bimomial theorem**
  $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k : (\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \to \mathbb{N}$

Chapter 4

Predicate abstraction

We extend the fundamental theory **PR** of primitive recursion *definitionally* by abstraction (sub)objects – *sets* – $\{A : \chi\} = \{a \in A : \chi(a)\}$ for p. r. *predicates* $\chi = \chi : A \to \mathbb{N}$, $a \in A$ a *bound variable.*

We get an (embedding) extension of **PR** into a constructive "set" theory **PRa** with *subsets* of cartesian powers of NNO $\mathbb{N}$. The extended primitive recursive theory gets all of the expected properties, see the **structure theorem** for theory **PRa** below, theory of *primitive recursion with scheme of predicate abstraction.*

4.1 Extension by predicate abstraction

We discuss a p. r. **abstraction scheme** as a definitional extension of **PR** into theory **PRa** of *p. r. decidable sets and p. r. maps inbetween,* decidable subsets of the objects of **PR**. The objects of **PR** are up to isomorphism

$$\mathbb{1},\ \mathbb{N}^1 \ =_{\text{def}}\ \mathbb{N},\ \mathbb{N}^{\boldsymbol{m}+1}\ =_{\text{def}}\ (\mathbb{N}^{\boldsymbol{m}} \times \mathbb{N})$$

Here – and always below – $\boldsymbol{m} \in \mathbf{PR}(\mathbb{1}, \mathbb{N})$ is a free metavariable , over the (natural) **number**s.

The extension **PRa** is given by adding schemes $(\text{Ext}_{\mathbf{Obj}})$, $(\text{Ext}_{\mathbf{Map}})$, and $(\text{Ext}_{=})$ below. Together they correspond to the *scheme of abstraction* in **set** theory, and they are referred below as *schemes* of *p. r. abstraction.*

Our first predicate-into-set *abstraction* scheme is

$$\chi : A \to \mathbb{N} \text{ a } \mathbf{PR}\text{-predicate:}$$

$$\text{sgn} \circ \chi = \chi : A \to \mathbb{N} \to \mathbb{N},$$

$$A \xrightarrow{\chi} \mathbb{N} \xrightarrow{\text{sgn}} \mathbb{N}, \quad \text{sgn} \circ \chi = \chi : A \to \mathbb{N}$$

$$(\text{Ext}_{\mathbf{Obj}}) \quad \overline{}$$

$$\{A : \chi\} \text{ set (of emerging theory } \mathbf{PRa})$$

Subset $\{A : \chi\} \subseteq A \cong \mathbb{N}^{\boldsymbol{n}}$ may be written alternatively, with *bound* variable a, as

$$\{A : \chi\} = \{a \in A : \chi(a)\}$$

Decidability remark: Object $A \cong \mathbb{N}^{\boldsymbol{m}}$ is countable, and therefore you can enumerate (the "elements" of) $\{a \in A : \chi(a)\}$ by enumeration of A and taking out of this enumeration those $a \in A$ for which $\chi(a) = \text{true}$.

But for the time being you cannot in general decide algorithmically if $\{A : \chi\}$ is empty or finite.

Nevertheless, set $\{A : \chi\}$ is a *legitimate set* in Cantor's sense, since for every thing ("element") "feststeht" – is said – if it belongs to $\{A : \chi\}$, this at least for "things" in the "mother set" $A \cong \mathbb{N}^{\boldsymbol{m}} \cong \mathbb{N}$.

The *maps* of $\mathbf{PRa} = \mathbf{PR} + \text{(abstr)}$ come in by

$$(\mathrm{Ext}_{\mathbf{Map}}) \quad \frac{\begin{array}{l} \{A : \chi\},\ \{B : \varphi\}\ \mathbf{PRa}\text{-sets}, \\ f : A \to B \text{ a } \mathbf{PR}\text{-map}, \\ \mathbf{PR} \vdash \chi(a) \implies \varphi\, f\,(a), \text{ i.e.} \\ [\chi \implies \varphi \circ f] =^{\mathbf{PR}} \mathrm{true}_A : A \xrightarrow{\Pi} \mathbb{1} \xrightarrow{1} \mathbb{N} \end{array}}{f \text{ is a } \mathbf{PRa}\text{-map } f : \{A : \chi\} \to \{B : \varphi\}}$$

In particular, if for predicates $\chi', \chi'' : A \to \mathbb{N}$

$$\mathbf{PR} \vdash [\chi'(a) \implies \chi''(a)] : A \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

then $\mathrm{id}_A : \{A : \chi'\} \to \{A : \chi''\}$ in $\mathbf{PRa}$ is called an *inclusion,* and written $\subseteq : A' = \{A : \chi'\} \to A'' = \{A : \chi''\}$ or $A' \subseteq A''$.

Note: For predicate (terms!) $\chi, \varphi : A \to \mathbb{N}$ such that $\mathbf{PR} \vdash \chi = \varphi : A \to \mathbb{N}$ (logically: such that $\mathbf{PR} \vdash [\chi \iff \varphi]$), we have

$$\{A : \chi\} \subseteq \{A : \varphi\} \text{ and } \{A : \varphi\} \subseteq \{A : \chi\}$$

but – in general – not *equality of sets.* We only get in this case

$$\mathrm{id}_A : \{A : \chi\} \xrightarrow{\cong} \{A : \varphi\}$$

as a **PRa** *isomorphism.*

So *inclusion* $\mathrm{id}_A : \{A : \chi'\} \subseteq \{A : \chi''\}$ above is formally only an inclusion up to isomorphism.

A posteriori, we introduce[1] the $0, 1$ *truth algebra* $\mathbb{2}$ as

$$\mathbb{2} \ =_{\mathrm{def}} \ \{0, 1\} \ =_{\mathrm{by\,def}} \ \{\alpha \in \mathbb{N} : \alpha = 0 \vee \alpha = \mathrm{s}\,0\}$$

with proto boolean operations on $\mathbb{N}$ restricting – in codomain and domain – to *boolean* operations on $\mathbb{2}$, $\mathbb{2} \times \mathbb{2}$ by definition below of cartesian product of sets within **PRa**.

PRa maps with common **PRa** domain and codomain are considered equal, if their values are equal on their defining *domain predicate.* This is expressed by the scheme

$$(\mathrm{Ext}_=) \quad \frac{f,\ g : \{A : \chi\} \to \{B : \varphi\}\ \textbf{PRa}\text{ maps}, \quad \textbf{PR} \vdash \chi(a) \implies [f(a) =_B g(a)]}{f = g : \{A : \chi\} \to \{B : \varphi\},}$$

explicitly:

$f =^{\mathbf{PRa}} g : \{A : \chi\} \to \{B : \varphi\}$, also noted
$((\chi, f), \varphi) =^{\mathbf{PRa}} ((\chi, g), \varphi)$ or
$\mathbf{PRa} \vdash f = g : \{A : \chi\} \to \{B : \varphi\}$

[1] following Reiter 1982

4.2 Arithmetical structure theorem

for theory **PRa**, of *primitive recursion with predicate abstraction:* [2]

PRa is a cartesian p. r. theory. Theory **PR** is cartesian p. r. embedded. Theory **PRa** has (universal) extensions of all of its predicates and a (preliminary) two-valued truth set as codomain of these predicates. In detail:

(i) **PRa** inherits associative map composition and identities from **PR**

(ii) **PRa** has **PR** fully embedded by

$$\langle f : A \to B \rangle \mapsto \langle f : \{A : \mathrm{true}_A\} \to \{B : \mathrm{true}_B\} \rangle$$

Such A are called *objects,* $\{A : \chi\} = \{a \in A : \chi(a)\}$ *sets.*

In less formal context we abbreviate embedded object $\{A : \mathrm{true}_A\}$ by A.

(iii) **PRa** has cartesian product

$$\{A : \chi\} \times \{B : \varphi\} \;=_{\mathrm{def}}\; \{A \times B : \chi \wedge \varphi : A \times B \to \mathbb{N} \times \mathbb{N} \xrightarrow{\wedge} \mathbb{N}\}$$

with projections and universal property inherited from **PR**.

(iv) The embedding **PR** $\longrightarrow$ **PRa** is a *cartesian functor:* it preserves products and their cartesian universal property with respect to the projections inherited from **PR**.

---

[2] cf. Reiter 1980

(v) **PRa** has *extensions* of its predicates, namely

$$\mathrm{Ext}\,[\varphi : \{A : \chi\} \to \mathbb{2}] \;=_{\mathrm{def}}\; \{A : \chi \wedge \varphi\} \subseteq \{A : \chi\}$$

characterised as (**PRa**)-*equalisers*

$$\mathrm{Equ}\,(\chi \,\wedge\, \varphi,\; \mathrm{true}_A) : \{A : \chi\} \to \mathbb{2}$$

[mutatis mutandis: within theory **PRa** we identify predicates $\chi = \mathrm{sgn} \circ \chi : A \to \mathbb{N} \to \mathbb{N}$ with maps $\chi : A \to \mathbb{2} \;=_{\text{by def}}\; \{0, 1\}$]

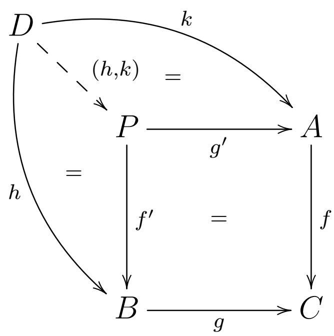
PRa has all *equalisers,* namely equalisers

$$\mathrm{Equ}[f, g] \;=_{\mathrm{def}}\; \{a \in A : \chi(a) \,\wedge\, f(a) =_B g(a)\}$$
$$= \mathrm{Ext}[=_B \circ\, (f, g) : A' \to B' \times B' \xrightarrow{=} \mathbb{2}]$$

of arbitrary **PRa** map pairs $f, g : A' = \{A : \chi\} \to B' = \{B : \varphi\}$

and hence all finite projective *limits,* in particular *pullbacks* which we will rely on later, and kernel pairs.

A *pullback,* of a map $f : A \to C$ *along* a map $g : B \to C$, also of g along f, is the square in



[We prefer the "set theoretical" way to construct first extension sets out of the cartesian category structure of fundamental theory **PR**, and we construct equalisers and the other finite limits on this basis. Another possibility – ROMÀN 1989 – is to add equalisers as *undefined notion* and to construct limits directly from these and cartesian product.]

The embedding preserves such limits as far as available already in **PR**. Equality *predicate* extends to cartesian products componentwise as

$$[(a,b) =_{A\times B} (a',b')] \ =_{\text{def}} \ [a =_A a'] \wedge [b =_B b'] : (A\times B)^2 \to \mathbb{2},$$

and to (predicative) subsets $\{A : \chi\}$ by restriction.

(vi) Arithmetical structure extends from **PR** to **PRa** i. e. **PRa** admits the *iteration* scheme as well as FREYD's *uniqueness* scheme: the iterated

$$f^{\S} : \{A : \chi\} \times \{\mathbb{N} : \text{true}_{\mathbb{N}}\} \to \{A : \chi\}$$

is just the *restricted* **PR**-map $f^{\S} : A \times \mathbb{N} \to A$, the uniqueness schemes follow from definition of $=^{\mathbf{PRa}}$ via **PRa**'s scheme $(\text{Ext}_=)$ above.

(vii) In particular our *equality predicate* $=_A : A^2 \to \mathbb{N}$ restricted to subsets $A' = \{A : \chi\} \subseteq A$ inherits all of the properties of equality on $\mathbb{N}$ and on the other fundamental objects.

(viii) **PRa** has (binary) *sums* (coproducts).

(ix) **PRa** has *coequalisers* of kernel pairs, of *equivalence predicates.*

(x) **Countability:** Each fundamental object A i. e. A a finite power of $\mathbb{N} \equiv \{\mathbb{N} : \text{true}_{\mathbb{N}}\}$, admits by CANTOR's isomorphism

$$\text{ct} = \text{ct}_{\mathbb{N}\times\mathbb{N}}(n) : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N}$$

a retractive count $\mathrm{ct}_A(n) : \mathbb{N} \to A$.

Problem: For which predicates $\chi : A \to \mathbb{2}$ (A fundamental) does theory **PRa** admit a retractive *count*

$$\mathrm{ct} = \mathrm{ct}_{\{A:\chi\}}(n) : \mathbb{N} \equiv \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\} \to \{A : \chi\}?$$

The difficulty is seen already in case $\emptyset_A \; =_{\text{by def}} \; \{A : \mathrm{false}_A\}$. A *sufficient condition* is $\{A : \chi\}$ to come with a *point,* $a_0 : \mathbb{1} \to \{A : \chi\}$, preferably $0_A : \mathbb{1} \to \{A : \chi\}$.

In this case: $\mathbf{PR} \vdash \chi(0_A)$ – we **call** set $\{A : \chi\}$ *zero-pointed.* If not, and point needed, we replace $\{A : \chi\}$ by subset of **PR** object A augmented by 0_A of A.

4.3 Proof of structure theorem

(i) For $f : \{A : \chi\} \to \{B : \varphi\},\; g : \{B : \varphi\} \to \{C : \psi\}$ in **PRa** we have

$$\mathbf{PR} \vdash \chi \implies \varphi\, f \implies \psi\, g\, f : A \to \mathbb{N}$$

whence $g \circ f : \{A : \chi\} \to \{C : \psi\}$ in **PRa**, asociativity of composition and neutrality of identitities are inherited from **PR**.

Compatibility of composition with $=^{\mathbf{PRa}}$: For

$$f =^{\mathbf{PRa}} f' : \{A : \chi\} \to \{B : \varphi\},$$
$$g =^{\mathbf{PRa}} g' : \{B : \varphi\} \to \{C : \psi\} \text{ in } \mathbf{PRa}$$

we show

$$g \circ f =^{\mathbf{PRa}} g \circ f' : \{A : \chi\} \to \{C : \psi\},$$
$$g' \circ f =^{\mathbf{PRa}} g \circ f : \{A : \chi\} \to \{C : \psi\} :$$

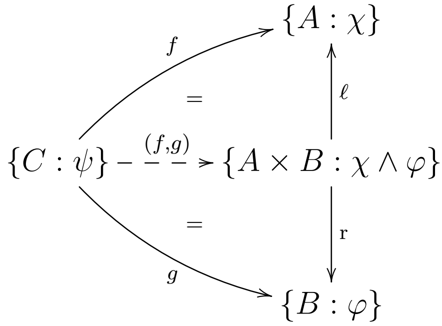
$$\frac{\mathbf{PR} \vdash \chi(a) \implies f(a) =_B f'(a) : A \to \mathbb{N}}{\mathbf{PR} \vdash \chi(a) \implies g\,f(a) =_C g\,f'(a) : A \to \mathbb{N},}$$

$$\frac{\begin{array}{l}\mathbf{PR} \vdash \chi(a) \implies \varphi\,f(a) : A \to \mathbb{N} \\ \mathbf{PR} \vdash \varphi(b) \implies g(b) =_C g'(b) : A \to \mathbb{N}\end{array}}{\mathbf{PR} \vdash \chi(a) \implies g\,f(a) =_C g'\,f(a) : A \to \mathbb{N}}$$

both by LEIBNIZ substitutivities with respect to = q. e. d.

(ii) The embedding assertion is obvious.

(iii) Assertion on the cartesian product: Consider induced-into-product DIAGRAM



$$\mathbf{PR} \vdash \psi(c) \implies \chi\, f(c) \wedge \varphi\, g(c)$$
$$\iff [\chi \wedge \varphi]\,(f, g)(c) \text{ q. e. d.}$$

(iv) Cartesian embedding assertion is obvious by construction of **PRa** over **PR**.

(v) Extensions of predicates etc: Proof is left to the reader as categorical **exercise** on construction of all finite limits out of binary products and *extensions of predicates,* in particular on construction of pullbacks.

(vi) Proof of critical iteration assertion: consider an endomorphism $f : \{A : \chi\} \to \{A : \chi\}$, so

$$\mathbf{PR} \vdash \chi \implies \chi\, f :$$
$$A \xrightarrow{(\chi, \chi\, f)} \mathbb{N} \times \mathbb{N} \xrightarrow{\implies} \mathbb{N}.$$

The iterated is the restriction of **PR** iterated $f^{\S} : A \times \mathbb{N} \to A$. Is it a **PRa** map $f^{\S} : \{A : \chi\} \times \{\mathbb{N} : \text{true}\} \to \{A : \chi\}$?

Apply Peano Induction P5 (within **PR**) to predicate

$\varphi = \varphi(a, n) \;=_{\text{def}}\; [\chi(a) \implies \chi\, f^n(a)] : A \times \mathbb{N} \to \mathbb{N} :$

$$\varphi(a, 0) = \text{true by anchoring } f^{\S}$$
$$[\varphi(a, n) \implies \varphi(a, \mathrm{s}\, n)]$$
$$= [[\chi(a) \implies \chi\, f^{\S}(a, n)] \implies [\chi(a) \implies \chi\, f^{\S}(a, \mathrm{s}\, n)]]$$
$$= [[\chi(a) \implies \chi\, f^{\S}(a, n)] \implies [\chi(a) \implies \chi\, f\, f^{\S}(a, n)]]$$
$$= \text{true}$$

the latter by $f : \{A : \chi\} \to \{A : \chi\}$ a **PRa** map:

$$\mathbf{PR} \vdash \chi\, f^{\S}(a, n) \implies \chi\, f\, f^{\S}(a, n)$$

and by boolean tautology.

Peano Induction then gives $\varphi = \varphi(a, n) = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$ i. e. $f^{\S} : \{A : \chi\} \times \{\mathbb{N} : \mathrm{true}\} \to \{A : \chi\}$ is in fact a **PRa** map.

Compatibility of iteration with **PRa**'s equality: for endo maps $f =^{\mathbf{PRa}} g : \{A : \chi\} \to \{A : \chi\}$ i. e.

$$\mathbf{PR} \vdash \chi(a) \implies f(a) = g(a) : A \to \mathbb{N}$$

We show

$$\mathbf{PR} \vdash \chi(a) \implies f^{\S}(a, n) = g^{\S}(a, n) : A \times \mathbb{N} \to \mathbb{N}$$

by Peano Induction on

$$\varphi(a, n) = [\chi(a) \implies f^{\S}(a, n) = g^{\S}(a, n)]$$

as follows:

anchor $\varphi(a, 0) = \mathrm{true}_A$ is trivial. Step is an analogon to step above:

$$\begin{aligned}
&[\varphi(a, n) \implies \varphi(a, \mathrm{s}\, n)] \\
&= [[\chi(a) \implies f^{\S}(a, n) = g^{\S}(a, n)] \\
&\qquad \implies [\chi(a) \implies f^{\S}(a, \mathrm{s}\, n) = g^{\S}(a, \mathrm{s}\, n)]] \\
&= [[\chi(a) \implies f^{\S}(a, n) = g^{\S}(a, n)] \\
&\qquad \implies [\chi(a) \implies f\, f^{\S}(a, n) = g\, g^{\S}(a, n)]] \\
&= \mathrm{true}
\end{aligned}$$

by $f \overset{\mathbf{PRa}}{=} g : \{A : \chi\} \to \{A : \chi\}$.

Peano Induction then gives $\varphi = \varphi(a, n) = \mathrm{true} : A \times \mathbb{N} \to \mathbb{N}$ i. e. in fact

$$f^{\S} \overset{\mathbf{PRa}}{=} g^{\S} : \{A : \chi\} \times \{\mathbb{N} : \mathrm{true}\} \to \{A : \chi\} \quad \text{q. e. d.}$$

(vii) restriction of equality predicates is obvious.

(viii) we have constructed in section on *Hilbert's infinite hotel* the sum $\mathbb{1} + \mathbb{N}$ just as $\mathbb{1} + \mathbb{N} \cong \mathbb{N}$ and reveal set $\mathbb{2} = \{0, 1\}$ in section below, on *2-valued set,* as coproduct/sum $\mathbb{2} \cong \mathbb{1} + \mathbb{1}$.

Define $\mathbb{N} + \mathbb{N} := \mathbb{N}$ with coproduct *injections*

$$\begin{aligned} \iota = \iota(n) &=_{\mathrm{def}} 2\,n : \mathbb{N} \to \mathbb{N} \text{ and} \\ \kappa = \kappa(n) &=_{\mathrm{def}} 2\,n + 1 \end{aligned}$$

$\mathbb{N}$ is the disjoint *union* of its even and its odd numbers. This gives the assertion since by Cantor isomorphy any (pointed) set of **PRa** is isomorphic to $\mathbb{N}$ or a predicative subset of $\mathbb{N}$.

(ix) **PRa** has quotients of equivalence pairs (and hence of kernel pairs) in form $A/\rho \;=_{\mathrm{def}}\; \{a \in A : a =_A \bar{a}\}$ where $\bar{a} \;=_{\mathrm{def}}\; \min\{\tilde{a} \leq_A a : \tilde{a}\,\rho\,a\}$ is the minimal representant of the equivalence class of a, minimal with respect to the linear well order $\leq_A : A \times A \to \mathbb{2}$ on A which is given by Cantor's isomorphism $\mathrm{ct}_A : \mathbb{N} \xrightarrow{\cong} A$, A a nested binary power of $\mathbb{N}$, and its codomain restriction to subsets $A' = \{A : \chi\}$ in **PRa**. In formal terms:

PRa admits the following scheme of forming quotients by equivalence predicates:

$$\rho : \{A : \chi\} \times \{A : \chi\} \to \mathbb{2}$$

an equivalence predicate in **PRa**

(QuotPred) ———————————————————

$$[a]_\rho \ =_{\text{def}} \ \min\{\tilde{a} \leq_A a : \tilde{a} \, \rho \, a\} : A \to A$$

$$\{A : \chi\}/\rho \ =_{\text{def}} \ \{a \in \{A : \chi\} : a =_A [a]_\rho\}$$

together with *quotient map*

$$\text{nat}_\rho = \text{nat}_\rho(a) \ =_{\text{def}} \ [a]_\rho : \{A : \chi\} \to \{A : \chi\}/\rho$$

$\text{nat}_\rho : \{A : \chi\} \to \{A : \chi\}/\rho$ has the universal properties of a coequaliser of **PRa** pair

$$\{(a', a'') \in \{A : \chi\}^2 : a' \, \rho \, a''\} \xrightarrow{\subseteq} A \times A \overset{\ell}{\underset{r}{\rightrightarrows}} A$$

[$[a]_\rho : \{A : \chi\} \to \{A : \chi\}$ is the *minimal representant* of the ρ equivalence class of a.]

Map pair above is the canonical *kernel pair* $\text{KP}[\text{nat}_\rho]$ of quotient $\text{nat}_\rho : \{A : \chi\} \to \{A : \chi\}/\rho$ **q. e. d.**

Chapter 5

Arithmetical logic

NNO $\mathbb{N}$ with truth value false $= 0$, and all successors working as truth value true, make out of $\mathbb{N}$ sort of boolean truth set allowing for a *protoboolean* logic and predicate calculus.

In the framework **PRa** of primitive recursion with predicate-into-subset abstraction we get the usual 2-element boolean algebra $\mathbb{2} = \{0, 1\} \subset \mathbb{N} \equiv \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}$ and the usual boolean logic and free-variables predicate calculus in categorical form.[1]

Set $\mathbb{2} = \{0, 1\}$ turns out to be a *sum/coproduct* $\mathbb{2} \cong \mathbb{1} + \mathbb{1}$ of the terminal object $\mathbb{1}$ with itself. The proof is by the full schema of primitive recursion.

The definition of the boolean operations on $\mathbb{2}$ is as usual out of *negation* $\neg\,\alpha = 1 \smallsetminus \alpha : \mathbb{2} \to \mathbb{2}$ and *conjunction* $\alpha \wedge \beta = \alpha \boldsymbol{\cdot} \beta : \mathbb{2} \times \mathbb{2} \to \mathbb{2}$, and gives $\mathbb{2}$ the structure of a *boolean algebra.*

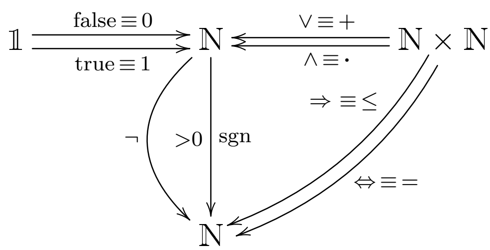
The free-variable form of the Peano axioms is shown as a theorem of

[1] This development is taken from Reiter 1982.

the theory **PR** of primitive recursion. Same for Leibniz' substitutivity into predicative equality.

5.1 Protoboolean Structure on the NNO

In the framework **GA** of Goodstein Arithmetic and primitive recursion **PR** we introduce on NNO $\mathbb{N}$ the following *proto boolean* structure:



[Successors are all viewed logically to represent truth value true.]

Object $\mathbb{N}$ admits definition of (boolean) "logical functions" by *truth tables* as does set $\mathbb{2}$ classically and – below – in theory $\mathbf{PRa} = \mathbf{PR} +$ (abstr) of primitive recursion with predicate abstraction.

Definition (recall): A **PR** map $\chi : A \to \mathbb{N}$ to be a *predicate* (on A) is to mean

$$\mathbf{PR} \vdash \ \chi = \mathrm{sgn} \circ \chi : A \to \mathbb{N} \to \mathbb{N} \text{ i. e.}$$
$$\mathbf{PR} \vdash \ [[\chi(a) = 0] \vee [\chi(a) = 1]] : A \to \mathbb{N} \text{ i. e.}$$
$$\mathbf{PRa} \vdash \ \chi = \chi : A \equiv \{A : \mathrm{true}_A\} \to \mathbb{2} \xrightarrow{\subseteq} \mathbb{N} \equiv \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}$$

PRa set $\mathbb{2}$ **defined** by

$$\mathbb{2} = \{0, 1\} \ =_{\text{by def}} \ \{\alpha \in \mathbb{N} : [\alpha = 0] \vee [\alpha = 1 = \mathrm{s}\,0]\}$$

$\mathbb{2}$ works as *coproduct/sum*

$\mathbb{2} \cong \mathbb{1} + \mathbb{1}$ with coproduct *injections*

$\mathbb{1} \xrightarrow{0} \mathbb{2} \xleftarrow{1} \mathbb{1}$

see next section.

Discreteness question: Do we have for $\chi : A \to \mathbb{N}$

$$\mathbf{PR} \vdash \ [\chi(a) \leq 1] = [[\chi(a) = 0] \vee [\chi(a) = 1]] : A \to \mathbb{N}?$$

We do not rely on this here.

5.2 2-valued set as coproduct/sum

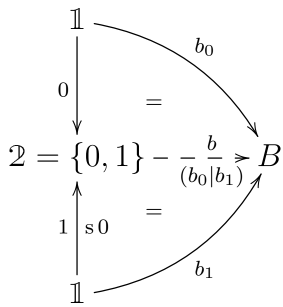
Within theory $\mathbf{PRa} = \mathbf{PR} +$ (abstr) set $\mathbb{2}$ comes as a *sum*
$\mathbb{1} \xrightarrow{0} \mathbb{2} \cong (\mathbb{1} + \mathbb{1}) \xleftarrow{1} \mathbb{1}$ over which cartesian product
$A \times \_$ *distributes:*

Coproduct Lemma for set $\mathbb{2}$

- Set $\mathbb{2} = \{0, 1\}$ inherits coproduct property

  $\mathbb{2} \cong \mathbb{1} + \mathbb{1}$ from $\mathbb{N} \cong \mathbb{1} + \mathbb{N}$:

For $b_0, b_1 : \mathbb{1} \to B$ in **PRa**, DIAGRAM:



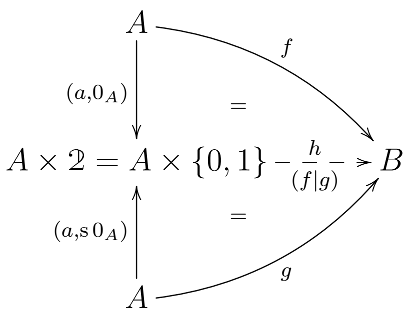
PR map

$$b =_{\text{def}} \text{pr}[b_0, b_1 \circ \text{r}_{\mathbb{1},\mathbb{N}} \circ (\Pi, \text{id})] : \mathbb{N} \to \mathbb{1} \times \mathbb{N} \to B$$

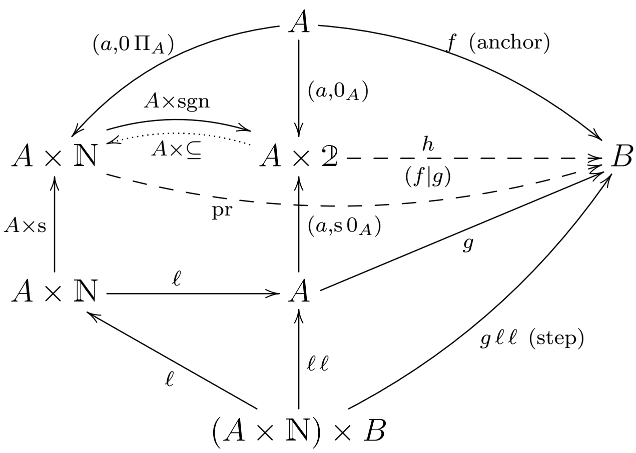
does the job, uniquely with respect to equality of **PRa**, **since**

- with general parameter set A in **PRa** in place of $\mathbb{1}$:

  $A \times \mathbb{2} \cong A + A$, DIAGRAM:



embedded in *full p. r.* (commuting) DIAGRAM



PRa map

$$\mathrm{pr} = \mathrm{pr}[f, \ell\,\ell\,g] : A \times \mathbb{N} \to B$$

is – full schema of primitive recursion – the unique map such that

$$\mathrm{pr}(a, 0) = f(a) \text{ as well as}$$
$$\mathrm{pr}(a, s\,n) = g\,\ell\,\ell((a, n), \mathrm{pr}(a, n)) = g(a)$$

whence – $A\times \subseteq : A\times 2 \to A\times\mathbb{N}$ having $A\times\mathrm{sgn} : A\times\mathbb{N} \to A\times 2$ as a retraction – $h = \mathrm{pr}\circ(A\times \subseteq) : A \times 2 \to A \times \mathbb{N} \to B$ is the unique commutative fill-in $(f|g) : A\times 2 \to B$ into the coproduct diagram **q. e. d.**

5.3 Boolean operations

Within theory **PRa** "the" boolean operations can be defined on $\mathbb{2} = \{0, 1\}$ by heritage from the arithmetical structure of NNO as follows:

- *truth values* $\mathrm{false} := 0,\ \mathrm{true} := 1 = \mathrm{s}\,0 : \mathbb{1} \to \mathbb{2} \subset \mathbb{N}$
- *negation*

  $\neg = \neg\,\alpha\ =_{\mathrm{def}}\ 1 \smallsetminus \alpha : \mathbb{2} \xrightarrow{(\subseteq,1)} \mathbb{N} \times \mathbb{N} \xrightarrow{\smallsetminus} \mathbb{N} \xrightarrow{\mathrm{sgn}} \mathbb{2}$

  where *signum* p. r. defined by

  $\mathrm{sgn}\,0 = 0,\ \mathrm{sgn}(\mathrm{s}\,n) = 1 = \mathrm{s}\,0$ i. e.

  $\mathrm{sgn}\,n = [n > 0] : \mathbb{N} \to \mathbb{N}$ p. r. decides on *positiveness.*

- *conjunction*

  $[\alpha \wedge \beta]\ =_{\mathrm{def}}\ \mathrm{sgn}(\alpha \cdot \beta) : \mathbb{2} \times \mathbb{2} \xrightarrow{\subseteq \times \subseteq} \mathbb{N} \times \mathbb{N} \to \mathbb{N} \xrightarrow{\mathrm{sgn}} \mathbb{2};$

- *disjunction*

  $[\alpha \vee \beta]\ =_{\mathrm{def}}\ \mathrm{sgn}(\alpha + \beta) : \mathbb{2} \times \mathbb{2} \xrightarrow{\subseteq \times \subseteq} \mathbb{N} \times \mathbb{N} \to \mathbb{N} \xrightarrow{\mathrm{sgn}} \mathbb{2};$

  as well as

- ***implication***

  $[\alpha \Rightarrow \beta] := [\alpha \le \beta] : \mathbb{2} \times \mathbb{2} \xrightarrow{\subseteq \times \subseteq} \mathbb{N} \times \mathbb{N} \to \mathbb{2};$

- *biimplication, logical equivalence*

$$[\alpha \iff \beta] := [\alpha \Rightarrow \beta] \wedge [\beta \Rightarrow \alpha] = [\alpha \le \beta] \wedge [\beta \le \alpha]$$
$$= [\alpha = \beta] : \mathbb{2} \times \mathbb{2} \xrightarrow{\subseteq \times \subseteq} \mathbb{N} \times \mathbb{N} \to \mathbb{2}$$

the latter predicate equation by antisymmetry of (weak) order predicative equality on $\mathbb{N}$.

- (relative) *complement* "α *but not* β"

  $[\alpha \smallsetminus \beta] = [\alpha \wedge \neg\beta]$:
  $\mathbb{2} \times \mathbb{2} \xrightarrow{\subseteq \times \subseteq} \mathbb{N} \times \mathbb{N} \to \mathbb{2}$

5.4 Formal extension by truth algebra

In Computer Science some consider it an advantage to separate the type of (boolean) *truth values* – BOOLEAN – from the type of natural numbers – UNSIGNED INTEGER, for the sake of (relative) context independence.

Let us **category equivalently extend** theory **PR** of primitive recursion into a theory $\mathbf{PR2} = \mathbf{PR} + \mathbf{2}$ as follows:

- formally add an object **2** to the **set** $\{\mathbb{1}, \mathbb{N}\}$ of **PR**'s basic objects: *Borrow* this object and its operations (intuitively) from **Logic.**

- add a map $\mathrm{ON} \equiv \top \equiv \mathrm{true} : \mathbb{1} \to \mathbf{2}$ to the basic maps of **PR**

  as well as a map $\smallsetminus : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$

  ['$\alpha \smallsetminus \beta$' is to mean α *but not* β.]

  - **define** $\langle \mathrm{OFF} : \mathbb{1} \to \mathbf{2} \rangle \equiv \boldsymbol{\phi} \equiv \mathrm{false}$
    $=_{\mathrm{def}}\ \mathrm{true} \smallsetminus \mathrm{true} = \smallsetminus \circ\ (\mathrm{true}, \mathrm{true}) : \mathbb{1} \to \mathbf{2} \times \mathbf{2} \to \mathbf{2}$
  - **define** negation $\neg = \langle \neg(\alpha) : \mathbf{2} \to \mathbf{2} \rangle = \mathrm{true} \smallsetminus \alpha$:
    $\mathbf{2} \to \mathbf{2} \times \mathbf{2} \to \mathbf{2}$

– **define** conjunction $\langle\wedge = \alpha\wedge\beta\rangle = \alpha\diagdown\neg\beta = \alpha\diagdown(\text{true}\diagdown\beta)$: $\mathbf{2}\times\mathbf{2}\to\mathbf{2}\times\mathbf{2}\to\mathbf{2}$

- **define** – as usual – the other boolean operations

  out of $\neg$ and $\wedge$, in particular

  $\text{NOR} = \wedge\circ(\neg\times\neg) : \mathbf{2}\times\mathbf{2}\to\mathbf{2}$

  $\vee = \neg\circ\text{NOR} = \neg\circ\wedge\circ(\neg\times\neg) : \mathbf{2}\times\mathbf{2}\to\mathbf{2}$

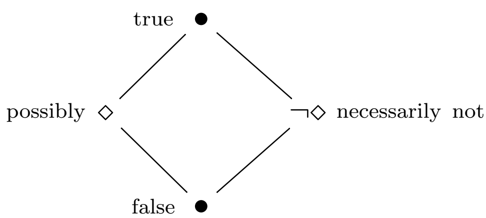
  $\Rightarrow\ =\ \vee\circ(\neg\circ\ell,\text{r}) : \mathbf{2}\times\mathbf{2}\to\mathbf{2}$

  $\Leftarrow\ =\ \vee\circ(\ell,\neg\circ\text{r}) : \mathbf{2}\times\mathbf{2}\to\mathbf{2}$

  $\Longleftrightarrow\ =\ \wedge\circ(\Leftarrow,\Rightarrow) : \mathbf{2}\times\mathbf{2}\to\mathbf{2}$

  logical equality $=_{\mathbf{2}}\ =_{\text{def}}\ \Longleftrightarrow\ : \mathbf{2}\times\mathbf{2}\to\mathbf{2}$

- Up to here, object **2** is introduced just as a *boolean algebra,* **example** for such a boolean algebra is



- For to make object **2** two-valued, **insert** into the generation process for theory **PR2** two additional “undefined” maps:

  a *2-values number interpretation* of **2**,

  $$\text{pret} = \text{pret}(\alpha) : \mathbf{2}\to\mathbb{N}$$

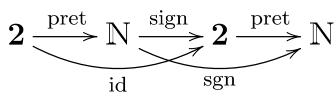
coming with a retractive "inverse", boolean *signum*

$$\mathrm{sign} = \mathrm{sign}(n) : \mathbb{N} \to \mathbf{2}$$

additional ("generic") equations

$$\mathrm{sign} \circ \mathrm{pret} = \mathrm{id}_{\mathbf{2}} : \mathbf{2} \to \mathbb{N} \to \mathbf{2} \qquad (\mathrm{pret}_1)$$
$$\mathrm{pret} \circ \mathrm{sign} = \mathrm{sgn} : \mathbb{N} \to \mathbb{N} \qquad (\mathrm{pret}_2)$$
$$\mathrm{sgn}(n) \;=_{\text{by def}}\; 1 \setminus (1 \setminus n) : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \qquad (\textbf{recall})$$

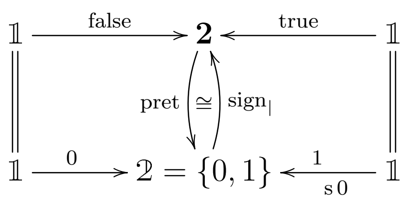
commutative DIAGRAM



2-anchoring Remark: Within theory $\mathbf{S} = \mathbf{PR2} + (\mathrm{abstr})$ below, these two maps restrict to a pair

$$\mathrm{pret} : \mathbf{2} \xrightarrow{\cong} \{0, 1\},$$
$$\mathrm{sign}_| : \{0, 1\} \xrightarrow{\cong} \mathbf{2}$$

of mutual inverse *isomorphisms*, compatible with the pertaining truth values, DIAGRAM



- build the "class" of objects of theory **PR2** by closure of the set $\{\mathbb{1}, \mathbb{N}, \mathbf{2}\}$ of basic objects against (binary) cartesian product;

- build the class of maps of theory **PR2** by closure of the above against *identic maps, terminal maps, left and right projections, composition, induced maps* as well as against *endo map iteration.*

- build the class of equations for theory **PR2** as the class of primitive recursive equations generated over the (additional) equations introduced above – in particular equations (pret_1) and (pret_2).

5.5 Constructive set theory S

The *boole-extended* theory **PR2** – conservative extension of fundamental p. r. theory **PR** – comes with the usual free-variables *boolean logic* and with an "induced" free-variables (boolean) *predicate calculus.*

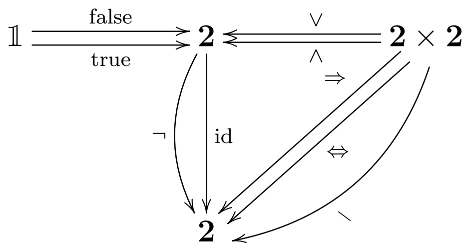
The *protoboolean* structure on NNO $\mathbb{N}$ has been turned above, within theory **PRa** and strengthenings, into a two-valued boolean algebra on set $\mathbb{2}$,

$$\mathbb{2} \ =_{\text{by def}} \ \{0, 1\} \ =_{\text{by def}} \ \{n \in \mathbb{N} : n = 0 \vee n = 1\}$$

and is turned within boolean (fundamental) p. r. theory **PR2** into "the" boolean algebra on object

$$\begin{aligned} &\mathbf{2} = \{\text{false} : \mathbb{1} \to \mathbf{2},\ \text{true} : \mathbb{1} \to \mathbf{2}\} \\ &= \{\alpha \in \mathbf{2} : \alpha = \text{false} \vee \alpha = \text{true}\} \\ &\equiv \{\phi, \top\} \equiv \{\text{OFF}, \text{ON}\} \end{aligned}$$

DIAGRAM for the latter



A **PR** *predicate* on an object A of **PR** has been **defined** as a **PR** map $\chi : A \to \mathbb{N}$ with $\mathrm{sgn} \circ \chi = \chi$.

A **PR2** predicate on an object A of **PR2** is **defined** as a **PR2** map $\chi = \chi(a) : A \to \mathbf{2}$.

Definition: Theory **PR2** of *boolean primitive recursion* has a (conservative, embedding) extension into theory

$$\mathbf{S} \;=_{\mathrm{def}}\; \mathbf{PR2} + (\mathrm{abstr})$$

of *boolean primitive recursion with predicate abstraction,* abstraction of **PR2** predicates $\chi : A \to \mathbf{2}$ into *subsets* $\{A : \chi\}$ – in complete (category-equivalent) parallel to the extension of fundamental theory **PR** of primitive recursion into theory $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ of primitive recursion with predicate abstraction, a **PR** predicate $\chi = \mathrm{sgn} \circ \chi : A \to \mathbb{N}$ giving a subset $\{A : \chi\}$ of A within **PRa**.

Theory **S** is called *p. r. constructive set theory.* Its objects $\{A : \chi\}$ are called *sets.* Its sets of form $\{A : \mathrm{true}_A\}$ are embedded *objects* of theory **PR2**, and are identified with these: $A \equiv \{A : \mathrm{true}_A\}$ for A in **PR2**.

Equivalence Remark: The mutually inverse **S** isomorphisms

$$\mathbf{2} \underset{\mathrm{sign}_|}{\overset{\mathrm{pret}}{\rightleftarrows}} \cong \{0,1\}$$

generate a natural functor equivalence between the Identity **functor** $\mathbf{ID} : \mathbf{S} \longrightarrow \mathbf{S}$ and the **Retraction/Coretraction functor**

$$\mathbf{S} \xrightarrow{\mathbf{pret}} \mathbf{PRa} \xrightarrow{\subseteq} \mathbf{S},$$

the categories **S** and **PRa** are (retractively) **equivalent:**

$$\mathbf{S} \cong \mathbf{PRa}$$

We write *maps* $f : \{A : \chi\} \to \{B : \varphi\}$
of theory $\mathbf{S} = \mathbf{PR2} + (\mathrm{abstr})$ as

$\langle((\chi, f) \times \varphi) : A \times B \to (\mathbf{2} \times B) \times \mathbf{2}\rangle :$
$\{A : \chi\} \to \{B : \varphi\},$
$\chi : A \to \mathbf{2},\ \varphi : B \to \mathbf{2},\ f : A \to B$ in **PR2**,
$\mathbf{PR2} \vdash [\chi(a) \implies (\varphi \circ f)(a)] : A \to \mathbf{2}$

Two such maps $f, \tilde{f} : \{A : \chi\} \to \{B : \varphi\}$ are equal in **S**,

$\mathbf{S} \vdash ((\chi, f) \times \varphi) = ((\chi, \tilde{f}) \times \varphi)$
iff $\mathbf{PR2} \vdash \chi \implies [\varphi \circ f =_B \varphi \circ \tilde{f}]$

Theory **S** admits a cartesian p. r. **Embedding** functor

$\mathbf{I} : \mathbf{PR2} \longrightarrow \mathbf{S}$ defined by
$\mathbf{I}\langle f : A \to B\rangle$
$= \langle((\mathrm{true}_A, f) \times \mathrm{true}_B) : \{A : \mathrm{true}_A\} \to \{B : \mathrm{true}_B\}\rangle$

We may abbreviate $\mathbf{I}\langle f : A \to B\rangle$ by $f : A \to B$.

Definition: In analogy to the case of theory $\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$ we call the objects of theory **PR2** – cartesian products of $\mathbb{1}, \mathbb{N}, \mathbf{2}$ – *objects,* and the objects of theory **S** (predicative) *subsets*

$\{A : \chi\} \subseteq A$ of **PR2** objects A – *sets.*

$\mathbf{2}$ has been added as an *object,* this *truth algebra object* is to replace logically two-element *set* $\{0, 1\} \subset \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}$ subset of **PRa**'s NNO.

5.6 Boolean logic on set theory S

Using the boolean operations on **2** above, a ***free-variables boolean predicate calculus*** is easily **defined,** making the set of **S** predicates on (any) object A into a boolean algebra:

- Overall negation:

  $\neg\,\varphi(a) = \neg \circ \varphi : A \to \mathbf{2} \to \mathbf{2}$

- Conjunction:

  $[\chi \wedge \varphi] = \wedge \circ (\chi, \varphi) : A \to \mathbf{2} \times \mathbf{2} \to \mathbf{2}$

- Disjunction:

  $[\chi \vee \varphi] = \vee \circ (\chi, \varphi) : A \to \mathbf{2} \times \mathbf{2} \to \mathbf{2}$

- Implication:

  $[\chi \implies \varphi] = \Rightarrow \circ (\chi, \varphi) : A \to \mathbf{2} \times \mathbf{2} \to \mathbf{2}$

- Equivalence:

  $[\chi \iff \varphi] \;=\; [\chi \Rightarrow \varphi] \wedge [\varphi \Rightarrow \chi] : A \to (\mathbf{2} \times \mathbf{2}) \overset{\wedge}{\to} \mathbf{2}$

'$\iff$' acts as equality on truth object $\mathbf{2}$

- Complement:

  $[\chi \setminus \varphi] \;=\; [\chi \wedge \neg\varphi] : A \to (\mathbf{2} \times \mathbf{2}) \to \mathbf{2}$

5.7 Map definition by case distinction

We construct in variable-free manner map definition

$$f = \mathrm{if}[\chi, (h|g)](a) = \begin{cases} h(a) \text{ if } \chi(a) \\ g(a) \text{ if } \neg\, \chi(a) \;\; \textit{“(otherwise)”} \end{cases}$$

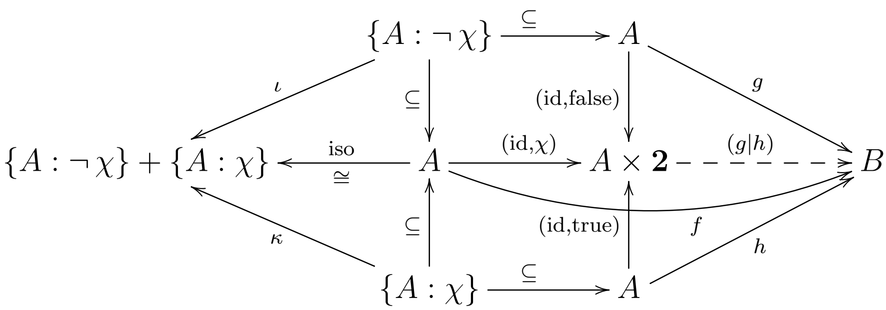
$$: A \to B$$

by *case distinction* – for given $h, g : A \to B$ and predicate $\chi : A \to \mathbf{2}$ on set A.

A **consequence** of $A \times \mathbf{2}$ to be the coproduct $A \times \mathbf{2} \cong A + A$ is in fact the following scheme of map definition by **case distinction:**

$$\text{(IF)}\quad \frac{\begin{array}{l}\chi : A \to \mathbf{2} \text{ p. r. predicate} \\ h, g : A \to B \text{ p. r. maps}\end{array}}{\begin{array}{l} f = \mathrm{if}[\chi, (h|g)] \quad \text{“if } \chi \text{ then } h \text{ else } g\text{”} \\ =_{\mathrm{def}} \;(g|h \circ \ell) \circ (\mathrm{id}_A, \chi) : A \to A \times \mathbb{N} \to B \\ \quad \text{satisfies – is characterised by –} \\ \neg\chi(a) \implies [f(a) = \mathrm{if}\,[\chi, (h|g)](a) = g(a)] \\ \chi(a) \implies [f(a) = \mathrm{if}\,[\chi, (h|g)](a) = h(a)] \end{array}}$$

Proof: Commuting DIAGRAM:



with $(g|h) : A \times \mathbf{2} \cong (A+A) \to B$ the induced map out of the coproduct, with *injections* $(\mathrm{id}_A, \mathrm{false}), (\mathrm{id}_A, \mathrm{true}) : A \to A \times \mathbf{2}$. Necessarily

$f = (g \circ \subseteq | h \circ \subseteq) \circ \mathrm{iso} : A \to \{A : \neg\chi\} + \{A : \chi\} \to B,$

and this $f : A \to B$ does the job. ι and κ are the injections into the sum – disjoint union – $\{A : \neg\chi\} + \{A : \chi\} \cong A$ **q. e. d.**

5.8 Peano induction

Peano's axioms read in categorical free-variables form[2] as

Peano theorem

- P1: *zero is a natural number:*

  $0 : \mathbb{1} \to \mathbb{N}$ is a map constant of $\mathbb{N}$, a *natural number* as such.

- P2: *to any natural number (free variable) n is assigned* a *successor:*

[2] cf. PFENDER/KRÖPLIN/PAPE 1994

This *assignment* is realised categorically by the *successor map* $\mathrm{s} = \mathrm{s}(n) : \mathbb{N} \to \mathbb{N}$.

Such successor $\mathrm{s}(n)$ is *unique:*

The notion 'map' is an undefined notion of theory **PR**, and as a **PR** map $\mathrm{s} : \mathbb{N} \to \mathbb{N};\ n \overset{\mathrm{s}}{\mapsto} \mathrm{s}(n)$, it is to make available a uniquely determined successor (to $n \in \mathbb{N}$ free.)

- P3: 0 *is not a successor:*

  This follows from $\mathrm{s}\,n > 0$ whence $\mathrm{s}\,n \neq 0$ by definition of $m = n$ and $m < n$ via $m \smallsetminus n$.

  Problematic: Without this negative **axiom,** *infinity* does not follow. Quotient ring $\mathbb{N}/(\boldsymbol{m})$ satisfies P1, P2, and P4, P5 below.

- P4: *equality* $\mathrm{s}(m) = \mathrm{s}(n)$ *implies* $m = n$:

  This is *injectivity* of successor map $\mathrm{s} : \mathbb{N} \to \mathbb{N}$.

  Definition: Call a map $f : A \to B$ *injective,* if

  $$f(a) = f(\tilde{a}) \implies a = \tilde{a} : A \times A \to \mathbf{2}$$

  holds true.

  The successor map $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ is in fact injective, since it admits the predecessor map $\mathrm{pre} : \mathbb{N} \to \mathbb{N}$ as a retraction, $\mathrm{pre} \circ \mathrm{s} = \mathrm{id}_{\mathbb{N}}$, and is therefore injective (**exercise:** injective=monomorphic).

- P5: Peano-***induction*** derived from *uniqueness* part (pr!) of *full* scheme (pr) of primitive recursion:

$$\text{(P5)}\quad \frac{\begin{array}{l} \varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbf{2} \ \ \text{predicate} \\ \varphi(a,0) = \text{true}_A(a) \ \ (\textit{anchor}) \\ [\varphi(a,n) \implies \varphi(a,\mathrm{s}\,n)] = \text{true}_{A\times\mathbb{N}} \ \ (\textit{induction step}) \end{array}}{\varphi(a,n) = \text{true}_{A\times\mathbb{N}} \ \ (\textit{conclusio}).}$$

Proof of Peano induction principle (P5) from *full scheme* (pr) of primitive recursion:[3]

For scheme (pr!) choose as anchor map

$$g = g(a) = \varphi(a,0) = \text{true}_A(a) : A \to \mathbf{2} \text{ and as step map}$$
$$h = h((a,n),b) = b \vee \varphi(a,\mathrm{s}\,n) : (A \times \mathbb{N}) \times \mathbb{N} \to \mathbf{2}$$

By (pr) we get a unique $f = f(a,n) : A \times \mathbb{N} \to \mathbf{2}$ which satifies

$$f(a,0) = \varphi(a,0) = \text{true}_A(a) \text{ and}$$
$$f(a,\mathrm{s}\,n) = h((a,n),f(a,n)) = f(a,n) \vee \varphi(a,\mathrm{s}\,n)$$

This works for $f = \text{true}_{A\times\mathbb{N}} : A \times \mathbb{N} \to \mathbf{2}$ as well as for $f = \varphi$, the

[3] Reiter 1982 and Pfender/Kröplin/Pape 1994

latter since

$$
\begin{aligned}
&\varphi(a,n) \vee \varphi(a,\mathrm{s}\,n) \\
&= (\varphi(a,n) \vee \varphi(a,\mathrm{s}\,n)) \wedge (\varphi(a,n) \implies \varphi(a,\mathrm{s}\,n)) \\
&\quad \text{by 2nd hypothesis} \\
&= \varphi(a,\mathrm{s}\,n) \text{ by boolean tautology} \\
&(\alpha \vee \beta) \wedge (\alpha \Rightarrow \beta) = \beta : \\
&\textit{test with } \beta = \text{false } \textit{and } \beta = \text{true.}
\end{aligned}
$$

q. e. d.

By replacing predicate φ with

$$\psi(a,n) := \bigwedge_{i \leq n} \varphi(a,i) : A \times \mathbb{N} \to \mathbf{2}$$

in this proof we get

Course of values induction principle

$$
\text{(P5)} \quad \frac{\begin{array}{l} \varphi = \varphi(a,n) : A \times \mathbb{N} \to \mathbf{2} \text{ predicate} \\ \varphi(a,0) = \text{true}_A(a) \ \ (\textit{anchor}) \\ [\bigwedge_{i \leq n} \varphi(a,i) \implies \varphi(a,\mathrm{s}\,n)] = \text{true}_{A \times \mathbb{N}} \quad (\textit{induction step}) \end{array}}{\varphi(a,n) = \text{true}_{A \times \mathbb{N}} \ \ (\textit{conclusio}).}
$$

Here predicate $\bigwedge_{i \leq n} \varphi(a, i) : A \times \mathbb{N} \to \mathbf{2}$ is p. r. **defined** by

$$\bigwedge_{i \leq 0} \varphi(a, i) = \varphi(a, 0) : A \to \mathbf{2}$$

$$\bigwedge_{i \leq \mathrm{s}\, n} \varphi(a, i) = \bigwedge_{i \leq n} \varphi(a, i) \wedge \varphi(a, \mathrm{s}\, n) : A \times \mathbb{N} \to \mathbf{2} \;\; \textbf{q. e. d.}$$

Diagonal induction principle

A predicate on two (free) natural numbers, which is true on the horizontal (half-)axis and on the vertical (half-)axis of the $\mathbb{N} \times \mathbb{N}$ grid, and whose truth spreads (everywhere) in diagonal direction, is globally true.

Formally, with a "passive" parameter (free variable) $a \in A$ added:

$$\text{(diagind)} \quad \frac{\begin{array}{l} \varphi = \varphi(a, (m, n)) : A \times (\mathbb{N} \times \mathbb{N}) \to \mathbf{2} \text{ predicate} \\ \varphi(a, (m, 0)) = \text{true} : A \times (\mathbb{N} \times \mathbb{1}) \to \mathbf{2} \\ \varphi(a, (0, n)) = \text{true} : A \times (\mathbb{1} \times \mathbb{N}) \to \mathbf{2} \\ [\varphi(a, (m, n)) \implies \varphi(a, (\mathrm{s}\, m, \mathrm{s}\, n))] = \text{true} : \\ \quad A \times (\mathbb{N} \times \mathbb{N}) \to \mathbf{2} \end{array}}{\varphi = \text{true} : A \times (\mathbb{N} \times \mathbb{N}) \to \mathbf{2}}$$

Proof:[4] Use the assertion $\varphi(a, (x \setminus n), (y \setminus n)) \implies \varphi(a, (x, y))$ proved for $n = \mathrm{s}\, 0$ by case distinction on $x \overset{?}{>} 0$ and $y \overset{?}{>} 0$, the general case being obtained from this case by Peano induction P5. The principle then follows by substitution of n for y.

[4] Pfender/Kröplin/Pape 1994

Chapter 6

Further Algebra on the NNO

Natural Numbers Object "NNO" $\mathbb{N} = \langle \mathbb{N}, 0, \mathrm{s} \rangle$ bears the structure

$$\mathbb{N} = \langle \mathbb{N}, 0, 1, +, \smallsetminus, \cdot, <, \leq, = \rangle$$

of a linearly ordered commutative integrity semiring with *truncated subtraction*

$a \smallsetminus b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

defined recursively by

$0 \smallsetminus 1 = 0$

$a \smallsetminus 0 = a$

$a \smallsetminus (n + 1) = (a \smallsetminus n) \smallsetminus 1$

and *equality predicate*

$$[a = b] = [a =_{\mathbb{N}} b] = [a \leq b] \wedge [b \leq a] : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$$

Maximum is **defined** as

$$\max(a, b) = a + (b \setminus a) = b + (a \setminus b) = \max(b, a),$$

minimum as

$$\min(a, b) = a \setminus (a \setminus b) = b \setminus (b \setminus a) = \min(b, a)$$

This is the (algebraic) quintessence of chapter on free-variables *Goodstein Arithmetic* **GA**.

$\mathbb{N}$ has *exponentiation* $a^n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ recursively **defined** by

$$a^0 = 1$$
$$a^{n+1} = a^n \cdot a$$

Exponentiation Lemma:

- $a^{m+n} = a^m \cdot a^n$
- $(a^m)^n = a^{m \cdot n}$
- $a^{m^n} \;=_{\text{def}}\; a^{(m^n)}$

Proof as **exercise.**

$\mathbb{N}$ has *faculty* $n! : \mathbb{N} \to \mathbb{N}$ recursively **defined** by

$$0! = 1$$
$$(n + 1)! = n! \cdot (n + 1)$$

Integer division

Integer division with remainder (Euclide)

$$(a \div b, a \text{ rem } b) : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N} \times \mathbb{N}$$

is characterised by

$$a \div b = \max\{c \le a : b \cdot c \le a\} : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}$$
$$a \text{ rem } b = a \smallsetminus (a \div b) \cdot b : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}$$

Here $\mathbb{N}_{>} \ =_{\text{def}} \ \{n \in \mathbb{N} : n > 0\}$

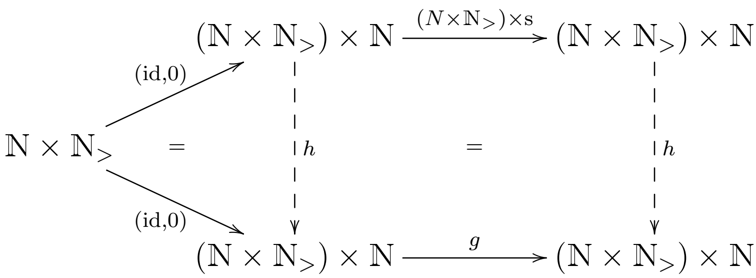
Explicitely, we **define**

$$\div = a \div b : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}$$

via *initialised iteration* $h = h((a,b),n)$ of

$$g = g((a,b),c) = \begin{cases} ((a,b),c) \text{ if } a < b, \\ ((a \smallsetminus b, b), c+1) \text{ if } a \ge b \end{cases}$$

in



$$a \div b \ =_{\text{def}} \ \mathrm{r}\, h((a,b),a) : \mathbb{N} \times \mathbb{N}_{>} \to (\mathbb{N} \times \mathbb{N}_{>})\mathbb{N} \to \mathbb{N}$$
$$a \,\mathrm{rem}\, b \ =_{\text{def}} \ \ell\ell\, h((a,b),a) = a \smallsetminus b \cdot (a \div b) : \mathbb{N} \times \mathbb{N}_{>} \to \mathbb{N}$$

The predicate $a|b : \mathbb{N}_{>} \times \mathbb{N} \to \mathbb{N}$, *a is a divisor of b*, *a divides b* is **defined** by

$$a|b = [(b \text{ rem } a) = 0]$$

Exercise: Construct the Gaussian algorithm for determination of the gcd of $a, b \in \mathbb{N}_>$ **defined** as

$$\gcd(a,b) = \max\{c \leq \min(a,b) : c|a \wedge c|b\} : \mathbb{N}_> \times \mathbb{N}_> \to \mathbb{N}_>$$

by iteration of mutual rem.

Primes

Define the predicate *is a prime* by

$$\mathbb{P}(p) = \bigwedge_{m=1}^{p} [m|p \implies m = 1 \vee m = p] : \mathbb{N} \to \mathbf{2} :$$

Only 1 and p divide p.

Write $\mathbb{P}$ for $\{n \in \mathbb{N} : \mathbb{P}(n)\} \subset \mathbb{N}$ too.

The (euclidean) count $p_n : \mathbb{N} \to \mathbb{N}$ of all primes is given by

$$\begin{aligned} p_0 &= 2, \\ p_{n+1} &= \min\{p \in \mathbb{N} : \mathbb{P}(p), p_n < p \leq \prod_q [q \leq p_n \wedge \mathbb{P}(q)]\} + 1 \\ &= \min\{p \in \mathbb{N} : \mathbb{P}(p), p < 2p_n\} : \mathbb{P} \to \mathbb{P} \end{aligned}$$

iterated binary product and iterated binary minimum.

The latter presentation is given by Bertrand's theorem.

Notes

(a) An NNO, within a cartesian closed category of **sets,** was first studied by Lawvere 1964.

(b) Eilenberg/Elgot 1970 iteration, here special case of one-successor iteration theory **PR**, is because of Freyd's uniqueness scheme (FR!) a priori stronger than classical free-variables *primitive recursive arithmetic* **PRA** in the sense of Smorynski 1977. If viewed as a conservative subsystem of **PM**, **ZF**, or **NGB** that **PRA** is stronger than our **PR**.

(c) Over *Elementary Topoi* (with their cartesian closed structure), Freyd 1970 characterised Lawvere's NNO by unique initialised iteration. Such Freyd's NNO has been called later, e.g. in Maietti 2010, *parametrised NNO.*

(d) Lambek/Scott 1986 consider in parallel a *weak NNO:* uniqueness of Lawvere's sequences $a : \mathbb{N} \to A$ not required. We need here uniqueness (of the initialised iterated) for proof of Goodstein's 1971 uniqueness rules basic for his development of p. r. arithmetic. Without the latter uniqueness requirement, the definition of parametrised (weak) NNO is equational.

(e) For uniqueness of the set of natural numbers (out of the Peano-axioms), classical set theory needs *higher order.* This corresponds here to the use of free meta-variables on *maps.*

(f) The idea to incorporate categorically truth set and free variables predicate logic into primitive recursive Arithmetic is in Reiter's dissertation 1982.

Chapter 7

Partiality

Maps $f : \{\mathbb{N}^{\boldsymbol{m}} : \chi\} \to \{\mathbb{N}^{\boldsymbol{n}} : \psi\}$ of *constructive set theory* **S** can be seen as *partial* p. r. maps $f : \mathbb{N}^{\boldsymbol{m}} \rightharpoonup \mathbb{N}^{\boldsymbol{n}}$ "but" with p. r. decided domain of defined arguments.

If you generalise this suitably to *domain of defined arguments* given as an **S** map into the source set of the partial map to be introduced, you arrive at the notion of a *partial p. r. map:* a *right unique correspondence* given as a *hook* of two p. r. maps, *correspondence* in the sense of Brinkmann/Puppe 1969.

General recursive maps/*algorithms* fit into the theory $\widehat{\mathbf{S}}$ of partial p. r. maps. Central question about these recursive maps/algorithms is definedness/*termination,* as theorem or as condition, see in particular *termination conditioned soundness* of ***evaluation,*** which fits into theory $\widehat{\mathbf{S}}$ as a *complexity controlle iteration (CCI)* `while` loop.

In classical **set** theory these domains of definition are usually given via existential quantification. But *we* want to avoid (non-constructive) formal (existential) quantification.

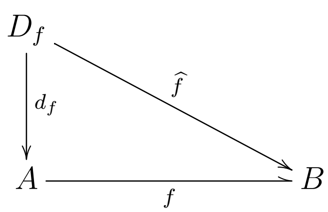
7.1 Partial p. r. maps

A *partial PR map* $f : A \rightharpoonup B$ is a pair

$$f = \langle d_f : D_f \to A,\ \widehat{f} : D_f \to B \rangle : A \rightharpoonup B$$

of **S**-maps. It consists of a p. r. *domain of defined arguments enumeration* $d_f : D_f \to A$ and a p. r. (calculation) *rule* $\widehat{f} : D_f \to B$ into the domain of *values* of f. **S** set D_f (*roof* of f) has the form $D_f = \{D : \delta_f\}$, $\delta_f : D \to \mathbf{2}$ a p. r. predicate.

Partial map DIAGRAM



Typical index domain $D_f = \mathbb{N}$. In general, as an **S** set, it has form

$$D_f = \{D : \delta_f\},\ \delta_f : D \to \mathbf{2} \text{ an } \mathbf{S} \text{ predicate.}$$

Usually $D_f = \{A \times B, \delta_f : A \times B \to \mathbf{2}\}$ for $f : A \rightharpoonup B$.

The pair $f = \langle d_f,\ \widehat{f} \rangle$ is to fullfill the *right-uniqueness condition*

$$d_f(\hat{a}) =_A d_f(\hat{a}') \implies \widehat{f}(\hat{a}) =_B \widehat{f}(\hat{a}')$$

Alternatively, for general diagonal monoidal frame, $f : A \rightharpoonup B$ is given by its *graph*

$$\gamma f : D_f \to A \times B$$
$$d_f = \ell\,\gamma f = \ell_{A,\mathbb{1}}\,(A \times \Pi)\,\gamma f :$$
$$D_f \to A \times B \to A \times \mathbb{1} \xrightarrow{\cong} A$$
$$\widehat{f} = \mathrm{r}\,\gamma f : D_f \to B$$

such that right-uniqueness condition is fullfilled for these $d_f,\ \widehat{f}$.

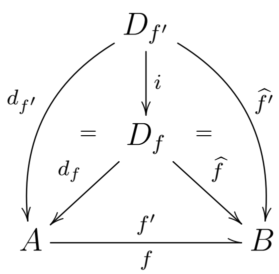
In both definitions, *graph* γf of $f : A \rightharpoonup B$ is

$$\gamma f = (d_f, \widehat{f}) = (d_f \times \widehat{f})\,\Delta_{D_f} : D_f \to A \times B$$

Typically, γf is just an inclusion

$$\gamma f : D_f = \{A \times B : \delta_f\} \xrightarrow{\subseteq} A \times B,$$
$$\delta_f : A \times B \to \mathbf{2} \text{ an } \mathbf{S} \text{ predicate}$$

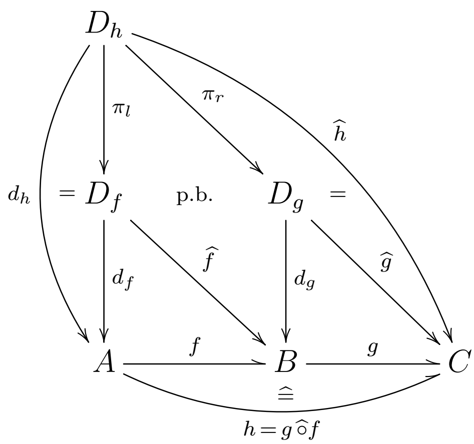
Graph inclusion $f' \mathbin{\widehat{\subseteq}} f$ of partial p. r. maps is given by an $\mathbf{S}$-map $i : D_{f'} \to D_f$ with



Equality of p. r. partials (enumeration):

$$\frac{f \mathbin{\widehat{\subseteq}} f', \ f' \mathbin{\widehat{\subseteq}} f}{f \ \widehat{=} \ f'}$$

Partial p. r. map *composition* $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$:



Pullback π_ℓ of d_g along $\widehat{f}$ is typically the *inverse image* of d_g under $\widehat{f}$. But the definability domains d_f, d_g, d_h need not to be monic (injective).

[The idea is from Brinkmann/Puppe 1969: They construct composition of *correspondences* this way via pullback.]

Remark: The *standard form* of the pullback D_h is

$$D_h = \{(\hat{a}, \hat{b}) \in D_f \times D_g : \widehat{f}(\hat{a}) \ =_B \ d_g(\hat{b})\}$$

with pullback-*projections*

$$\pi_l = \ell \circ \subseteq : D_h \to D_f \times D_g \to D_f \text{ and}$$
$$\pi_r = \mathrm{r} \circ \subseteq : D_h \to D_f \times D_g \to D_g$$

In a sense, the pullback D_h represents the inverse image $D_h = \overset{-1}{f}[D_g]$, more precisely: $[D_h \xrightarrow{\ell} D_f] = \overset{-1}{\widehat{f}}\,[D_g \xrightarrow{d_g} B]$.

Composition $h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C$ gives a *well-defined* partial p. r. map h, since for $(\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h$ free

$$d_h(\hat{a}, \hat{b}) =_A d_h(\hat{a}', \hat{b}') \iff d_f(\hat{a}) =_A d_f(\hat{a}')$$
$$\implies \widehat{f}(\hat{a}) =_B \widehat{f}(\hat{a}') \ (f \text{ well-defined})$$
$$\iff \widehat{f}\,\ell(\hat{a}, \hat{b}) = \widehat{f}\,\ell(\hat{a}', \hat{b}')$$
$$\implies d_g(\mathrm{r}(\hat{a}, \hat{b})) =_B d_g(\mathrm{r}(\hat{a}', \hat{b}'))$$
$$(\ (\hat{a}, \hat{b}), (\hat{a}', \hat{b}') \in D_h,\ \text{p.b. commutes})$$
$$\iff d_g(\hat{b}) =_B d_g(\hat{b}') \implies \widehat{g}(\hat{b}) =_C \widehat{g}(\hat{b}')$$
$$\implies \widehat{h}(\hat{a}, \hat{b}) = \widehat{g}(\hat{b}) =_C \widehat{g}(\hat{b}') = \widehat{h}(\hat{a}', \hat{b}') : D_h \times D_h \to \mathbb{2}$$

Obviously, partial **S**-map, $\widehat{\mathbf{S}}$-map $\mathrm{id}_A^{\widehat{\mathbf{S}}} =_{\mathrm{def}} \langle (\mathrm{id}_A, \mathrm{id}_A) : A \to A^2 \rangle : A \rightharpoonup A$ works as *identity* for set A with respect to composition $\widehat{\circ}$ for (emerging) theory $\widehat{\mathbf{S}}$

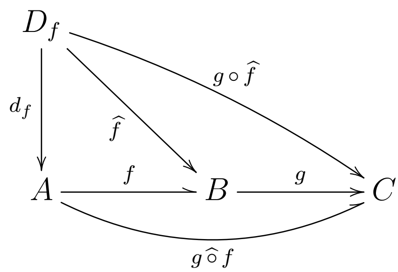
If one of two $\widehat{\mathbf{S}}$ maps to be composed is an **S**-map, $\widehat{\mathbf{S}}$-*composition* becomes simpler:

Mixed Composition Lemma:

(i) For $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$, and $g : B \to C$ in **S** :

$$g \,\widehat{\circ}\, f = \langle (d_f, g \circ \widehat{f}) : D_f \to A \times C \rangle : A \rightharpoonup C$$

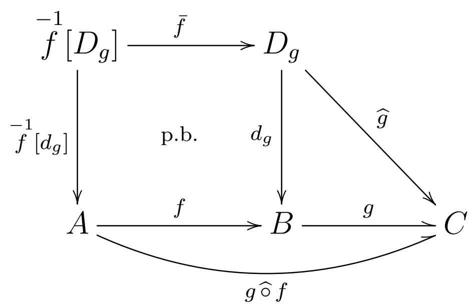
in DIAGRAM form:



(ii) For $f : A \to B$ in $\mathbf{S}$, $g : B \rightharpoonup C$ in $\widehat{\mathbf{S}}$:

$$g \mathbin{\widehat{\circ}} f = \langle (\overset{-1}{f}[d_g], \widehat{g} \circ \bar{f}) : \overset{-1}{f}[D_g] \to A \times C \rangle : A \rightharpoonup C,$$

as DIAGRAM:



Proof: Left as a (category theory) **exercise.**

7.1.1 Structure theorem for p. r. partials

Constructive p. r. set theory $\mathbf{S}$ carries theory $\widehat{\mathbf{S}}$ of *partial p. r. maps* over $\mathbf{S}$ which comes with the following structure:

(i) $\widehat{\mathbf{S}}$ carries a canonical structure of a *diagonal symmetric monoidal category*, with composition $\widehat{\circ}$ and identities introduced above, monoidal product $\times$ extending $\times$ of $\mathbf{S}$, *association*

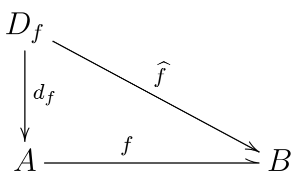
$$\mathrm{ASS} : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C),$$

symmetry $\Theta : A \times B \xrightarrow{\cong} B \times A$,

and *diagonal* $\Delta : A \to A \times A$

inherited from $\mathbf{S}$.

(ii) The defining diagram for an $\widehat{\mathbf{S}}$-map – namely

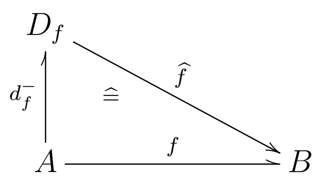


partial map DIAGRAM

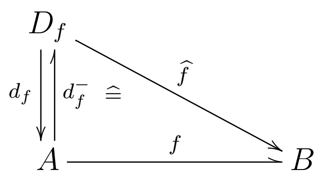
is a commuting $\widehat{\mathbf{S}}$ diagram.

Conversely, the *minimised opposite* $\widehat{\mathbf{S}}$-map $d_f^- : A \rightharpoonup D_f$ to

$\mathbf{S}$ map $d_f : D_f \to A$ fullfills



Put together:



basic partial map DIAGRAM

(iii) $\widehat{\mathbf{S}}$ clearly inherits from $\mathbf{S}$ *retractive pairing:*

For $h : C \rightharpoonup A \times B$ in $\widehat{\mathbf{S}}$

$$h \mathrel{\widehat{=}} (\ell \mathbin{\widehat{\circ}} h, \mathrm{r} \mathbin{\widehat{\circ}} h) : C \rightharpoonup A \times B$$

where for $f : C \rightharpoonup A\ ,\ g : C \rightharpoonup B$

$$(f, g) \ =_{\mathrm{def}} \ (f \times g) \mathbin{\widehat{\circ}} \Delta_C :$$
$$C \to C \times C \rightharpoonup A \times B$$
with *diagonal* $\Delta_C : C \to C \times C$ of $\mathbf{S}$

This equation guarantees *uniqueness* of the "*induced*" $(f, g) : C \rightharpoonup A \times B$, but (f, g) does not satisfy (both of) the *cartesian equations* $\ell \mathbin{\widehat{\circ}} (f, g) \mathrel{\widehat{=}} f$ *and* $\mathrm{r} \mathbin{\widehat{\circ}} (f, g) \mathrel{\widehat{=}} g$ except f and g have *equal domains of definition* i. e. if $i : D_f \to D_g$, $j : D_g \to D_f$ are available such that $d_g \circ\ i = d_f : D_f \to A$ as well as $d_f \circ\ j = d_g : D_g \to A$.

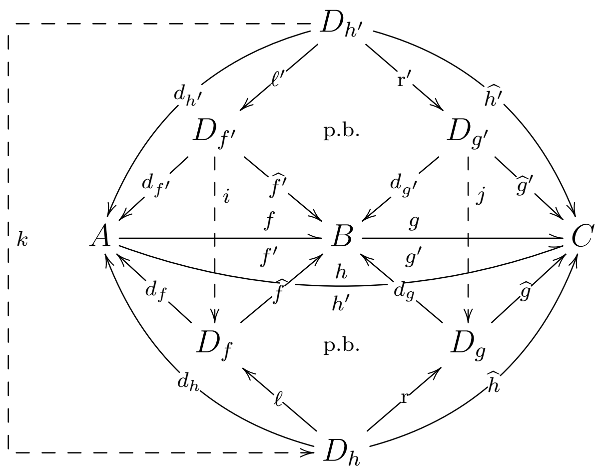
Note: Primitive recursive iteration of p. r. *partials* is not considered express, by reasons to be discussed within the section on content driven loops.

7.1.2 Proof of structure theorem for p. r. partials

Proof of assertion (i):

We first give to $\widehat{\mathbf{S}}$ the structure of a diagonal monoidal category and verify the defining properties of this structure:

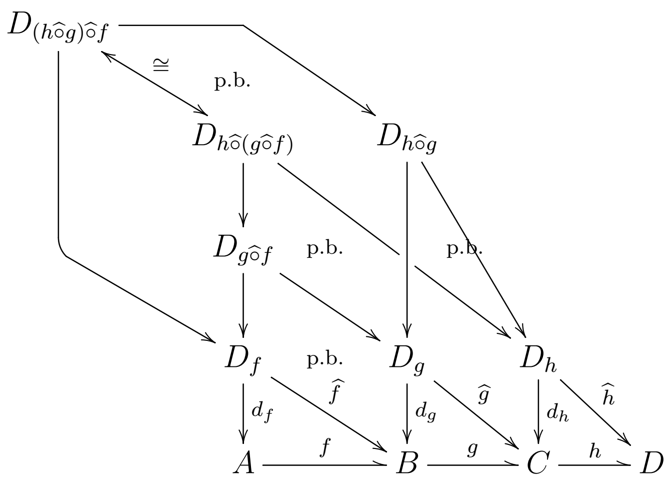
Composition $\widehat{\circ}$ introduced above – by pullback – is compatible with $\widehat{\subseteq}$ and hence also with $\widehat{=}$ since for $f' \mathbin{\widehat{\subseteq}} f : A \rightharpoonup B$ and $g' \mathbin{\widehat{\subseteq}} g : B \rightharpoonup C$ we are given "inclusions" $i : D_{f'} \to D_f$ and $j : D_{g'} \to D_g$ such that for $h = g \mathbin{\widehat{\circ}} f : A \rightharpoonup B \rightharpoonup C$ and $h' = g' \mathbin{\widehat{\circ}} f' : A \rightharpoonup B \rightharpoonup C$ compatibility DIAGRAM below commutes with (unique) $k : D_{h'} \to D_h$ in $\mathbf{S}$, *induced into the pullback* D_h by $i \circ \ell' : D_{h'} \to D_{f'} \to D_f$ and $j \circ \mathrm{r}' : D_{h'} \to D_{g'} \to D_g$



Compatibility DIAGRAM[a] of $\widehat{\circ}$ with $\subseteq$

[a] F. Herrmann

For proving associativity of (partial) composition $\widehat{\circ}$, consider



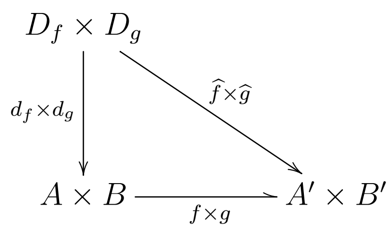
Associativity DIAGRAM for $\widehat{\circ}$ – via *nested pullbacks*

Here the standard form of isomorphism $D_{(h\circ g)\circ f} \xrightarrow{\cong} D_{h\circ(g\circ f)}$ is restriction of *association isomorphism*

$$\text{ASS} : (A \times B) \times C \xrightarrow{\cong} A \times (B \times C)$$

to an isomorphism $D_{(h\circ g)\circ f} \xrightarrow{\cong} D_{h\circ(g\circ f)}$

The (monoidal) product $f \times g : A \times B \rightharpoonup A' \times B'$ of partial maps is given *componentwise* as the *hook*



In particular, cylindrification with a set A is the hook

$$\begin{array}{ccc} A \times D_g & & \\ {\scriptstyle A\times d_g}\downarrow & \searrow^{A\times\widehat{g}} & \\ A \times B & \xrightarrow[A\times g]{} & A \times B' \end{array}$$

Cylindrification preserves inclusion $f' \mathrel{\widehat{\subseteq}} f : A \rightharpoonup B$ given by
$i : D'_f \to D_f$, since

$$C \times i : D_{C\times f'} = C \times D_{f'} \to C \times D_f = D_{C\times f}$$

gives the inclusion $C \times f' \mathrel{\widehat{\subseteq}} C \times f : C \times A \rightharpoonup C \times B$.

Hence in particular, cylindrification preserves (partial) equality $f' \mathrel{\widehat{=}} f$ defined by $f' \mathrel{\widehat{\subseteq}} f$ and $f \mathrel{\widehat{\supseteq}} f'$ being given simultaneously.

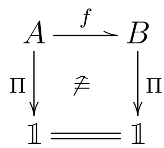
As for $\mathbf{S}$, the product of maps is given alternatively by composition of cylindrifications:

$$(\times_{\widehat{\mathbf{S}}}) \quad \frac{f : A \rightharpoonup A',\ g : B \rightharpoonup B' \text{ in } \widehat{\mathbf{S}}}{\begin{array}{l} (f \times g) \ =_{\text{def}} \ (f \times B') \mathbin{\widehat{\circ}} (A \times g) : \\ \quad A \times B \rightharpoonup A \times B' \rightharpoonup A' \times B' \\ \mathrel{\widehat{=}} (A' \times g) \mathbin{\widehat{\circ}} (f \times B) : \\ \quad A \times B \rightharpoonup A' \times B \rightharpoonup A' \times B' \end{array}}$$

It extends the *cartesian* product of $\mathbf{S}$ into a *bifunctor* again, on theory $\widehat{\mathbf{S}}$. Within $\widehat{\mathbf{S}}$, this product looses its universal property, essentially since already $[\Pi_A : A \to \mathbb{1}]_{A \text{ in } \mathbf{S}}$ looses *naturality,*[1] within $\widehat{\mathbf{S}}$:

[1] Budach/Hoehncke 1975, *half-terminal category.* Reichel 1987

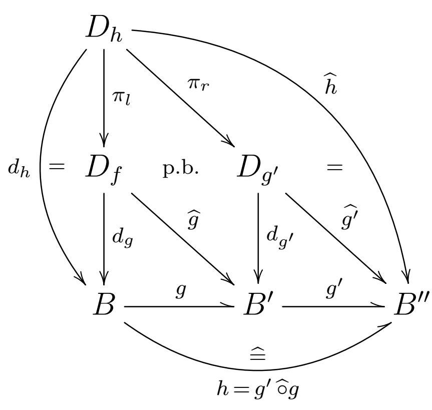
In general, *domain of definition* $d_f : D_f \to A$ of a *partial*
$f = (d_f, \widehat{f}) : A \rightharpoonup B$
does not cover the whole of *domain* A, whence



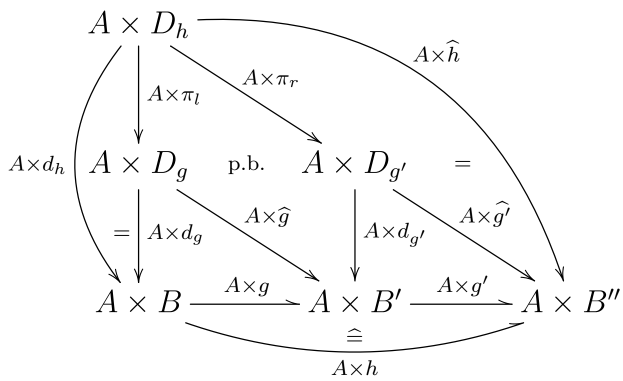
Proof of bifunctoriality of $\times$ in $\widehat{\mathbf{S}}$:
The point here is functoriality of cylindrification:

$$\langle g : B \to B' \rangle \mapsto \langle A \times g : A \times B \to A \times B' \rangle$$

For partial maps $\langle (d_g, \widehat{g}) : D_g \to B \times B' \rangle : B \rightharpoonup B'$ and $\langle (d_{g'}, \widehat{g'}) : D'_g \to B' \times B'' \rangle : B' \rightharpoonup B''$, and a ("cylindrifying") set A, recall the following defining $\mathbf{S}/\widehat{\mathbf{S}}$ DIAGRAM for g, g', and $h := g' \,\widehat{\circ}\, g$:



Functorial – and pullback preserving – cylindrification with set A inside $\mathbf{S}$ leads to



Functoriality DIAGRAM for theory $\widehat{\mathbf{S}}$

The "global" argument for functoriality of cylindrification in $\widehat{\mathbf{S}}$ (and hence for bifunctoriality of $\times$) now reads:

Both $A \times D_h$ and $D_{(A\times g')\,\widehat{\circ}\,(A\times g)}$ are *projective limits* of the lower-two-rows part of the $\mathbf{S}$ DIAGRAM when coming with their respective *cones.* Therefore they admit a "comparing" *natural isomorphism,* and that's what is sufficient for functoriality of cylindrification within theory $\widehat{\mathbf{S}}$.

$\widehat{\mathbf{S}}$ inherits from $\mathbf{S}$ transposition

$$\Theta = \Theta_{A,B}(a,b) \ =_{\text{def}} \ (b,a) = (\mathrm{r},\ell) : \\ A \times B \xrightarrow{\cong} B \times A$$

as well as diagonal

$$\Delta = \Delta_A(a) \ =_{\text{def}} \ (a,a) = (\mathrm{id},\mathrm{id}) : \\ A \to A \times A$$

and association

$$\begin{aligned} &\mathrm{ASS} = \mathrm{ASS}_{A,B,C}((a,b),c) \ =_{\mathrm{def}} \ (a,(b,c)) = (\ell\ell,(\mathrm{r}\ell,\mathrm{r})) : \\ &\quad ((A \times B) \times C) \xrightarrow{\cong} (A \times (B \times C)) \end{aligned}$$

It is obvious that $\widehat{\mathbf{S}}$ inherits *naturality* of the *transformation* families ASS, Θ, and Δ.

Using these natural transformations, we get (from functoriality of cylindrification) in fact *bifunctoriality* of (binary) *product* $\times$ within theory $\widehat{\mathbf{S}}$. This shows assertion (i) of the **Structure theorem.**

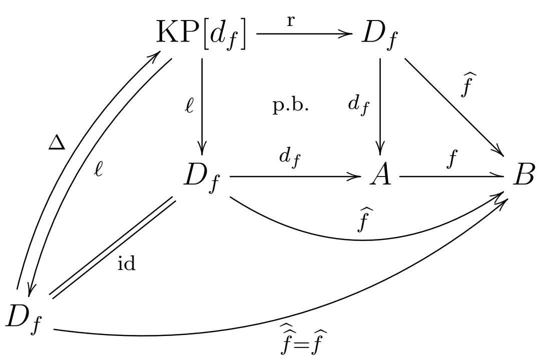
For **proof** of first half of assertion (ii), namely

$$f \mathbin{\widehat{\circ}} d_f \mathrel{\widehat{=}} \widehat{f} : A \rightharpoonup B$$

for given partial

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$$

consider the following $\mathbf{S}/\widehat{\mathbf{S}}$ diagram:



Partial Map Definition DIAGRAM

This diagram shows downwards inclusion

$$f \mathbin{\widehat{\circ}} d_f = (\ell, \widehat{f} \circ \mathrm{r}) \mathrel{\widehat{\subseteq}} \widehat{f} = (\mathrm{id}_{D_f}, \widehat{f}) : D_f \rightharpoonup B$$

via $\ell : \mathrm{KP}[d_f] \xrightarrow{\ell} D_f$ with $\widehat{f}$ embedded as its graph $(\mathrm{id}_{D_f}, \widehat{f})$.

The opposite (graph) inclusion $\Delta : D_f \to \mathrm{KP}[d_f]$, given by reflexivity of *kernel pair* $\mathrm{KP}[d_f]$, is immediate.

For **proof** of second $\widehat{\mathbf{S}}$-equality of assertion (ii), define opposite to $d_f : D_f \to A$ as

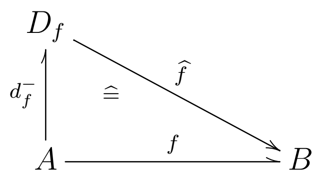
$$d_f^- \;=_{\mathrm{def}}\; \langle (d_f, [\ ]_{\widehat{f}}) : D_f \to A \times D_f \rangle : A \rightharpoonup D_f$$

made *right-unique* by selecting D_f *minimal* $\widehat{f}$ *equivalence representant*

$$[\ ]_{\widehat{f}} = [\alpha]_{\widehat{f}} \;=_{\mathrm{def}}\; \min_{D_f}\{\alpha' \le \alpha : \widehat{f}(\alpha') \;=_B\; \widehat{f}(\alpha)\} : D_f \to D_f$$

minimal with respect to Cantor-order on $\mathbf{S}$-set D_f supposed pointed, by $\hat{a}_0 : \mathbb{1} \to D_f$ say.

Get in fact the commuting $\widehat{\mathbf{S}}$-diagram



This finishes the proof of (ii) and hence of the **structure theorem** for partial p. r. map theory $\widehat{\mathbf{S}}$ **q. e. d.**

For our consistency considerations below, we strongly rely on

7.1.3 Totality Lemma

(i) For a partial p. r. map

$$f = \langle (d_f\,, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B, \text{ in } \widehat{\mathbf{S}}$$

the following statements are **equivalent:**

(a) $f : A \rightharpoonup B$ is (an embedded) "total" p. r. map, an **S** map.

(b) its defined-arguments enumeration
$d_f = d_f(\hat{a}) : D_f \to A$ is a retraction.

(c) $d_f : D_f \to A$ admits minimised opposite

$$d_f^- = d_f^-(a) = \mu\{\hat{a} : d_f(\hat{a}) = a\} : A \rightharpoonup D_f$$

as an embedded **S** coretraction $d_f^- : A \to D_f$, "minimum" μ taken with respect to a Cantor ordering of (countable) D_f .

(ii) The first factor $f : A \rightharpoonup B$ in an $\widehat{\mathbf{S}}$ composition

$$h = g \,\widehat{\circ}\, f : A \rightharpoonup B \rightharpoonup C$$

when giving an (embedded) **S**-map $h : A \to C$ is itself an (embedded) **S**-map:

A first p. r.-partial-composition factor of a (total) p. r. map is itself (total) p. r.

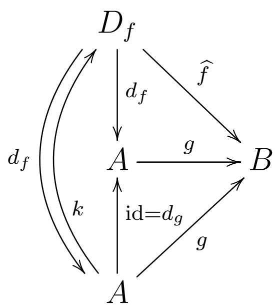
(iii) Therefore any coretraction of theory $\widehat{\mathbf{S}}$ is an **S**-map.[2]

[2]J. Sablatnig has pointed to a serious **problem** with this assertion when taking for coretraction the non-p. r. Ackermann function and as retraction its (partial) opposite, **problem** see below.

Proof:

(i) (a) $\rightarrow$ (b): If $\mathbf{S}$-map $k : A \to D_f$ establishes $\widehat{\mathbf{S}}$ graph inclusion $\langle \mathrm{id}, g \rangle \mathrel{\widehat{\subseteq}} f = \langle d_f, \widehat{f} \rangle$, then $k : A \to D_f$ is a coretraction to $d_f : D_f \to A$ within $\mathbf{S}$ – f *is defined on all of* A –

DIAGRAM



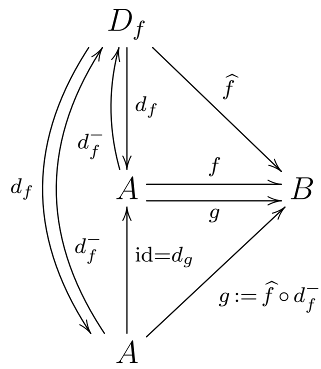
(b) $\rightarrow$ (c) : Then embedded $\mathbf{S}$ map

$$\begin{aligned} d_f^- = d_f^-(a) \ &=_{\text{by def}} \ \mu\{\hat{a} : d_f(\hat{a}) = a\} : A \rightharpoonup D_f \\ &\widehat{=} \ \min\{\hat{a} \in D_f : \hat{a} \le k(a) \ \wedge \ d_f(\hat{a}) = a\} : A \to D_f \end{aligned}$$

is a coretraction to $d_f : D_f \to A$, as minimised coretraction constructed out of coretraction $k : A \to D_f$ to d_f .

For less obvious (c) $\rightarrow$ (a) consider the following $\mathbf{S}/\widehat{\mathbf{S}}$ DIAGRAM, with embedded $\mathbf{S}$ map

$$g = \langle d_g, g \rangle := \langle \mathrm{id}_A, \widehat{f} \circ d_f^- \rangle : A \rightharpoonup B$$



We **show** for this g :

If d_f^- is an **S** coretraction to d_f, **then** $g \mathrel{\widehat{\subseteq}} f$ via d_f^- :

left triangle $d_f \circ d_f^- = \mathrm{id}_A = d_g$

and outer triangle $\widehat{f} \circ d_f^- = g$

as well as $f \mathrel{\widehat{\subseteq}} g$, the latter since

$$d_g \circ d_f = \mathrm{id}_A \circ d_f = d_f \qquad \text{(domain comparison)}$$

and – rule comparison **assertion** –

$$g \circ d_f(\hat{a}) = \widehat{f} \circ d_f^- \circ d_f(\hat{a}) = \widehat{f}(\hat{a}) : D_f \to B \qquad (\bullet)$$

(First retraction d_f, then coretraction d_f^-, followed by rule $\widehat{f}$).

We show ($\bullet$) by right uniqueness of $f = \langle d_f, \widehat{f} \rangle : A \rightharpoonup B$, namely

$$\frac{\mathbf{S} \vdash [d_f(\hat{a}) =_A d_f \circ d_f^- \circ d_f(\hat{a}) =_A d_f((d_f^- \circ d_f)(\hat{a}))] : D_f \to A \times A \stackrel{=}{\to} \mathbf{2}}{\mathbf{S} \vdash [\widehat{f}(\hat{a}) =_B \widehat{f}((d_f^- \circ d_f)(\hat{a})) : D_f \to B \times B \stackrel{=}{\to} \mathbf{2}}$$

Postcedent gives remaining **S** equation

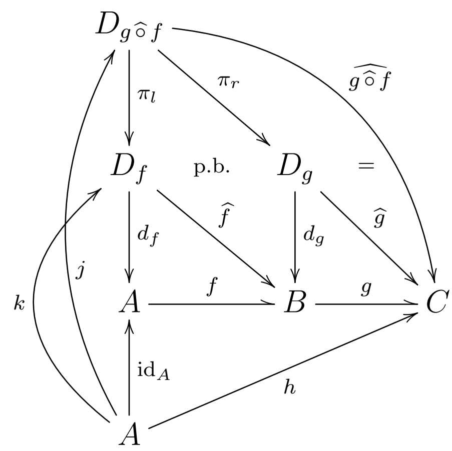
$$g \circ d_f = \widehat{f} \circ (d_f^- \circ d_f) = \widehat{f} : D_f \to A \qquad (\bullet)$$

for $\widehat{\mathbf{S}}$ inclusion $f \mathrel{\widehat{\subseteq}} g \equiv \langle \mathrm{id}_A, g \rangle$, by **equality definability** for theory **S**.

(ii) For $f : A \rightharpoonup B,\ g : B \rightharpoonup C$ given consider – with notation introduced for defined-arguments enumerations and rules – the DIAGRAM below, showing their "total" composition

$$h = \langle (\mathrm{id}_A, h) : A \to A \times C \rangle : A \to C$$

This DIAGRAM enriches earlier composition DIAGRAM by the data of h and comparison **S** map $j : A \to D_{g \,\widehat{\circ}\, f}$ which establishes *graph inclusion* $h \mathrel{\widehat{\subseteq}} g \mathbin{\widehat{\circ}} f : A \rightharpoonup C$ in



composition-total DIAGRAM for $\widehat{\mathbf{S}}$

Define $k := \pi_l \circ j : A \to D_{g \widehat{\circ} f} \to D_f$ having coretraction property $d_f \circ k = \mathrm{id}_A : A \to D_f \to A$ inherited from comparison property of $j : A \to D_{g \widehat{\circ} f}$. This proves the **Lemma,** by assertion (i), (c) $\to$ (a).

7.1.4 A counterexample?

Problem with 3rd assertion of the **lemma:**

Take for $f : \mathbb{N} \to \mathbb{N}$ the *partial p. r,* not primitive recursive, diagonalised Ackermann function $f = f(a) := \Psi(a, a) : \mathbb{N} \rightharpoonup \mathbb{N}$ – cf. **Appendix A** – and for $g = g(b) : \mathbb{N} \rightharpoonup \mathbb{N}$ the (partial) function inverse to f – given **set** theoretially by the *opposite graph* $\{(f(a), a) : a \in \mathbb{N}\}$.

Then $g \widehat{\circ} f \mathrel{\widehat{=}} \mathrm{id}_{\mathbb{N}}$ is primitive recursive – and first composition factor (coretraction) f is not!

The objection works in case of **set** theory, where maps, and partial maps can be defined as (even *actually* infinite) *lists* of argument/value pairs.

But if you want to **define** the list-defined retraction g above as a partial p. r. (!) map $g : \mathbb{N} \rightharpoonup \mathbb{N}$ within theory $\widehat{\mathbf{S}}$, you are lead – in the setting of the **lemma** – to try as "retraction" $g : \mathbb{N} \rightharpoonup \mathbb{N}$ a *partial map* of form

$$
\begin{aligned}
& g = \langle (d_g, \widehat{g}) : D_g \to \mathbb{N} \times \mathbb{N} \rangle : \mathbb{N} \rightharpoonup \mathbb{N} \quad \text{to have } \mathbf{S} \text{ components} \\
& D_g \ =_{\text{def}} \ \{(b, a) \in \mathbb{N} \times \mathbb{N} : \delta_g(b, a)\} \subseteq \mathbb{N} \times \mathbb{N}, \quad \textit{(opposite) graph,} \\
& \delta_g = \delta_g(b, a) : \mathbb{N} \times \mathbb{N} \to \mathbf{2} \quad \text{a p. r.(!) predicate} \\
& d_g = d_g(b, a) \ =_{\text{def}} \ b = \ell \circ \subseteq : \\
& \qquad \{\mathbb{N} \times \mathbb{N} : \delta_g\} \xrightarrow{\subseteq} \mathbb{N} \times \mathbb{N} \xrightarrow{\ell} \mathbb{N} \\
& \text{(p. r.) } \textit{defined arguments enumeration,} \text{ and} \\
& \widehat{g} = \widehat{g}(b, a) \ =_{\text{def}} \ \min\{a' \le a : \delta_g(b, a')\} : \\
& \qquad \{\mathbb{N} \times \mathbb{N} : \delta_g\} \subseteq \mathbb{N} \times \mathbb{N} \to \mathbb{N} \\
& \text{p. r. } \textit{rule}
\end{aligned}
$$

"The" choice for *graph* predicate δ_g would be, in present *opposite-to-Ackermann* case, opposite predicate

$$
\begin{aligned}
& \delta_g = \delta_g(b, a) := [\Psi(b, b) = a] : \\
& \quad \mathbb{N} \times \mathbb{N} \to (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}) \xrightarrow{\Psi \times \mathbb{N}} \mathbb{N} \times \mathbb{N} \xrightarrow{=} \mathbf{2}
\end{aligned}
$$

opposite to the Ackermann graph

$$\delta_f(a, b) = [\Psi(a, a) = b] : \mathbb{N} \times \mathbb{N} \to \mathbf{2}$$

of first factor $f : \mathbb{N} \rightharpoonup \mathbb{N}$ in the composition $g \mathbin{\widehat{\circ}} f \ \widehat{=} \ \mathrm{id}_{\mathbb{N}} : \mathbb{N} \rightharpoonup \mathbb{N} \rightharpoonup \mathbb{N}$.

But graph predicate $\delta_g : \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{2}$ is **not** primitive recursive, not in **S** as required for a graph predicate to define a *p. r.(!)-partial map:*

The Ackermann function Ψ is recursive, total, but not *primitive* recursive, since $\Psi(a, a)$ grows too fast, see **Appendix A** and references there. Ψ is only *double recursive,* admits resolution just into a *Complexity Controlled Iteration.* So graph $D_g = \{\mathbb{N} \times \mathbb{N} : \delta_g\}$ of *opposite partial* map $g = \langle (d_g, \widehat{g}) : D_g \to \mathbb{N} \times \mathbb{N} \rangle : \mathbb{N} \rightharpoonup \mathbb{N}$ would not be primitive recursive. g would not be partial *p. r.*, not in $\widehat{\mathbf{S}}$.

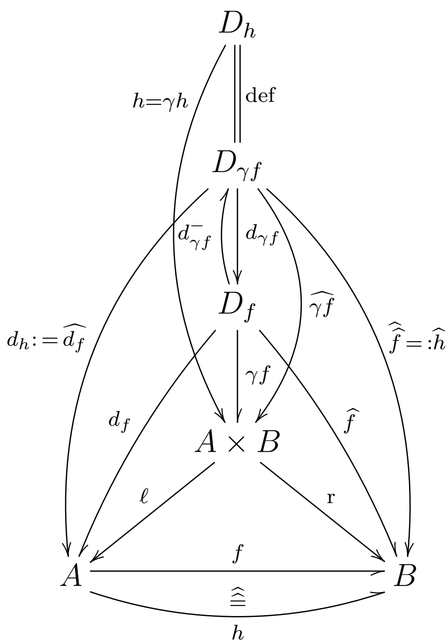
To **summarise:** Ackermann-opposite g (as tried "naturally" above), cannot be partial *p. r.*(!). It is not in our frame theory $\widehat{\mathbf{S}} \supset \mathbf{S}$ as required in the *Totality Lemma* being discussed. The Ackermann function itself *is* partial p.r. It is not invertible in our constructive context based on *primitive recursion,* not a coretraction in $\widehat{\mathbf{S}}$.

This discusion shows the technical, subtle character of the **lemma:** it bears on the difference of (partial) recursive maps in **set** theoretical power/complexity vs. frame $\widehat{\mathbf{S}}$ of partial p. r. maps – with p. r. graph predicates in particular. We rely on this distinction in consistency discussion. In **Appendix A** we discuss "the" alleged counterexample again, in terms of (Computer Science) `while` loops.

Note again: In the framework of this book, all sets are **S** sets, p. r. predicative subsets of **PR2** objects, subsets of cartesian products of $\mathbb{1}$, **2**, and $\mathbb{N}$. So the only *partial* maps $f : A \rightharpoonup B$ available in the context of this book have graph set D_f in **S**, they are partial *p. r.* maps.

7.2 Partial partial maps

For reduction of *partial* partial p. r. maps to (just) partial p. r. maps consider DIAGRAM



Closure DIAGRAM for extension by partial maps

The diagram shows a *partial* partial p. r. map

$$f = \langle \gamma f : D_f \rightharpoonup A \times B \rangle : A \rightharpoondown B$$

defined by its (partial) graph $\gamma f : D_f \rightharpoonup A \times B$ in turn defined as an **S**-map

$$\gamma f = (d_{\gamma f}, \widehat{\gamma f}) : D_{\gamma f} \to A \times B$$

As p. r. partial representant of *partial* p. r. partial map $f : A \rightharpoondown B$ take the $\widehat{\mathbf{S}}$-map $h : A \rightharpoonup B$ given by the *frame* in the DIAGRAM above:

$$\begin{aligned} h &= \langle (d_h, \widehat{h}) : D_h \to A \times B \rangle : A \rightharpoonup B \\ &=_{\text{def}} \ \langle (\widehat{d}_f, \widehat{\widehat{f}}) : D_{\gamma f} \to A \times B \rangle : A \rightharpoonup B \\ &=_{\text{by def}} \ \langle \widehat{\gamma f} : D_{\gamma f} \to A \times B \rangle : A \rightharpoonup B \end{aligned}$$

This shows: *Partial* partial p. r. maps are (represented by) partial p. r. maps – and so on: *partial partial* partial p. r. maps by partial p. r. maps etc.

This gives in particular representation of an arbitrarily nested `while` loop by one "flat" `while` loop with (one) p. r. *control* predicate controlling *iteration* of (one) p. r. endomorphism; for `while` loops as partial p. r. maps see section *content driven loops.*

7.3 Recursion without quantifiers

We **define** μ-recursion within the free-variables framework of partial p. r. maps as follows:

Given an $\mathbf{S}$ predicate $\varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbf{2}$, the $\widehat{\mathbf{S}}$-map

$$\mu\varphi = \langle (d_{\mu\varphi}, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

is to have (**S**-)components

$$
\begin{aligned}
&D_{\mu\varphi} \ =_{\text{def}} \ \{A \times \mathbb{N} : \varphi\} \subseteq A \times \mathbb{N} \\
&d_{\mu\varphi} = d_{\mu\varphi}(a,n) \ =_{\text{def}} \ a = \ell \circ \subseteq : \\
&\qquad \{A \times \mathbb{N} : \varphi\} \xrightarrow{\subseteq} A \times \mathbb{N} \xrightarrow{\ell} A \text{ and} \\
&\widehat{\mu}\varphi = \widehat{\mu}\varphi(a,n) \ =_{\text{def}} \ \min\{m \leq n : \varphi(a,m)\} : \\
&\qquad \{A \times \mathbb{N} : \varphi\} \subseteq A \times \mathbb{N} \to \mathbb{N}
\end{aligned}
$$

Comment:

- This definition of $\mu\varphi : A \rightharpoonup \mathbb{N}$ is a *static* one. The subset-enumeration of *defined arguments* is here given just by the "problem" $\varphi \subset A \times \mathbb{N}$ itself: **Assume** you know already an $a \in A$ coming with a "solution" $n \in \mathbb{N} : (a,n) \in \varphi$. **Then** $\mu\varphi(a)$ is defined, and $\mu\varphi(a)$ is the minimal $m \leq n$ such that $\varphi(a,m)$.

- If you want to make visible the defined arguments enumeration by a p. r. enumeration $d : \mathbb{N} \to A$, you may take codomain restriction $\mathbb{N} \to \{A \times \mathbb{N} : \varphi\}$ of Cantor count $\text{ct} : \mathbb{N} \to A \times \mathbb{N}$ followed by left projection, enumerating those arguments $a \in A$ for which "terminating" n are "given".

- No need – and in general no "direct" possibility – to *decide,* for a given $a \in A$, **if** a is of form $a = d_{\mu\varphi}(a,n)$ with $(a,n) \in D_{\mu\varphi}$ i. e. if *exists* $n \in \mathbb{N}$ such that $\varphi(a,n)$. In particular, if

  $$D_{\mu\varphi} = \{A \times \mathbb{N} : \varphi\} = \emptyset_{A \times \mathbb{N}},$$

  then $d_{\mu\varphi}$ as well as $\widehat{\mu}\varphi$ are empty maps.

μ-**Lemma:** $\widehat{\mathbf{S}}$ admits the following (free-variables) scheme (μ) combined with ($\mu!$) – *uniqueness* – as a characterisation of the μ-operator $\langle \varphi : A \times \mathbb{N} \to \mathbf{2} \rangle \mapsto \langle \mu\varphi : A \rightharpoonup \mathbb{N} \rangle$ above:

$$(\mu) \quad \frac{\varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbf{2}\ \mathbf{S} \text{ map ("predicate"),}}{\begin{array}{l} \mu\varphi = \langle (d_{\mu\varphi},\, \widehat{\mu}\varphi) : D_{\mu\varphi} \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N} \\ \qquad \text{is an } \widehat{\mathbf{S}}\text{-map such that} \\ \mathbf{S} \vdash \varphi(d_{\mu\varphi}(\hat{a}), \widehat{\mu}\varphi(\hat{a})) = \mathrm{true}_{D_{\mu\varphi}} : D_{\mu\varphi} \to \mathbf{2}, \\ + \text{ "argumentwise" minimality:} \\ \mathbf{S} \vdash [\varphi(d_{\mu\varphi}(\hat{a}), n) \implies \widehat{\mu}\varphi(\hat{a}) \le n] : D_{\mu\varphi} \times \mathbb{N} \to \mathbf{2} \end{array}}$$

as well as uniqueness by *maximal extension*:

$$(\mu!) \quad \frac{\begin{array}{l} f = f(a) : A \rightharpoonup \mathbb{N} \text{ in } \widehat{\mathbf{S}} \text{ such that} \\ \mathbf{S} \vdash \varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) = \mathrm{true}_{D_f} : D_f \to \mathbf{2} \\ \mathbf{S} \vdash \varphi(d_f(\hat{a}), n) \implies \widehat{f}(\hat{a}) \le n : D_f \times \mathbb{N} \to \mathbf{2} \end{array}}{\mathbf{S} \vdash f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N} \quad \text{(inclusion of graphs)}}$$

[Requiring this maximality of $\mu\varphi$ is *necessary* since – for example – (μ) alone is fulfilled already by the *empty* partial function $\emptyset_A : A \rightharpoonup \mathbb{N}$]

Proof of $\mu\varphi : A \rightharpoonup \mathbb{N}$ to satisfy upper, "existence" part "(μ)" of the scheme is straigthforward by definition of $\mu\varphi$. What remains to be proved is uniqueness-by-maximal-extension scheme ($\mu!$) :

Let a partial map

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

be given such that f fullfills the antecedent of scheme $(\mu!)$. Then the **S** map

$$j = j(\hat{a}) := (d_f(\hat{a}), \widehat{f}(\hat{a})) : D_f \to A \times \mathbb{N}$$

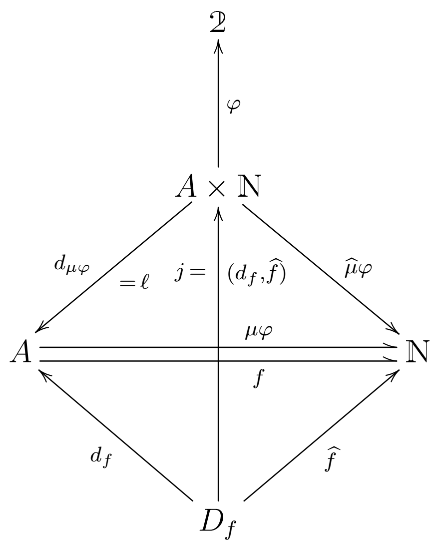
defines in fact, by the first premise on f, namely

$$\varphi(d_f(\hat{a}), \widehat{f}(\hat{a})) = \mathrm{true}_{D_f}(\hat{a}) : D_f \to \mathbb{2}$$

an **S**-map $j : D_f \to \{A \times \mathbb{N} : \varphi\}$ which establishes the wanted graph inclusion

$$j : [f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}]$$

as shows the following (commuting) $\mathbf{S}/\widehat{\mathbf{S}}$-DIAGRAM:



μ-applied-to-**S**-predicates DIAGRAM

Here, by definition of $\widehat{\mu}\varphi = \widehat{\mu}\varphi(a,n) : D_{\mu\varphi} = A \times \mathbb{N} \to \mathbb{N}$ we have in particular

$$\begin{aligned}&\widehat{\mu}\varphi \circ j(\hat{a}) = \widehat{\mu}\varphi(d_f(\hat{a}), \widehat{f}(\hat{a}))\\ &= \min\{m \leq d_f(\hat{a}) : \varphi(d_f(\hat{a}), m)\} : D_f \to A \times \mathbb{N} \to \mathbb{N}\\ &= \widehat{f}(\hat{a}) : D_f \to \mathbb{N}\end{aligned}$$

The latter by assumed *minimum property* of

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times \mathbb{N} \rangle : A \rightharpoonup \mathbb{N}$$

Together with (trivial)

$$d_{\mu\varphi} \circ j = \ell_{A,\mathbb{N}} \circ (d_f, \widehat{f}) = d_f : D_f \to A \times \mathbb{N} \to A$$

this gives in fact (remaining) *graph-inclusion* $f \mathrel{\widehat{\subseteq}} \mu\varphi : A \rightharpoonup \mathbb{N}$ via $j = (d_f, \widehat{f}) : D_f \to D_{\mu\varphi} = A \times \mathbb{N}$ **q. e. d.**

Remark: Within Peano-Arithmétique **PA** and hence also within **set** theory, our $\mu\varphi : A \rightharpoonup \mathbb{N}$ equals

$$\mu\varphi = \langle (\subseteq, \widehat{\mu}\varphi) : \hat{A} \to A \times \mathbb{N} \rangle : A \supset \hat{A} \to \mathbb{N}$$

with $\hat{A} = \{\hat{a} \in A : \exists n\, \varphi(\hat{a}, n)\}$, and $\widehat{\mu}\varphi(\hat{a}) = \min\{m \in \mathbb{N} : \varphi(\hat{a}, m)\} : \hat{A} \to \mathbb{N}$ i. e. it is given there by the classical – partial – minimum definition. But this definition lacks *constructivity* since $\hat{A} \subseteq A$ is not p. r. decidable apriori.

What about the *converse direction* to μ-**Lemma** above? In fact:

Partial p. r. ≡ μ-recursion, Instance of Church's Thesis: Any *partial* **S**-map

$$f = \langle (d_f, \widehat{f}) : D_f \to A \times B \rangle : A \rightharpoonup B$$

is *represented* – within theory $\widehat{\mathbf{S}}$ – by an " $\widehat{=}$ " equal μ-**recursive** $\widehat{\mathbf{S}}$-map

$$g = (\widehat{f} \circ \mathrm{count}_{D_f}) \mathbin{\widehat{\circ}} \mu\varphi_f :$$
$$A \rightharpoonup \mathbb{N} \to D_f \to B$$
$$\varphi_f = \varphi_f(a, n) : A \times \mathbb{N} \to \mathbf{2} \text{ suitable, namely}$$
$$\varphi_f = \varphi_f(a, n) \ =_{\mathrm{def}} \ [a =_A \ d_f \circ \mathrm{count}_{D_f}(n)] :$$
$$A \times \mathbb{N} \to \mathbf{2} \text{ (p. r.)}$$

$\mathrm{count}_{D_f} : \mathbb{N} \to D_f$ being a Cantor type (p. r.) *count* of D_f .

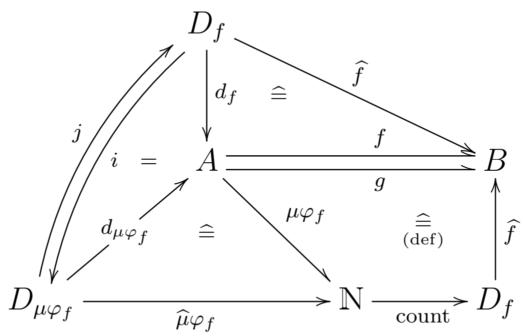
Remark:

$$\mathrm{count}_{D_f} = \mathrm{count}_{D_f}(n) : \mathbb{N} \to D_f = \{\mathbb{X} : D_f : \mathbb{X} \to \mathbb{2}\}$$

is easily constructed if D_f comes with a *point,* $\hat{a}_0 : \mathbb{1} \to D_f$ say. If not – or if you cannot name such point – just add one, namely injection $\iota : \mathbb{1} \to \mathbb{1} + D_f$ into the sum, replace D_f by $\mathbb{1} + D_f$, A by $\mathbb{1} + A$, B by $\mathbb{1} + B$, d_f by $\mathbb{1} + d_f : \mathbb{1} + D_f \to \mathbb{1} + A$, $\widehat{f}$ by $\mathbb{1} + \widehat{f} : \mathbb{1} + D_f \to \mathbb{1} + B$, and keep track of the added point.

D_f is "now" pointed, and admits – because of this – a retraction $\mathrm{count}_{D_f} : \mathbb{N} \to D_f$ by linear (well) order on D_f inherited from that of $\mathbb{X}$ and anchored at D_f's *point,* "defined element" $\hat{a}_0 : \mathbb{1} \to D_f \subseteq \mathbb{X}$.

Proof of partials to be μ-recursive maps: Consider the following $\mathbf{S}/\widehat{\mathbf{S}}$-DIAGRAM:



Partial p. r. map $\equiv$ μ-recursion DIAGRAM

All sets and (partial) maps in this DIAGRAM have been defined above with the exception of **S** *comparison maps* $i : D_f \to D_{\mu_f}$ and j in the other direction.

We define these two maps "suitably" by

$D_{\mu\varphi_f} \ =_{\text{by def}} \ \{A \times \mathbb{N} : \varphi_f\}$

$\ =_{\text{by def}} \ \{(a, n) : d_f \circ \text{count}_{D_f}(n) \ =_A \ a\},$

$i = i(\hat{a}) \ =_{\text{def}} \ (d_f(\hat{a}), \min\{m \leq n : d_f(\text{count}_{D_f}) \ =_A \ d_f(\hat{a})\} :$

$D_f \to D_{\mu\varphi_f}$

and

$j = j(a, n) \ =_{\text{def}} \ \text{count}_{D_f}(\min\{m \leq n : d_f(\text{count}(m)) = a\}) :$

$A \times \mathbb{N} \supseteq D_{\mu\varphi_f} \to D_f$

By definition of $\varphi_f : A \times \mathbb{N} \to \mathbb{2}$ and then – general for such a predicate, see above – of

$$\mu\varphi_f = \langle(d_{\mu\varphi_f}, \widehat{\mu}\varphi_f) : D_{\mu\varphi} \to A \times \mathbb{N}\rangle : A \rightharpoonup \mathbb{N}$$

and – eventually – (alleged) *representant*

$$g =_{\text{def}} \widehat{f} \circ \text{count}_{D_f} \,\widehat{\circ}\, \mu\varphi_f : A \rightharpoonup \mathbb{N} \to D_f \to B$$

of f, this $\widehat{\mathbf{S}}$-DIAGRAM commutes; μ-recursive *representant* involves just (two) **S**-maps, namely p. r. retraction count $= \text{count}_{D_f} : \mathbb{N} \to D_f$ and *rule* $\widehat{f} : D_f \to B$ (given), as well as one genuinely μ-recursive map $\mu\varphi_f : A \rightharpoonup \mathbb{N}$: μ-recursion applied to **S**-predicate $\varphi_f : A \times \mathbb{N} \to \mathbf{2}$. Commutativity of this $\widehat{\mathbf{S}}$-DIAGRAM shows

$$i : [f \,\widehat{\subseteq}\, g : A \rightharpoonup B],\ j : [g \,\widehat{\subseteq}\, f : A \rightharpoonup B]$$

and hence $f \,\widehat{=}\, g : A \rightharpoonup B$.

An arbitrary *partial* p. r. map $f : A \rightharpoonup B$ in $\widehat{\mathbf{S}}$ admits within $\widehat{\mathbf{S}}$ a representation $g : A \rightharpoonup B$, obtained via suitable **S**-map(s) and one μ-recursive one, $\mu\varphi_f : A \rightharpoonup \mathbb{N}$, defined in turn "over" the **S**-predicate $\varphi_f : A \times \mathbb{N} \to \mathbf{2}$ above **q. e. d.**

Corollary: define theory $\mu\mathbf{S}$ over **S** and within $\widehat{\mathbf{S}}$ by closure of **S** under the μ-operator – applied to **S**-predicates – *merged* with monoidal-theory closure. Then this *subtheory* $\mu\mathbf{S}$ is in fact isomorphic to theory $\widehat{\mathbf{S}}$ as a diagonal monoidal theory: $\mathbf{S} \subset \mu\mathbf{S} \cong \widehat{\mathbf{S}}$.

Both theories have cartesian p. r. theory **S** embedded as a diagonal monoidal subcategory, and the embedding is compatible with the isomorphism $\mu\mathbf{S} \cong \widehat{\mathbf{S}}$.

Our **conclusion** so far is:

- We can *eliminate formal existential quantification* – as well as (individual, formal) *variables* – from the theory of μ-recursion by interpreting the μ-operator into theory $\widehat{\mathbf{S}} \supset \mathbf{S}$ of *partial* p. r. maps.

- Conversely, the μ-operator when applied to **S**-predicates: p. r. predicates $\varphi = \varphi(a, n) : A \times \mathbb{N} \to \mathbb{2}$, *generates* all $\widehat{\mathbf{S}}$-morphisms – *partial* **S**-maps – out of **S** via necessarily formally *partial* composition with suitable **S**-maps.

7.4 Content driven loops

By a *content driven* loop we mean an *iteration* of a given *step endo map* whose number of performed steps is not known at *entry time* into the *loop* – as is the case for a p. r. iteration $f^{\S}(a, n) : A \times \mathbb{N} \to A$ with *iteration number* $n \in \mathbb{N}$ – , but whose (re) entry into a "new" endo step $f : A \to A$ depends on *content* $a \in A$ reached so far:

This *(re) entry* or *exit* from the loop is now *controlled* by an **S** predicate $\chi = \chi(a) : A \to \mathbf{2}$.

Example: A `while` loop $\mathrm{wh}\,[\chi : f] : A \rightharpoonup A$ for given p. r. *control* predicate $\chi = \chi(a) : A \to \mathbf{2}$ and *(looping) step* endo $f : A \to A$.

Classically, *with* variables, such $\mathrm{wh} = \mathrm{wh}\,[\chi : f]$ would be "defined" – in *pseudocode* – by

$\mathrm{wh}(a) :=$

$[a' := a;$

`while` $\chi(a')$

`do` $a' := f(a')$ `od`;

`result` $:= a']$

The formal version of this – within a *classical,* element based setting

–, is the following partial-(PEANO)-map characterisation:

$$\mathrm{wh}(a) = \mathrm{wh}[\chi : f](a) = \begin{cases} a \text{ if } \neg\chi(a) \\ \mathrm{wh}(f(a)) \text{ if } \chi(a) \end{cases} \quad : A \rightharpoonup A$$

But can this *dynamical* or *bottom up* "definition" be converted into a p. r. *enumeration* of a suitable *graph* "of all *argument-value pairs*" in terms of an $\widehat{\mathbf{S}}$-morphism

$$\begin{aligned} &\mathrm{wh} = \mathrm{wh}[\chi : f] \\ &= \langle (d_{\mathrm{wh}}, \widehat{\mathrm{wh}}) : D_{\mathrm{wh}} \to A \times A \rangle : A \rightharpoonup A? \end{aligned}$$

In fact, we can give such suitable static **definition** of $\mathrm{wh} = \mathrm{wh}[\chi : f] : A \rightharpoonup A$ within $\widehat{\mathbf{S}} \supset \mathbf{S}$ as follows:

$$\begin{aligned} &\mathrm{wh} \;=_{\mathrm{def}}\; f^{\S} \,\widehat{\circ}\, (\mathrm{id}_A, \mu\varphi_{\,[\chi:f]}) \\ &= f^{\S} \,\widehat{\circ}\, (A \times \mu\varphi_{\,[\chi:f]}) \,\widehat{\circ}\, \Delta_A : \\ &A \to A \times A \rightharpoonup A \times \mathbb{N} \to A, \text{ where} \\ &\varphi = \varphi_{\,[\chi:f]}(a, n) \;=_{\mathrm{def}}\; \neg\,\chi\, f^{\S}(a, n) : A \times \mathbb{N} \to A \to \mathbb{2} \to \mathbb{2} \end{aligned}$$

Within a quantified arithmetical theory like **PA**, this $\widehat{\mathbf{S}}$-definition of $\mathrm{wh}\,[\chi : f] : A \rightharpoonup A$ fullfills the classical characterisation quoted above, as is readily shown by Peano-Induction "on" $n := \mu\varphi_{\,[\chi:f]}\,(a) : A \rightharpoonup \mathbb{N}$, at least within **PA** and its extensions.

[Classically, *partial definedness* of this – *dependent* – induction parameter n causes no problem: use a *case distinction* on definedness of $\mu\varphi_{[\chi:f]}(a)$"$\in$"$\mathbb{N}$. Even in our quantifier-free context such *dependent induction* on a *partial* dependent induction parameter is available.]

In this generalised sense, we have – within theory $\widehat{\mathbf{S}} \supset \mathbf{S}$ – **all** `while` loops, at least those with *control* $\chi : A \to \mathbf{2}$ and *step* endo $f : A \to A$ within $\mathbf{S}$.

It is obvious that such $\mathrm{wh}[\chi : f] : A \rightharpoonup A$ is in general only *partial* – as is trivially exemplified by integer division by *divisor* 0 which would be endlessly subtracted from the dividend, although in this case *control* and *step* are both p. r.

By the classical characterisation of these `while` loops above, we are motivated for its generalisation to the $\mathbf{S}/\widehat{\mathbf{S}}$ case:

Characterisation Theorem for `while` loops *over* $\mathbf{S}$ within theory $\widehat{\mathbf{S}}$: For $\chi : A \to 2$ (*control*) and $f : A \to A$ (*step*) both $\mathbf{S}$-maps, `while` loop $\mathrm{wh} = \mathrm{wh}[\chi : f] : A \rightharpoonup A$ (as defined above) is characterised by the following *implications* within $\widehat{\mathbf{S}}$:

$$\widehat{\mathbf{S}} \vdash [\neg\,\chi \circ a \implies \mathrm{wh} \,\widehat{\circ}\, a = a] : A \rightharpoonup \mathbf{2} \text{ and}$$
$$\widehat{\mathbf{S}} \vdash [\chi \circ a \implies \mathrm{wh} \,\widehat{\circ}\, a = \mathrm{wh} \,\widehat{\circ}\, f \circ a] : A \rightharpoonup \mathbf{2}$$

where use of "sort of" free variable 'a' is to help intuition, *formally* a is just another name for $\mathrm{id}_A : A \to A$.

That $\mathrm{wh} \mathrel{\widehat{=}} \mathrm{wh} \,\widehat{\circ}\, a : A \rightharpoonup A$ fullfills the implications of (alleged) characterisation is obvious. We omit the proof of wh to be unique with these properties within theory $\widehat{\mathbf{S}}$.

7.5 A further case of Church's Thesis

- The concept of a partial p. r. map is equivalent to that of a μ-recursive (partial) map. It is another – free-variables, formally:

variable-free – notion of a *general recursive (partial) map.*

All this in (and over) the categorical framework of *cartesian p. r. theory* **S** with (scheme of) abstraction of its predicates – as well as with equality *predicates* on its sets.

- Same for `while` loops $\mathrm{wh} = \mathrm{wh}\,[\chi : f] : A \rightharpoonup \mathbf{2}$: They obviously *generate* all μ-recursive (partial) maps: For given p. r. predicate $\varphi : A \times \mathbb{N} \to \mathbf{2}$

$$\mu\varphi \mathrel{\widehat{=}} \mathrm{r} \,\widehat{\circ}\, \mathrm{wh}\,[\neg\,\varphi : (A \times \mathrm{s})] :$$
$$A \times \mathbb{N} \rightharpoonup A \times \mathbb{N} \to \mathbb{N}$$

satisfies the characteristic implications for the μ-operator.

Therefore the `while`-operator wh generates all *partial* maps in $\widehat{\mathbf{S}} \supset \mathbf{S}$, even in just one step out of predicate/endo pairs

$\chi : A \to \mathbf{2}$ and $f : A \to A$ in **S**,

see reduction of partial partial p. r. maps to partial p. r. maps.

- Theory $\widehat{\mathbf{S}}$ is closed under the `while` operator, as it is – and because it is – under the μ-operator.

- A formal consequence of the last two assertions is in particular a fact known since long time to Computer Scientists: "one `while` loop is enough", starting from suitable `for` loop programs to define **S**-maps $\chi : A \to \mathbb{2}$ and $f : A \to A$, "data" for `while` loop $\mathrm{wh}\,[\chi : f] : A \rightharpoonup A$.

  Since `for` loops – equivalent to p. r. maps – can in turn be written as (trivial) `while` loops, `while` **closure** of the fundamental maps: 0, s as well as substitutions – *logical functions* in the sense of EILENBERG/ELGOT 1970 – reaches all of $\mu\mathbf{S}$, but presumably not in `while` nesting depth 1, as is the case when starting with all `for` loops. My guess: for such a one-step closure by the `while` operator you need *case distinctions,* and these come in here – formally – as p. r. maps on their own right, namely as *induced* maps out of a *sum* $A \xrightarrow{\iota} A + B \xleftarrow{\kappa} B$

From a logical point of view, there are – at least – the following

Arithmetics Complexity Problems

- Does theory **PR** admit *strict, consistent* strengthenings or is it a *simple theory,* will say that it admits its given notion of equality and the indiscrete (inconsistency) equality as only "congruences"?, cf. a simple *group* which has as *normal subgroups* only itself and $\{1\}$. Because of reasons to be explained later, my guess is: **PR** *admits* non-trivial strengthenings, in particular I suppose that the p. r. *trace* of **PA** is a strict strengthening of **PR**. But this only, if **PA** is consistent.

- Already at start we possibly have such a strengthening: If free-variables ("free variables" in the classical sense) *primitive recursive arithmetic* **PRA** is defined to have as its terms all map terms obtainable by the (full) scheme of primitive recursion, and as formulae just the *defining equations* for the maps introduced by that scheme, then I see no way to prove all of the usual semiring equations for $\mathbb{N}$:

We *need* Freyd's uniqueness (FR!) of the *initialised iterated:* From this HORN clause we can show (!) in particular GOODSTEIN's uniqueness rules U_1 to U_4 upon which *his* proof of the semiring properties of $\mathbb{N}$ is based. He takes these rules as **axioms.**

My guess is – if I have understood right the definition of **PRA**, that **PR** is a strict strengthening of **PRA** at least if there is no "underground" connection to the set theoretic view of maps as (possibly infinite) *argument-value lists.*

- **Conjecture:** *Iterative descent* theory $\pi\mathbf{R}$ in subsequent chapters, defined over theory **PR** by **axiom** of *non-infinite iterative descent,* is a *simple* p. r. theory.

  At least this should be the case for (formally) stronger theory $\Omega\mathbf{R}$ of *complexity controlled iteration* with *complexity values* in (linearily) ordered semiring $\Omega = \mathbb{N}[\omega_1, \omega_2, \ldots]$ of polynomials in several variables.

Part II

EVALUATION

Chapter 8

Evaluation

We consider codes and coding of p. r. maps, more precisely: of maps (and predicates) of theory **S** of primitive recursion with specific boolean truth algebra **2** and predicate-into-subset abstraction. We *evaluate* these map codes on their (fitting) arguments back into theory **S**. This coding and evaluation takes place in p. r. theory **S** as well as in finite iterative descent theory $\pi\mathbf{R} = \mathbf{S} + (\pi)$ which strengthens theory **S**.

Evaluation is introduced as a CCI, a *Complexity Controlled Iteration,* a special `while` loop which cannot loop endlessly as such – additional **axiom** (π) below.

Evaluation ε of **PR2** map codes turns out to be *objective* – as far as terminating – it *reflects* "concrete" map codes $\ulcorner f : A \to B \urcorner$ into the respective maps: $\varepsilon(\ulcorner f \urcorner, \_\,) = f$.

S's notion of *equality* between *maps* has an "internal" homologue: enumerated *internal* **S** equality $f \check{\doteq} g$ between *codes.*

Arithmetically central **theorem,** on *termination conditioned soundness,* lets evaluation turn each internal equality of **S** into an objective predicative equality, provided that *deduction tree* evaluation terminates on the (internal) deduction tree for that internal equation.

Iterative descent theory $\pi\mathbf{R}$ is **defined** by adding the **axiom** schema of *non-infinite descent* of CCI's: Complexity Controlled Iterations.

8.1 Universal sets

8.1.1 Strings and polynomials

Strings $a = a_0\, a_1\, \ldots\, a_n$ of natural numbers are coded as *prime power products*

$$2^{a_0} \cdot 3^{a_1} \cdot \ldots \cdot p_n^{a_n} \in \mathbb{N}_> = \mathbb{N}_{>0} \subset \mathbb{N}$$

iteratively defined as

$$((2^{a_0} \cdot 3^{a_1}) \cdot \ldots) \cdot p_n^{a_n} \in \mathbb{N}_{>0}$$

Euclidean *projection* family

$$\pi = \pi_j(a) : \mathbb{N} \times \mathbb{N}_> \to \mathbb{N},$$

is characterised by

$$a = p_0^{\pi_0(a)} \cdot p_1^{\pi_1(a)} \cdot \ldots \cdot p_a^{\pi_a(a)}$$

It *evaluates/interpretes* “code” $a \in \mathbb{N}_>$ into string

$\pi_0(a)\, \pi_1(a) \ldots \pi_a(a)$,

in general many trailing zeros.

Strings are identified with/interpreted as “their” *polynomials*

$p(X) \equiv 0$ or

$$p(X) = \sum_{j=0}^{n} a_j X^n = a_0 X^0 + \ldots + a_n X^n, \ a_n \neq 0,$$

degree $\deg p(X) = n$

$p(\omega) \equiv 0$ or

$$p(\omega) = \sum_{j=0}^{n} a_j \omega^j = a_0 + a_1 \omega^1 + \ldots + a_n \omega^n, \ a_n > 0,$$

ω an indeterminate for (arbitrarily) big natural numbers.

Addition (and truncated subtraction as well as equality) are defined *coefficientwise*, and product as Cauchy product (folding)

$$p(X) \cdot q(X) = (\sum_{i=0}^{m} a_i X^i) \cdot (\sum_{j=0}^{n} b_j X^j) \ =_{\mathrm{def}} \ \sum_{k=0}^{m+n} a_i b_{k \smallsetminus i} X^k$$

What we need in the sequel is special product

$$p(\omega) \cdot \omega = (\sum_{j=0}^{n} \omega^j) \cdot \omega = \sum_{j=0}^{n} a_j \omega^{j+1}$$

Order of polynomials is first by *degree,* second by *pivot coefficient,* and then – if these are equal – by comparison of the two polynomials with their equal pivot monomes removed, recursively, down to the zero polynomial (which has no degree).

Call $\mathbb{N}[\omega]$ the linearily ordered semiring of (coefficient strings) of these polynomials.

The linear order has – intuitively and *formally* within **set** theory – only *finite descending chains.*

8.1.2 Internal numerals

Numeralisation family ν is p. r. defined within **S** by

$$\nu(\text{false}) = \nu_{\mathbf{2}}(\text{false}) = \ulcorner\text{false}\urcorner :$$
$$\mathbb{1} \equiv \{\mathbb{1} : \text{true}_{\mathbb{1}}\} \to \mathbf{2}^{\mathbb{1}} \subset \text{PR2} \subset \mathbb{N}$$
gödel number of false
$$\nu(\text{true}) = \nu_{\mathbf{2}}(\text{true}) = \ulcorner\text{true}\urcorner : \mathbb{1} \to \mathbf{2}^{\mathbb{1}} \subset \text{PR2}$$
gödel number of true
$$\nu(0) = \ulcorner 0 \urcorner : \mathbb{1} \to \mathbb{N}^{\mathbb{1}} \subset \text{PR2}$$
gödel number, `utf8` code of 0
$$\nu(1) = \ulcorner(\urcorner \ulcorner \text{s}\urcorner \ulcorner \circ \urcorner \ulcorner 0 \urcorner \ulcorner)\urcorner$$
$$= \ulcorner(\urcorner * \ulcorner \text{s}\urcorner * \ulcorner \circ \urcorner * \ulcorner 0 \urcorner * \ulcorner)\urcorner : \mathbb{1} \to \mathbb{N}^{\mathbb{1}}$$
string concatenation of symbol codes
$$\nu(n+1) = \langle \ulcorner \text{s} \urcorner \odot \nu(n) \rangle \in \mathbb{N}^{\mathbb{1}}$$

where $\odot \equiv \ulcorner\circ\urcorner,\ \langle \equiv \ulcorner(\urcorner,\ \rangle \equiv \ulcorner)\urcorner$

This internal numeralisation distributes the "elements" of **2** and numbers of NNO $\mathbb{N}$ over $\mathbb{N} \equiv \{\mathbb{N} : \text{true}_{\mathbb{N}}\}$, with suitable gaps to receive in particular the codes of any other symbols of object language **S**.

Numeralisation extends to all objects A of **PR2** and then to the sets of **S** recursively as follows:

$$\begin{aligned}
&\nu_{\mathbb{1}} = \ulcorner \mathrm{id}_{\mathbb{1}} \urcorner : \mathbb{1} \to \mathbb{1}^{\mathbb{1}} \\
&\ \subset \mathrm{PR2} \subset \mathrm{S} \subset \mathbb{N} \equiv \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\} \\
&\nu_{A\times B} = \nu_{A\times B}(a,b) = \langle \nu_A(a); \nu_B(b) \rangle : \\
&\quad A \times B \to (A \times B)^{\mathbb{1}} \subset \mathrm{PR2} \subset \mathrm{S} \\
&\nu_{\{A:\chi\}}(a) = \nu_A(a) : \{A : \chi\} \to \{A : \chi\}^{\mathbb{1}} \subset \mathrm{S}
\end{aligned}$$

Numerals predicate Lemma

Enumeration $\nu : \mathbb{N} \to \mathbb{N}$ (out of **PR2**) defines a characteristic p. r. *image* predicate $\mathrm{im}[\nu] : \mathbb{N} \to \mathbf{2}$ (out of **PR2**), and by this **S** set

$$\begin{aligned}
&\dot{\mathbb{N}} = \nu\mathbb{N} = \{\mathbb{N} : \mathrm{im}[\nu]\} \subset \mathbb{N} \equiv \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}
\end{aligned}$$

of (enumerated) internal numerals

Proof: Use iterative '$\vee$' for definition of

$$\begin{aligned}
&\mathrm{im}[\nu] : \mathrm{im}[\nu](c) \\
&\ = [c = \nu(0)] \vee [c = \nu(1)] \vee [c = \nu(2)] \vee \ldots \vee [c = \nu(n)] \\
&\ = \max\{n : \nu(n) \ \leq \ c\} : \mathbb{N} \to \mathbb{N}
\end{aligned}$$

$\nu : \mathbb{N} \to \mathbb{N}$ has retractive codomain restriction

$$\dot{\nu} : \mathbb{N} \to \dot{\mathbb{N}} = \{\mathbb{N} : \mathrm{im}[\nu]\}$$

and is an iso with p. r. inverse

$$\dot{\nu}^{-1} = \dot{\nu}^{-1}(c) = \min\{n : n \leq c \wedge \nu(n) = c\} : \dot{\mathbb{N}} \xrightarrow{\cong} \mathbb{N} \ \ \mathbf{q.\,e.\,d.}$$

Extend these **definitions** to numeralisation – within **S** – of **PR2** products:

A, B **PR2** objects, $\nu_A : A \to A^{\mathbb{1}},\ \nu_B : B \to B^{\mathbb{1}}$ given
(first given for $A = B = \mathbb{N}$ in **PR2**)

$\nu_{A\times B}(a,b) \ =_{\text{def}}\ \langle \nu_A(a); \nu_B(b)\rangle : A \times B \to (A \times B)^{\mathbb{1}}$

Retractive codomain restriction $\dot{\nu} : \mathbb{N} \to \dot{\mathbb{N}}$ is extended to **PR2** products as follows:

A, B **PR2** objects, $\nu_A : A \to A^{\mathbb{1}},\ \nu_B : B \to B^{\mathbb{1}}$
$\dot{A} = \nu A = \{\mathbb{N} : \mathrm{im}[\nu_A]\},\ \dot{B} = \nu B = \{\mathbb{N} : \mathrm{im}[\nu_B]\}$
$\dot{\nu}_A : A \xrightarrow{\cong} \dot{A} = \nu A,\ \ \dot{\nu}_B : B \xrightarrow{\cong} \dot{B} = \nu B$ given
(first given for $A = B = \mathbb{N}$)

$\mathrm{im}[\nu_{A\times B}](c) = \max\{n : \nu(n)\ \le\ c\} : \mathbb{N} \to \mathbb{N}$
$\dot{\nu}^{-1}_{A\times B}(c)$
$= \mathrm{ct}_{A\times B} \min\{n \le c : \nu_{A\times B}(n) = c\} :$
$\quad \nu(A \times B) \xrightarrow{\cong} A \times B \equiv \{A \times B : \mathrm{true}_{A\times B}\}$
$\subset \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}$

Numeralisation extension to S sets

Extend numeralisation definition to predicative subsets by

$$\begin{aligned}
\nu_{\{A:\chi\}}(a) &=_{\text{def}} \nu_A(a) : \{A:\chi\} \to \{A:\chi\}^{\mathbb{1}} \subset \{\mathbb{N} : \text{true}\} \\
\dot{\nu}\{A:\chi\} &=_{\text{def}} \{\mathbb{N} : \text{im}[\nu_{\{A:\chi\}}]\} \subset \{A:\chi\}^{\mathbb{1}} \text{ where}
\end{aligned}$$

$$\text{im}[\nu_{\{A:\chi\}}](c) =_{\text{def}} \vee_{n \le c}\,[c = \nu_{\{A:\chi\}}(n)] : \mathbb{N} \to \mathbf{2},$$

χ supposed pointed, $\chi(\boldsymbol{a}_0) = \text{true}$

for a given point $\boldsymbol{a}_0 : \mathbb{1} \to A$, usually

$$\boldsymbol{a}_0 = 0_A,\ 0_{\mathbb{N}} = 0,\ 0_{\mathbb{1}} = \text{id}_{\mathbb{1}},\ 0_{A \times B} = (0_A, 0_B)$$

$$\dot{\nu}^{-1}_{\{A:\chi\}}(c) = \text{ct}_{\{A:\chi\}} \min\{n \le c : \nu_{\{A:\chi\}}(n) = c\} :$$

$$\nu(\{A:\chi\}) \xrightarrow{\cong} \{A:\chi\} \subset \text{S} \subset \{\mathbb{N} : \text{true}_{\mathbb{N}}\}$$

$\dot{\nu}^{-1}_{\{A:\chi\}} : \nu\{A:\chi\}$ in fact inverse to

$$\nu_{\{A:\chi\}} : \{A:\chi\} \xrightarrow{\cong} \nu\{A:\chi\} \subset \{\mathbb{N} : \text{true}_{\mathbb{N}}\}$$

8.1.3 Universal set of internal pairs

Define *universal sets*

$$\mathbb{X} = \{\mathbb{N} : \mathbb{X}\} = \dot{\bigcup}_{A \text{ in } \mathbf{PR}} \dot{A} \subset \mathbb{N} \text{ and}$$

$$\mathbb{X}_{\mathbf{2}} = \dot{\bigcup}_{A \text{ in } \mathbf{PR2}} \dot{A} = \bigcup_{\{A:\chi\} \text{ in } \mathbf{S}} \nu\{A:\chi\} \subset \mathbb{N}$$

of all *numerals* and (possibly nested) *numpairs/logic numpairs* first by p. r. enumeration.

Here is the enumeration of $\mathbb{X}_{\mathbf{2}}$ (extending that of $\mathbb{X}$):

$$\nu_{\mathbf{2}}(\text{false}) = \ulcorner\text{false}\urcorner,\ \nu_{\mathbf{2}}(\text{true}) = \ulcorner\text{true}\urcorner \in \mathbb{X}_{\mathbf{2}}$$
$$\nu(0) \ =_{\text{by def}}\ \ulcorner 0 \urcorner \in \mathbb{X} \subset \mathbb{X}_{\mathbf{2}}$$
$$n \in \mathbb{N} \Rightarrow \nu(s\,n)) = \langle \ulcorner\text{s}\urcorner \odot \nu(n)\rangle \in \mathbb{X} \subset \mathbb{X}_{\mathbf{2}}$$
$$x \in \mathbb{X} \wedge y \in \mathbb{X} \Rightarrow \langle x; y\rangle \in \mathbb{X} \subset \mathbb{X}_{\mathbf{2}}$$
$$x \in \mathbb{X}_{\mathbf{2}} \wedge y \in \mathbb{X}_{\mathbf{2}} \Rightarrow \langle x; y\rangle \in \mathbb{X}_{\mathbf{2}}$$

These enumerations have characteristic p. r. image predicates $\mathbb{X} = \mathbb{X}(c) : \mathbb{N} \to \mathbf{2}$, $\mathbb{X}_{\mathbf{2}} = \mathbb{X}_{\mathbf{2}}(c) : \mathbb{N} \to \mathbf{2}$ defined as follows:

$$\mathbb{X}(c) = \begin{cases} \text{true if } \vee_{n \le c}\, \text{ct}_{\mathbb{X}}(n) = c \\ \text{false otherwise, i. e. if } \wedge_{n \le c}\, \text{ct}_{\mathbb{X}}(n) \ne c \end{cases}$$
$$\mathbb{X}_{\mathbf{2}}(c) = \begin{cases} \text{true if } \vee_{n \le c}\, \text{ct}_{\mathbb{X}_{\mathbf{2}}}(n) = c \\ \text{false otherwise, i. e. if } \wedge_{n \le c}\, \text{ct}_{\mathbb{X}_{\mathbf{2}}}(n) \ne c \end{cases}$$

$\text{ct}_{\mathbb{X}} : \mathbb{N} \to \mathbb{N}$, $\text{ct}_{\mathbb{X}_{\mathbf{2}}} : \mathbb{N} \to \mathbb{N}$ are the p. r. enumeration/counting processes given by cyclic application of the rules above generating $\mathbb{X}$, $\mathbb{X}_{\mathbf{2}}$ as (predicative) sets:

Variable $c \in \mathbb{N}$ works in fact as an upper bound, since obviously $\text{ct}_{\mathbb{X}}(n), \text{ct}_{\mathbb{X}_{\mathbf{2}}}(n) > n,\ n \in \mathbb{N}$ free.

8.2 Gödelisation, map coding

Since boolean categorical p. r. theory **S** comes formally without variables and quantification, we can code **S** maps into NNO $\mathbb{N}$ simply by their LaTeX `unicode` source codes, the Byte strings seen as (binary) natural numbers, namely arrows $\ulcorner f \urcorner : \mathbb{1} \to \mathbb{N}$, *numbers* $\ulcorner f \urcorner \in \mathbf{S}(\mathbb{1}, \mathbb{N})$.

These codes enumerate internal theory $\mathrm{S} \subset \mathbb{N}$, in fact a predicative subset of $\mathbb{N}$ since later enumeration cycles insert longer code strings.

On the way are enumerated, predicatively defined, **internal hom sets,** code sets B^A into which are inserted the codes $\ulcorner f \urcorner$ for $f \in \mathbf{S}(A, B)$.

- Codes of basic maps

$\ulcorner \mathrm{ba} \urcorner \in \mathbb{N}$ for $\mathrm{ba} \in \mathrm{bas} = \mathrm{bas}_{\mathbf{PR2}} = \mathrm{bas}_{\mathbf{PR}} \cup \mathrm{bas}_{\mathbf{2}}$

$\mathrm{bas}_{\mathbf{PR}} = \{0, \mathrm{s}, \mathrm{id}_A, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B \text{ in } \mathbf{PR}\}$

$= \{0, \mathrm{s}, \mathrm{id}_A, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B \ \mathbf{PR} \text{ objects}\}$:

$\ulcorner 0 \urcorner = \mathtt{unicode}[0] \in \mathbb{N}^{\mathbb{1}} \subset \mathrm{PR2} \subset \mathbb{N}$,

$\ulcorner 0 \urcorner \in \mathrm{PR2}$ is a **PR2** *map code in set* $\mathbb{N}^{\mathbb{1}}$ *of map codes from* $\mathbb{1}$ *to* $\mathbb{N}$.

Analogously for the other basic map codes of **PR2** :

$\ulcorner \mathrm{s} \urcorner = \mathtt{unicode}[\backslash\mathtt{mathrm}\{\mathtt{s}\}] \in \mathbb{N}^{\mathbb{N}} \subset \mathrm{PR2} \subset \mathbb{N}$

$\ulcorner \mathrm{id} \urcorner_A = \ulcorner \mathrm{id}_A \urcorner \in A^A$

$\ulcorner \Pi \urcorner_A = \ulcorner \Pi_A \urcorner \in \mathbb{1}^A$

$\ulcorner \ell \urcorner_{A,B} = \ulcorner \ell_{A,B} \urcorner \in A^{A \times B}$

$\ulcorner \mathrm{r} \urcorner_{A,B} = \ulcorner \mathrm{r}_{A,B} \urcorner \in B^{A \times B} \subset \mathrm{PR2}$

and for the maps in

$\mathrm{bas}_{\mathbf{2}} = \{\mathrm{true}, \smallsetminus, \mathrm{sign}, \mathrm{pret}\}$:

$\ulcorner \mathrm{true} \urcorner \in \mathbf{2}^{\mathbb{1}}$

$\ulcorner \searrow \urcorner \in \mathbf{2}^{\mathbf{2}\times\mathbf{2}}$

$\ulcorner \mathrm{sign} \urcorner \in \mathbf{2}^{\mathbb{N}}$

$\ulcorner \mathrm{pret} \urcorner \in \mathbb{N}^{\mathbf{2}}$

- Coding map composition of **PR2** $\subset$ **S**:

  With $\odot = \ulcorner \circ \urcorner$

$$\frac{f : A \to B,\ g : B \to C}{\ulcorner (g \circ f) \urcorner = \langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle \in C^A}$$

  internal composition:

$$\frac{f \in B^A,\ g \in C^B}{\langle g \odot f \rangle = \ulcorner (\urcorner g \ulcorner \circ \urcorner f \ulcorner) \urcorner \in C^A}$$

  $\langle g \odot f \rangle \in \mathbb{N}$ is recognised as code $\ulcorner (g \circ f) \urcorner$ of the composition of maps g with f if f is "already" recognised as $f = \ulcorner f \urcorner$ and g as $g = \ulcorner g \urcorner$, recursively.

  Similar for the code cases below, this defines coding as an injective meta operation, and map code sets B^A by p. r. enumeration, turned a posteriori into predicative subsets of

  $\mathbb{N} \equiv \mathbf{I}\,\mathbb{N} = \{\mathbb{N} : \mathrm{true}_{\mathbb{N}}\}$ NNO of **S**.

- Coding **PR2** induced maps: with $\langle ; \rangle = \ulcorner (,) \urcorner$

$$\frac{f : C \to A,\ g : C \to B}{\ulcorner(f,g)\urcorner = \langle \ulcorner f \urcorner ; \ulcorner g \urcorner \rangle \in (A \times B)^C}$$

internal inducing:

$$\frac{f \in A^C,\ g \in B^C}{\langle f ; g \rangle = \ulcorner(\urcorner f \ulcorner , \urcorner g \ulcorner)\urcorner \in (A \times B)^C}$$

- Coding **PR2** map products (redundant): with $\# = \ulcorner\times\urcorner$

$$\frac{f : A \to A',\ g : B \to B'}{\ulcorner(f \times g)\urcorner = \langle \ulcorner f \urcorner \# \ulcorner g \urcorner \rangle \in (B \times B')^{A \times A'}}$$

Internal map product:

$$\frac{f \in A'^A,\ g \in B'^B}{\langle f \# g \rangle = \ulcorner(\urcorner f \ulcorner \times \urcorner g \ulcorner)\urcorner \in (A' \times B')^{A \times B}}$$

- Coding **PR2** endomap iteration: with $\$ = \ulcorner\S\urcorner$

$$\frac{f : A \to A}{\ulcorner f^{\S} \urcorner = \ulcorner f \urcorner^{\$} \in A^{A \times \mathbb{N}}}$$

internal iteration:

$$\frac{f \in A^A}{f^{\$} = f^{\ulcorner \S \urcorner} \in A^{A \times \mathbb{N}}}$$

- coding **S** maps between **S** abstraction sets

$$\frac{f : \{A : \chi\} \to \{B : \varphi\},\ \chi : A \to \mathbf{2},\ \varphi : B \to \mathbf{2} \text{ in } \mathbf{S}}{\begin{array}{l} \ulcorner((\chi, f), \varphi)\urcorner = \langle\langle\ulcorner\chi\urcorner; \ulcorner f\urcorner\rangle; \ulcorner\varphi\urcorner\rangle \in \{B : \varphi\}^{\{A:\chi\}} \\ \text{where } \{B : \varphi\}^{\{A:\chi\}} = \{\langle\langle\ulcorner\chi\urcorner; f\rangle; \ulcorner\varphi\urcorner\rangle : \\ f \in B^A \wedge \ulcorner\chi\urcorner \ulcorner\Rightarrow\urcorner\langle\ulcorner\varphi\urcorner \odot f \ulcorner=_A\urcorner \ulcorner\mathrm{true}_A\urcorner\rangle\} \\ \subset ((\mathbf{2}^A \times B^A) \times \mathbf{2}^B) \subset ((\mathrm{PR2} \times \mathrm{PR2}) \times \mathrm{PR2}) \end{array}}$$

Internal composition, internal map inducing into products as well as internal iteration map of internal endomaps for theory $\mathbf{S} = \mathbf{PR2}+$ (abstr) in place of **PR2** is readily obtained from the above.

8.3 Internal, arithmetised equality

Definition: The objective equality of **S** has an *internal-equality* (enumeration) analogon – a *list*

$$\begin{aligned}
&\mathrm{eq} = \mathrm{eq}(k) \ = \ \check{\doteq}_k : \mathbb{N} \to \mathrm{S} \times \mathrm{S} \subset \mathbb{N} \times \mathbb{N} \\
&k \mapsto \langle f \check{\doteq}_k g \rangle, \ \ k \in \mathbb{N} \text{ free}, \\
&\quad f = \ell \circ \mathrm{eq}(k), \ \ g = \mathrm{r} \circ \mathrm{eq}(k) : \mathbb{N} \to \mathrm{PR2} \ \textit{dependent} \text{ variables:} \\
&\quad \text{we have written } f \check{\doteq}_k g \text{ for} \\
&\mathrm{eq}(k) = (f, g) \in \mathrm{PR2} \times \mathrm{PR2} \subset \mathbb{N} \times \mathbb{N}, \\
&\quad k, f, g \in \mathbb{N} \text{ free}
\end{aligned}$$

This list $\langle f \check{\doteq}_k g \rangle : \mathbb{N} \to \mathrm{PR2} \times \mathrm{PR2}$ is given by "spiral form" p. r. count of internal deduction trees, for **example**

$$\mathrm{dtree}_k \quad = \quad \Uparrow \frac{f \check{\doteq}_k h}{\Uparrow \dfrac{f \check{\doteq}_i g}{\mathrm{dtree}_{ii} \quad \mathrm{dtree}_{ji}} \qquad \Uparrow \dfrac{g \check{\doteq}_j h}{\mathrm{dtree}_{ij} \quad \mathrm{dtree}_{jj}}}$$

$$i, j < k, \ \ ii, ji < i, \ \ ij, jj < j$$

Extra case of internally equal restrictions

$$f \check{\doteq}_k g \in \mathrm{S}(\{A : \chi\}, \{B : \varphi\})$$

$$\mathrm{dtree}_k \ = \ \frac{\langle\langle \ulcorner\chi\urcorner ; f\rangle ; \ulcorner\varphi\urcorner\rangle \check{\doteq}_k^{\mathbf{a}} \langle\langle \ulcorner\chi\urcorner ; g\rangle ; \ulcorner\varphi\urcorner\rangle}{\langle \ulcorner\chi\urcorner \ulcorner\Rightarrow\urcorner \langle f \ulcorner=_B\urcorner g\rangle\rangle \check{\doteq}_i^{\mathbf{PR2}} \ulcorner\mathrm{true}_A\urcorner}$$

The **PR2** deduction tree cases are modified by replacing internal **PR2** maps

$f \in B^A \subset \mathrm{PR2}$ by internal **S** maps

$\langle\langle \ulcorner \mathrm{true}_A \urcorner; f\rangle; \ulcorner \mathrm{true}_B \urcorner\rangle \in \{B : \ulcorner \mathrm{true}_B \urcorner\}^{\{A : \ulcorner \mathrm{true}_A \urcorner\}}$.

The internal deduction trees are counted in lexicographic order so that in particular a branch of such a tree is counted before that tree, and so that (internal) equations going into a first *proof* (deduction tree) for/of a given (internal) equation appear in the (spiral) list eq with earlier indices than the (internal) equation considered, c. f. the transitivity deduction tree above.

In the **proof** of *termination conditioned soundness* below, such (internal) deduction trees are – top down – *substituted* with (free-variable) *arguments.* The only problematic case of this *argumentation* arises in case of *compatibility of composition with equality.*

8.4 Numeralisation naturality

For constructive set theory **S** as well as its strengthening $\pi\mathbf{R}$ below, consider the (covariant) *constructive* S *internal* hom *functor*

$\hom(A, \_) = (\_)^A : \mathbf{S} \longrightarrow \mathbf{S}$ at *set* A, **defined** on *sets* by
$B^A = B^A/\check{=}$, equality $\check{=}$ the enumerated internal equality
$\check{=} : \mathbb{N} \to B^A \times B^A$ of **S**.
On maps $g : B \to B'$ internal hom functor $(\_)^A$ is **defined** by
$g^A = g^A(f) = \ulcorner g \urcorner \odot f : B^A \to B'^A$.
Compatibility with internal notion of equality:

$$f \check{=}_i \tilde{f} \implies g^A(f) = \ulcorner g \urcorner \odot f \check{=}_{k(i,\ulcorner g \urcorner)} \ulcorner g \urcorner \odot \tilde{f} = g^A(\tilde{f})$$

by internal Leibniz substitutivity.

$(\_)^A$ is a ***functor,*** since it preserves identies:

$$(\mathrm{id}_B)^A(f) = \ulcorner \mathrm{id}_B \urcorner \odot f \,\check{=}\, f = \mathrm{id}_{B^A}(f)$$

and preserves composition:

$$\begin{aligned}&(g' \circ g)^A(f) = \ulcorner g' \circ g \urcorner \odot f \\ &\check{=} (\ulcorner g' \urcorner \odot \ulcorner g \urcorner) \odot f \,\check{=}\, \ulcorner g' \urcorner \odot (\ulcorner g \urcorner \odot f) \\ &= g'^A(g^A(f)) = (g'^A \circ g^A)(f)\end{aligned}$$

Naturality Lemma

- Family $\nu_A : A \to A^{\mathbb{1}} \ = \ A^{\mathbb{1}}/\check{=} \ = \ A^{\mathbb{1}}/\check{=}^{\mathbf{a}}$

  is a natural transformation, from identity functor $\mathrm{ID}_{\mathbf{S}}$ to (constructive) internal hom functor $(\_)^A : \mathbf{S} \to \mathbf{S}$,

  will say: for $f : A \to B$ in $\mathbf{S}$

  $$\nu_B \circ f = \ulcorner f \urcorner \odot \nu_A = f^{\mathbb{1}} \circ \nu_A$$

  In diagram form:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \downarrow{\scriptstyle \nu_A} & = & \downarrow{\scriptstyle \nu_B} \\ A^{\mathbb{1}} & \xrightarrow{\ f^{\mathbb{1}}\ } & B^{\mathbb{1}} \end{array} \qquad (*)$$

$$\begin{array}{ccccc} a & & \xmapsto{\qquad f \qquad} & & f(a) \\ \downarrow{\scriptstyle \nu_A} & & & & \downarrow{\scriptstyle \nu_B} \\ \nu_A(a) & \xmapsto{\ f^{\mathbb{1}}\ } & \ulcorner f \urcorner \odot \nu_A(a) & \check{=} & \nu_B f(a) \end{array}$$

- For restrictrion $\dot{f} : \dot{A} \to \dot{B}$ of $f^{\mathbb{1}} : A^{\mathbb{1}} \to B^{\mathbb{1}}$ this gives a natural equivalence

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \dot{\nu}_A} \downarrow {\scriptstyle \cong} & = & {\scriptstyle \dot{\nu}_B} \downarrow {\scriptstyle \cong} \\ \dot{A} & \xrightarrow{\quad \dot{f} \quad} & \dot{B} \end{array} \qquad (**)$$

Proof of naturality by structural recursion on $f : A \to B$ in $\mathbf{S}$:

- **Anchor** cases
  - $f = \mathrm{id} : A \to A$:

    $$(\nu_A \circ \mathrm{id}_A)(a) = \nu_A(a) \stackrel{\vee}{=} \ulcorner \mathrm{id}_A \urcorner \odot (\nu_A(a))$$

  - $f = 0 : \mathbb{1} \to \mathbb{N}$:

    $$\nu \circ 0 = \nu(0) = \ulcorner 0 \urcorner \stackrel{\vee}{=} \ulcorner 0 \urcorner \odot \ulcorner \mathrm{id}_{\mathbb{1}} \urcorner$$

  - **non-trivial case** $f = \mathrm{s} : \mathbb{N} \to \mathbb{N}$:

    $$(\nu \circ \mathrm{s})(a) = \nu(\mathrm{s}(a)) \ =_{\text{by def}} \ \ulcorner \mathrm{s} \urcorner \odot \nu(a)$$

  - $f = \mathrm{false} : \mathbb{1} \to \mathbf{2}$:

    $$\nu \circ \mathrm{false} = \nu(\mathrm{false}) = \ulcorner \mathrm{false} \urcorner \stackrel{\vee}{=} \ulcorner \mathrm{false} \urcorner \odot \ulcorner \mathrm{id}_{\mathbb{1}} \urcorner$$

  - $f = \mathrm{true} : \mathbb{1} \to \mathbf{2}$: dito

– $f = \mathrm{sign} : \mathbb{N} \to \mathbf{2}$:

DIAGRAM

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\mathrm{sign}} & \mathbf{2} \\ \downarrow{\scriptstyle \nu} & \dot{\simeq} & \downarrow{\scriptstyle \nu_{\mathbf{2}}} \\ \mathbb{N}^{\mathbb{1}} & \xrightarrow{\ulcorner \mathrm{sign} \urcorner^{\mathbb{1}}} & \mathbf{2}^{\mathbb{1}} \end{array}$$

This diagram commutes in fact with respect to internal equality '$\dot{\simeq}$' since

$$\begin{aligned} &\nu_{\mathbf{2}} \circ \mathrm{sign}(0) = \nu_{\mathbf{2}}(\mathrm{false}) = \ulcorner \mathrm{false} \urcorner, \text{ and} \\ &\ulcorner \mathrm{sign} \urcorner^{\mathbb{1}} \circ \nu(0) \\ &= \ulcorner \mathrm{sign} \urcorner \odot \nu_{\mathbb{N}}(0) = \ulcorner \mathrm{sign} \circ 0 \urcorner = \ulcorner \mathrm{false} \urcorner \text{ likewise} \end{aligned}$$

as well as

$$\begin{aligned} &\nu_{\mathbf{2}} \circ \mathrm{sign}(\mathrm{s}\, n) = \nu_{\mathbf{2}}(\mathrm{true}) = \ulcorner \mathrm{true} \urcorner, \text{ and} \\ &\ulcorner \mathrm{sign} \urcorner \odot \nu(\mathrm{s}\, n) = \ulcorner \mathrm{sign} \urcorner \odot \ulcorner \mathrm{s}\, n \urcorner \\ &= \ulcorner \mathrm{sign} \circ \mathrm{s}(n) \urcorner = \ulcorner \mathrm{true} \urcorner \text{ likewise} \end{aligned}$$

– $f = \mathrm{pret} : \mathbf{2} \to \mathbb{N}$: DIAGRAM

$$\begin{array}{ccc} \mathbf{2} & \xrightarrow{\mathrm{pret}} & \mathbb{N} \\ \downarrow{\scriptstyle \nu_{\mathbf{2}}} & \dot{\simeq} & \downarrow{\scriptstyle \nu} \\ \mathbf{2}^{\mathbb{1}} & \xrightarrow{\ulcorner \mathrm{pret} \urcorner^{\mathbb{1}}} & \mathbb{N}^{\mathbb{1}} \end{array}$$

This diagram commutes in fact with respect to internal equality '$\check{=}$' since

$$\begin{aligned}
&\nu(\mathrm{pret}(\mathrm{false})) = \nu(0) = \ulcorner 0 \urcorner, \\
&\ulcorner \mathrm{pret} \urcorner^{\mathbb{1}}(\nu_{\mathbf{2}}(\mathrm{false})) = \ulcorner \mathrm{pret} \urcorner \odot \ulcorner \mathrm{false} \urcorner \\
&= \ulcorner \mathrm{pret}(\mathrm{false}) \urcorner = \ulcorner 0 \urcorner;
\end{aligned}$$

same for true and $1 = \mathrm{s}\,0$ in place of false and 0 respectively.

- $f = \diagdown : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ (relative complement):
  This case follows from the above and cases below, since

$$\begin{array}{ccc}
\mathbf{2} \times \mathbf{2} & \xrightarrow{\diagdown} & \mathbf{2} \\
{\scriptstyle \mathrm{pret} \times \mathrm{pret}} \downarrow & = & \uparrow {\scriptstyle \mathrm{sign}} \\
\mathbb{N} \times \mathbb{N} & \xrightarrow{\diagdown} & \mathbb{N}
\end{array}$$

- $f = \Pi : A \to \mathbb{1}$:

$$\begin{aligned}
&\nu_{\mathbb{1}} \circ \Pi_A(a) = \nu_{\mathbb{1}} \circ \mathrm{id}_{\mathbb{1}} = \nu_{\mathbb{1}} = \ulcorner 0 \urcorner \\
&\check{=} \ulcorner \Pi_A \urcorner \odot \nu_A(a)
\end{aligned}$$

- $\ell : A \times B \to A$:

$$\begin{aligned}
&(\nu_A \circ \ell_{A,B})(a, b) = \nu_A(a) \\
&= \ulcorner \ell_{A,B} \urcorner \odot \langle \nu_A(a); \nu_B(b) \rangle \\
&\check{=} \ulcorner \ell_{A,B} \urcorner \odot (\nu_{A \times B}(a, b))
\end{aligned}$$

- $\mathrm{r} : A \times B \to B$: symmetrical.

- **Map composition** $g \circ f : A \to B \to C$: combine the two commuting squares for f and for g into commuting rectangle for

$g \circ f$:

$$\nu_C \circ (g \circ f)(a) = (\nu_C \circ g)(f(a))$$
$$\check{\doteq} \ulcorner g \urcorner \odot \nu_B(f(a)) \text{ recursively}$$
$$\check{\doteq} \ulcorner g \urcorner \odot \langle \ulcorner f \urcorner \odot \nu_A(a) \rangle \text{ recursively}$$
$$\check{\doteq} \langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle \odot \nu_A(a)$$
$$\check{\doteq} \ulcorner g \circ f \urcorner \odot \nu_A(a) \;\; \text{q. e. d. in this case}$$

- **Induced map** $(f, g) : C \to A \times B$ into a product:

$$\nu_{A\times B} \circ (f, g)(c) = \langle \nu_A \# \nu_B \rangle (f(c), g(c))$$
$$= \langle \nu_A(f(c)); \nu_B(g(c)) \rangle$$
$$\check{\doteq} \langle \ulcorner f \urcorner \odot \nu_C(c); \ulcorner g \urcorner \odot \nu_C(c) \rangle \text{ recursively}$$
$$= \langle \ulcorner f \urcorner ; \ulcorner g \urcorner \rangle \odot \nu_C(c) = \ulcorner (f, g) \urcorner \odot \nu_C(c)$$

q. e. d. in this case

- **Iterated** $f^{\S}(a, n) : A \times \mathbb{N} \to A$ of (already tested) endo $f : A \to A$: Straight forward by recursion on n, since iteration is iterated composition, as follows:

  - $n = 0$:

$$\nu_A \circ f^{\S}(a, 0) = (\nu_A \circ \mathrm{id}_A)(a) = \nu_A(a)$$
$$\check{\doteq} \ulcorner \mathrm{id} \urcorner \odot \nu_A(a) \check{\doteq} \ulcorner f^{\S}(a, 0) \urcorner \odot \nu_A$$

– induction step:

$$\nu_A \circ f^{\S}(a, s\,n) = \nu_A \circ f \circ f^{\S}(a, n)$$
$$\check{=} \ulcorner f \urcorner \odot (\nu_A \circ f^{\S}(a, n))$$
by hypothesis on f
$$\check{=} \ulcorner f \urcorner \odot (\ulcorner f^{\S} \urcorner \odot (\nu_{A\times\mathbb{N}}(a, n))$$
by induction hypothesis
$$\check{=} (\ulcorner f \urcorner \odot \ulcorner f^{\S} \urcorner) \odot (\nu_{A\times\mathbb{N}}(a, n))$$
$$= (\ulcorner f \urcorner \odot \ulcorner f \urcorner^{\$}) \odot (\nu_{A\times\mathbb{N}}(a, n))$$
$$\check{=} \ulcorner f \urcorner^{\$} \odot \nu_{A\times\mathbb{N}}(a, \mathrm{s}\,n)$$
$$\check{=} \ulcorner f^{\S} \urcorner \odot \nu_{A\times\mathbb{N}}(a, \mathrm{s}\,n)$$

– f of **S**-restriction form $f : \{A : \chi\} \to \{B : \varphi\}$:

The corresponding naturality diagram commutes as restriction of the naturality diagram for $f : A \to B$,

in detail:

$$\nu_{\{A:\chi\}}(a) \ =_{\text{by def}} \ \nu_A(a) \ \in A^{\mathbb{1}} \text{ anyway}$$
but more than that:
$$\chi(a) \implies \ulcorner \chi \urcorner \odot \nu_A(a) \check{=} \nu_{\mathbf{2}}(\chi(a)) \text{ by the above}$$
$$= \nu_{\mathbf{2}}(\text{true}) = \ulcorner \text{true} \urcorner, \text{ whence in fact}$$
$$\nu_{\{A:\chi\}}(a) \in \{A : \chi\}^{\mathbb{1}}, \text{ same way:}$$
$$\nu_{\{B:\varphi\}}(f(a)) \in \{B : \varphi\}^{\mathbb{1}}$$

and then – restriction –

$$\begin{aligned}&\nu_{\{B:\varphi\}}(f(a))\\ &= (\nu_B \circ f)(a) \check{\doteq} \ulcorner f \urcorner \odot \nu_A(a)\\ &= \ulcorner f \urcorner \odot \nu_{\{A:\chi\}}(a)\end{aligned}$$

q. e. d.

8.5 Complexity controlled Iteration CCI

In sections below on evaluation of map codes on suitable arguments we rely on those `while` loops which are given by *Complexity Controlled Iteration* in the sense of the following schema (CCI) :

$c = c(a) : A \to \mathbb{N}[\omega]$ *complexity*

$f = f(a) : A \to A$ *predecessor* endo

$[c(a) > 0 \implies c\,f(a) < c(a)]$ (*descent*)

$\wedge\ [c(a) = 0 \implies f(a) = a]$ (*stationarity*)

put together: $\mathrm{CCI}[c : f]$

(CCI) ―――――――――――――――――――

$\mathrm{wh}[c > 0 : f] : A \rightharpoonup A$

$=$ `while`$[c(a) > 0]$ `do` $a := f(a)$ `od`, formally:

$D_{\mathrm{wh}} = \{(a, m) \in A \times \mathbb{N} : c\,f^m(a) = 0\}$,

$d_{\mathrm{wh}}(a, m) = a : D_{\mathrm{wh}} \to A$

$\widehat{\mathrm{wh}}(a, m) = f^m(a) : D_{\mathrm{wh}} \to A$

Question is ***termination,*** dependent on $a \in A$ or for $a \in A$ free.

In subsequent chapters we will obtain **objectivity** and *termination conditioned* **soundness** for the formally *partial* CCI ***evaluation*** to come.

Examples:

- A CCI $\mathrm{wh}[c > 0 : f] : A \rightharpoonup A$ with order values in $\mathbb{N} \subset \mathbb{N}[\omega]$ is a *primitive recursive* map, namely

$$\mathrm{wh}[c > 0 : f]$$
$$= f^{\S}(a, \min\{m \leq n : f^m(a) = 0\}) : A \times \mathbb{N} \to A$$

- **evaluation** below of p. r. map codes will be a CCI, with complexity values in ordinal $\mathbb{N}[\omega]$.

- **Counterexamples:** the `while` loop

$$[\ a := 0;\ n := 1;$$
$$\texttt{while}\ a < 1/3\ \texttt{do} \begin{cases} a := a + 3 \cdot 10^{-n}; \\ n := n + 1 \end{cases}$$
$$\texttt{result} := a\ ]$$

This `while` loop approximating $1/3$ does not come with complexity control, it would loop endlessly.

The arctan(1) Leibniz series which approximates but does not reach non-algebraic (geometric) number $\pi/4$ in finite time, is another `while` loop which is not a CCI. This loop could be controlled by a positive rational descending complexity, of the argument to become smaller than a prescribed

$$\varepsilon = 1/n_0,\ n_0 \in \mathbb{N}_> = \{n \in \mathbb{N} : n > 0\}.$$

8.6 Iterative descent theory

Iterative non-infinite-complexity-descent theory $\pi\mathbf{R}$ is **defined** as *strengthening* of boolean theory **S** of primitive recursion with predicate-into-subject abstraction, by the following additional **axiom schema:**

$c : A \to \mathbb{N}[\omega],\ p : A \to A$

data of a complexity controlled iteration – CCI –

with complexity values in polynomial ordered semiring $\mathbb{N}[\omega]$:

$[c(a) = 0 \Rightarrow p(a) = a] \wedge [c(a) > 0 \Rightarrow c\,p(a) < c(a)] : A \to \mathbf{2};$

$\psi = \psi(a) : A \to \mathbf{2}$ a *"negative" test* predicate:

$\psi(a) \implies c\,p^n(a) > 0,\ a \in A,\ n \in \mathbb{N}$ free

(**non**-termination for all a)

(π) ———————————————

$\psi = \mathrm{false}_A : A \to \mathbf{2}$

Non-infinite iterative descent: "**Only** the overall **false** predicate implies overall **non**-termination of CCI."

Comment: At first look, this **axiom** (π) may look *bizarre.* In order to approximate *termination* of map code evaluation to come – in particular within a framework without formal quantification – I came up with this "double negation" inference of implications, at poster session of Vienna conference 2006 celebrating Gödel's 100th birthday.

Special case of **axiom** (π) above, $A := \mathbb{N}$, number $\boldsymbol{a} : \mathbb{1} \to \mathbb{N}$

substituted to $a \in A = \mathbb{N}$, gives

$$\begin{array}{ll} & \mathrm{CCI}[c : \mathbb{N} \to \mathbb{N}[\omega],\ p : \mathbb{N} \to \mathbb{N}] \\ & \boldsymbol{a} : \mathbb{1} \to \mathbb{N}\ (\textbf{number}) \\ & [c(\boldsymbol{a}) = 0 \Rightarrow p(\boldsymbol{a}) = \boldsymbol{a}] \wedge [c(\boldsymbol{a}) > 0 \Rightarrow c\,p(\boldsymbol{a}) < c(\boldsymbol{a})]; \\ & \psi : \mathbb{1} \to \mathbf{2}\ (\textit{truth value})\ \text{s.t.} \\ & \psi \implies c\,p^n(\boldsymbol{a}) > 0,\ n \in \mathbb{N}\ \text{free} \\ (\pi\mathbb{1}) & \overline{\hspace{20em}} \\ & \psi = \text{false} : \mathbb{1} \to \mathbf{2} \end{array}$$

Only truth value false *can imply infinite descent of* $\mathbb{N}[\omega]$ *chains.*

If object $\mathbb{1}$ should be a *separator* object for the theory, then schema $(\pi\mathbb{1})$ would already entail **axiom** (π). This *is* the case for **set** theory extensions **T** of theory $\pi\mathbf{R}$.

8.7 Equality definability revisited

Boolean p. r. theory **S** admits the following schema:

$$\begin{array}{ll} & f, g : A \to \mathbb{N}\ \text{in}\ \mathbf{S}, \\ & \mathbf{S} \vdash [f(a) = g(a)] : A \to \mathbb{N} \times \mathbb{N} \to \mathbf{2} \\ (\text{EqDef}) & \overline{\hspace{15em}} \\ & \mathbf{S} \vdash f = g : A \to \mathbb{N},\ \textit{algebraically:} \\ & f =^{\mathbf{S}} g : A \to \mathbb{N} \end{array}$$

Equality definability extends to **S**-map pairs $f, g : A \to B$ with common codomain a cartesian product $B \cong \mathbb{N}^{\boldsymbol{m}}$ or even B an arbitrary set of theory **S**.

Proof by commutativity $\max(m,n) = \max(n,m) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, see proof of this result for Goodstein Arithmetic **GA**.

8.8 Iterative map code evaluation

8.8.1 Map code evaluation as CCI

Definition first of **PR2 evaluation**

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathbf{PR2}} = \boldsymbol{\varepsilon}(f, a) : \mathrm{PR2} \times \mathbb{X}_{\mathbf{2}} \rightharpoonup \mathrm{PR2} \times \mathbb{X}_{\mathbf{2}} \xrightarrow{\mathrm{r}} \mathbb{X}_{\mathbf{2}}$$

by *Complexity Controlled Iteration* (CCI)

$$\mathtt{while}\ \boldsymbol{c}\, f > 0\ \mathtt{do}\ (f, x) := \boldsymbol{e}(f, x)\ \mathtt{od}$$

where $\boldsymbol{c} = \boldsymbol{c}\, f : \mathrm{PR2} \to \mathbb{N}[\omega]$ will be a suitable map code *complexity* within the linearily ordered semiring $\mathbb{N}[\omega]$ of polynomials in one variable ω with coefficients in $\mathbb{N}$.

Iteration of *evaluation step*

$$\boldsymbol{e} = \boldsymbol{e}(f, x) : \mathrm{PR2} \times \mathbb{X}_{\mathbf{2}} \to \mathrm{PR2} \times \mathbb{X}_{\mathbf{2}}$$

is to descend this map code complexity $\boldsymbol{c}$ eventually down to $0 \in \mathbb{N}[\omega]$, and to give evaluation result as value in right component $\mathbb{X}_{\mathbf{2}}$ upon reaching complexity 0.

Iterative evaluation of theory PR2 within S

evaluation step

$$\boldsymbol{e} = \boldsymbol{e}\,(f, a) = (\boldsymbol{e}_{\text{map}}(f, a), \boldsymbol{e}_{\text{arg}}(f, a)) :$$
$$\text{PR2} \times \mathbb{X}_{\mathbf{2}} \longrightarrow \text{PR2} \times \mathbb{X}_{\mathbf{2}}$$

$\boldsymbol{e}_{\text{arg}}(f, a)$ *is the intermediate argument obtained by one evaluation step applied to the pair* (f, a)*, and* $\boldsymbol{e}_{\text{map}}(f, a)$ *is the remaining map code still to be evaluated on intermediate argument* $\boldsymbol{e}_{\text{arg}}(f, a)$*, same then iteratively applied to pair* $(\boldsymbol{e}_{\text{map}}, \boldsymbol{e}_{\text{arg}})$. Here $a \in \dot{A} \subset \mathbb{X}_{\mathbf{2}}$ free, set A arbitrary in **PR2**, numeral version $\dot{A}$ subset of universal (numerals) set $\mathbb{X}_{\mathbf{2}} = \dot{\bigcup}_{A \text{ in } \mathbf{PR2}} \dot{A} \quad [= \bigcup_{\{A:\chi\} \text{ in } \mathbf{S}} \nu\{A : \chi\} \subset \mathbb{N}]$.

This evaluation step $\boldsymbol{e}$ is **defined** by recursive case distinction, *controlled* by $\mathbb{N}[\omega]$-valued descending **complexity**

$$\boldsymbol{c} = \boldsymbol{c}\, f \;\in \mathbb{N}[\omega],$$

in turn p. r. defined the time being by

$$\boldsymbol{c}\,\ulcorner \text{id}_A \urcorner := 0, \;\; A \text{ in } \mathbf{PR2}$$
$$\boldsymbol{c}\,\ulcorner \text{ba} \urcorner := 1$$
$$\text{ba} \in \text{bas} \setminus \{\text{id}\} = \{0, \text{true}, \text{s}, \Pi, \ell, \text{r}, \setminus, \text{sign}, \text{pret}\}$$
$$\boldsymbol{c}\,\langle g \odot f \rangle := \boldsymbol{c}\, f + \boldsymbol{c}\, g + 1$$
$$\boldsymbol{c}\,\langle f; g \rangle := \boldsymbol{c}\, f + \boldsymbol{c}\, g + 1$$
$$\boldsymbol{c}\,\langle f \# g \rangle := \boldsymbol{c}\, f + \boldsymbol{c}\, g + 1$$
$$\boldsymbol{c}\, f^{\$} \text{ see below.}$$

evaluation step $\boldsymbol{e} = \boldsymbol{e}(f, a)$ is p. r. defined (and is iteration complexity-controlled) as follows:

- **Basic map cases:**
  - case of an identity:

$$\begin{aligned}&\boldsymbol{e}\,(\ulcorner \mathrm{id}_A \urcorner, \dot a) := (\ulcorner \mathrm{id}_A \urcorner, \dot a)\\&\boldsymbol{c}\,\ulcorner \mathrm{id}_A \urcorner = 0\\&\textit{stationary}\end{aligned}$$

  - remaining basic map cases:

$$\begin{aligned}&\boldsymbol{e}\,(\ulcorner \mathrm{ba} \urcorner, \dot a) := (\ulcorner \mathrm{id} \urcorner, \dot{\mathrm{ba}}\, a),\\&\quad A = \mathrm{Dom}\,\mathrm{ba},\ \ B = \mathrm{Codom}\,\mathrm{ba},\\&\quad \mathrm{ba} \in \mathrm{bas} \smallsetminus \{\mathrm{id}\}\\&\quad = \{0, \mathrm{s}, \mathrm{true}, \smallsetminus, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B\ \textbf{PR2}\ \text{objects}\},\\&\boldsymbol{c}(\ulcorner \mathrm{id} \urcorner) = 0 < \boldsymbol{c}(\ulcorner \mathrm{ba} \urcorner) = 1,\ \ \mathrm{ba} \in \mathrm{bas} \smallsetminus \{\mathrm{id}\}\end{aligned}$$

- **Composition cases,** $\dot a \in \dot A \subset \mathbb{X}_{\mathbf{2}}$ free:
  - identity subcase:

$$\begin{aligned}&\boldsymbol{e}(g \odot \ulcorner \mathrm{id}_A \urcorner, \dot a) := (g, \dot a),\\&\boldsymbol{c}\,g < \boldsymbol{c}\,g + 0 + 1 = \boldsymbol{c}\,\langle g \odot \ulcorner \mathrm{id}_A \urcorner\rangle\end{aligned}$$

  - for $f \in B^A \subset \mathrm{S},\ \ g \in C^B \subset \mathrm{S},\ \ \dot a \in \dot A,\ \ \boldsymbol{c}\,f > 0:$

$$\begin{aligned}&\boldsymbol{e}\,(g \odot f, \dot a) = (\boldsymbol{e}_{\mathrm{map}}(g \odot f, \dot a), \boldsymbol{e}_{\mathrm{arg}}(g \odot f, \dot a))\\&\ := (g \odot \boldsymbol{e}_{\mathrm{map}}(f, \dot a), \boldsymbol{e}_{\mathrm{arg}}(f, \dot a))\end{aligned}$$

Complexity descent:

$$\begin{aligned}
&\boldsymbol{c}\ \boldsymbol{e}_{\text{map}}(g \odot f, \dot{a}) \\
&= \boldsymbol{c}\,(g \odot \boldsymbol{e}_{\text{map}}(f, \dot{a})) \\
&= \boldsymbol{c}\,\boldsymbol{e}_{\text{map}}(f, \dot{a}) + \boldsymbol{c}\,g + 1 \\
&< \boldsymbol{c}\,f + \boldsymbol{c}\,g + 1 \\
&= \boldsymbol{c}\,\langle g \odot f\rangle
\end{aligned}$$

- **Cases of an induced:**
  - identities case:

$$\begin{aligned}
&\boldsymbol{e}(\langle \ulcorner \mathrm{id}_C \urcorner ; \ulcorner \mathrm{id}_C \urcorner \rangle, \dot{c}) := (\ulcorner \mathrm{id}_{C\times C} \urcorner, \langle \dot{c}; \dot{c} \rangle), \\
&\boldsymbol{c}\,\ulcorner \mathrm{id}_{C\times C} \urcorner = \boldsymbol{c}(\ulcorner \mathrm{id} \urcorner) = 0 \\
&< 1 = \boldsymbol{c}(\langle \ulcorner \mathrm{id}_C \urcorner ; \ulcorner \mathrm{id}_C \urcorner \rangle)
\end{aligned}$$

  - case $f \in A^C$, $g \in B^C$, not both equal to $\ulcorner \mathrm{id}_C \urcorner$:

$$\begin{aligned}
&\boldsymbol{e}\,(\langle f; g\rangle, \dot{c}) \\
&:= (\langle \boldsymbol{e}_{\text{map}}(f, \dot{c}); \boldsymbol{e}_{\text{map}}(g, \dot{c})\rangle, \langle \boldsymbol{e}_{\text{arg}}(f, \dot{c}); \boldsymbol{e}_{\text{arg}}(g), \dot{c}\rangle), \\
&\boldsymbol{c}\,\boldsymbol{e}_{\text{map}}(\langle f; g\rangle, \dot{c}) \\
&= \boldsymbol{c}\,\boldsymbol{e}_{\text{map}}(f, \dot{c}) + \boldsymbol{c}\,\boldsymbol{e}_{\text{map}}(g, \dot{c}) + 1 \\
&< \boldsymbol{c}\,f + \boldsymbol{c}\,g + 1 = \boldsymbol{c}\,\langle f; g\rangle,
\end{aligned}$$

since in this case $\boldsymbol{c}\,f > 0$ *and/or* $\boldsymbol{c}\,g > 0$,

and therefore $\boldsymbol{c}\,\boldsymbol{e}_{\text{map}}\,f < \boldsymbol{c}\,f$

and/or $\boldsymbol{c}\,\boldsymbol{e}_{\text{map}}\,g < \boldsymbol{c}\,g$

- Case $f \# g \in (A' \times B')^{A\times B}$, $\langle \dot{a}; \dot{b}\rangle \in \langle A \# B\rangle$:
  analogous (and redundant).

- Iteration case: For endomap code $f \in A^A$ and $\dot{a} \in \dot{A}$,

  $\boldsymbol{e}\,(f^{\$}, \langle \dot{a}; \ulcorner 0 \urcorner \rangle) := (f^0, \dot{a})$ as well as
  $\boldsymbol{e}\,(f^{\$}, \langle \dot{a}; \nu(\mathrm{s}\,n) \rangle) := (\langle f \odot f^n \rangle, \dot{a})$,
  where $f^0 := \ulcorner \mathrm{id} \urcorner$,
  $f^{\mathrm{s}\,n} := \langle f \odot f^n \rangle$ recursively,
  code expansion

Complexity extension:

$$\boldsymbol{c}\,f^{\$} := (\boldsymbol{c}\,f + 1) \cdot \omega \;\in\; \mathbb{N}[\omega]$$

$\mathbb{N}[\omega]$ the well-ordered semiring of polynomials in one indeterminate over $\mathbb{N}$.

In this "acute" iteration case we have

Complexity descent

$$\boldsymbol{c}\,f^0 = \boldsymbol{c}\,\ulcorner \mathrm{id} \urcorner = 0 < (\boldsymbol{c}\,f + 1) \cdot \omega = \boldsymbol{c}\,f^{\$},$$

and further inductively

$$\begin{aligned} \boldsymbol{c}\,f^{\mathrm{s}\,n} &= \boldsymbol{c}\,\langle f \odot \langle f \ldots f \rangle \ldots \rangle \\ &= \boldsymbol{c}\,f \cdot \mathrm{s}\,n + n \\ &< (\boldsymbol{c}\,f + 1) \cdot (n + 1) \\ &< (\boldsymbol{c}\,f + 1) \cdot \omega = \boldsymbol{c}\,f^{\$} \end{aligned}$$

Explication: In this case $\boldsymbol{c}$ takes values within the linearily ordered semiring $\mathbb{N}[\omega] \supset \mathbb{N}$ of polynomials in one indeterminate

ω, ω thought to represent (arbitrarily) big natural numbers. So in fact $\boldsymbol{c}(f^{\mathrm{s}\,n}) < \boldsymbol{c}(f^{\$})$, since the former polynomial has lower degree than the latter.

Linear order of polynomials $p, q \in \mathbb{N}[\omega]$ is defined hierarchically by first comparison of the *degrees* of p and q, second in case of equal degrees by comparison of pivot coefficients, and third if the pivot monomials are equal, recursively by comparison of the polynomials p and q with the two pivot monomials deleted.

Evaluation extension to theory S

Evaluation $\boldsymbol{\varepsilon}^{\mathbf{a}} : \mathrm{S} \times \mathbb{X}_{\mathbf{2}} \to \mathbb{X}_{\mathbf{2}}$ is **defined** as (purely) formal extension of above **PR2** evaluation $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathbf{PR2}}$ by

- **complexity**

$$\begin{aligned}
&\boldsymbol{c}^{\mathbf{a}} = \boldsymbol{c}^{\mathbf{a}}\langle\langle \ulcorner\chi\urcorner; f\rangle; \langle \ulcorner\chi\urcorner\rangle\rangle \\
&\quad =_{\mathrm{def}} \ =_{\mathrm{def}} \ \boldsymbol{c}\langle\langle \ulcorner\mathrm{true}_A\urcorner; f\rangle; \ulcorner\mathrm{true}_A\urcorner\rangle \setminus 8 \\
&\quad = ((((3+0)+1)+\boldsymbol{c}(f))+1) \setminus 8 =\in \mathbb{N}[\omega] \\
&\qquad \text{in particular} \\
&\boldsymbol{c}^{\mathbf{a}}(\ulcorner\mathrm{id}_{\{A:\chi\}}\urcorner) = \boldsymbol{c}^{\mathbf{a}}\langle\langle \ulcorner\chi\urcorner; \ulcorner\mathrm{id}_A\urcorner\rangle; \langle \ulcorner\chi\urcorner\rangle\rangle = 0
\end{aligned}$$

evaluation step

$$\begin{aligned}
&\boldsymbol{e}^{\mathbf{a}} = \boldsymbol{e}^{\mathbf{a}}\langle\langle \ulcorner\chi : A \to \mathbf{2}\urcorner; f\rangle; \langle \ulcorner\chi : B \to \mathbf{2}\urcorner\rangle\rangle \\
&\quad =_{\mathrm{def}} \ \langle\langle \mathrm{true}_A; \boldsymbol{e}(f\rangle; \ulcorner\mathrm{true}_B\urcorner\rangle\rangle
\end{aligned}$$

Then descent of complexity $\boldsymbol{c}^{\mathbf{a}}$ with application of step $\boldsymbol{e}^{\mathbf{a}}$ follows readily from descent of **PR2** complexity $\boldsymbol{c}$ with **PR2** evaluation step $\boldsymbol{e}$.

Evaluation $\boldsymbol{\varepsilon}^{\mathbf{a}}$ defined as CCI of $\boldsymbol{e}^{\mathbf{a}}$ clearly extends evaluation $\boldsymbol{\varepsilon}$ of theory **PR2**.

Notation: We note evaluation complexity $\boldsymbol{c}^{\mathbf{a}}$ and evaluation step $\boldsymbol{e}^{\mathbf{a}}$ of theory **S** simply as

$$\boldsymbol{c} = \boldsymbol{c}\langle\langle\ulcorner\chi\urcorner; f\rangle; \langle\ulcorner\chi\urcorner\rangle\rangle : \mathrm{S} \times \mathbb{X}_{\mathbf{2}} \to \mathrm{N}[\omega]$$

$$\boldsymbol{e} = \boldsymbol{e}(\langle\langle\ulcorner\chi\urcorner; f\rangle; \langle\ulcorner\chi\urcorner\rangle\rangle, \dot{a}) : \mathrm{S} \times \mathbb{X}_{\mathbf{2}} \to \mathrm{S} \times \mathbb{X}_{\mathbf{2}}$$

giving – just below – partial evaluation map

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}(\langle\langle\ulcorner\chi\urcorner; f\rangle; \langle\ulcorner\chi\urcorner\rangle\rangle, \dot{a}) : \mathrm{S} \times \mathbb{X}_{\mathbf{2}} \rightharpoonup \mathbb{X}_{\mathbf{2}}$$

8.8.2 Evaluation resolution

Evaluation definition

- Evaluation $\dot{\boldsymbol{\varepsilon}}$ of **S** map code variable $f \in B^A \subset \mathrm{S}$ on (fitting) arguments $\dot{a} \in \dot{A} \subset \mathbb{X}$, is (formally partial) **defined,** by the ***complexity controlled iteration*** (CCI)

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}(f, \dot{a}) := \begin{cases} \texttt{init}\, \Big\{ (h, x) := (f, \dot{a}) \\ * \\ \texttt{while}\, [\boldsymbol{c}(h) > 0] \\ \texttt{do}\ (h, x) := \boldsymbol{e}\,(h, x)\ \texttt{od} \\ * \\ \texttt{result} := x \in \dot{B} \subset \mathbb{X} \end{cases}$$

which in fact always **terminates** within quantified theories $\mathbf{T}$ (with finite descent in $\mathbb{N}[\omega]$), and cannot iterate infinitely within theory $\pi\mathbf{R}$ – **axiom** (π).

- **Define** (natural) *evaluation family*

  $\varepsilon = \varepsilon_{A,B} = \varepsilon_{A,B}(f, a) : B^A \times A \rightharpoonup B$ by
  $\varepsilon_{A,B}(f, a) = \dot{\nu}_B^{-1}(\dot{\varepsilon}(f, \dot{\nu}_A(a)),$
  $\dot{\nu}$ image-restricted internal numeralisation

8.8.3 Dominated characterisation of evaluation

With **abbreviation**

$[m \ \mathit{defs} \ \varepsilon(f, \dot{a})]$ for $\ell\, \mathbf{e}^m(f, \dot{a}) = \ulcorner \mathrm{id} \urcorner$:
termination in at most m steps,

Family

$\varepsilon = \varepsilon_{A,B}(f, a) : B^A \times A \rightharpoonup B,\ A, B$ *sets* in $\mathbf{S}$,

is **characterised** within theory $\pi\mathbf{R}$ by

- $\varepsilon(\ulcorner \mathrm{ba} \urcorner, a) = \mathrm{ba}(a),\ a \in A = \mathrm{Dom}(\mathrm{ba})$

- $[m \ \mathit{defs} \ \dot{\varepsilon}(g \odot f, \dot{a})]$
  $\Longrightarrow [m \ \mathit{defs} \ \dot{\varepsilon}(f, \dot{a})] \wedge [m \ \mathit{defs} \ \dot{\varepsilon}(g, \dot{\varepsilon}(f, \dot{a}))] \wedge$
  $\varepsilon(g \odot f, a) = \varepsilon(g, \varepsilon(f, a))$:

  If m defines *left hand iteration* $\dot{\varepsilon}$, *then evaluations on right hand side terminate in (at most)* m *evaluation steps* $\mathbf{e}$ *too, equal result*

- $[m \ \mathit{defs} \ \dot{\varepsilon}(\langle f; g\rangle, \dot{c})]$
  $\implies \varepsilon(\langle f; g\rangle, c) = \langle \varepsilon(f, c); \varepsilon(g, c)\rangle;$

- $[m \ \mathit{defs} \ \dot{\varepsilon}(f \# g, \langle \dot{a}; \dot{b}\rangle)]$
  $\implies \varepsilon(\langle f \# g\rangle, (a, b)) = (\varepsilon(f, a), \varepsilon(g, b));$

- $\varepsilon(f^{\$}, (a, 0)) = a;$

  $[m \ \mathit{defs} \ \dot{\varepsilon}(f^{\$}, \langle \dot{a}, \nu(\mathrm{s}\, n)\rangle)]$
  $\implies [m \ \mathit{defs}$ both ε below$] \wedge$
  $\varepsilon(f^{\$}, (a, \mathrm{s}\, n)) = \varepsilon(f, \varepsilon(f^{\$}, (a, n));$

- Global evaluation $\dot{\varepsilon}$ *doesn't iterate infinitely* within theory $\pi\mathbf{R}$, and upon termination it terminates with all the properties above of evaluation *family* $\varepsilon = \varepsilon_{A,B}, \ A, B$ in $\mathbf{S}$.

Proof by Peano induction on $m \in \mathbb{N}$ free, via case distinction on codes h, and arguments appearing in the different cases of asserted conjunction:

• Case $(h, a), \ h = \mathrm{ba} \in \mathrm{bas}$ *(basic)*,
$\mathrm{Dom}[\mathrm{ba}] = A$ (*say*), $\mathrm{Codom}[\mathrm{ba}] = B$

$$\begin{aligned}
\varepsilon_{A,B}(\ulcorner \mathrm{ba} \urcorner, a) &= \dot{\nu}_B^{-1}(\dot{\varepsilon}(\ulcorner \mathrm{ba} \urcorner, \dot{\nu}_A(a)) \\
&= \dot{\nu}_B^{-1}(\mathrm{ba}(\dot{\nu}_A(a)) \quad \text{by definition of } \dot{\varepsilon} \\
&= (\dot{\nu}_B^{-1} \circ \dot{\nu}_B \circ \mathrm{ba})(a) \quad \text{by naturality of } \nu \\
&= \mathrm{ba}(a) \in B
\end{aligned}$$

• Case $(h, a) = (g \odot f, a)$
– subcase $f = \ulcorner \mathrm{id}_A \urcorner$: obvious

– non-trivial subcase f not an identity code:

$m+1$ *defs* $\dot{\boldsymbol{\varepsilon}}(g \odot f, a) \implies$

$\boldsymbol{\varepsilon}_{A,C}(g \odot f, a)$

$= \dot{\nu}_C^{-1} \boldsymbol{e}^{\S}((g \odot \boldsymbol{e}_{\mathrm{map}}(f, \dot{\nu}_A(a)), \boldsymbol{e}_{\mathrm{arg}}(f, \dot{\nu}_A(a))), m)$

$= \boldsymbol{\varepsilon}(g, \boldsymbol{\varepsilon}(\boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a)))$

by induction hypothesis on m

$\implies$

$m+1$ *defs* $\dot{\boldsymbol{\varepsilon}}(g, \dot{\boldsymbol{\varepsilon}}(\boldsymbol{e}_{\mathrm{map}}(f, \dot{a}), \boldsymbol{e}_{\mathrm{arg}}(f, \dot{a}))) \wedge$

$\boldsymbol{\varepsilon}(g, \boldsymbol{\varepsilon}(\boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a))) = \boldsymbol{\varepsilon}(g, \boldsymbol{\varepsilon}(f, a))$

- Case $(h, c) = (\langle f; g \rangle, c)$: Analogous to the above; easier, since here f and g have common domain.
- Product-of-maps case is redundant, covered by the above.
- Case $(h, z) = (f^{\$}, (a, 0))$: obvious
- Case $(h, z) = (f^{\$}, (a, \mathrm{s}\, n))$:

$m+1$ *defs* $\dot{\boldsymbol{\varepsilon}}(f^{\$}, (a, \nu(\mathrm{s}\, n))) \implies$

$m+1$ *defs* all (implicit) instances of $\dot{\boldsymbol{\varepsilon}}$ below, and

$\boldsymbol{\varepsilon}(f^{\$}, (a, \mathrm{s}\, n))$

$= \mathrm{r}\, \boldsymbol{e}^m(\boldsymbol{e}(f^{\$}, (a, \mathrm{s}\, n))$ by hypothesis on m

$= \boldsymbol{\varepsilon}(f^{[n+1]}, a) = \boldsymbol{\varepsilon}(f \odot f^{[n]}, a)$

$= \boldsymbol{\varepsilon}(f, \boldsymbol{\varepsilon}(f^{[n]}, a))$ by hypothesis on m

$= \boldsymbol{\varepsilon}(f, \boldsymbol{\varepsilon}(f^{\$}, (a, n)))$ by hypothesis on m

q. e. d.

8.8.4 Evaluation objectivity

Evaluation ε is *objective*, i. e. for each *single*, (meta free) $f : A \to B$ in theory $\pi\mathbf{R}$ given

Formally partial

$\varepsilon(\ulcorner f \urcorner, a) = \varepsilon \widehat{\circ} (\ulcorner f \urcorner, a) : \mathbb{1} \times A \xrightarrow{\ulcorner f \urcorner \times A} B^A \times A \xrightarrow{\varepsilon} B$

is in fact p. r. represented and (then) satisfies

$\pi\mathbf{R} \vdash \varepsilon(\ulcorner f \urcorner, a) = f(a) : A \to B$

symbolically:

$\varepsilon(\ulcorner f \urcorner, \_) = f$ (*map* ***reflection***)

Proof by structural recursion on $f : A \to B$:

- $f \in$ bas one of the basic maps of theory $\mathbf{S}$:

  Assertion given by definition of ε.

- composition:

$$\begin{aligned}
&\varepsilon(\ulcorner g \circ f \urcorner, a) = \varepsilon(\ulcorner g \urcorner \odot \ulcorner f \urcorner, a) \\
&= \varepsilon(\ulcorner g \urcorner, \varepsilon(\ulcorner f \urcorner, a)) \\
&= \varepsilon(\ulcorner g \urcorner, f(a)) \text{ by hypothesis on } f \\
&= g(f(a)) \text{ by hypothesis on } g \\
&= (g \circ f)(a)
\end{aligned}$$

- case of an induced map $(f, g) : C \to A \times B$ analogous.

- case of an iterated map $f^{\S} : A \times \mathbb{N} \to A$

$$\boldsymbol{\varepsilon}(\ulcorner f^{\S} \urcorner, (a, 0))) = \boldsymbol{\varepsilon}(\ulcorner f \urcorner^{\$}, (a, 0)))$$
$$= a = f^{\S}(a, 0)$$
and further recursively
$$\boldsymbol{\varepsilon}(\ulcorner f^{\S} \urcorner, (a, \mathrm{s}\, n))) = \boldsymbol{\varepsilon}(\ulcorner f \urcorner^{\$}, (a, \mathrm{s}\, n)))$$
$$= \boldsymbol{\varepsilon}(\ulcorner f \urcorner, \boldsymbol{\varepsilon}(\ulcorner f \urcorner^{\$}, (a, n)))$$
$$= \boldsymbol{\varepsilon}(\ulcorner f \urcorner, \boldsymbol{\varepsilon}(\ulcorner f^{\S} \urcorner, (a, n)))$$
$$= \boldsymbol{\varepsilon}(\ulcorner f \urcorner, f^{\S}(a, n)) \text{ by hypothesis on } n$$
$$= f(f^{\S}(a, n)) \text{ by structural hypothesis on } f$$
$$= f^{\S}(a, \mathrm{s}\, n)$$

- case of a predicatively restricted map

$$((\chi, f), \varphi) : \{A : \chi\} \to \{B : \varphi\}$$

$$\boldsymbol{\varepsilon}(\langle\langle \ulcorner \chi \urcorner; \ulcorner f \urcorner\rangle; \ulcorner \varphi \urcorner\rangle, a)$$
$$= ((\boldsymbol{\varepsilon}(\ulcorner \chi \urcorner, a), \boldsymbol{\varepsilon}(\ulcorner f \urcorner, a)), \boldsymbol{\varepsilon}(\ulcorner \varphi \urcorner, a))$$
$$= ((\chi, f), \varphi)$$

all of that within descent theory $\pi\mathbf{R}$ (in fact already in $\mathbf{S}$) **q. e. d.**

8.9 Soundness metamathematically

Metamathematical soundness theorem

- For $\pi\mathbf{R}$ maps, i. e. $\mathbf{S}$ maps $f, g : A \to B$ and (any) **number** $\boldsymbol{k} : \mathbb{1} \to \mathbb{N}$

$$\frac{\mathbf{S} \vdash \ulcorner f \urcorner \check{=}^{\pi}_{\boldsymbol{k}} \ulcorner g \urcorner}{\pi\mathbf{R} \vdash f = g}$$

whence in particular:

- For an $\mathbf{S}, \pi\mathbf{R}$ predicate $\varphi : A \to \mathbf{2}$ and (any) **number** $\boldsymbol{k} \in \mathbf{PR}(\mathbb{1}, \mathbb{N})$

$$\frac{\mathbf{S} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\boldsymbol{k}, \ulcorner \varphi \urcorner)}{\pi\mathbf{R} \vdash \varphi}$$

Here $\mathrm{Prov}_{\pi\mathbf{R}}(k, \varphi)$ **means:** $k \in \mathbb{N}$ is index for an internal *proof* of map code $\varphi \in \mathbf{2}^A$. It is **defined** by

$$\mathrm{Prov}_{\pi\mathbf{R}}(k, \varphi) = [\varphi \check{=}^{\pi}_{k} \ulcorner \mathrm{true}_A \urcorner]$$

Proof (of first assertion) by external (metamathematical) course-of-values Peano induction on $\boldsymbol{k} \in \mathbf{PR}(\mathbb{1}, \mathbb{N})$:

- **case** that $\boldsymbol{k}$ points to the internalised/coded version of an equational **axiom, example** associativity of composition:

$$\frac{\mathbf{S} \vdash \ulcorner h \circ (g \circ f) \urcorner \check{=}_{\boldsymbol{k}} \ulcorner (h \circ g) \circ f \urcorner}{\pi\mathbf{R} \vdash h \circ (g \circ f) = (h \circ g) \circ f}$$

Here the postcedent holds in itself.

Analogously for the other equational cases.

- **case** that $\boldsymbol{k}$ points to the conclusion $\ulcorner f \urcorner \check{\doteq}_{\boldsymbol{k}} \ulcorner h \urcorner$ of an internalised transitivity,

$$\Uparrow \quad \frac{\ulcorner f \urcorner \check{\doteq}_{\boldsymbol{k}} \ulcorner h \urcorner}{\ulcorner f \urcorner \check{\doteq}_{\boldsymbol{i}} \ulcorner g \urcorner \ \wedge \ \ulcorner g \urcorner \check{\doteq}_{\boldsymbol{j}} \ulcorner h \urcorner}$$

  Then, because of induction hypothesis on $\boldsymbol{i}, \boldsymbol{j} < \boldsymbol{k}$:

  $\mathbf{S} \vdash \ f = g$ and $\mathbf{S} \vdash \ g = h$
  whence $\mathbf{S} \vdash \ f = h$
  q. e. d. in this transitivity case.

- **case** that $\boldsymbol{k}$ points to the conclusion of an internalised composition-with-equality,

$$\Uparrow \quad \frac{\ulcorner g \circ f \urcorner \check{\doteq}_{\boldsymbol{k}} \ulcorner \tilde{g} \circ \tilde{f} \urcorner}{\ulcorner f \urcorner \check{\doteq}_{\boldsymbol{i}} \ulcorner \tilde{f} \urcorner \ \wedge \ \ulcorner g \urcorner \check{\doteq}_{\boldsymbol{j}} \ulcorner \tilde{h} \urcorner}$$

  Then, because of induction hypothesis on $\boldsymbol{i}, \boldsymbol{j} < \boldsymbol{k}$:

  $\mathbf{S} \vdash \ f = \tilde{f}$ and $\mathbf{S} \vdash \ g = \tilde{g}$
  whence
  $\mathbf{S} \vdash \ g \circ f = \tilde{g} \circ \tilde{f}$
  q. e. d. in this compatibility case.

- **case** of compatibility of forming the *induced* with equality: analogous.

- **case** of Freyd's uniqueness of the initialised iterated:

$$\Uparrow \quad \frac{\ulcorner h \urcorner \check{=}_{\boldsymbol{k}} \ulcorner g^{\S} \circ (f \times \mathrm{id}_{\mathbb{N}}) \urcorner}{\ulcorner h \circ (\mathrm{id}_A, 0) \circ \Pi_A \urcorner \check{=}_{\boldsymbol{i}} \ulcorner f \urcorner \\ \wedge \ \ulcorner h \circ (\mathrm{id}_A \times \mathrm{s}) \urcorner \check{=}_{\boldsymbol{j}} \ulcorner g \circ h \urcorner}$$

By hypothesis on $\boldsymbol{i}$ and $\boldsymbol{j}$

$$\mathbf{S} \vdash \ h \circ (\mathrm{id}_A, 0) = f \text{ and } \mathbf{S} \vdash \ h \circ (A \times \mathrm{s}) = g \circ h$$

Freyd's uniqueness on the objective level finally gives

$$\mathbf{S} \vdash \ h = g^{\S} \circ (f \times \mathrm{id}_{\mathbb{N}})$$

q. e. d. in this case, the last case for $\mathbf{S} := \mathbf{PR2}$ and hence the last to be considered for its *definitional,* conservative extension $\mathbf{S} = \mathbf{PR2} + (\mathrm{abstr})$.

- **case** of *iterative descent,* for $\pi\mathbf{R}$: **Let**

$$\pi\mathbf{R} \vdash \Uparrow \quad \frac{\ulcorner \psi \urcorner \check{=}^{\pi}_{\boldsymbol{k}} \ulcorner \mathrm{false}_A \urcorner}{\ulcorner [[c = 0] \implies [p = \mathrm{id}_A]] \\ \wedge \ [c > 0 \implies c \circ p < c] \urcorner \\ \check{=}^{\pi}_{\boldsymbol{i}} \ulcorner \mathrm{true}_A \urcorner] \\ \wedge \ [\psi \implies [c \circ p^{\S} > 0] \check{=}^{\pi}_{\boldsymbol{j}} \ulcorner \mathrm{true}_{A \times \mathbb{N}} \urcorner]}$$

By hypothesis on $\boldsymbol{i}, \boldsymbol{j} < \boldsymbol{k}$ the premissae infer

$$\pi\mathbf{R} \vdash [c(a) = 0 \implies p(a) = a]$$
$$\wedge [c(a) > 0 \implies c \circ p(a) < c(a)] : A \to \mathbf{2} \text{ (descent)}$$
as well as
$$\pi\mathbf{R} \vdash \psi(a) \implies c \circ p^{\S}(a, n) > 0 : A \times \mathbb{N} \to \mathbf{2} \text{ (test)}$$

But (objective) **axiom** (π) of *non-infinite descent*

– which constitutes theory $\pi\mathbf{R}$ over $\mathbf{S}$ –

infers from the above

$$\pi\mathbf{R} \vdash \psi = \mathrm{false}_A : A \to \mathbf{2} \ \text{ q. e. d.}$$

The postcedents above exhaust all theorems of theory $\pi\mathbf{R}$. This **proves** the theorem.

We approach **soundness** on objective mathematical level as follows.

8.10 Termination conditioned soundness

For p. r. theory $\mathbf{S}$ and its internal notion of equality
$\check{\doteq} \ = \ \check{\doteq}_k : \mathbb{N} \to \mathrm{S} \times \mathrm{S}$,
dtree_k the k th internal equation deduction tree of $\mathbf{S}$, we have:

(i) *Termination-conditioned "inner" evaluation soundness:*

With $\mathbf{S}$ sets A, B, with $k \in \mathbb{N}$ free, and map codes $f, g \in B^A \subset \mathrm{S} \subset \mathbb{N}$ free, argument $a \in A$ free

$$\mathbf{S} \vdash [m\ defs\ \boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_k/a)] \implies [f \check{\doteq}_k g \implies \boldsymbol{\varepsilon}(f, a) = \boldsymbol{\varepsilon}(g, a)] \qquad (\bullet)$$

If *an internal p. r. deduction-tree for internal equality of f and g is available, and* ***if*** *on this tree – top down argumented with a in A – tree evaluation* ***terminates,*** *will say: iteration of evaluation step* $\boldsymbol{e}_{\mathrm{dt}}$ *becomes* ***stationary*** *after a finite number m of steps,* ***then*** *equality of evaluation of f and g on this argument is the consequence.*

By substitution of *concrete* codes, codes $\ulcorner f \urcorner, \ulcorner g \urcorner$ of **PR** maps $f, g : A \to B$ into free $f, g \in B^A$, we get from the above

(ii) *Termination-conditioned "concrete" evaluation soundness, reflection:*

For $\mathbf{S}$ maps $f, g : A \to B$:

$$\begin{aligned} \mathbf{S} \vdash\ & [\ulcorner f \urcorner \check{\doteq}_k \ulcorner g \urcorner \wedge m\ defs\ \boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_k/a)] \\ & \implies f(a) = \boldsymbol{\varepsilon}(\ulcorner f \urcorner, a) = \boldsymbol{\varepsilon}(\ulcorner g \urcorner, a) = g(a) \\ & k \in \mathbb{N},\ m \in \mathbb{N},\ a \in A \text{ free} \\ & \implies f = g : A \to B \end{aligned}$$

Internal equality '$\check{\doteq}$' is ***reflected*** *into objective equality '='.*

($\boldsymbol{\varepsilon}_{\mathrm{dt}}$ termination conditioned).

(iii) Specialisation in (i) to $f = \varphi \in \mathbf{2}^A$ an internal predicate, and substitution of $\ulcorner \mathrm{true}_A \urcorner \in \mathbf{2}^A$ give

Termination-conditioned logical internal soundness:

$$\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{S}}(k, \varphi) \wedge m \textit{ defs } \boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies \boldsymbol{\varepsilon}(\varphi, a)$$

If *tree-evaluation of an internal* **S** *deduction tree for an internal p. r. predicate* $\varphi \in \mathbf{2}^A$ *– the tree* argumented *with* $a \in A$ *–* ***terminates*** *after a finite number* m *of evaluation steps,* ***then*** $\boldsymbol{\varepsilon}(\varphi, a)$ *is the consequence,* within **S** as well as within its strengthening $\pi\mathbf{R}$ (and in **set** theory).

(iv) Specialisation of (iii) to case $\ulcorner \varphi \urcorner \in \mathbf{2}^A$ a concrete p. r. predicate code we get

Termination-conditioned logical objective soundness, reflection:

$$\mathbf{S} \vdash \mathrm{Prov}_{\mathbf{S}}(k, \ulcorner \varphi \urcorner) \wedge m \textit{ defs } \boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies \varphi(a)$$

If *tree-evaluation of an internal* **S** *deduction tree for a free variable p. r. predicate* $\varphi : A \to \mathbf{2}$ *– the tree* argumented *with* $a \in A$ *–* ***terminates*** *after a finite number* m *of evaluation steps,* ***then*** $\varphi(a)$ *is the consequence,* within **S** as well as within its strengthening $\pi\mathbf{R}$ (and in **set** theory).

Remark to **proof** below: in present case of frame theory **S** (and of stronger theory $\pi\mathbf{R}$) we have to *control* all evaluation step iterations, and we do that by control of iterative evaluation $\boldsymbol{\varepsilon}_{\mathrm{dt}}$ of whole *argumented deduction trees,* whose recursive definitions will be – merged – part of this proof.

Proof of – basic – *termination-conditioned soundness* (i) i. e. of implication (•) in the theorem, is by induction on deduction tree enumerating index $k \in \mathbb{N}$ of sequence $[\mathrm{dtree}_k]_{k\in\mathbb{N}}$, starting with (flat) $\mathrm{dtree}_0 = \langle \ulcorner \mathrm{id}_{\mathbb{1}} \urcorner \check{=}_0 \ulcorner \mathrm{id}_{\mathbb{1}} \urcorner \rangle$. Count is first by depth of trees, and second by lexicographical order. $m \in \mathbb{N}$ is to dominate argumented-deduction-tree evaluation $\boldsymbol{\varepsilon}_{\mathrm{dt}}$ to be recursively defined below: *condition m defs* $\boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_k/a)$ with respect to complexity $\boldsymbol{c}_{\mathrm{dt}}$, and step $\boldsymbol{e}_{\mathrm{dt}}$.

We argue by *recursive case distinction* on the form of the top up-to-two layers of argumented deduction tree dtree_k/x at hand.

We first treat the case of theory **PR2** and its internal deduction trees.

Flat **super case** $\mathrm{depth}(\mathrm{dtree}_k) = 0$, i. e. super case of *unconditioned,* axiomatic (internal) *equation* $f \check{=}_k g$:

The first involved of these cases is *associativity* of (internal) *composition,* with abbreviation $\odot$ for $\ulcorner\circ\urcorner$:

$$\mathrm{dtree}_k \ = \ \langle\langle h \odot g\rangle \odot f\rangle \check{=}_k \langle h \odot \langle g \odot f\rangle\rangle$$

In this case – no need of a recursion on k –

$$\mathbf{PR2} \vdash$$

$$m \ \mathit{defs}\ \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies$$

$$[m \ \mathit{defs}\ \varepsilon(\langle h \odot g\rangle \odot f, a)]$$

$$\wedge\ [m \ \mathit{defs}\ \varepsilon(\langle h \odot g\rangle, \varepsilon(f, a))$$

$$\wedge\ [m \ \mathit{defs}\ \varepsilon(h, \varepsilon(g, \varepsilon(f, a)))$$

$$\wedge\ [m \ \mathit{defs}\ \varepsilon(h, \varepsilon(\langle g \odot f\rangle, a))$$

$$\wedge\ [m \ \mathit{defs}\ \varepsilon(\langle h \odot \langle g \odot f\rangle\rangle, a)]$$

$$\wedge$$

$$\varepsilon(\langle h \odot g\rangle \odot f, a) = \varepsilon(\langle h \odot g\rangle, \varepsilon(f, a))$$

$$= \varepsilon(h, \varepsilon(g, \varepsilon(f, a)))$$

$$= \varepsilon(h, \varepsilon(\langle g \odot f\rangle, a)) = \varepsilon(h \odot \langle g \odot f\rangle, a)$$

This proves assertion (•) in present *associativity-of-composition* case.

Analogous **proof** for the other flat, equational cases, namely reflexivity of equality, left and right neutrality of id family, the boolean equations for object **2**, Godement's equations for the induced map as well as retractive pairing and distributivity of composition over forming the induced map:

Godement's equations $\ell \circ (f, g) = f,\ \mathrm{r} \circ (f, g) = g$, with ';' abbreviating '$\ulcorner, \urcorner$':

$$m \ \mathit{defs}\ \varepsilon \text{ etc.} \implies$$

$$\varepsilon(\ulcorner \ell \urcorner \odot \langle f; g\rangle, c) = \varepsilon(\ulcorner \ell \urcorner, \varepsilon(\langle f; g\rangle, c))$$

$$= \varepsilon(\ulcorner \ell \urcorner, (\varepsilon(f, c), \varepsilon(g, c))) = \varepsilon(f, c),$$

analogously for composition with right projection

Analogous proof for cases of *retractive pairing* and distributivity of composition over forming the induced map. Here are the **proofs** of

(●) for the last equational cases, with $^{\$}$ abbreviating $\ulcorner\S\urcorner$:

Anchor case statement for the internal iterated $f^{\$}$:

$$\mathrm{dtree}_k \;=\; \langle f^{\$} \odot \langle \ulcorner \mathrm{id}_A \urcorner ; \ulcorner 0 \urcorner \odot \ulcorner \Pi_A \urcorner \rangle \,\check{\doteq}_k\, \ulcorner \mathrm{id}_A \urcorner \rangle$$

is straight forward, as follows:

$$\begin{aligned} &\varepsilon(\langle f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner ; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle\rangle, a) \\ &= \varepsilon(f^{\$}, \varepsilon(\langle \ulcorner \mathrm{id} \urcorner ; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle), a)) \\ &= \varepsilon(f^{\$}, (a, 0)) = a = \varepsilon(\ulcorner \mathrm{id} \urcorner, a) \end{aligned}$$

Iteration step, case of *genuine iteration equation*
$\mathrm{dtree}_k \;=\; \langle f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle \,\check{\doteq}_k\, f \odot f^{\$} \rangle$:

PR2 $\vdash$ *m defs* $\varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k/(a,n)) \implies$

m defs all instances of ε *below,* and:

$$\begin{aligned} &\varepsilon(f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle, (a, n)) && (1) \\ &= \varepsilon(f^{\$}, \varepsilon(\langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle, (a, n))) \\ &= \varepsilon(f^{\$}, (a, \mathrm{s}\, n)) \\ &= \varepsilon(f^{[\mathrm{s}\, n]}, a) \quad \text{(by definition of } \varepsilon \text{ step } e\text{)} \\ &= \varepsilon(f \odot f^{[n]}, a) \\ &= \varepsilon(f, \varepsilon(f^{\$}, (a, n)) \\ &= \varepsilon(f \odot f^{\$}, (a, n)) && (2) \end{aligned}$$

Proof of termination-conditioned soundness for the remaining *deep,* genuine Horn cases: for dtree_k, Horn type (at least) at *deduction* of *root:*

Transitivity-of-equality case: with map code variables $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ we start with argument-free (implicational) deduction tree

$$\text{dtree}_k \quad = \quad \Uparrow\frac{\Uparrow\dfrac{\text{dtree}_{ii} \quad \text{dtree}_{ji}}{\boldsymbol{f} \stackrel{\vee}{\doteq}_i \boldsymbol{g}} \qquad \Uparrow\dfrac{\text{dtree}_{ij} \quad \text{dtree}_{jj}}{\boldsymbol{g} \stackrel{\vee}{\doteq}_j \boldsymbol{h}}}{\boldsymbol{f} \stackrel{\vee}{\doteq}_k \boldsymbol{h}}$$

It is argumented with argument $a \in A$ (free) say, recursively spread down:

$$\text{dtree}_k/a \quad = \quad \frac{\dfrac{\text{dtree}_{ii}/x_{ii} \quad \text{dtree}_{ji}/x_{ji}}{\boldsymbol{f}/a \stackrel{\vee}{\doteq}_i \boldsymbol{g}/a} \qquad \dfrac{\text{dtree}_{ij}/x_{ij} \quad \text{dtree}_{jj}/x_{jj}}{\boldsymbol{g}/a \stackrel{\vee}{\doteq}_j \boldsymbol{h}/a}}{\boldsymbol{f}/a \stackrel{\vee}{\doteq}_k \boldsymbol{h}/a}$$

Spreading down arguments from upper level down to 2nd level must/is given explicitly, further arguments spread down is then recursive by the type of deduction (sub)trees $\text{dtree}_i, \text{dtree}_j,\ \ i, j < k$.

By induction hypothesis on i, j we have for tree evaluation $\varepsilon_{\mathrm{dt}}$:

$$
\begin{aligned}
& f \,\check{=}_k\, h \wedge m \text{ } \mathit{defs} \text{ } \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k/a) \\
& \quad \Longrightarrow m \text{ } \mathit{defs} \text{ } \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_i/a), \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_j/a) \wedge \\
& \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_i/a) = \langle \ulcorner \mathrm{id} \urcorner / \varepsilon(f, a) \,\check{=}_i\, \ulcorner \mathrm{id} \urcorner / \varepsilon(g, a) \rangle \\
& \wedge \ \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_j/a) = \langle \ulcorner \mathrm{id} \urcorner / \varepsilon(g, a) \,\check{=}_j\, \ulcorner \mathrm{id} \urcorner / \varepsilon(h, a) \rangle \\
& \quad \Longrightarrow \varepsilon(f, a) = \varepsilon(g, a) \wedge \varepsilon(g, a) = \varepsilon(h, a) \\
& \quad\ \text{by induction hypothesis on } i, j < k \\
& \quad \Longrightarrow \varepsilon(f, a) = \varepsilon(h, a)
\end{aligned}
$$

and this is what we wanted to show in present transitivity of equality case.

[Transitivity axiom for equality is a main reason for necessity to consider (argumented) deduction trees: intermediate map code equalities '$\check{=}$' in a transitivity chain must be each evaluated, and pertaining deduction trees may be of arbitrary high evaluation complexity]

Case of **symmetry** axiom scheme for equality is obvious.

Compatibility Case of composition with equality

$$
\mathrm{dtree}_k/a \quad = \quad \frac{\langle g \odot f \rangle / a \,\check{=}_k\, \langle g \odot \tilde{f} \rangle / a}{\dfrac{f/a \,\check{=}_j\, \tilde{f}/a}{\mathrm{dtree}_{ij}/a \quad \mathrm{dtree}_{jj}/a}}
$$

By induction hypothesis on $j < k$

$$m \ \textit{defs} \ \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies$$
$$m \ \textit{defs} \ \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_j/a) \wedge \varepsilon(f, a) = \varepsilon(\tilde{f}, a) \implies$$
$$\varepsilon(g \odot f, a) = \varepsilon(g, \varepsilon(f, a)) = \varepsilon(g, \varepsilon(\tilde{f}, a))$$
$$= \varepsilon(g \odot \tilde{f}, a)$$

by dominated characterisic equations for ε and Leibniz' substitutivity into equality q. e. d. in this first compatibility case.

Spread down arguments is more involved in

Case of composition with equality in second composition factor: argument spread down merged with tree evaluation $\varepsilon_{\mathrm{dt}}$ and proof of result:

$$\mathrm{dtree}_k/a \quad = \quad \cfrac{\langle g \odot f \rangle/a \,\check{\doteq}_k\, \langle \tilde{g} \odot f \rangle/a}{\cfrac{g \,\check{\doteq}_i\, \tilde{g}}{\mathrm{dtree}_{ii} \quad \mathrm{dtree}_{ji}}}$$

[Here dtree_i is not (yet) provided with argument, it *is* argumented during top down tree evaluation]

$$m \ \textit{defs} \ \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies$$
$$[m \ \textit{defs} \text{ all instances of } \varepsilon \text{ below}] \wedge$$
$$\varepsilon(g \odot f, a) = \varepsilon(g, \varepsilon(f, a)) = \varepsilon(\tilde{g}, \varepsilon(f, a)) \qquad (*)$$
$$= \varepsilon(\tilde{g} \odot f, a)$$

$(*)$ holds by Leibniz' substitutivity and

$$m \ \mathit{defs}\ \boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies$$
$$m \ \mathit{defs}\ \boldsymbol{\varepsilon}_{\mathrm{dt}}(\mathrm{dtree}_i/\boldsymbol{\varepsilon}(f, a))$$
$$\wedge\ m \ \mathit{defs}\ \boldsymbol{\varepsilon}(g, \boldsymbol{\varepsilon}(f, a)) = \boldsymbol{\varepsilon}(\tilde{g}, \boldsymbol{\varepsilon}(f, a))$$

by induction hypothesis on $i < k$.

This proves assertion $(\bullet)$ in this 2nd compatibility case.

Compatibility case of internal formation of the induced map with internal equality

$$f \check{\doteq}_i \tilde{f},\ g \check{\doteq}_j \tilde{g} \implies \langle f; g\rangle \check{\doteq}_k \langle \tilde{g}; \tilde{f}\rangle :$$

$$m \ \mathit{defs}\ \boldsymbol{\varepsilon}(\langle f; g\rangle, c) \wedge m \ \mathit{defs}\ \boldsymbol{\varepsilon}(\langle \tilde{f}; \tilde{g}\rangle, c)$$
$$\implies m \ \mathit{defs}\ \boldsymbol{\varepsilon}(f, c),\ \boldsymbol{\varepsilon}(g, c),\ \boldsymbol{\varepsilon}(\tilde{f}, c),\ \boldsymbol{\varepsilon}(\tilde{g}, c) \quad \wedge$$
$$\boldsymbol{\varepsilon}(\langle f; g\rangle, c) = (\boldsymbol{\varepsilon}(f, c), \boldsymbol{\varepsilon}(g, c)) = (\boldsymbol{\varepsilon}(\tilde{f}, c), \boldsymbol{\varepsilon}(\tilde{g}, c))$$
by hypothesis $f \check{\doteq}_i \tilde{f},\ g \check{\doteq}_j \tilde{g}$
$$= \boldsymbol{\varepsilon}(\langle \tilde{f}; \tilde{g}\rangle, c)$$

Same for compatibility of internal cartesian map product with equality (redundant).

Case of Freyd's (internal) uniqueness of the *initialised iterated,* is case

$$\mathrm{dtree}_k/(a, n) \quad = \quad \frac{h/(a, n) \check{\doteq}_k \langle g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner\rangle/(a, n)\rangle}{t_i \qquad\qquad\qquad\qquad t_j}$$

where

$$\mathrm{root}\,(t_i) = \langle h \odot \langle \ulcorner \mathrm{id} \urcorner; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner\rangle/a \check{\doteq}_i f/a\rangle,$$
$$\mathrm{root}\,(t_j) = \langle h \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{r} \urcorner\rangle/(a, n) \check{\doteq}_j \langle g \odot h\rangle/(a, n)\rangle$$

Comment: h is an internal *comparison candidate* fullfilling the same internal p. r. equations as (internal) initialised iterated

$\langle g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner \rangle \rangle$.

It should be – **is**: *soundness* – evaluated equal to the latter, on $A \times \mathbb{N}$.

Soundness **proof** in this case

$$h \odot \langle \ulcorner \mathrm{id} \urcorner, 0 \rangle \check{\doteq}_i f \wedge h \odot \langle \ulcorner \mathrm{id} \urcorner \# \mathrm{s} \rangle \check{\doteq}_j g \odot h$$
$$\implies h \check{\doteq}_k g^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# f \rangle$$

is the following one, by (structural) recursion on k :

$$m \text{ } \mathit{defs} \text{ } \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k / a) \implies$$
$$[m \text{ } \mathit{defs} \text{ all instances of } \varepsilon \text{ below}] \wedge$$
$$\varepsilon(h, (a, 0)) = \varepsilon(f, a)$$
$$= \varepsilon(g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner \rangle, (a, 0)) \qquad \text{(hypothesis on } i < k)$$
$$(\# = \ulcorner \times \urcorner \text{ the internal cartesian map code product})$$

as well as – *induction on n* –

$$\varepsilon(h, (a, \mathrm{s}\, n))$$
$$= \varepsilon(h, \varepsilon(\langle \ulcorner \mathrm{id}_A \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle, (a, n)))$$
$$= \varepsilon(h \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle, (a, n))$$
$$= \varepsilon(g \odot h, (a, n)) \qquad \text{(hypothesis on } j < k)$$
$$= \varepsilon(g, \varepsilon(h, (a, n)))$$
$$= \varepsilon(g, \varepsilon(g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner \rangle, (a, n))) \qquad (*)$$
$$= \varepsilon(g \odot \langle g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner \rangle \rangle, (a, n))$$
$$= \varepsilon(g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner \rangle, (a, \mathrm{s}\, n))$$

(∗) by induction hypothesis on n and since evaluation ε preserves predicative equality '=' (Leibniz)

Termination-condtioned-soundness **extension** to theory $\mathbf{S} = \mathbf{PR2}+$ (abstr)

Case of internally equal restrictions

$$f \,\check{=}_k\, g \in \mathrm{S}(\{A : \chi\}, \{B : \varphi\})$$

of internal **PR2** maps $f, g \in \mathrm{PR2}(A, B)$:

$$\mathrm{dtree}_k \;=\; \frac{\langle\langle \ulcorner\chi\urcorner ; f\rangle ; \ulcorner\varphi\urcorner\rangle \,\check{=}^{\mathbf{a}}_k\, \langle\langle \ulcorner\chi\urcorner ; g\rangle ; \ulcorner\varphi\urcorner\rangle}{\langle \ulcorner\chi\urcorner \ulcorner\Rightarrow\urcorner \langle f \ulcorner =_B \urcorner g\rangle\rangle \,\check{=}^{\mathbf{PR2}}_i\, \ulcorner \mathrm{true}_A \urcorner}$$

Here

$$
\begin{aligned}
& m \; \mathit{defs} \; \varepsilon_{\mathrm{dt}}(\mathrm{dtree}_k/a) \implies \\
& [m \; \mathit{defs} \text{ all instances of } \varepsilon \text{ below}] \wedge : \\
& \varepsilon(\langle \ulcorner\chi\urcorner \ulcorner\Rightarrow\urcorner \langle f \ulcorner =_B \urcorner g\rangle\rangle, a) = \mathrm{true} \\
& \iff [\chi(a) \Rightarrow [\varepsilon(f, a) =_B \varepsilon(g, a)] \\
& \iff \varepsilon(\ulcorner \mathrm{true}_A \urcorner, a) = \mathrm{true} \\
& \implies \\
& \varepsilon(\langle\langle \ulcorner\chi\urcorner ; f\rangle ; \ulcorner\varphi\urcorner\rangle, a) \\
& = ((\chi(a), \varepsilon(f, a)), \varphi(a)) = \varepsilon(\langle\langle \ulcorner\chi\urcorner ; g\rangle ; \ulcorner\varphi\urcorner\rangle, a)
\end{aligned}
$$

q. e. d. in this restriction case.

Cases of internal composition, induced maps into products, as well as iteration of internal **S** maps are obtained directly by formal map restriction in the corresponding **PR2** cases.

q. e. d. *Termination conditioned p. r. soundness Theorem.*

Comment: Already for stating the evaluations, we needed the – categorical, free-variables theories **PR**, **PR2**, **S** of primitive recursion, as well as – for "termination", even in classial frame **T** – p. r. complexities within $\mathbb{N}[\omega]$. Since this type of *soundness* is a corner stone in our approach, the above complicated categorical combinatorics seem to be appropriate for the constructive framework of iterative descent theory $\pi\mathbf{R}$ below, although "terminology used is not in the mainstream of category theory and logic."

Part III

CONSISTENCY

Chapter 9

Predicates decidability

This chapter is *logically* central: In forgoing chapter on evaluation we have strengthened (boolean) p. r. theory **S** of *primitive recursion with predicate-into-subset abstraction* into iterations.

Theory $\pi\mathbf{R}$ turns out to be ***sound*** over **S**, by *termination-conditioned* soundness of theory **S**.

Within $\pi\mathbf{R}$ we define for each **PR2** predicate an alleged decision algorithm to decide on *counterexamples vs. overall validity.* Discussion of that decision algorithm leads to *decidablity* of all p. r. predicates, within/by theory $\pi\mathbf{R}$.

Consistency *provability* of "any" theory can be stated as a p. r. predicate, decidable within $\pi\mathbf{R}$ (and extensions like in particular **set** theory).

9.1 Relative soundness

From *termination-conditioned* soundness of theory **S** we get

Internal/*arithmetised* S consistency
framed by descent theory $\pi\mathbf{R}$:

For *iterative descent* theory $\pi\mathbf{R} = \mathbf{S} + (\pi)$, **axiom** (π) stating non-infinite iterative descent in *ordinal* $\mathbb{N}[\omega]$ we have

$\pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{S}}$ i. e. "necessarily" in *free-variables* form:

$\pi\mathbf{R} \vdash \neg\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbf{2},\ k \in \mathbb{N}$ free :

Theory $\pi\mathbf{R}$ derives *that no* $k \in \mathbb{N}$ *is the internal* **S**-Proof *index for* $\ulcorner\mathrm{false}\urcorner$.

Recall: For p. r. theory **S** predicate $\mathrm{Prov}_{\mathbf{S}}$ is defined as

$$\mathrm{Prov}_{\mathbf{S}}(k, \varphi) = [\varphi \,\check{=}^{\mathbf{S}}_{k}\, \ulcorner\mathrm{true}_A\urcorner] : \mathbb{N} \times \mathbf{2}^A \to \mathbf{2}$$

Proof by *termination-conditioned soundness of* **S** :
By objective logic assertion (iv) of that **theorem,** with
$\varphi = \varphi(a) := \mathrm{false}(a) = \mathrm{false} : \mathbb{1} \to \mathbf{2}$, we get:

Evaluation-effective internal inconsistency of **S**
– i. e. availability of an *evaluation-terminating* internal *deduction tree* of $\ulcorner\mathrm{false}\urcorner$ –
implies false :

$$\mathbf{S},\ \pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\mathrm{false}\urcorner) \wedge \boldsymbol{c}_{\mathrm{dt}}\, \boldsymbol{e}^{m}_{\mathrm{dt}}(\mathrm{dtree}_k/\mathrm{false}) = 0$$
$$\implies \mathrm{false}$$

Contraposition to this, still with $k, m \in \mathbb{N}$ free:

$$\pi\mathbf{R} \vdash \text{ true } \implies$$
$$\neg\text{Prov}_{\mathbf{S}}(k, \ulcorner\text{false}\urcorner) \vee \boldsymbol{c}_{\text{dt}}\ \boldsymbol{e}_{\text{dt}}^{m}(\text{dtree}_k/\text{false}) > 0$$

i. e. by free-variables (boolean) tautology:

$$\pi\mathbf{R} \vdash \text{ Prov}_{\mathbf{S}}(k, \ulcorner\text{false}\urcorner) \implies \boldsymbol{c}_{\text{dt}}\ \boldsymbol{e}_{\text{dt}}^{m}(\text{dtree}_k/\text{false}) > 0$$

For k "fixed", the conclusion of this implication – m free – means infinite descent in $\mathbb{N}[\omega]$ of iterative argumented deduction-tree evaluation $\boldsymbol{\varepsilon}_{\text{dt}}$ on $\text{dtree}_k/\text{false}$, which is excluded intuitively.

Formally it is excluded within our theory $\pi\mathbf{R}$ taken as frame:

We apply non-infinite-descent scheme (π) to $\boldsymbol{\varepsilon}_{\text{dt}}$ which is given by *step* $\boldsymbol{e}_{\text{dt}}$ and complexity $\boldsymbol{c}_{\text{dt}}$ – the latter descends (this is *argumented-tree evaluation descent*) with each application of $\boldsymbol{e}_{\text{dt}}$ as long as complexity $0 \in \mathbb{N}[\omega]$ is not ("yet") reached. We combine this with – choice of – *overall "negative"* condition

$$\psi = \psi(k) := \text{Prov}_{\mathbf{S}}(k, \ulcorner\text{false}\urcorner) : \mathbb{N} \to \mathbf{2},\ \ k \in \mathbb{N} \text{ free}$$

and get – by that scheme (π) – overall negation of this (overall) *excluded* predicate ψ, namely

$$\pi\mathbf{R} \vdash \neg\text{Prov}_{\mathbf{S}}(k, \ulcorner\text{false}\urcorner) : \mathbb{N} \to \mathbf{2},\ \ k \in \mathbb{N} \text{ free, i. e.}$$
$$\pi\mathbf{R} \vdash \text{Con}_{\mathbf{S}} \quad \textbf{q. e. d.}$$

So "slightly" strengthened theory $\pi\mathbf{R} = \mathbf{S} + (\pi)$ derives the free-variables consistency formula for theory $\mathbf{S}$ of primitive recursion with 2-valued truth object and predicate abstraction.

[Scheme (π) holds in **set** theory, since there $\mathbb{N}[\omega]$ is an *ordinal.*]

As is well known, consistency *provability* and soundness of a theory are strongly tied together. We get in fact even

Theorem on S-to-$\pi\mathbf{R}$ relative soundness:

- for an **S** predicate $\varphi = \varphi(a) : A \to \mathbf{2}$ we have
  $\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) \implies \varphi(a) : \mathbb{N} \times A \to \mathbf{2}$
- in particular we get for **S**-maps $f,\ g : A \to B$
  $\pi\mathbf{R} \vdash \ \ulcorner f\urcorner \check{\doteq}_k \ulcorner g\urcorner \implies [f(a) = g(a)]$

Proof of first assertion is a slight generalisation of proof of **internal consistency** of **S** framed by $\pi\mathbf{R}$ as follows – take predicate φ instead of false, and use *termination-conditioned soundness,* assertion (iv) on *termination-conditioned objective logical soundness* directly:

$$\begin{aligned}
&\mathbf{S},\ \pi\mathbf{R} \vdash \\
&\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) \wedge \boldsymbol{c}_{\mathrm{dt}}\, \boldsymbol{e}^m_{\mathrm{dt}}(\mathrm{dtree}_k/a) = 0 \\
&\ \implies \varphi(a) : (\mathbb{N} \times \mathbb{N}) \times A \to \mathbf{2} \\
&k, m \in \mathbb{N},\ \ a \in A \text{ all free}
\end{aligned}$$

Boolean free-variables calculus tautology

$$\begin{aligned}
&[\alpha \wedge \beta \Rightarrow \gamma] \iff [\neg[\alpha \Rightarrow \gamma] \Rightarrow \neg\beta] \\
&(\text{test with } \beta = \text{false as well as with } \beta = \text{true})
\end{aligned}$$

gives from this, still with k, m, a free:

$$\begin{aligned}
\pi\mathbf{R} \vdash \neg&[\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) \implies \varphi(a)] \\
&\implies \boldsymbol{c}_{\mathrm{dt}}\, \boldsymbol{e}^m_{\mathrm{dt}}(\mathrm{dtree}_k/\mathrm{false}) > 0.
\end{aligned}$$

As before, apply non-infinite descent scheme (π) to $\varepsilon_{\mathrm{dt}}$ in combination with – choice of – *overall "negative"* condition

$$\psi(k,a) := \neg[\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) \implies \varphi(a)] : \mathbb{N} \times A \to \mathbf{2}$$

and get – scheme (π) – overall negation of this (overall) *excluded* predicate $\psi(k,a)$, namely

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) \implies \varphi(a) : \mathbb{N} \times (\mathbb{1} \times A) \to \mathbf{2},$$

proof index $k \in \mathbb{N}$ and argument $a \in A$ free

q. e. d. for first assertion.

For **proof** of the second assertion, take in the above

$$\varphi = \varphi(a) := [f(a) = g(a)] : A \to B \times B \to \mathbf{2}$$

and get

$$\begin{aligned} \pi\mathbf{R} \vdash \ulcorner f\urcorner \check{=}_k \ulcorner g\urcorner & \\ & \implies \mathrm{Prov}_{\mathbf{S}}(j(k), \ulcorner[f = g]\urcorner) \\ & \qquad [j : \mathbb{N} \to \mathbb{N} \text{ suitable p. r.}] \\ & \implies [f(a) = g(a)] : \mathbb{N} \times A \to \mathbf{2} \quad \textbf{q. e. d.} \end{aligned}$$

9.2 An alleged partial decision algorithm

As the kernel of decision of an $\mathbf{S}$ predicate $\chi : A \to \mathbf{2}$ by iterative descent theory $\pi\mathbf{R}$ we introduce an (a priori partial) μ-recursive *decision algorithm* $\nabla\chi$ for χ : *counterexample vs. proof.* Without restriction of generality $\chi = \chi(n) : \mathbb{N} \to \mathbf{2}$.

As a partial p. r. map $\nabla\varphi : \mathbb{1} \rightharpoonup \mathbf{2}$ is given by three $\mathbf{S}$ data:

- its *index domain for defined arguments*

$$D_{\nabla\varphi} \text{ of form } D_{\nabla\varphi} = \{k : \delta_\varphi(k)\}$$
$$D_{\nabla\varphi} =_{\text{def}} \{k : \neg\varphi(k) \vee \text{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner)\} \subseteq \mathbb{N}$$
$$\text{“}k \text{ counterexample or } \mathbf{S}\text{-}\textit{proof}\text{”}$$

- its *defined arguments enumeration*

$$d_{\nabla\varphi} =_{\text{def}} \Pi : D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\} \xrightarrow{\subseteq} \mathbb{N} \xrightarrow{\Pi} \mathbb{1}$$

(not a priori a retraction or empty,)

- and its *rule* $\widehat{\nabla} = \widehat{\nabla}_\varphi : D_{\nabla\varphi} \to \mathbf{2}$ defined by

$$\widehat{\nabla}_\varphi = \widehat{\nabla}_\varphi(k) =_{\text{def}} \begin{cases} \text{false if } \neg\varphi(k) \\ \text{true if } \text{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) \end{cases}$$
$$: D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\} \to \mathbf{2}$$

$\widehat{\nabla}_\varphi : D_{\nabla\varphi} \to \mathbf{2}$ is in fact a well defined rule, since by the above S-to-$\pi\mathbf{R}$ *objective soundness* we have

$$\pi\mathbf{R} \vdash \text{Prov}_{\mathbf{S}}(k, \ulcorner\chi\urcorner) \implies \varphi(n) : \mathbb{N} \times \mathbb{N} \to \mathbf{2},$$
$$k, n \in \mathbb{N} \text{ free}$$

whence case disjointness of the alternative within $D_{\nabla\varphi}$.

Remark: This taken together means intuitively within $\pi\mathbf{R}$:

$$\nabla\varphi = \begin{cases} \text{false if } \neg\varphi(\mu\delta_\varphi) \\ \quad \textit{counterexample found} \\ \text{true if } \mathrm{Prov}_{\mathbf{S}}(\mu\delta_\varphi, \ulcorner\varphi\urcorner) \\ \quad \mathbf{S} \textit{ proof found} \\ \text{undefined } \textit{otherwise} \end{cases}$$

$$: \mathbb{1} \rightharpoonup \mathbf{2}$$

From the above we get the following complete (metamathematical)

πR case distinction for p. r. predicates

- **first case,** termination: $D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\}$ has at least one (total) p. r. *point,* namely

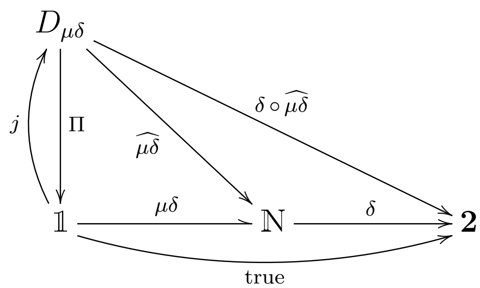
$$\mu\delta_\varphi : \mathbb{1} \to D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\}$$
$$=_{\text{by def}} \{k \in \mathbb{N} : \neg\varphi(k) \vee \mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner)\}, \text{with}$$
$$\pi\mathbf{R} \vdash \delta_\varphi \circ \mu\,\delta_\varphi : \mathbb{1} \to \mathbf{2}$$

This $\mu\delta_\varphi$ is formally partial, we state the present **termination case** as follows:

$$\mu\delta_\varphi : \mathbb{1} \rightharpoonup D_{\nabla\varphi} \subseteq \mathbb{N} \text{ with } \delta_\varphi \,\widehat{\circ}\, \mu\,\delta_\varphi \mathrel{\widehat{=}} \text{true i. e. with}$$

$$\delta_\varphi \,\widehat{\circ}\, \mu\,\delta_\varphi \overset{\Pi}{\subseteq} \text{true and true} \overset{j_\varphi}{\subseteq} \delta_\varphi \,\widehat{\circ}\, \mu\,\delta_\varphi$$

DIAGRAM:



$D_{\mu\delta} = D_{\mu\delta_\varphi} \subseteq \mathbb{N}$,
and $j = j_\varphi : \mathbb{1} \to D_{\mu\delta} = D_{\mu\delta_\varphi}$

suitable, to be found by (external, in present case terminating) count of maps and equations, suitable for

$$\delta_\varphi \,\hat{\circ}\, \mu\delta_\varphi = \delta_\varphi \circ \widehat{\mu\delta_\varphi} \circ j_\varphi = \mathrm{true} : \mathbb{1} \to \mathbf{2}$$

and hence, by **Totality Lemma:**

$k_0 := \mu\delta_\varphi : \mathbb{1} \to D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\} \subseteq \mathbb{N}$ total p. r. and
$\pi\mathbf{R} \vdash \ \delta_\varphi(k_0) = \delta_\varphi \circ \mu\,\delta_\varphi = \mathrm{true} : \mathbb{1} \to D_{\nabla\varphi} \to \mathbf{2}$

Subcases of this termination case are:

– negative (total) **subcase:**

$$\pi\mathbf{R} \vdash \neg\varphi\, k_0 \qquad (\mathbf{1}.\mathbf{1})$$

$k_0 : \mathbb{1} \to \mathbb{N}$ (minimal) counterexample

[Then $\pi\mathbf{R} \vdash \ \nabla\varphi = \mathrm{false} : \mathbb{1} \to \mathbf{2}$]

– positive (total) **subcase:**

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{S}}(k_0, \ulcorner\varphi\urcorner) \qquad (\mathbf{1.2})$$

$$k_0 : \mathbb{1} \to \mathbb{N} \ \text{(first)}\ \mathbf{S}\ \textit{proof}$$

[Then $\pi\mathbf{R} \vdash \nabla\varphi = \mathrm{true} : \mathbb{1} \to \mathbf{2}$]

These two subcases are in fact ***disjoint,*** disjoint by PR2-to-$\pi\mathbf{R}$ soundness.

By substitution of $k_0 = \mu\delta_\chi$ for $k \in \mathbb{N}$ free, we get in present **subcase:**

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{S}}(k_0, \ulcorner\varphi\urcorner) \implies \varphi(n) : \mathbb{N} \to \mathbf{2}$$

whence

$$\pi\mathbf{R} \vdash \varphi(n) \wedge \mathrm{Prov}_{\mathbf{S}}(k_0, \ulcorner\varphi\urcorner) : \mathbb{N} \to \mathbf{2} \qquad (\mathbf{1.2^+})$$

- **2nd case:** $\nabla\varphi$ does not terminate, $\pi\mathbf{R}$-derivably:

$$\pi\mathbf{R} \vdash D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\} = \emptyset_{\mathbb{N}} \ \text{i.e.}$$

$$\pi\mathbf{R} \vdash \delta_\varphi = \delta_\varphi(k) = \mathrm{false}$$

in particular

$$\pi\mathbf{R} \vdash (k)\neg\varphi(k) = \mathrm{false} : \mathbb{N} \to \mathbf{2}$$

whence

$$\pi\mathbf{R} \vdash (n)\varphi(n) : \mathbb{N} \to \mathbf{2}$$

in this second case as well as

$$\pi\mathbf{R} \vdash (k)\neg\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) : \mathbb{N} \to \mathbf{2}$$

Hence this 2nd case reads:

$$\pi\mathbf{R} \vdash (n)\varphi(n) \wedge (k)\neg\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner) : \mathbb{N} \times \mathbb{N} \to \mathbf{2} \quad (\mathbf{2})$$

"φ is $\pi\mathbf{R}$-derivable but not $\mathbf{S}$-*provable*": case of $\pi\mathbf{R}/\mathbf{S}$ *incompleteness,* case of *non-conservation* of extension $\pi\mathbf{R}$ of $\mathbf{S}$ at φ.

Remains **3rd case:**

$D_{\nabla\varphi} = \{\mathbb{N} : \delta_\varphi\}$ may be not empty, but has no *concrete numbers:* for all $\mathbf{S}$ points $\boldsymbol{k} : \mathbb{1} \to \mathbb{N} : \boldsymbol{k} \notin D_\chi$. This can/must be expressed ("metamathematically") by

$\pi\mathbf{R} \vdash \delta_\varphi(\boldsymbol{k}) \neq \text{true}$ i. e.

$\pi\mathbf{R} \vdash \neg[\delta_\varphi(\boldsymbol{k}) = \text{true}] : \mathbb{1} \to \mathbf{2}$ i. e.

$\pi\mathbf{R} \vdash \delta_\varphi(\boldsymbol{k}) = \text{false} : \mathbb{1} \to \mathbb{N} \to \mathbf{2}$

for $\boldsymbol{k} \in \mathbf{PR}(\mathbb{1}, \mathbb{N})$ arbitrary (number)

[2nd case just above is stronger than, contained in, latter 3rd case.]

Inequality Remark: For $f, g : A \to B$ p. r. maps, inequality $f \neq g$ *between* maps is **not** directly expressed as a formula of theories $\mathbf{S}$ and $\pi\mathbf{R}$.

Related is *predicative* inequality

$$[f \neq g] = \neg[f = g]$$
$$= \neg[f = g](a) : A \xrightarrow{(f,g)} B \times B \xrightarrow{=} \mathbf{2} \xrightarrow{\neg} \mathbf{2}$$

meaning $(\forall a \in A)[f(a) \neq g(a)]$ (!), and "just" for $A = \mathbb{1}$

$$(\exists a \in A)[f(a) \neq g(a)] \text{ i. e.}$$

$f \neq g$ is just for p. r. *points* the classical inequality of maps.

Decidability by iterative descent theory

Each $\mathbf{S}$ predicate $\varphi = \varphi(n) : \mathbb{N} \to \mathbf{2}$ gives rise to the following **complete case distinction** within, by iterative descent theory $\pi\mathbf{R}$:

$$\pi\mathbf{R} \vdash \neg\varphi(\mu\delta_\varphi) : \mathbb{1} \to \mathbb{N} \to \mathbf{2}$$

defined counterexample

or else

$$\pi\mathbf{R} \vdash \varphi \circ \boldsymbol{n} : \mathbb{1} \to \mathbb{N} \to \mathbf{2},$$

$\boldsymbol{n} : \mathbb{1} \to \mathbb{N}$ arbitrary (**number**) in $\mathbf{S}(\mathbb{1}, \mathbb{N})$

concrete theorems

Proof:

First alternative is just **subcase** $(\mathbf{1.1})$ in the complete disjunction above.

For the remaining alternative merge entailment $(\mathbf{1.2}^+)$ of **subcase** $(\mathbf{1.2})$ with **case** $(\mathbf{3})$ **number**wise:

$$\pi\mathbf{R} \vdash [\varphi(\boldsymbol{n}) \wedge \mathrm{Prov}_{\mathbf{S}}(k_0, \ulcorner\varphi\urcorner)] \vee [\varphi(\boldsymbol{n}) \wedge \neg\mathrm{Prov}_{\mathbf{S}}(k_0, \ulcorner\varphi\urcorner)]$$
$$k_0 = \mu\delta_\varphi \in \mathbf{PR}(\mathbb{1}, \mathbb{N}),\ \boldsymbol{n} \in \mathbf{PR}(\mathbb{1}, \mathbb{N}) \text{ arbitrary}$$

and get from this in joint **case** $(\mathbf{1.2})\dot{\vee}(\mathbf{3})$, alternative

$$\pi\mathbf{R} \vdash \varphi(\boldsymbol{n}),\ \boldsymbol{n} \in \mathbf{PR}(\mathbb{1}, \mathbb{N}) \text{ arbitrary } \mathbf{q.\,e.\,d.}$$

Comment: The key argument for this decidability is PR2-to-$\pi\mathbf{R}$ *soundness.*

Decidability extension

The decidability theorem above generalises to decidability of arbitrary **S** predicates as follows:

Each **S** predicate $\varphi = \varphi(a) : A \to \mathbf{2}$ gives rise to the following **complete case distinction** within, by iterative descent theory $\pi\mathbf{R}$:

$\pi\mathbf{R} \vdash \neg\varphi(\mu\delta_{\varphi \circ \mathrm{ct}_A}) : \mathbb{1} \to \mathbb{N} \to \mathbf{2}$

defined counterexample

or else

$\pi\mathbf{R} \vdash \varphi \circ \boldsymbol{a} : \mathbb{1} \to A \to \mathbf{2}$ *theorems,*

$\boldsymbol{a} : \mathbb{1} \to A$ an arbitrary point in $\mathbf{S}(\mathbb{1}, A)$,

$\mathrm{ct}_A : \mathbb{N} \to A$ the (retractive) Cantor count

of **S** object A

Proof of Decidability Corollary: substitute in the decidability theorem predicate $\varphi = \varphi(n) : \mathbb{N} \to \mathbf{2}$ by $\varphi \circ \mathrm{ct}_A : \mathbb{N} \to A \to \mathbf{2}$ **q. e. d.**

9.3 General consistency decidability

For constructive set theory **S** and quantified arithmetical theories **T** (with only finite descent in $\mathbb{N}[\omega]$) as well as iterative descent theory $\pi\mathbf{R}$, we discuss the pertaining free-variable *consistency* formula/predicate

$$\gamma = \gamma(k) = \neg\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbf{2}$$

and **get** by p. r. predicate **decidability** within iterative descent theory $\pi\mathbf{R}$

Consistency decidability for arithmetical theory $\mathbf{S}$ by iterative descent theory $\pi\mathbf{R}$:

For cartesian p. r. constructive set theory $\mathbf{S}$ and for p. r. consistency predicate

$$\mathrm{Con}_{\mathbf{S}} = \mathrm{Con}_{\mathbf{S}}(k) = \neg\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbf{2}$$

we have – first alternative –

$\pi\mathbf{R} \vdash \neg\mathrm{Con}_{\mathbf{S}}$ i. e. $\pi\mathbf{R} \vdash \mathrm{Prov}_{\mathbf{S}}(\mu\gamma, \ulcorner\mathrm{false}\urcorner)$

a concrete contradiction

or else – second alternative:

$\pi\mathbf{R} \vdash \mathrm{Con}_{\mathbf{S}}(\boldsymbol{k}),\ \boldsymbol{k}$ arbitrary in $\mathbf{S}(\mathbb{1}, \mathbb{N})$

i. e.

$\pi\mathbf{R} \vdash \neg\mathrm{Prov}_{\mathbf{S}}(\boldsymbol{k}, \ulcorner\mathrm{false}\urcorner),\ \boldsymbol{k}$ arbitrary (**number**)

no *(concrete) contradiction:*

(concrete) ***consistency*** **q. e. d.**

9.4 Self-Consistency

(i) If consistent, then theory $\pi\mathbf{R}$ does not derive its own *inconsistency formula:*

$$\pi\mathbf{R} \nvdash \neg\mathrm{Con}_{\pi\mathbf{R}}$$

(ii) Iterative descent theory $\pi\mathbf{R}$ is *self-consistent:*

$$\pi\mathbf{R} \vdash \mathrm{Con}_{\pi\mathbf{R}}$$

Indirect **proof** of assertion (i): Suppose we would have inconsistency derivation alternative in *consistency decision* above:

$$\pi\mathbf{R} \vdash \neg\mathrm{Con}_{\pi\mathbf{R}} \text{ i.e.}$$
$$\pi\mathbf{R} \vdash \neg\gamma \mathbin{\widehat{\circ}} \mu D_\gamma \mathrel{\widehat{=}} \mathrm{true},$$
$$\gamma = \gamma(k) := \neg\mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner\mathrm{false}\urcorner) : \mathbb{N} \to \mathbf{2}$$

By **Totality Lemma**

$$\mu D_\gamma : \mathbb{1} \to D_\gamma \subseteq \mathbb{N} \text{ (“total”) p.r. and}$$
$$\pi\mathbf{R} \vdash \neg\gamma \circ \mu D_\gamma$$

By choice of γ this is to say

$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\mu D_\gamma, \ulcorner\mathrm{false}\urcorner),$$
$$\mu D_\gamma : \mathbb{1} \to \mathbb{N} \text{ in } \mathbf{S}, \text{ a } \textbf{number } \boldsymbol{k}_0 : \mathbb{1} \to \mathbb{N},$$
$$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\boldsymbol{k}_0, \ulcorner\mathrm{false}\urcorner)$$

whence, by **metamathematical soundness theorem:**

$$\pi\mathbf{R} \vdash \mathrm{false}$$

contradiction to assumed *consistency* of theory $\pi\mathbf{R}$.

This proves assertion (i).

Main assertion (ii) then follows by **consistency decidability** for theory $\pi\mathbf{R}$.

[An inconsistent theory derives everything, in particular its own consistency and inconsistency formulae.]

q. e. d.

Chapter 10

Soundness

Soundness of p. r. set theory $\mathbf{S}$, soundness of $\mathbf{S}$ within itself, would mean – logically – that availability $\mathrm{Prov}_{\mathbf{S}}(k, \ulcorner\varphi\urcorner)$ of an $\mathbf{S}$ internal *proof* (index) k for the code $\ulcorner\varphi\urcorner$ of a predicate $\varphi = \varphi(a) : A \to \mathbf{2}$ implies (within $\mathbf{S}$) overall validity $\varphi = \mathrm{true}_A : A \to \mathbf{2}$.

Soundness of *iterative descent theory* $\pi\mathbf{R}$ is a consequence of injectivity of all (internal) numeralisations

$$\nu_A : A \to A^{\mathbb{1}}/\check{=}^{\pi}$$

We derive that general injectivity from (particular) injectivity of

$$\nu_{\mathbf{2}} : \mathbf{2} \to \mathbf{2}^{\mathbb{1}}/\check{=}^{\pi}$$

by naturality of transformation $\nu = [\nu_A]_A$.

The latter injectivity is shown in section below on *iterative soundness* to follow from (already established) ***self-consistency*** of theory $\pi\mathbf{R}$.

10.1 Iterative soundness

We get for iterative descent theory $\pi\mathbf{R}$

- Soundness: for $\pi\mathbf{R}$ maps $f,\ g : A \to B$

  $$\pi\mathbf{R} \vdash [\ulcorner f \urcorner \check{=}^{\pi}_{k} \ulcorner g \urcorner] \implies f(a) =_B g(a)$$

- This **entails** in particular *logical soundness* of $\pi\mathbf{R}$:

  For any p. r. predicate $\varphi = \varphi(a) : A \to \mathbf{2}$

  $$\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \varphi \urcorner) \implies \varphi(a)$$

  $k \in \mathbb{N}$ free, meaning *exists* k, and $a \in A$ free, meaning *for all* a.

- **Conclusion:** (Derivable) **Truth** = ***Provability*** for constructive “set” theory $\pi\mathbf{R}$ taken as Arithmetics as well as Foundations.

Proof:

- Numeralisation

  $\nu_{\mathbf{2}} : \mathbf{2} = \{\mathrm{false}, \mathrm{true}\}$

  $= \{\alpha \in \mathbf{2} : \alpha = \mathrm{false} \vee \alpha = \mathrm{true}\} \to \mathbf{2}^{\mathbb{1}}$

  is injective, since for $\alpha, \beta \in \mathbf{2}$ free

  $$\pi\mathbf{R} \vdash \ulcorner \alpha \urcorner \check{=}^{\pi} \ulcorner \beta \urcorner \implies :$$

  $$[\alpha \neq \beta$$
  $$\implies [\alpha = \mathrm{false} \wedge \beta = \mathrm{true}] \vee [\alpha = \mathrm{true} \wedge \beta = \mathrm{false}]$$
  $$\implies \ulcorner \mathrm{false} \urcorner \check{=}^{\pi} \ulcorner \mathrm{true} \urcorner \vee \ulcorner \mathrm{true} \urcorner \check{=}^{\pi} \ulcorner \mathrm{false} \urcorner$$
  $$\iff \mathrm{Prov}_{\pi\mathbf{R}}(k_0, \ulcorner \mathrm{false} \urcorner)$$
  $$\implies \text{false by self-consistency of system } \pi\mathbf{R}.]$$

whence

$$\pi\mathbf{R} \vdash \ulcorner\alpha\urcorner \mathrel{\check{\doteq}^{\pi}} \ulcorner\beta\urcorner \implies \alpha = \beta : \mathbf{2} \times \mathbf{2} \to \mathbf{2},$$

injectivity of $\nu_{\mathbf{2}} : \mathbf{2} \to \mathbf{2}^{\mathbb{1}}$.

- $\nu = \nu(n) : \mathbb{N} \to \mathbb{N}^{\mathbb{1}}$ is injective:

$$\begin{aligned}
&\nu(m) \mathrel{\check{\doteq}^{\pi}} \nu(n) \\
&\implies \ulcorner{=}\urcorner \odot \nu_{\mathbb{N}\times\mathbb{N}}(m,n) = \\
&\qquad \ulcorner{=}\urcorner \odot \langle \nu(m); \nu(n)\rangle \mathrel{\check{\doteq}^{\pi}} \ulcorner \mathrm{true} \urcorner \\
&\qquad \text{the latter by internal substitutivity} \\
&\qquad \text{into predicative equality } = \\
&\iff \nu[m = n] \mathrel{\check{\doteq}^{\pi}} \ulcorner \mathrm{true} \urcorner = \nu\,\mathrm{true} \in \mathbf{2}^{\mathbb{1}} \\
&\qquad \text{by } \nu\text{-}\textit{naturality} \\
&\implies [m = n] = \mathrm{true} \iff m = n \\
&\qquad \text{by injectivity of } \nu_{\mathbf{2}}
\end{aligned}$$

- By ν-naturality, the injectivity above carries over to all numeralisations

$$\nu_C : C \to C^{\mathbb{1}},\ C \text{ an } \mathbf{S} \text{ set}$$

namely

  - from ν_A, ν_B to $\nu_{A\times B} = \nu_{A\times B}(a,b) = \langle \nu_A(a); \nu_B(b)\rangle$ by

$$\begin{aligned}
&\nu_{A\times B}(a,b) \mathrel{\check{\doteq}^{\pi}} \nu_{A\times B}(\tilde{a},\tilde{b}) \\
&\iff \nu_A(a) \mathrel{\check{\doteq}^{\pi}} \nu_A(\tilde{a}) \wedge \nu_B(b) \mathrel{\check{\doteq}^{\pi}} \nu_B(\tilde{b}) \\
&\implies a = \tilde{a} \wedge b = \tilde{b} \iff (a,b) = (\tilde{a},\tilde{b})
\end{aligned}$$

– finally from ν_A to $\nu_{\{A:\chi\}}$ by restriction.

- Soundness **proof:** Use compatibility of internal composition with internal equality, naturality of transformation $\nu = [\nu_A]_A$ and injectivity of ν_B as follows:

$$
\begin{aligned}
\pi\mathbf{R} \vdash \ulcorner f \urcorner \check{=}^{\pi} \ulcorner g \urcorner \; [\in B^A] \\
&\implies \ulcorner f \urcorner \odot \nu_A(a) \check{=}^{\pi} \ulcorner g \urcorner \odot \nu_A(a) \\
&\implies \nu_B(f(a)) \check{=}^{\pi} \nu_B(g(a)) \\
&\implies f(a) = g(a), \;\; a \in A \text{ free}
\end{aligned}
$$

- **Conclusion** means just the conjunction of **proof internalisation**

$$
\frac{\varphi =^{\pi\mathbf{R}}_{\boldsymbol{k}} \mathrm{true}_A}{\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(\boldsymbol{k}, \ulcorner \varphi \urcorner)} \quad \boldsymbol{k} \in \mathbf{PR}(\mathbb{1}, \mathbb{N}) \text{ “meta-free”}
$$

and (logical) soundness

$$
\begin{aligned}
\pi\mathbf{R} \vdash \mathrm{Prov}_{\pi\mathbf{R}}(k, \ulcorner \varphi \urcorner) &\implies \varphi(a) = \mathrm{true}_A(a) \\
& k \in \mathbb{N}, \;\; a \in A \text{ both free, meaning} \\
& \text{“\textit{exists} } k \text{ s.t. \textit{for all} } a\text{”}
\end{aligned}
$$

q. e. d.

10.2 Completeness

By **self-consistency** theory $\pi\mathbf{R}$ admits the following *ω-completeness* schema of *test by all (internal) numerals:*

$$\text{(}\omega\text{-Comp)}\quad \frac{\begin{array}{l}\varphi = \varphi(a) : A \to \mathbf{2} \text{ predicate} \\ k = k(a) : A \to \mathbb{N} \text{ p. r. "suitable" such that} \\ \pi\mathbf{R} \vdash \operatorname{Prov}_{\pi\mathbf{R}}(k(a), \ulcorner\varphi\urcorner \odot \nu_A(a))\end{array}}{\pi\mathbf{R} \vdash \varphi}$$

[The converse is given by proof-internalisation.]

Proof: By ν naturality – within $\pi\mathbf{R}$ – the antecedent gives

$$\begin{array}{l}\pi\mathbf{R} \vdash \operatorname{Prov}_{\pi\mathbf{R}}(k'(a), \nu_{\mathbf{2}} \circ \varphi(a)), \\ \quad k' : A \to \mathbb{N} \text{ suitable p. r. i. e. such that} \\ \pi\mathbf{R} \vdash \nu_{\mathbf{2}} \circ \varphi \,\check{=}^{\pi}_{k'(a)} \ulcorner\text{true}\urcorner = \nu_{\mathbf{2}} \circ \text{true}_A\end{array}$$

whence by $\pi\mathbf{R}$ self-consistency, namely by injectivity of $\nu_{\mathbf{2}}$ within $\pi\mathbf{R}$:

$$\pi\mathbf{R} \vdash \varphi = \text{true}_A \quad \textbf{q. e. d.}$$

Interpretation: The $\nu_A(a), a \in A$ are jointly epic, νA lies *dense* in $[\mathbb{1}, A]_{\pi\mathbf{R}}$. object $\mathbb{1}$ is a **separator,** all of this with respect to $\pi\mathbf{R}$ maps (on object language level): use (ω-Comp) and *equality definability* for separation of maps $f, g : A \to B$.

Problems

(1) Is axiom scheme (π) redundant, $\pi\mathbf{R} \cong \mathbf{S}$? Certainly not, since isotonic maps from lexicographically ordered $\mathbb{N} \times \mathbb{N}, \ldots, \mathbb{N}[\omega]$ to $\mathbb{N}$ are not available. Evaluation is Ackermann recursive, not *primitive* recursive.

(2) Is theory $\pi\mathbf{R}$ consistent relative to theories $\mathbf{PR}$, $\mathbf{PRa}$, $\mathbf{PR2}$, $\mathbf{S}$? Presumably yes, "since" it is self-consistent. Is it even a conservative extension of $\mathbf{PR}$? Presumably no, see (1) above.

(3) Can we get *inner* soundness for theory $\pi\mathbf{R}$? I. e. is evaluation $\varepsilon : B^A \times A \to B$ compatible with $\pi\mathbf{R}$'s internal equality,

$$B^A \ni f \check{=}^{\pi} g \implies \varepsilon(f, a) = \varepsilon(g, a)?$$

For the time being we have only *objective* (evaluation) soundness:

$$\begin{aligned} &\text{For } f, g : A \to B \text{ in } \pi\mathbf{R} \\ &\ulcorner f \urcorner \check{=}^{\pi} \ulcorner g \urcorner \\ &\implies \varepsilon(\ulcorner f \urcorner, a) = f(a) = g(a) = \varepsilon(\ulcorner g \urcorner, a) \end{aligned}$$

This is the one considered by mathematical logicians.

Inner soundness (of *evaluation*) is a challenging open problem with present approach.

(4) Can we ***assume*** consistently that object $\mathbb{1}$ is a *generator* for category $\pi\mathbf{R}$, i. e. that any given (metamathematical) p. r. map $F : \pi\mathbf{R}(\mathbb{1}, \mathbb{N}) \to \pi\mathbf{R}(\mathbb{1}, \mathbb{N})$ comes with an $\pi\mathbf{R}$ map $f : \mathbb{N} \to \mathbb{N}$

such that f represents F within $\pi\mathbf{R}$? Will say:

$$F = \pi\mathbf{R}(\mathbb{1}, f) : \pi\mathbf{R}(\mathbb{1}, \mathbb{N}) \to \pi\mathbf{R}(\mathbb{1}, \mathbb{N})$$
$$(\mathbb{1} \xrightarrow{\boldsymbol{n}} \mathbb{N}) \mapsto F(\boldsymbol{n}) = (\mathbb{1} \xrightarrow{\boldsymbol{n}} \mathbb{N} \xrightarrow{f} \mathbb{N})$$

This would solve a **question** asked to Erich Kähler in 1964:

Aber Sie benutzen doch schon natürliche Zahlen zur Beschreibung der Mengenlehre, mit der Sie die natürlichen Zahlen begründen wollen?

Kähler's answer: *diese Frage wird später beantwortet werden.*

Appendix A
Ackermann recursion as CCI

Following PÉTER 1967 and EILENBERG/ELGOT 1970 we discuss an Ackermann type function named Ψ originally given by a *double recursion.* For to separate the two recursion variables we start with a (p. r.) candidate $\Phi = \Phi(m) : \mathbb{N} \to [\mathbb{N}, \mathbb{N}]$ for *constructive conjugation* of Ψ. Ackermann function Ψ then is obtained by complexity controlled iterative evaluation of Φ. Function Ψ turns out this way to be given within theory $\pi\mathbf{R}$ by an – intuitively terminating – complexity controlled `while` loop which is not primitive recursive.

An Ackermann double recursion

Define an auxiliary unary map-code valued primitive recursive function $\Phi = \Phi(m) : \mathbb{N} \to \ulcorner \mathbb{N}^{\mathbb{N}} \urcorner \xrightarrow{\text{nat}} [\mathbb{N}, \mathbb{N}]$ as follows:

- **anchor:**
  $\Phi(0) = \ulcorner \mathrm{id} \urcorner : \mathbb{1} \to [\mathbb{N}, \mathbb{N}]$ and
  $\Phi(1) = \ulcorner \mathrm{s} \urcorner : \mathbb{1} \to [\mathbb{N}, \mathbb{N}]$

- **recursion:**

$$\Phi(m+1) = \Phi(m)^{\$} \odot \langle \ulcorner 1 \urcorner ; \ulcorner \mathrm{s} \urcorner \rangle$$
$$= \Phi(m)^{\$} \odot \langle \ulcorner \mathrm{s}\,(0\,\Pi_{\mathbb{N}}) \urcorner ; \ulcorner \mathrm{s} \urcorner \rangle : \qquad (1)$$
$$\mathbb{N} \to [\mathbb{N} \times \mathbb{N}, \mathbb{N}] \times [\mathbb{N}, \mathbb{N} \times \mathbb{N}] \xrightarrow{\odot} [\mathbb{N}, \mathbb{N}]$$

Applicate evaluation ε to unary function $\Phi = \Phi(m) : \mathbb{N} \to [\mathbb{N}, \mathbb{N}]$ and get **binary function** $\Psi = \Psi(m, n) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ in $\pi\mathbf{R}$ as

$$\Psi(m, n) \; =_{\mathrm{def}} \; \varepsilon(\Phi(m), n) \qquad (2)$$

and hence – double recursion –

- $\Psi(0, n) = \varepsilon(\Phi(0), n) = \varepsilon(\ulcorner \mathrm{id} \urcorner, n) = n$

- $\Psi(1, n) = \varepsilon(\Phi(1), n) = \varepsilon(\ulcorner \mathrm{s} \urcorner, n) = \mathrm{s}\,n = n + 1$

$$[\,\Psi(2, n) = \varepsilon(\Phi(2), n) = \varepsilon(\Phi(1)^{\$} \odot \langle \ulcorner 1 \urcorner ; \ulcorner \mathrm{s} \urcorner \rangle, n)$$
$$= \varepsilon(\ulcorner \mathrm{s} \urcorner^{\$}, \varepsilon(\langle \ulcorner 1 \urcorner ; \ulcorner \mathrm{s} \urcorner \rangle, n)) = \varepsilon(\ulcorner \mathrm{s} \urcorner^{\$}, (1, \mathrm{s}\,n))$$
$$= \mathrm{s}^{\S}(1, \mathrm{s}\,n) = 1 + (n + 1) = 2 + n\,]$$

- $\Psi(m + 1, 0) = \varepsilon(\Phi(m + 1), 0)$

$$= \varepsilon(\Phi(m)^{\$} \odot \langle \ulcorner 1 \urcorner ; \ulcorner \mathrm{s} \urcorner \rangle, 0)$$
$$= \varepsilon(\Phi(m)^{\$}, (1, \mathrm{s}\,0))$$
$$= \varepsilon(\Phi(m), \varepsilon(\Phi(m)^{\$}, (1, 0)))$$
$$= \varepsilon(\Phi(m), 1) = \Psi(m, 1) \qquad (3)$$

- $\Psi(m+1, n+1) = \boldsymbol{\varepsilon}(\Phi(m+1), n+1)$

  $=_{\text{by def}}\ \boldsymbol{ev}(\Phi(m)^{\$} \odot \langle \ulcorner 1 \urcorner; \ulcorner \mathrm{s} \urcorner \rangle, n+1)$

  $= \boldsymbol{\varepsilon}(\Phi(m)^{\$}, \boldsymbol{\varepsilon}(\langle \ulcorner 1 \urcorner; \ulcorner \mathrm{s} \urcorner \rangle, n+1))$

  $= \boldsymbol{\varepsilon}(\Phi(m)^{\$}, (1, n+2)) = \boldsymbol{\varepsilon}(\Phi(m)^{[n+2]}, 1) \qquad (4)$

  $= \boldsymbol{\varepsilon}(\Phi(m), \boldsymbol{\varepsilon}(\Phi(m)^{[n+1]}, 1))$

  $= \boldsymbol{\varepsilon}(\Phi(m), \boldsymbol{\varepsilon}(\Phi(m)^{\$}, (1, n+1)))$

  $= \boldsymbol{\varepsilon}(\Phi(m), \boldsymbol{\varepsilon}(\Phi(m)^{\$} \odot \langle \ulcorner 1 \urcorner; \ulcorner \mathrm{s} \urcorner \rangle, n))$

  $=_{\text{by def}}\ \boldsymbol{ev}(\Phi(m), \boldsymbol{\varepsilon}(\Phi(m+1), n))$

  $= \Psi(m, \Psi(m+1, n)) \qquad (5)$

Note: The Ackermann type double recursive "function" $\psi(m, n) = \Psi(m+1, n)$ is just PÉTER's 1967 *number theoretic function* ψ *which is not primitive recursive,* identic to function $\Psi = \Psi(x, y)$ in EILENBERG/ELGOT 1970 Appendix A, which is shown in a different way to be recursive but not primitive recursive. The latter authors define family Ψ_m by $\Psi_0(n) = n+1$ and $\Psi_{m+1}(n)$ iteratively by ${\Psi_m}^{\S}(1, n+1)$. Introduction of map-code valued p. r. map Φ above formalises this definition within the framework of (recursive) theory $\pi\mathbf{R}$ of non-infinite iterative (complexity) descent.

Iterative resolution

Double recursive "function" $\Psi : \mathbb{N} \times \mathbb{N}$ is represented as a – "quasi terminating" recursive – map in iterative descent theory $\pi\mathbf{R}$. Ψ has form

$$\Psi = \Psi(m,n) \ =_{\text{by def}} \ \boldsymbol{ev}(\Phi(m),n) = \boldsymbol{\varepsilon}\,\widehat{\circ}\,(\Phi(m),n) :$$
$$\mathbb{N}\times\mathbb{N} \xrightarrow{\Phi\times\mathrm{id}_{\mathbb{N}}} [\mathbb{N},\mathbb{N}]\times\mathbb{N} \xrightarrow{\boldsymbol{\varepsilon}} \mathbb{N}.$$

In fact Φ is primitive recursive and evaluation $\boldsymbol{\varepsilon}$ is defined as a complexity controlled iteration, within theory $\pi\mathbf{R}$.

Double recursive property (characterisation?) of Ψ

$$\Psi(0,n) = \boldsymbol{\varepsilon}(\Phi(0),n) = \boldsymbol{\varepsilon}(\ulcorner\mathrm{id}\urcorner,n) = n$$
$$\Psi(1,n) = \boldsymbol{\varepsilon}(\Phi(1),n) = \boldsymbol{\varepsilon}(\ulcorner\mathrm{s}\urcorner,n) = n+1$$
$$\Psi(m+1,0) = \Psi(m,1) \qquad \text{by (3) above}$$
$$\Psi(m+1,n+1) = \Psi(m,\Psi(m+1,n)) \qquad \text{by (5) above}$$

q. e. d.

Majorant

$\Psi\,\Delta(n) = \Psi(n,n) : \mathbb{N} \to \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ majorises any p. r. function $f = f(n) : \mathbb{N} \to \mathbb{N}$, intuitively since it starts from successor $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ and encounters all iteration nesting depths n and all arguments $n \in \mathbb{N}$, hence it cannot be primitive recursive, since any primitive recursive map has limited iteration nesting depth.

Péter 1967 as well as Eilenberg/Elgot 1970 show this result for function $\psi = \psi_(m,n) = \Psi(m+1,n) : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ directly from its double recursive definition.

As an injective map (with non-empty domain $\mathbb{N}$) $\Psi\,\Delta : \mathbb{N} \to \mathbb{N}$ is a coretraction in **sets**, but it does not admit a retraction in theory $\widehat{\mathbf{S}}$

nor in $\Omega\widehat{\mathbf{R}}$: If so, it would be primitive recursive, by last assertion of **Totality Lemma** in chapter on *Partiality.*

But let us discuss here an a priori possible **(counter)example** of an $\Omega\widehat{\mathbf{R}}$ retraction $g : \mathbb{N} \rightharpoonup \mathbb{N}$ to the (diagonalised) Ackermann function $f = \Psi \,\widehat{\circ}\, \Delta : \mathbb{N} \rightharpoonup \mathbb{N}$.

As a `while` loop g must have the form

$$g = \mathrm{wh}[\chi : h] : \mathbb{N} \rightharpoonup \mathbb{N} \text{ with}$$
$$\chi = \chi(a) : \mathbb{N} \to \mathbf{2},\ \ h : \mathbb{N} \to \mathbb{N},$$

both primitive recursive, this by section on *Partial partial maps* – giving reduction of an arbitrarily nested `while` loop to a `while` loop with `while` nesting depth 1, and (hence) with control predicate and endomap to be iterated both p. r.: *One* `while` *loop is sufficient.*

For $g = \mathrm{wh}[\chi : h] : \mathbb{N} \rightharpoonup \mathbb{N}$ a retraction to

$$f = \Psi \,\widehat{\circ}\, \Delta = \boldsymbol{\varepsilon} \,\widehat{\circ}\, (\Phi \times \mathbb{N}) \circ \Delta : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N},$$

(see above), it is nearby to choose

$$\chi = \chi(b) = \min\{b' : b' \geq f(b)\} = \mu\{b' : b' \geq \boldsymbol{ev} \,\widehat{\circ}\, (\Phi \times \mathbb{N}) \circ \Delta(b)\}$$
$$h = h(b) = b + 1 : \mathbb{N} \to \mathbb{N}$$

But this control predicate χ would **not** be primitive recursive, since evaluation $\boldsymbol{\varepsilon}$ is not p. r.

So this partial map g, a natural candidate, is not a retraction to Ackermann's $f = \Psi \,\widehat{\circ}\, \Delta : \mathbb{N} \rightharpoonup \mathbb{N}$, at least not within $\widehat{\mathbf{S}}$ nor in $\pi\widehat{\mathbf{R}}$. In fact there can be (consistently) no counterexample against the

Totality Lemma, we have proved it in the framework of theory **S** and also for stronger theory $\pi\mathbf{R}$.

A *logical* possibility for construction of a **recursive** but not primitive recursive sequence is to p. r. enumerate all p. r. map codes:

$$\mathrm{enum} = \mathrm{enum}(n) : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$$

and to evaluate at enumeration index n:

$$E = E(n) = \boldsymbol{\varepsilon}(\mathrm{enum}(n), n)$$

$$: \mathbb{N} \xrightarrow{(\mathrm{enum},\mathrm{id})} \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \xrightarrow{\boldsymbol{\varepsilon}} \mathbb{N}$$

If this (formally partial recursive) function would be *primitive* recursive, it would make up a *code self-evaluation* of theory **S** and hence it would formalise the liar-paradoxon into an antinomy, see **Appendix B** below.

Sequence $\Psi(n, n) : \mathbb{N} \xrightarrow{\Delta} \mathbb{N} \times \mathbb{N} \xrightarrow{\Psi} \mathbb{N}$ above is an "equipotent" subsequence of $E = E(n) : \mathbb{N} \to \mathbb{N}$.

Appendix B
Witnessed termination?

Theory extension $\tau\mathbf{R}$ of $\pi\mathbf{R}$, of *witnessed finite complexity controlled iterative descent* is **defined** over theory $\mathbf{S} = \mathbf{PR} + \mathbf{2} +$ (abstr) by the following additional **axiom schema:**

$c : A \to \mathbb{N}[\omega],\ p : A \to A]$

data of a complexity controlled iteration – CCI –

with complexity values in polynomial semiring $\mathbb{N}[\omega]$:

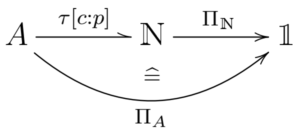
$$[c(a) = 0 \Rightarrow p(a) = a] \wedge [c(a) > 0 \Rightarrow c\,p(a) < c(a)]$$

(τ) ———————————————

For $\tau = \tau[c : p] = \tau[c : p](a)$

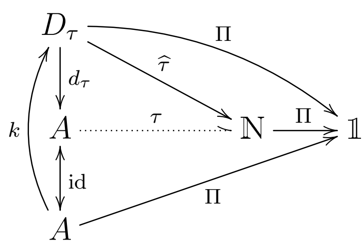
$=_{\text{def}}\ \mu\{n : c\,p^n(a) = 0\} : A \rightharpoonup \mathbb{N}$

is to hold



(•)

As a commutative **S** diagram (•) reads



$d_\tau : D_\tau$ accounts for $\Pi_{\mathbb{N}} \,\hat{\circ}\, \tau \,\widehat{\subseteq}\, \Pi_A$, and $k : A \to D_\tau$ accounts for $\Pi_A \,\widehat{\subseteq}\, \Pi_{\mathbb{N}} \,\hat{\circ}\, \tau : A \rightharpoonup \mathbb{1}$, all of this within theory $\tau\widehat{\mathbf{R}}$ of partials over p. r. theory $\tau\mathbf{R}$ – a theory strengthening **S** – and makes domain enumeration $d_\tau : D_\tau \to A$ into an **S** retraction. Therefore, by first assertion of **Totality Lemma** for $\tau\widehat{\mathbf{R}}$, $\tau = \langle d_\tau, \widehat{\tau}\rangle$ must be an embedded (***total***)

p. r. map $\tau : A \to \mathbb{N}$.

Comment: Operator 'τ' is here a particular instance of BOURBAKI's existence witnessing operator ("témoin"), of Hilbert's iota:

$$p^{\tau(a)}(a) = p^{\S}(a, \tau[c:p](a)) = 0 : A \to \mathbb{N}[\omega]$$

It witnesses *termination* of CCI concerned: Complexity controlled iteration.

With $\mathrm{O} = \mathbb{N}[\omega]$ (or O an arbitrary polynomial ordinal) we have **axiom schema**

$$c : A \to \mathrm{O},\ p : A \to A] \text{ a } \mathrm{CCI}_{\mathrm{O}} :$$

$$(\tau_{\mathrm{O}}) \quad \frac{[c(a) = 0 \Rightarrow p(a) = a] \wedge [c(a) > 0 \Rightarrow c\,f(a) < c(a) \in \mathrm{O}]}{\begin{array}{l} \tau = \tau[c:p] = \tau_{\mathrm{O}}[c:p](a) : A \to \mathbb{N} \text{ s.t.} \\ p^{\tau(a)}(a) = p^{\S}(a, \tau_{\mathrm{O}}[c:p](a)) = 0 : A \to \mathrm{O} \end{array}}$$

Self-evaluation Question: Does $\tau_{\mathrm{O}}\mathbf{R}$ admit code self-evaluation (and is therefore inconsistent)? **Yes:**

Boil down partially defined, complexity controlled evaluation

$$\varepsilon = \varepsilon_{A,B}(f, a) : B^A \times A \rightharpoonup B \text{ within } \widehat{\mathbf{S}}$$

to ***total*** p. r. evaluation

$$\varepsilon = \varepsilon_{A,B}(f, a) : B^A \times A \to B \text{ within } \tau_{\mathrm{O}}\mathbf{R}.$$

Define a "Cretian" map, *truth value* liar $: \mathbb{1} \to \mathbf{2}$ – called '*liar*' because it equals its own negation – as follows:

Let $\mathrm{ct} : \mathbb{N} \to \mathbf{2}^{\mathbb{N}}$ be the – primitive recursive – *count* of all predicate codes on $\mathbb{N}$; it comes with a primitive recursive (!) inverse isomorphism $\mathrm{ct}^{-1} : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$:

Enumerative cyclic construction of PR2 map term codes gives strictly greater codes (in lexicographic order), by each later application of a given map term constructing axiom: *basics, composed, induced, iterated.*

With *negated self-evaluation*

$$\delta \ =_{\mathrm{def}} \ \neg \circ \boldsymbol{ev} \circ (\mathrm{ct}, \mathrm{id}_{\mathbb{N}}) : \mathbb{N} \xrightarrow{(\mathrm{ct},\mathrm{id})} \mathbf{2}^{N} \times \mathbb{N} \xrightarrow{\varepsilon} \mathbf{2} \xrightarrow{\neg} \mathbf{2}$$

(*evaluation* $\boldsymbol{\varepsilon} : \mathbf{2}^{\mathbb{N}} \times \mathbb{N} \to \mathbf{2}$ *is here total p. r.*)

Consider p. r. map (truth value) $\mathrm{liar} : \mathbb{1} \to \mathbf{2}$,

$$\begin{aligned}
\mathrm{liar} \ &=_{\mathrm{def}} \ \delta \circ \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner \\
&=_{\text{by def}} \ \neg \circ \boldsymbol{ev} \circ (\mathrm{ct}, \mathrm{id}_{\mathbb{N}}) \circ \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner \\
&= \neg \circ \boldsymbol{ev} \circ (\mathrm{ct}\,\mathrm{ct}^{-1} \ulcorner \delta \urcorner, \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \\
&= \neg \circ \boldsymbol{ev}(\ulcorner \delta \urcorner, \mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \\
&= \neg \circ \delta(\mathrm{ct}^{-1} \circ \ulcorner \delta \urcorner) \qquad (\textit{objectivity of } \boldsymbol{\varepsilon}) \\
&=_{\text{by def}} \ \neg\,\mathrm{liar} : \mathbb{1} \to \mathbf{2} \to \mathbf{2}
\end{aligned}$$

q. e. d. contradiction within theory $\tau_{\mathrm{O}}\mathbf{R}$, in particular within $\tau_{\mathbb{N}[\omega]}\mathbf{R}$.

Corollary

As extensions of inconsistent theories $\tau_{\mathbb{N}[\omega]}\mathbf{R}$ with *witnessed termination* of complexity controlled iteration $\mathrm{CCI} = \mathbf{CCI}_{\mathbb{N}[\omega]}$ the following theories are all **inconsistent:**

- **set** theories as in particular **PM, ZF,** and **NGB** and their first-order parts, all of these first taken with **axiom** of choice **AC**; and then also without **AC**, since Gödel has shown *consistency* of **AC relative** to these **set** theories.

- Peano Arithmetic **PA** + **AC** with (countable) **axiom** of choice.

  Question: Is (countable) **AC** relative consistent over classical, quantified Peano Arithmetic **PA**? If so, then this **PA** itself would be inconsistent.

Appendix C
History Highlights

I 360° Babylonian # of days of | year

highly *super-perfect:*

$$1+2+3+4+5+6=21 \ +8+9+10=48 \ +60=108 \ +72= 180+80=260 \ +90=350 \ +120=470 \ +180=650 \ >> 360° = |-180°|+180° = 180° east + 360°/2 \ \ west\ green\, which = 360\,000 \ nautic\ miles =?? yards =?? feet$$

$$\text{II} \ perfect \ 28 = 1+2+4+7+14 \ days\ of \ | \ month \ *13 = 364 \ +1+1/4-1/100+1/400 = | \ gregorian\ year$$

$+.0\xi \ minutes = |$ **astronomic** *year*

ξ *weakly increasing*

III $EGYPT \ 3^2+4^2=5^2$

IIII=IV $\Pi Y \Theta A \Gamma OPAS \ A*A+B*B=\Gamma*\Gamma$

$$a^2+b^2=c^2$$

$$5^2+12^2=13^2 \ etc.\,?$$

$ratios\ \mathcal{Q} = \mathcal{N}/\mathcal{N}$

$music of the Sphairas$

$GREEK$ *constructions with compass and* $RULEr$

V $\Pi\Lambda AT\Omega/\Sigma\Omega KPATHS/\Theta AI\Theta ETOS\ \sqrt{2}$ *irrational*

VI $EYK\Lambda I\Delta HS\ GREEK\ GeoMETRIK$ *and* $Number$

VII $Diophant/GREEK$ *polynomials, diophantine equations*

$\nearrow\ Hilbert 10th\, problem$

VIII ***O*** Hesse/India/Siddharta/Buddha $\mathcal{OM}$

go west transformed into arabic zero 0

goes typewriter zero `0` =()

VIIII=IX $Cardano/Tartaglia$ *radicals* $\sqrt[4]{a}/\sqrt[3]{a}$

X $DECARTES$ *cartesian coordinates* :

number pairs resp. *triples*

for description of points and curves in $\mathbb{Q} \times \mathbb{Q}$

and $(\mathbb{Q} \times \mathbb{Q}) \times \mathbb{Q}$

X+? GÖDel

Index of Notation

	chapter 1 Cartesian language
$\mathbb{1}$	terminal object one
$\mathbb{N}$	Natural Numbers Object NNO
CA	cartesian category theory
$\times$	cartesian product of sets and of maps
0	zero constant $0 : \mathbb{1} \to \mathbb{N}$
s	successor function $s : \mathbb{N} \to \mathbb{N}$
id	identity map
$\circ$	map composition
Π	terminal map
ℓ/r	left/right projection
var	(free) variable
Ax	axiom

chapter 2 Primitive recursion

$\mathbf{PRa} = \mathbf{PR} + (\text{abstr})$	p. r. theory with predicate abstraction into subsets
$f^{\S}$	iteration of endo map f
(FR!)	Freyd's uniqueness of the initialised iterated
(pr)	full schema of primitive recursion
$\text{pr}[g, h]$	p. r. map defined out of anchoring g and step map h
T	classical, quantified arithmetical theory, in particular **set** theory
PM, **ZF**, **NGB**	Principia Mathematica, Zermelo/Fraenkel and v. Neumann/Gödel/Bernays **set** theories
$\epsilon_{A,B}$	axiomatic, higher order evaluation
$\cong$	isomorphy
	resp.. natural equivalence of functors

chapter 3 Algebra and order

pre	predecessor map
$a \searrow b$	truncated subtraction
U_1 to U_4	Goodstein uniqueness rules
sgn	(arithmetical) sign
V_4	derived Goodstein uniqueness rule
$(a \doteq b),\ [a = b]$	individual equality, equality *predicate*
Δ	diagonal map
$a^{\uparrow n}$	superexponentiation

chapter 4 Predicate abstraction

$\vdash$	(a theory) derives
pre	predecessor map
$\{A : \chi\}$	subset abstracted from predicate χ
$\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$	p. r. theory $\mathbf{PR}$ + abstraction subsets
$2 = \{0, 1\} \subset \mathbb{N}$	2-element subset of NNO $\mathbb{N}$
$f =^{\mathbf{S}} g$	equality *between* $\mathbf{S}$ maps of theory $\mathbf{S}$
$\mathbb{1} + \mathbb{N}$	sum/coproduct of objects
A/ρ	quotient set by an equivalence predicate

chapter 5 Arithmetical logic

$\alpha \setminus \beta$	relative complement: α but not β
$\mathbf{2} = \{\text{false}, \text{true}\}$	boolean algebra (in logical terms)
$\text{pret} : \mathbf{2} \to \mathbb{N}$	interpretation of truth values as numbers
$\text{sign} : \mathbb{N} \to \mathbf{2}$	logical signum of natural numbers
$\mathbf{PR2} = \mathbf{PR} + \mathbf{2}$	theory **PR** enriched by boolean algebrea **2**
$\mathbf{S} = \mathbf{PR2} + (\text{abstr})$	theory **PR2** enriched
	by predicate abstraction into subsets
S	constructive p. r. “set” theory
$f = \text{if}[\chi, (h\|g)](a) : A \to B$	definition of a map by case distinction
P1 to P5	Peano axioms, here theorems, in particular
P5	Peano induction

chapter 7 Partiality

$f : A \rightharpoonup B$	partial map
$f' \,\widehat{\subseteq}\, f$	graph inclusion
$f \;\widehat{=}\; f'$	equality of partial maps
$g \,\widehat{\circ}\, f$	partial map composition
p. b.	pull back
$\widehat{\mathbf{S}}$, $\widehat{\mathbf{S}}$	theory of p. r. partials over **S**
Ψ	Ackermann function
(μ)	schema of μ-recursion
wh $[\chi : f]$	`while` loop
PA	classical Peano Arithmetic
PRA	classical free-variables p. r. Arithmetic

chapter 8 Evaluation

ω	indeterminate for (arbitrarily) big natural numbers
$\mathbb{N}[\omega]$	polynomials over $\mathbb{N}$ in one indeterminate ω
(π)	descent axiom schema
$\pi\mathbf{R}$	theory of non-infinite iterative descent
$\ulcorner f \urcorner$	code of map f
$\odot = \ulcorner \circ \urcorner$	internal composition on map code sets
$\# = \ulcorner \times \urcorner$	internal map code cartesian product
$\$ = \ulcorner \S \urcorner$	internal iteration operator
B^A	map code set
num	(objective) numeralisation
ν	internal numeralisation
$\mathbb{X}$	universal set
(EqDef)	Equality definability schema
$\boldsymbol{\varepsilon}$	map code evaluation
$\boldsymbol{c}$	map code complexity
$\boldsymbol{e}$	evaluation step to be iterated
m *defs*	m defines
$\check{=}$	internal, arithmetised equality
$\mathrm{Prov}_{\mathbf{S}}$	internal **S** *proof*
$\boldsymbol{\varepsilon}_{\mathrm{dt}}$	deduction tree evaluation

chapter 9 Predicates decidability

$\pi\mathbf{R} = \mathbf{S} + (\pi)$	iterative non-infinite-descent theory
∇	predicate decision operator
$\mathrm{Con}_{\mathbf{T}}$	consistency formula/predicate for a theory $\mathbf{T}$
$D_{\nabla\chi}$	domain set for partial decision of predicate χ
δ_χ	predicate defining subset $D_{\nabla\chi} = \{\mathbb{N} : \delta_\chi\}$

chapter 10 Soundness

$\hom(A, \_) = (\_)^A$	constructive internal hom functor

Appendix B

$\tau[c:p]$	termination index for iteration of step p controlled by complexity c
$\tau\mathbf{R} = \tau_{\mathbb{N}[\omega]}\mathbf{R}$	theory of witnessed finite iterative descent
$\pi\mathbf{R}$	theory of non-infinite iterative descent
PM	Principia Mathematica
ZF	Zermelo/Fraenkel **set** theory
AC	axiom of choice
NGB	von Neumann-Gödel-Bernays **set** theory
PA	(classical) Peano Arithmetic
AC	axiom of choice

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Index