Structure preserving condensed forms for pairs of Hermitian matrices and matrix valued functions

Lena Wunderlich *

December 7, 2006†

Abstract

The study of matrix pairs or pairs of matrix valued functions is often motivated by applications from linear differential-algebraic equations. In many applications from mechanics or control theory the underlying matrices are symmetric or Hermitian. We study structure preserving condensed forms for pairs of Hermitian matrices and pairs of Hermitian matrix functions. Furthermore, we show how we can derive a structure preserving equivalent strangeness-free system from differential-algebraic equations using the derivative array approach.

1 Introduction

The study of matrix pairs is often motivated by the analysis of initial value problems for linear differential-algebraic equations (DAEs) [12] with variable coefficients

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \quad t \in [t_0, t_1], \tag{1}$$

with $E, A \in \mathcal{C}([t_0, t_1], \mathbb{C}^{m,n})$ and $f \in \mathcal{C}([t_0, t_1], \mathbb{C}^n)$ together with an initial condition

$$x(t_0) = x_0 \in \mathbb{C}^n. \tag{2}$$

Here, $\mathcal{C}^k([t_0, t_1], \mathbb{C}^{m,n})$ denotes the set of k-times continously differentiable functions from the interval $[t_0, t_1]$ to the complex vector space $\mathbb{C}^{m,n}$. Based on canonical forms of the underlying matrix pair one gets existence and uniqueness results for linear differential-algebraic equations. In many applications from mechanics or control theory the coefficient matrices $E(t)$ and $A(t)$ are structured, e.g., they are symmetric or Hermitian [1, 6, 14, 18, 19]. Thus, in this paper we study structure preserving condensed forms for pairs of Hermitian matrices (E, A) and pairs of Hermitian matrix functions $(E(t), A(t))$, arising in differential-algebraic systems with Hermitian coefficient matrices. We show that in certain cases it is possible to preserve the symmetry of the system matrices during the transformation process and to obtain structure preserving condensed forms. To achieve this, we cannot use general equivalence transformations, but have to restrict to congruence transformations in order to preserve the Hermitian structure of the system.

There are several reasons why it is important to consider symmetric systems and structure preserving canonical forms. Usually, the structure of the matrices reflects a physical property of the system that should be preserved. For example, the algebraic structure of the

*Institut für Mathematik, MA 4-5 TU Berlin, Straße des 17. Juni 136, D-10623 Berlin; wunder@math.tu-berlin.de. This author was supported by *Deutsche Forschungsgemeinschaft*, through Research Grant ME 790/14-2.

†Slightly revised version of February 16, 2006.

problem forces the eigenvalues to lie in certain regions in the complex plane (e.g., on the unit circle or on the real axis) or to occur in different kind of pairings. If we operate on such a system with the usual equivalence transformations which destroy the structure, these physical properties are obscured and if we use numerical methods for the solution of such problems, then we might get physically meaningless results as rounding errors can cause eigenvalues to wander out of their required region, see [4]. Another important aspect is the fact that linear system solvers for structured systems have a much better performance than for general un-symmetric systems. So, preserving known structures is always advantageous. In the analysis of linear differential-algebraic equations (1) an approach based on computing the invariants of the underlying matrix pair under equivalence transformations is used, see [12]. In this approach a property called the strangeness and the strangeness index are of special importance. The index of a DAE describes the degree of difficulty to solve the DAE analytically as well as numerically and is needed to determine the smoothness that is required for the inhomogeneity to guarantee the existence of a classical solution. Using the so-called derivative array approach it is possible to derive an equivalent strangeness-free system, i.e., a system consisting only of decoupled differential and algebraic equations having a strangeness index 0.
The paper is organized as follows. In Section 2 we will consider pairs of constant Hermitian matrices arising from linear differential-algebraic equations with constant coefficients and derive structure preserving condensed forms. In Section 3 we then extend the results as far as possible to pairs of Hermitian matrix valued functions arising in variable coefficient systems. We will see that it is only possible to obtain a structure preserving condensed form if certain additional assumptions hold and that a structure preserving strangeness-free system only exists if the strangeness index is $\mu \leq 1$. Finally, in Section 4 we present a structure preserving derivative array approach to derive an equivalent strangeness-free differential-algebraic systems for constant coefficient systems.

2 Structure preserving condensed forms for matrix pairs

In this section we study the case of Hermitian matrix pairs

$$(E, A), \tag{3}$$

or equivalently matrix pencils $\alpha E - \beta A$, as they arise in the analysis of linear differential-algebraic systems with constant Hermitian coefficients

$$E\dot{x}(t) = Ax(t) + f(t), \quad t \in [t_0, t_1], \tag{4}$$

where $E, A \in \mathbb{C}^{n \times n}$ are Hermitian, i.e., $E = E^H$ and $A = A^H$ and $f \in \mathcal{C}([t_0, t_1], \mathbb{C}^n)$. Here, A^H denotes the conjugate transpose of a matrix A. We want to give a canonical form for the matrix pair (3) under structure preserving congruence transformations.

Definition 1. *[9] Two pairs of Hermitian matrices (E_i, A_i), $i = 1, 2$, with $E_i, A_i \in \mathbb{C}^{n,n}$ are called* (strongly) congruent *if there exists a nonsingular matrix $P \in \mathbb{C}^{n,n}$, such that*

$$E_2 = P^H E_1 P, \quad A_2 = P^H A_1 P. \tag{5}$$

Clearly this defines an equivalence relation. The canonical form for matrix pairs under general equivalence transformations is the well-known Kronecker canonical form, see e.g. [5]. In the Hermitian case we have a Hermitian version of the Kronecker canonical form under congruence transformations.

Theorem 2. *Let $E, A \in \mathbb{C}^{n,n}$ be Hermitian. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n,n}$ such that*

$$P^H(\alpha E - \beta A)P = diag(\Delta_{\delta_1}, \ldots, \Delta_{\delta_p}, \Theta_{\eta_1}, \ldots, \Theta_{\eta_q}, \Lambda_{\rho_1}, \ldots, \Lambda_{\rho_v}, \Psi_{\sigma_1}, \ldots, \Psi_{\sigma_w}), \tag{6}$$

where

(a) Δ_{δ_j} *is an* $(2\delta_j - 1) \times (2\delta_j - 1)$*-block,* $\delta_j \in \mathbb{N}$ *of the form* $\alpha \begin{bmatrix} 0 & L_1 \\ L_1^T & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & L_2 \\ L_2^T & 0 \end{bmatrix}$,

$$\text{with} \quad L_1 = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix},$$

(b) Θ_{η_j} *is an* $\eta_j \times \eta_j$*-bidiagonal block,* $\eta_j \in \mathbb{N}_0$ *with sign* $\varepsilon \in \{-1, 1\}$ *of the form*

$$\alpha\varepsilon Z - \beta\varepsilon \hat{J}(\lambda_j) = \alpha\varepsilon \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} - \beta\varepsilon \begin{bmatrix} & & & \lambda_j \\ & & \lambda_j & 1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ \lambda_j & 1 & & \end{bmatrix},$$

corresponding to a Jordan block for a real eigenvalue λ_j*,*

(c) Λ_{ρ_j} *is a* $2\rho_j \times 2\rho_j$*-Jordan block,* $\rho_j \in \mathbb{N}$ *of the form* $\alpha \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix} - \beta \begin{bmatrix} 0 & \hat{J}(\lambda_j) \\ \hat{J}(\bar{\lambda}_j) & 0 \end{bmatrix}$, *with* $Z, \hat{J}(\lambda_j)$ *as in (b), corresponding to a pair of complex conjugate eigenvalues,*

(d) Ψ_{σ_j} *is a* $\sigma_j \times \sigma_j$*-nilpotent block,* $\sigma_j \in \mathbb{N}$ *with sign* $\varepsilon \in \{-1, 1\}$ *of the form*

$$\alpha\varepsilon \begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & \end{bmatrix} - \beta\varepsilon \begin{bmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{bmatrix},$$

associated with the eigenvalue ∞*.*

Proof. See [17]. □

The numerical computation of this canonical form is an ill-conditioned problem as small rounding errors can radically change the kind and number of the Kronecker blocks. However, if we restrict perturbations to be symmetric then the sensitivity with respect to structured perturbations may be much smaller than with respect to unstructured perturbations, see [2, 8].

Example 3. *Consider the matrix* $A = \begin{bmatrix} 1-\varepsilon & 0 \\ 0 & 1+\varepsilon \end{bmatrix}$ *with small* ε*, eigenvalues* $\lambda_{1,2} = 1 \pm \varepsilon$ *and corresponding orthogonal eigenvectors* $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$*. A symmetric perturbation* $\delta\tilde{A} = \begin{bmatrix} \tilde{\varepsilon}_1 & \tilde{\varepsilon}_2 \\ \tilde{\varepsilon}_2 & \tilde{\varepsilon}_3 \end{bmatrix}$ *only varies the eigenvalues slightly, but does not change the eigenspace of the matrix. But if we consider a non-symmetric perturbation* $\delta A = \begin{bmatrix} \varepsilon & \varepsilon \\ 0 & -\varepsilon \end{bmatrix}$ *then* $A + \delta A = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$ *has a Jordan block for the double eigenvalue* $\lambda = 1$ *with eigenvector* v_1*. One of the orthogonal eigenvectors of the original matrix* A *has vanished. Thus, a small perturbation can radically change the eigenspace of the matrix.*

We can give another condensed form for pairs of Hermitian matrices which is easier to compute numerically using rank decisions based on orthogonal transformations (e.g., singular value decomposition or rank revealing QR-decompositions [7]). To derive this new condensed form we use the following factorizations for Hermitian matrices, namely the singular value decomposition (SVD), the inertia revealing factorization due to Sylvesters Law of Inertia and the Echelon form of an Hermitian matrix, see e.g. [7, 5, 13].

Lemma 4. *Let $A \in \mathbb{C}^{n,n}$ be Hermitian with* $\operatorname{rank} A = r$. *Then*

1. *there exists a unitary matrix $P \in \mathbb{C}^{n,n}$ such that*
$$P^H A P = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$
with nonsingular and diagonal $\Sigma \in \mathbb{C}^{r,r}$,

2. *there exists a nonsingular matrix $V \in \mathbb{C}^{n,n}$ such that*
$$V^H A V = \begin{bmatrix} I_{r_p} & 0 & 0 \\ 0 & -I_{r_n} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$
where $(r_p, r_n, n-r)$ denotes the inertia of A, i.e., the number of positive, negative and zero eigenvalues with $r = r_n + r_p$,

3. *there exists a unitary matrix $U \in \mathbb{C}^{n,n}$ and a nonsingular matrix $W \in \mathbb{C}^{n,n}$ such that*
$$U^H A W = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}$$
is in Echelon form.

Proof. For the first two factorizations see e.g. [7, 13]. To prove the third part we first perform a singular value decomposition of A. Multiplication with a nonsingular matrix $Q = \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix}$ yields

$$P^H A P Q = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where P is unitary and $W = PQ$ is nonsingular. □

Now, we can derive a condensed form for pairs of Hermitian matrices. At first, we will allow only unitary transformations.

Theorem 5. *Let $E, A \in \mathbb{C}^{n,n}$ be Hermitian and let*

$$\begin{aligned} &T \text{ be a basis of } \operatorname{kernel} E, \\ &T' \text{ be a basis of } \operatorname{cokernel} E, \\ &V \text{ be a basis of } \operatorname{corange}(T^H A T). \end{aligned}$$

Then there exists a unitary matrix $P \in \mathbb{C}^{n,n}$ such that the matrix pair (E, A) is strongly congruent to an Hermitian matrix pair of the form

$$\left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{12}^H & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & \Sigma_s & 0 \\ A_{12}^H & A_{22} & A_{23} & 0 & 0 \\ A_{13}^H & A_{23}^H & \Sigma_a & 0 & 0 \\ \Sigma_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right), \quad \begin{matrix} s \\ d \\ a \\ s \\ u \end{matrix} \tag{10}$$

where the blocks Σ_a *and* Σ_s *are invertible and diagonal,* $\begin{bmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{bmatrix}$ *is nonsingular and the quantities*

(a) $r = \operatorname{rank} E$,
(b) $a = \operatorname{rank}(T^H A T)$,
(c) $s = \operatorname{rank}(V^H T^H A T')$,
(d) $d = r - s$,
(e) $u = n - r - a - s$

are invariant under the congruence relation (5).

Proof. To derive the condensed form (10) we use the following sequence of congruence transformations of the form (7) with unitary transformation matrices

$$(E, A) \sim \left(\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{bmatrix} \right) \sim \left(\begin{bmatrix} \Sigma_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^H & \Sigma_a & 0 \\ A_{13}^H & 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{12}^H & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & \Sigma_s & 0 \\ A_{12}^H & A_{22} & A_{23} & 0 & 0 \\ A_{13}^H & A_{23}^H & \Sigma_a & 0 & 0 \\ \Sigma_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

□

If we also allow non-unitary but nonsingular transformations, then we can reduce the matrix pair further.

Theorem 6. *Let* $E, A \in \mathbb{C}^{n,n}$ *be Hermitian and let*

$$\begin{aligned} &T \textit{ be a basis of } \operatorname{kernel} E, \\ &T' \textit{ be a basis of } \operatorname{cokernel} E, \\ &Z \textit{ be a basis of } \operatorname{cokernel}(T^H A T'). \end{aligned}$$

Then there exists a nonsingular matrix $P \in \mathbb{C}^{n,n}$ *such that the matrix pair* (E, A) *is strongly congruent to an Hermitian matrix pair* $(\tilde{E}, \tilde{A})$ *of the form*

$$\left(\begin{bmatrix} I_{s_d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{s_d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{w_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{w_n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{d_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{d_n} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right. \tag{11a}$$

$$
\left.\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_d} & 0 & 0 & 0\\
0 & \tilde A_{22} & 0 & 0 & \tilde A_{25} & \tilde A_{26} & 0 & 0 & I_{s_d} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{w_p} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{w_n} & 0\\
0 & \tilde A_{25}^H & 0 & 0 & \tilde A_{55} & \tilde A_{56} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \tilde A_{26}^H & 0 & 0 & \tilde A_{56}^H & \tilde A_{66} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 & 0 & 0\\
I_{s_d} & I_{s_d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I_{w_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I_{w_n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right), \tag{11b}
$$

and the quantities

(a)	$r = \operatorname{rank} E$	*(rank)*
(b)	a_p	*(number of pos. eigenvalues of* T^HAT*)*
(c)	a_n	*(number of neg. eigenvalues of* T^HAT*)*
(d)	$a = a_n + a_p = \operatorname{rank}(T^HAT)$	*(algebraic part)*
(e)	w_p	*(number of pos. eigenv. of* $Z^H(T')^HET'Z$*)*
(f)	w_n	*(number of neg. eigenv. of* $Z^H(T')^HET'Z$*)*
(g)	s_d	*(number of zero eigenvalues of* $Z^H(T')^HET'Z$*)*
(h)	$s = s_p + s_n$	*(strangeness)*
(i)	$d = d_n + d_p = r - s$	*(differential part)*
(j)	$u = n - r - a - s$	*(undetermined part)*

with $w_p = s_p - s_d$*,* $w_n = s_n - s_d$ *and* u *is the width of the last column in* (11)*, are invariant under the congruence relation* (5)*.*

Proof. The proof is constructive by the following sequence of congruence transformations. First, we perform an inertia revealing factorization (8) of E, where $(p, q, n-r)$ is the inertia of E and $r = p + q$ is the rank of E

$$
(E, A) \sim \left(\begin{bmatrix} I_p & 0 & 0\\ 0 & -I_q & 0\\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33} \end{bmatrix} \right).
$$

Next, we perform an inertia revealing factorization of A_{33} with inertia $(a_p, a_n, n-a)$ and $a = a_p + a_n$ is the rank of A_{33}

$$
\sim \left(\begin{bmatrix} I_p & 0 & 0 & 0 & 0\\ 0 & -I_q & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15}\\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25}\\ A_{31} & A_{32} & I_{a_p} & 0 & 0\\ A_{41} & A_{42} & 0 & -I_{a_n} & 0\\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix} \right)
$$

$$
\sim \left(\left[\begin{array}{c|c|cc|c} I_p & 0 & 0 & 0 & 0\\ \hline 0 & -I_q & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{c|c|cc|c} A_{11} & A_{12} & 0 & 0 & A_{15}\\ \hline A_{21} & A_{22} & 0 & 0 & A_{25}\\ \hline 0 & 0 & I_{a_p} & 0 & 0\\ 0 & 0 & 0 & -I_{a_n} & 0\\ \hline A_{51} & A_{52} & 0 & 0 & 0 \end{array}\right] \right).
$$

Now, we transform A_{15} to Echelon form, i.e. we determine U_{15} unitary and V_{15} nonsingular such that $A_{15} = U_{15} \begin{bmatrix} I_{p_0} & 0 \\ 0 & 0 \end{bmatrix} V_{15}^H$

$$
\sim \left(\left[\begin{array}{cc|c|cc|cc}
I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{p-p_0} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & -I_q & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right], \left[\begin{array}{cc|c|cc|cc}
A_{11} & A_{12} & A_{13} & 0 & 0 & I_{p_0} & 0 \\
A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & 0 \\ \hline
A_{31} & A_{32} & A_{33} & 0 & 0 & A_{36} & A_{37} \\ \hline
0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 \\ \hline
I_{p_0} & 0 & A_{63} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{73} & 0 & 0 & 0 & 0
\end{array} \right] \right)
$$

$$
\sim \left(\left[\begin{array}{cc|c|cc|cc}
I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{p-p_0} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & -I_q & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right], \left[\begin{array}{cc|c|cc|cc}
0 & 0 & 0 & 0 & 0 & I_{p_0} & 0 \\
0 & A_{22} & A_{23} & 0 & 0 & 0 & 0 \\ \hline
0 & A_{32} & A_{33} & 0 & 0 & A_{36} & A_{37} \\ \hline
0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 \\ \hline
I_{p_0} & 0 & A_{63} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{73} & 0 & 0 & 0 & 0
\end{array} \right] \right).
$$

Next, we transform A_{37} to Echelon form, i.e., we determine U_{37} unitary and V_{37} nonsingular such that $A_{37} = U_{37} \begin{bmatrix} I_{q_0} & 0 \\ 0 & 0 \end{bmatrix} V_{37}^H$

$$
\sim \left(\left[\begin{array}{cc|cc|cc|ccc}
I_{p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{p-p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{q-q_0} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right], \right.
$$

$$
\left. \left[\begin{array}{cc|cc|cc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & I_{p_0} & 0 & 0 \\
0 & A_{22} & 0 & A_{24} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{q_0} & 0 \\
0 & A_{42} & 0 & A_{44} & 0 & 0 & A_{47} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 & 0 \\ \hline
I_{p_0} & 0 & 0 & A_{74} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right] \right).
$$

Further, we transform A_{47} to Echelon form, i.e., we determine U_{47} unitary and V_{47} nonsingular such that $A_{47} = U_{47} \begin{bmatrix} I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} V_{47}^H$

$$
\sim \left(\left[\begin{array}{ccc|ccc|cc|cccc}
I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{p_0-p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{p-p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_{q-q_0-p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \right.
$$

$$
\left. \left[\begin{array}{ccc|ccc|cc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p_0-p_1} & 0 & 0 \\
0 & 0 & A_{33} & 0 & A_{35} & A_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{q_0} & 0 \\
0 & 0 & A_{53} & 0 & A_{55} & A_{56} & 0 & 0 & I_{p_1} & 0 & 0 & 0 \\
0 & 0 & A_{63} & 0 & A_{65} & A_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 & 0 & 0 \\
\hline
I_{p_1} & 0 & 0 & 0 & I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{p_0-p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \right),
$$

Finally, permutations yield

$$
\sim \left(\left[\begin{array}{ccc|ccc|cc|cccc}
I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{p_0-p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{p-p_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_{q-q_0-p_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right], \right.
$$

$$
\left. \left[\begin{array}{ccc|ccc|cc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p_1} & 0 & 0 & 0 \\
0 & A_{22} & 0 & 0 & A_{25} & A_{26} & 0 & 0 & I_{p_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p_0-p_1} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{q_0} & 0 \\
0 & A_{52} & 0 & 0 & A_{55} & A_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{62} & 0 & 0 & A_{65} & A_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 & 0 & 0 \\
\hline
I_{p_1} & I_{p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{p_0-p_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{q_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \right),
$$

and with $s_d = p_1$, $w_p = p_0 - p_1$, $w_n = q_0$, $d_p = p - p_0$, $d_n = q - q_0 - p_1$ the form (11) follows. $\square$

The relationship between the invariants of the Kronecker canonical form (6) and the invariants of the form (10) can be seen if we first transform to Kronecker canonical form and then treat the single blocks separately. Then we can determine the invariants for each block.

Lemma 7. *Consider a matrix pair (E, A) in Kronecker canonical form (6). Then we have the following characteristic quantities for the different Kronecker blocks.*

1. *Kronecker block Δ_δ:*
$$r = 2\delta - 2,\ a = 0,\ s = \begin{cases} 1 & \text{for } \delta \neq 1 \\ 0 & \text{for } \delta = 1 \end{cases},\ d = \begin{cases} 2\delta - 3 & \text{for } \delta \neq 1 \\ 2\delta - 2 & \text{for } \delta = 1 \end{cases},\ u = \begin{cases} 0 & \text{for } \delta \neq 1 \\ 1 & \text{for } \delta = 1 \end{cases}.$$
2. *Kronecker block Θ_η: $r = \eta,\ a = 0,\ s = 0,\ d = \eta,\ u = 0$.*
3. *Kronecker block Λ_ρ: $r = 2\rho,\ a = 0,\ s = 0,\ d = 2\rho,\ u = 0$.*
4. *Kronecker block Ψ_σ:*
$$r = \sigma - 1,\ a = \begin{cases} 1 & \text{for } \sigma = 1 \\ 0 & \text{for } \sigma \neq 1 \end{cases},\ s = \begin{cases} 0 & \text{for } \sigma = 1 \\ 1 & \text{for } \sigma \neq 1 \end{cases},\ d = \begin{cases} \sigma - 1 & \text{for } \sigma = 1 \\ \sigma - 2 & \text{for } \sigma \neq 1 \end{cases},\ u = 0.$$

Proof. 1. For Kronecker blocks Δ_δ we have $(E, A) = \left(\begin{bmatrix} 0 & L_1 \\ L_1^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & L_2 \\ L_2^T & 0 \end{bmatrix} \right)$. Therefore, we get $T = e_1^{(2\delta-1)}$ as basis of kernel E and $T' = (e_2^{(2\delta-1)}, \dots, e_{2\delta-1}^{(2\delta-1)})$ as basis of cokernel E and $V = [1]$ as basis of corange $(T^H A T)$.

2. For Kronecker blocks Θ_η we have $(E, A) = (Z, \hat{J}(\lambda_j))$. Therefore, we get $T = \emptyset$ as basis of kernel E, $T' = I_\eta$ as basis of cokernel E and $V = \emptyset$ as basis of corange $(T^H A T)$. Thus, we have $r = \eta$, $a = 0$, $s = 0$, $d = \eta$ and $u = 0$.

3. For Kronecker blocks Λ_ρ we have $(E, A) = \left(\begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}, \begin{bmatrix} 0 & \hat{J}(\lambda_j) \\ \hat{J}(\bar{\lambda}_j) & 0 \end{bmatrix} \right)$. We get $T = \emptyset$, $T' = I_{2\rho}$ and $V = \emptyset$.

4. For Kronecker blocks Ψ_σ we have $(E, A) = \left(\begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 1 & \\ \iddots & & \\ 1 & & \end{bmatrix} \right)$.

Thus, we get $T = e_1^{(\sigma)}$, $T' = (e_2^{(\sigma)}, \dots, e_\sigma^{(\sigma)})$, $T^H A T = \begin{cases} [1] & \text{for } \sigma = 1 \\ \emptyset & \text{for } \sigma \neq 1 \end{cases}$,

$$V = \begin{cases} \emptyset & \text{for } \sigma = 1 \\ [1] & \text{for } \sigma \neq 1 \end{cases} \text{ and } V^H T^H A T' = \begin{cases} \emptyset & \text{for } \sigma = 1 \\ \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} & \text{for } \sigma \neq 1 \end{cases}.$$

□

The following example illustrates how the characteristic quantities of a matrix pair in Kronecker canonical form (6) can be obtained from the characteristic quantities of each block.

Example 8. *Consider the matrix pair*

$$(E, A) = \left(\left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right], \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \right), \tag{12}$$

in Kronecker canonical form (6) *consisting of a block* Δ_2 *and a block* Θ_2 *(with eigenvalue* $\lambda = 1$*). The pair* (12) *is congruent to the pair*

$$(\tilde{E}, \tilde{A}) = \left(\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array} \right] \right),$$

in condensed form (10) *with characteristic values* $r = 4 = 2+2, \quad a = 0, \quad s = 1 = 1+0, \quad d = 3 = 1+2, \quad u = 0$. *The complete characteristic values are given as the sum of the characteristic values of the two separate blocks.*

3 Structure preserving condensed forms for pairs of matrix functions

In this section we consider pairs of Hermitian matrix functions

$$(E(t), A(t)), \tag{13}$$

as they arise in linear differential-algebraic systems with variable coefficients (1), where $m = n$ and $E, A \in \mathcal{C}([t_0, t_1], \mathbb{C}^{n,n})$ are Hermitian.
In [9] it has been posed as an open problem to derive a canonical form for system (1), where $E(t), A(t)$ are Hermitian. Here, we will show that it is possible to derive a structure preserving condensed form under certain extra assumptions.
For the numerical solution of differential-algebraic equations it is usually important to consider local quantities which are numerically computable and which give information on the global behavior of the solution in the neighborhood of a fixed point $\hat{t} \in [t_0, t_1]$. Thus, local equivalence is the basis for global equivalence and therefore we first like to have a local condensed form for pairs of Hermitian matrix functions (13) at a fixed point $\hat{t}$. To study local equivalence, we modify the definition of congruence.

Definition 9. *[9] Two pairs of Hermitian matrices* (E_i, A_i)*,* $E_i, A_i \in \mathbb{C}^{n,n}$*,* $i = 1, 2$ *are called* (locally) congruent *if there exist matrices* $P, B \in \mathbb{C}^{n,n}$*, with* P *nonsingular such that*

$$E_2 = P^H E_1 P, \quad A_2 = P^H A_1 P - P^H E_1 B, \tag{14}$$

and E_2, A_2 *are again Hermitian.*

We can observe that not any matrix B in (14) will lead again to an Hermitian pair. In order to obtain an Hermitian matrix pair we have to demand that $P^H E_1 B = B^H E_1 P$, e.g., we can choose B such that $E_1 B$ vanishes.
A condensed form under congruence transformation (14) restricting to unitary transformations is given in the following Theorem.

Theorem 10. *Let* $E, A \in \mathbb{C}^{n,n}$ *be Hermitian and let*

$$\begin{aligned} &T \text{ be a basis of } \operatorname{kernel} E, \\ &T' \text{ be a basis of } \operatorname{cokernel} E, \\ &V \text{ be a basis of } \operatorname{corange}(T^H A T). \end{aligned}$$

Then there exists a unitary matrix $P \in \mathbb{C}^{n,n}$ *and a matrix* $B \in \mathbb{C}^{n,n}$*, such that the matrix pair* (E, A) *is locally congruent to an Hermitian matrix pair of the form*

$$\left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{12}^H & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_{13} & \Sigma_s & 0 \\ 0 & 0 & A_{23} & 0 & 0 \\ A_{13}^H & A_{23}^H & \Sigma_a & 0 & 0 \\ \Sigma_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right), \quad \begin{matrix} s \\ d \\ a \\ s \\ u \end{matrix} \tag{15}$$

where the blocks Σ_a *and* Σ_s *are nonsingular and diagonal,* $\begin{bmatrix} E_{11} & E_{12} \\ E_{12}^H & E_{22} \end{bmatrix}$ *is nonsingular and the quantities*

(a) $r = \operatorname{rank} E$*,*
(b) $a = \operatorname{rank} (T^H AT)$*,*
(c) $s = \operatorname{rank} (V^H T^H AT')$*,*
(d) $d = r - s$*,*
(e) $u = n - r - a - s$

are invariant under the congruence relation (14)*.*

Proof. The proof is similar to the proof of Theorem 5 only that the matrix B allows to eliminate the block $\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{bmatrix}$. Following the proof of Theorem 5 by choosing $B = 0$ in every step we get

$$(E, A) \sim \left(\begin{bmatrix} \Sigma_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^H & \Sigma_a & 0 \\ A_{13}^H & 0 & 0 \end{bmatrix} \right).$$

If we now choose $P = I$ and $B = \begin{bmatrix} \Sigma_r^{-1} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ we get

$$\begin{aligned} (E, A) &\sim \left(\begin{bmatrix} \Sigma_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{12}^H & \Sigma_a & 0 \\ A_{13}^H & 0 & 0 \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{12}^H & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_{13} & \Sigma_s & 0 \\ 0 & 0 & A_{23} & 0 & 0 \\ A_{13}^H & A_{23}^H & \Sigma_a & 0 & 0 \\ \Sigma_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right). \end{aligned}$$

□

If we also allow non-unitary but nonsingular transformations we can reduce the matrix pair further, see also [9].

Theorem 11. *Let* $E, A \in \mathbb{C}^{n,n}$ *be Hermitian and let*

T *be a basis of* $\operatorname{kernel} E$,
T' *be a basis of* $\operatorname{cokernel} E$,
Z *be a basis of* $\operatorname{cokernel} (T^H AT')$.

Then there exist matrices $P, B \in \mathbb{C}^{n,n}$, with P nonsingular such that the matrix pair (E, A) is locally congruent to an Hermitian matrix pair of the form

$$
\left(\begin{bmatrix}
I_{s_d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -I_{s_d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{w_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{w_n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{d_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I_{d_n} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \right. \tag{16a}
$$

$$
\left. \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_d} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s_d} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{w_p} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{w_n} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{a_p} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -I_{a_n} & 0 & 0 & 0 & 0 \\
I_{s_d} & I_{s_d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{w_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{w_n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \right), \tag{16b}
$$

and the quantities

(a)	$r = \operatorname{rank} E$	*(rank)*
(b)	a_p	*(number of pos. eigenvalues of $T^H AT$)*
(c)	a_n	*(number of neg. eigenvalues of $T^H AT$)*
(d)	$a = a_n + a_p = \operatorname{rank}(T^H AT)$	*(algebraic part)*
(e)	w_p	*(number of pos. eigenv. of $Z^H(T')^H ET'Z$)*
(f)	w_n	*(number of neg. eigenv. of $Z^H(T')^H ET'Z$)*
(g)	s_d	*(number of zero eigenvalues of $Z^H(T')^H ET'Z$)*
(h)	$s = s_p + s_n$	*(strangeness)*
(i)	$d = d_n + d_p = r - s$	*(differential part)*
(j)	$u = n - r - a - s$	*(undetermined part)*

with $w_p = s_p - s_d$, $w_n = s_n - s_d$ and u is the width of the last column in (16)*, are invariant under the congruence relation* (14)*.*

Proof. The proof is similar to the proof of Theorem 6 only that the matrix B allows to eliminate the entries $\tilde{A}_{ij}$, $i, j = 1, \dots, 6$ in the upper 6×6-block of $\tilde{A}$ in (11). If we choose $P = I$ and $B = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 \\ -A_{12}^H & -A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ after the third transformation step in the proof of Theorem 6 we get

$$
\begin{aligned}
&(E,A)\\
&\sim \left(\begin{bmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{15} \\ A_{12}^H & A_{22} & 0 & 0 & A_{25} \\ 0 & 0 & I_{a_p} & 0 & 0 \\ 0 & 0 & 0 & -I_{a_n} & 0 \\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 \\ A_{12}^H & A_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)\\
&\sim \left(\begin{bmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & A_{15} \\ 0 & 0 & 0 & 0 & A_{25} \\ 0 & 0 & I_{a_p} & 0 & 0 \\ 0 & 0 & 0 & -I_{a_n} & 0 \\ A_{51} & A_{52} & 0 & 0 & 0 \end{bmatrix} \right).
\end{aligned}
$$

Then we proceed as in the proof of Theorem 6. □

We can now apply the local condensed form (15) or the local canonical form (16), respectively, to the matrix pair (13) for each fixed value $\hat{t} \in [t_0, t_1]$. Then we obtain integer valued functions $d, a, s, u : [t_0, t_1] \to \mathbb{N}_0$ and we assume that the regularity assumptions

$$d(t) \equiv d, \quad a(t) \equiv a, \quad s(t) \equiv s, \quad u(t) \equiv u, \quad \text{for all } t \in [t_0, t_1] \tag{17}$$

hold. To derive a global condensed form for pairs of Hermitian matrix functions we need the concept of global congruence.

Definition 12. *Two pairs of Hermitian matrix functions $(E_i(t), A_i(t))$, with $E_i, A_i \in \mathcal{C}([t_0, t_1], \mathbb{C}^{n,n})$, $i = 1, 2$ are called* (globally) congruent *if there exists a pointwise nonsingular matrix function $P \in \mathcal{C}^1([t_0, t_1], \mathbb{C}^{n,n})$ such that*

$$E_2(t) = P^H(t)E_1(t)P(t), \quad A_2(t) = P^H(t)A_1(t)P(t) - P^H(t)E_1(t)\dot{P}(t), \tag{18}$$

and $E_2(t), A_2(t)$ are again Hermitian.

We can observe that the matrix pair $(E_2(t), A_2(t))$ in (18) is only Hermitian again if

$$P^H(t)E_1(t)\dot{P}(t) = \dot{P}^H(t)E_1(t)P(t).$$

This condition holds for example in the special case where $E_1(t)\dot{P}(t) = 0$. Further, we have the following factorization for Hermitian matrix functions.

Lemma 13. *Let $E \in \mathcal{C}^k([t_0, t_1], \mathbb{C}^{n,n})$, $k \in \mathbb{N}_0 \cup \{\infty\}$ be Hermitian, with* $\operatorname{rank} E(t) = r$ *for all $t \in [t_0, t_1]$. Then there exists a pointwise unitary (and therefore nonsingular) function $P \in \mathcal{C}^k([t_0, t_1], \mathbb{C}^{n,n})$ such that*

$$P^H(t)E(t)P(t) = \begin{bmatrix} \Delta(t) & 0 \\ 0 & 0 \end{bmatrix}$$

with pointwise nonsingular and Hermitian $\Delta \in \mathcal{C}^k([t_0, t_1], \mathbb{C}^{r,r})$.

Proof. See [12, 15, 16]. □

To derive a global condensed form, we additionally need the following Assumption and Lemma.

Assumption 1. *Let $E \in \mathcal{C}^1([t_0,t_1],\mathbb{C}^{n,n})$ be Hermitian, with* $\operatorname{rank} E(t) = r$ *for all* $t \in [t_0,t_1]$. *There exists a matrix $Q \in \mathbb{C}^{n,r}$ (time-independent) such that the columns of Q form an orthogonal basis of* $\operatorname{range} E$.

Lemma 14. *Let $E \in \mathcal{C}^1([t_0,t_1],\mathbb{C}^{n,n})$ be Hermitian, with* $\operatorname{rank} E(t) = r$ *for all $t \in [t_0,t_1]$ and let $P \in \mathcal{C}^1([t_0,t_1],\mathbb{C}^{n,n})$ be a pointwise unitary function such that*

$$P^H(t)E(t)P(t) = \begin{bmatrix} \Delta(t) & 0 \\ 0 & 0 \end{bmatrix},$$

with pointwise nonsingular and Hermitian $\Delta \in \mathcal{C}^1([t_0,t_1],\mathbb{C}^{r,r})$ (see Lemma 13). If Assumption 1 hold and

$$\operatorname{kernel} E \subseteq \operatorname{kernel} \dot{E},$$

then we have

$$E(t)\dot{P}(t) = 0.$$

Proof. We write P as $P = \begin{bmatrix} U & V \end{bmatrix}$ with $U \in \mathcal{C}^1([t_0,t_1],\mathbb{C}^{n,r})$ and $V \in \mathcal{C}^1([t_0,t_1],\mathbb{C}^{n,n-r})$. The columns of U and V form orthogonal bases of $\operatorname{range} E$ and $\operatorname{corange} E = \operatorname{kernel} E^H = \operatorname{kernel} E$, respectively. As the basis of $\operatorname{range} E$ is time-independent due to Assumption 1 $U \in \mathbb{C}^{n,r}$ is a constant matrix and thus it holds that

$$E\dot{U} = 0.$$

Furthermore, we have

$$\begin{aligned} EV &= 0, \\ E\dot{V} + \dot{E}V &= 0 \end{aligned}$$

for all $t \in [t_0,t_1]$. Thus, it holds that

$$E\dot{V} = 0 \Leftrightarrow \dot{E}V = 0.$$

As V is a basis of $\operatorname{kernel} E$ and $\operatorname{kernel} E \subseteq \operatorname{kernel} \dot{E}$ it follows that $\dot{E}V = 0$. □

Then we get the following global condensed form for pairs of Hermitian matrix functions.

Theorem 15. *Let the pair $(E(t),A(t))$ be sufficiently smooth and Hermitian with* $\operatorname{rank} E(t) = r$ *for all $t \in [t_0,t_1]$ and suppose that Assumption 1 hold and* $\operatorname{kernel} E \subseteq \operatorname{kernel} \dot{E}$. *Further, suppose that the regularity assumptions* (17) *hold. Then the pair $(E(t),A(t))$ is (globally) congruent to a pair of Hermitian matrix functions of the form*

$$\left(\begin{bmatrix} E_{11}(t) & E_{12}(t) & 0 & 0 & 0 \\ E_{12}^H(t) & E_{22}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \Delta_s^H(t) & 0 \\ 0 & A_{22}(t) & 0 & 0 & 0 \\ 0 & 0 & \Delta_a(t) & 0 & 0 \\ \Delta_s(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right), \quad \begin{matrix} s \\ d \\ a \\ s \\ u \end{matrix} \tag{19}$$

where the block matrices $\begin{bmatrix} E_{11}(t) & E_{12}(t) \\ E_{12}^H(t) & E_{22}(t) \end{bmatrix}$, $\Delta_s(t)$ and $\Delta_a(t)$ are pointwise nonsingular.

Proof. We give a constructive proof using Lemma 13 and Lemma 14. First, we determine a pointwise unitary matrix function P_1, such that

$$E_1(t) := P_1^H(t)E(t)P_1(t) = \begin{bmatrix} \Delta_r(t) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Delta_r(t) \in \mathcal{C}([t_0, t_1], \mathbb{C}^{r,r})$ is Hermitian and pointwise nonsingular and $E(t)\dot{P}_1(t) = 0$. Then

$$A_1(t) := P_1^H(t)A(t)P_1(t) - P_1^H(t)E(t)\dot{P}_1(t) = P_1^H(t)A(t)P_1(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{12}^H(t) & A_{22}(t) \end{bmatrix}.$$

As $\operatorname{rank} A_{22}(t) = a$ is constant in $[t_0, t_1]$, we can determine a pointwise unitary matrix function $\hat{P}_2(t)$ such that $\hat{P}_2^H(t)A_{22}(t)\hat{P}_2(t) = \begin{bmatrix} \Delta_a(t) & 0 \\ 0 & 0 \end{bmatrix}$, with $\Delta_a \in \mathcal{C}([t_0, t_1], \mathbb{C}^{a,a})$ pointwise nonsingular. Then we perform a transformation to the whole system with $P_2^H(t) = \begin{bmatrix} I_r & 0 \\ 0 & \hat{P}_2(t) \end{bmatrix}$ from the left and $P_2(t)$ from the right yielding

$$E_2(t) := P_2^H(t)E_1(t)P_2(t) = \begin{bmatrix} \Delta_r(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_2(t) := P_2^H(t)A_1(t)P_2(t) - P_2^H(t)E_1(t)\dot{P}_2(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ A_{12}^H(t) & \Delta_a(t) & 0 \\ A_{13}^H(t) & 0 & 0 \end{bmatrix},$$

where $E_1(t)\dot{P}_2(t) = 0$. We can now eliminate the blocks $A_{12}(t)$ and $A_{12}^H(t)$ with a transformation
$P_3^H(t) = \begin{bmatrix} I_r & -A_{12}(t)\Delta_a(t)^{-1} & 0 \\ 0 & I_a & 0 \\ 0 & 0 & I_{n-r-a} \end{bmatrix}$ from the left and $P_3(t)$ from the right yielding

$$E_3(t) := P_3^H(t)E_2(t)P_3(t) = \begin{bmatrix} \Delta_r(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_3(t) := P_3^H(t)A_2(t)P_3(t) - P_3^H(t)E_2(t)\dot{P}_3(t) = \begin{bmatrix} A_{11}(t) & 0 & A_{13}(t) \\ 0 & \Delta_a(t) & 0 \\ A_{13}^H(t) & 0 & 0 \end{bmatrix},$$

where again $E_2(t)\dot{P}_3(t) = 0$. As $\operatorname{rank} A_{13}(t) = s$ for all $t \in [t_0, t_1]$, we can find pointwise unitary matrix functions $\hat{P}_4(t)$ and $\hat{Q}_4(t)$ such that $\hat{P}_4(t)A_{13}^H(t)\hat{Q}_4^H(t) = \begin{bmatrix} \Delta_s(t) & 0 \\ 0 & 0 \end{bmatrix}$, with pointwise nonsingular $\Delta_s(t) \in \mathcal{C}([t_0, t_1], \mathbb{C}^{s,s})$ and we set $P_4^H(t) = \begin{bmatrix} \hat{Q}_4(t) & 0 & 0 \\ 0 & I_a & 0 \\ 0 & 0 & \hat{P}_4(t) \end{bmatrix}$.

Thus, we get

$$E_4(t) := P_4^H(t)E_3(t)P_4(t) = \begin{bmatrix} \hat{Q}_4(t)\Delta_r(t)\hat{Q}_4^H(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} A_4(t) &:= P_4^H(t)A_3(t)P_4(t) - P_4^H(t)E_3(t)\dot{P}_4(t) \\ &= \begin{bmatrix} \hat{Q}_4(t)A_{11}(t)\hat{Q}_4^H(t) & 0 & \hat{Q}_4(t)A_{13}(t)\hat{P}_4^H(t) \\ 0 & \Delta_a(t) & 0 \\ \hat{P}_4(t)A_{13}^H(t)\hat{Q}_4^H(t) & 0 & 0 \end{bmatrix} - \begin{bmatrix} \hat{Q}_4(t)\Delta_r(t)\dot{\hat{Q}}_4^H(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

As $\hat{Q}_4(t)\Delta_r(t)\hat{Q}_4^H(t) = \begin{bmatrix} E_{11}(t) & E_{12}(t) \\ E_{12}^H(t) & E_{22}(t) \end{bmatrix}$ is nonsingular and because $\operatorname{kernel} E \subseteq \operatorname{kernel} \dot{E}$ implies that $\operatorname{kernel} \Delta_r \subseteq \operatorname{kernel} \dot{\Delta}_r$ it follows from Lemma 14 that $\Delta_r(t)\dot{\hat{Q}}_4^H(t) = 0$. Thus, the resulting pair is again Hermitian. We write the system as

$$\left(\begin{bmatrix} E_{11}(t) & E_{12}(t) & 0 & 0 & 0 \\ E_{12}^H(t) & E_{22}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11}(t) & A_{12}(t) & 0 & \Delta_s^H(t) & 0 \\ A_{12}^H(t) & A_{22}(t) & 0 & 0 & 0 \\ 0 & 0 & \Delta_a(t) & 0 & 0 \\ \Delta_s(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right),$$

and eliminate certain blocks to get

$$\left(\begin{bmatrix} E_{11}(t) & E_{12}(t) & 0 & 0 & 0 \\ E_{12}^H(t) & E_{22}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \Delta_s^H(t) & 0 \\ 0 & A_{22}(t) & 0 & 0 & 0 \\ 0 & 0 & \Delta_a(t) & 0 & 0 \\ \Delta_s(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

□

Under the assumptions of Theorem 15 we can now transform the Hermitian pair (13) into global condensed form (19). In order to obtain a strangeness-free normal form as in the non-symmetric case, we would have to eliminate the 'strange' coupled parts. This means that we would have to use the derivative of certain algebraic equations associated with the fourth block row of (19) to eliminate certain derivatives arising in the first and second block row of (19). In general, a structure preserving strangeness-free formulation for variable coefficient systems only exists if the strangeness index $\mu \leq 1$. In the following we distinguish two different cases. In the first case we assume that the system is strangeness-free (corresponding to a strangeness index $\mu = 0$), i.e., $s = 0$ in the global condensed form (19). Then the differential-algebraic system reduces to the Hermitian system

$$\begin{aligned} E_{22}(t)\dot{x}_1(t) &= A_{22}(t)x_1(t) + f_1(t), \\ 0 &= \Delta_a(t)x_2(t) + f_2(t), \\ 0 &= f_3(t). \end{aligned}$$

As second case we assume that $\mu = 1$, i.e., $s > 0$ in the global condensed form (19). The matrix pair can be written as differential-algebraic system

$$
\begin{aligned}
E_{11}(t)\dot{x}_1(t) + E_{12}(t)\dot{x}_2(t) &= \Delta_s^H(t)x_4(t) + f_1(t),\\
E_{12}^H(t)\dot{x}_1(t) + E_{22}(t)\dot{x}_2(t) &= A_{22}(t)x_2(t) + f_2(t),\\
0 &= \Delta_a(t)x_3(t) + f_3(t),\\
0 &= \Delta_s(t)x_1(t) + f_4(t),\\
0 &= f_5(t).
\end{aligned}
\tag{20}
$$

If we use the derivative of the forth equation in (20) to eliminate the terms with $\dot{x}_1$ in the first two equations of (20) we get

$$
\begin{aligned}
E_{12}(t)\dot{x}_2(t) &= \Delta_s^H(t)x_4(t) + \tilde{f}_1(t),\\
E_{22}(t)\dot{x}_2(t) &= A_{22}(t)x_2(t) + \tilde{f}_2(t),\\
0 &= \Delta_a(t)x_3(t) + f_3(t),\\
0 &= \Delta_s(t)x_1(t) + f_4(t),\\
0 &= f_5(t),
\end{aligned}
$$

where $\tilde{f}_1(t) = f_1(t) + E_{11}(t)\frac{d}{dt}\left(\Delta_s^{-1}(t)f_4(t)\right)$ and $\tilde{f}_2(t) = f_2(t) + E_{12}^H(t)\frac{d}{dt}\left(\Delta_s^{-1}(t)f_4(t)\right)$. Now, we can see that existence and uniqueness of solutions depend on the rank of $\begin{bmatrix} E_{12}(t) \\ E_{22}(t) \end{bmatrix}$. If $E_{22}(t)$ is invertible then all informations concerning existence and uniqueness are given and no further steps are needed, which is the case in a system with strangeness index $\mu = 1$. Therefore, we assume that $E_{22}(t)$ is invertible. Then we can eliminate the block $E_{12}(t)$ and get the following system

$$
\begin{aligned}
0 &= \Delta_s^H(t)x_4(t) - E_{12}(t)E_{22}^{-1}(t)A_{22}(t)x_2(t) + \tilde{f}_1(t),\\
E_{22}(t)\dot{x}_2(t) &= A_{22}(t)x_2(t) + \tilde{f}_2(t),\\
0 &= \Delta_a(t)x_3(t) + f_3(t),\\
0 &= \Delta_s(t)x_1(t) + f_4(t),\\
0 &= f_5(t).
\end{aligned}
\tag{21}
$$

As we have assumed that the strangeness index μ is equal to 1, it follows that $E_{12}(t)E_{22}^{-1}(t)A_{22}(t) = 0$ (there should be no coupling between differential and algebraic equations anymore) and we get the strangeness-free system

$$
\begin{aligned}
0 &= \Delta_s^H(t)x_4(t) + \tilde{f}_1(t),\\
E_{22}(t)\dot{x}_2(t) &= A_{22}(t)x_2(t) + \tilde{f}_2(t),\\
0 &= \Delta_a(t)x_3(t) + f_3(t),\\
0 &= \Delta_s(t)x_1(t) + f_4(t),\\
0 &= f_5(t),
\end{aligned}
$$

which is again Hermitian. Rearanging and renaming the matrices and vector-valued functions finally yields the strangeness-free Hermitian system

$$
\begin{aligned}
\dot{\hat{x}}_1(t) &= \hat{A}_{11}(t)\hat{x}_1(t) + \hat{f}_1(t),\\
0 &= \hat{x}_2(t) + \hat{f}_2(t),\\
0 &= \hat{f}_3(t).
\end{aligned}
$$

Theorem 16. *If $E_{22}(t)$ in* (19) *is nonsingular then the system* (1) *has strangeness index $\mu \leq 1$.*

Proof. The Theorem follows from the previous discussion. □

For systems with strangeness index $\mu > 1$ either $E_{22}^{-1}(t)$ does not exist or the block $E_{12}(t)E_{22}^{-1}(t)A_{22}(t)$ in (21) does not vanish identically. Thus, in general we cannot preserve the symmetry of the strangeness-free system in this case. From the previous discussion we get some more results.

Corollary 17. *Consider a linear differential-algebraic equation* (1) *with Hermitian coefficient matrices $(E(t), A(t))$, where $E(t)$ is positive semidefinite. Then the system has strangeness index $\mu \leq 1$.*

Proof. The Corollary follows directly from Theorem 16. If $E(t)$ is positive semidefinite then the matrix $\begin{bmatrix} E_{11}(t) & E_{12}(t) \\ E_{12}^H(t) & E_{22}(t) \end{bmatrix}$ in (19) is positive definite and thus $E_{22}(t)$ is positive definite and therefore nonsingular. □

Example 18. *Consider the symmetric system*

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & t \\ 0 & t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

in an interval $[t_0, t_1]$ with $t_0 > 0$ and characteristic values $\mu = 1$, $d_\mu = 1$, $a_\mu = 2$, $u_\mu = 0$. The system is already in global condensed form (19)*. Further transformations yield the pair*

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ t & 0 & 0 \end{bmatrix} \right),$$

where $E_{22}(t) = t$ is nonsingular for all $t \neq 0$. The differentiation-and-elimination step together with a permutation yields the strangeness-free system

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & t \\ 0 & t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 + \frac{1}{t}\dot{f}_3 - \frac{1}{t^2} f_3 \\ f_3 \end{bmatrix}.$$

4 A Structure Preserving Derivative Array Approach

In the numerical solution of differential-algebraic equations it was suggested in [3, 11, 10, 12] to transform the system into an equivalent strangeness-free differential-algebraic system. In this section we show how this can be done in a structure preserving way using the so-called derivative array approach.

In the general approach described in [12] we consider a linear differential-algebraic system (1) with well-defined strangeness index μ. Then it is possible to determine an equivalent strangeness-free differential-algebraic system using derivative arrays. The idea is to write the original DAE together with the first μ derivatives into a large system and then choose suitable projectors of the right dimension to extract the differential and the algebraic parts. The *derivative arrays* or *inflated differential-algebraic equations* are defined by

$$\mathcal{M}_l \dot{z}_l(t) = \mathcal{N}_l z_l(t) + g_l(t), \quad l = 0, \ldots, \mu, \tag{22}$$

where for $i, j = 0, \dots, l$ we have

$$\begin{aligned}
[\mathcal{M}_l]_{i,j} &:= \binom{i}{j} E^{(i-j)} - \binom{i}{j+1} A^{(i-j-1)}, \quad i, j = 0, \dots, l, \\
[\mathcal{N}_l]_{i,j} &:= \begin{cases} A^{(i)} & \text{for } i = 0, \dots, l,\ j = 0, \\ 0 & \text{otherwise,} \end{cases} \\
[z_l]_j &:= x^{(j)}, \\
[g_l]_i &:= f^{(i)}.
\end{aligned} \tag{23}$$

For differential-algebraic systems (1) with well-defined strangeness index μ and $u_\mu = 0$ (i.e., the initial value problem for consistent initial conditions has a unique solution), it was shown in [11] that the pair $(E(t), A(t))$ satisfies the following theorem.

Theorem 19. *There exist integers μ, a_μ, and d_μ such that the inflated pair $(\mathcal{M}_\mu, \mathcal{N}_\mu)$ associated with (E, A) has the following properties:*

1. *For all $t \in [t_0, t_1]$ it holds that $\operatorname{rank} \mathcal{M}_\mu = (\mu+1)n - a_\mu$, such that there exists a smooth matrix function Z_2 with orthonormal columns and size $((\mu+1)n, a_\mu)$ satisfying $Z_2^H \mathcal{M}_\mu = 0$.*
2. *For all $t \in [t_0, t_1]$ it holds that $\operatorname{rank} Z_2^H \mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}^H = a_\mu$, such that there exists a smooth matrix function T_2 with orthonormal columns and size (n, d_μ), $d_\mu = n - a_\mu$, satisfying $Z_2^H \mathcal{N}_\mu \begin{bmatrix} I_n & 0 & \dots & 0 \end{bmatrix}^H T_2 = 0$.*
3. *For all $t \in [t_0, t_1]$ it holds that $\operatorname{rank} E T_2 = d_\mu$, such that there exists a smooth matrix function Z_1 with orthonormal columns and size (n, d_μ) yielding that $Z_1^H E$ has constant rank d_μ.*

Using this Theorem it is possible to derive an equivalent strangeness-free system by using only local information from the inflated pair $(\mathcal{M}_\mu, \mathcal{N}_\mu)$.
In the variable coefficient case a structure preserving strangeness-free formulation only exists if $\mu \leq 1$. To obtain a strangeness-free formulation in this case we can use the procedure described in Section 3. For the constant coefficient case we can use the derivative array approach and modify the procedure described in [12] to obtain a structure preserving strangeness-free system from the inflated pair.
In the case of constant coefficient systems of the form (4), the inflated system (22) has the form

$$\begin{bmatrix} E & 0 & \cdots & \cdots & 0 \\ -A & E & 0 & \cdots & 0 \\ 0 & -A & E & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & -A & E \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ \vdots \\ x^{(\mu+1)} \end{bmatrix} = \begin{bmatrix} A & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & \cdots & & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ \vdots \\ x^{(\mu)} \end{bmatrix} + \begin{bmatrix} f \\ \dot{f} \\ \vdots \\ \vdots \\ f^{(\mu)} \end{bmatrix}. \tag{24}$$

At first, we decompose the matrix Z_2 from Theorem 19 into $Z_2 = \begin{bmatrix} Z_{2,1} \\ Z_{2,2} \end{bmatrix}$ with $Z_{2,1}$ and $Z_{2,2}$ of sizes $n \times a_\mu$ and $\mu n \times a_\mu$, such that

$$\begin{bmatrix} Z_{2,1}^H & Z_{2,2}^H \end{bmatrix} \mathcal{M}_\mu = 0, \quad \text{and} \quad \operatorname{rank} Z_{2,1}^H A = a_\mu,$$

where

$$\operatorname{rank}\left(\begin{bmatrix} Z_{2,1}^H & Z_{2,2}^H \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix}\right) = \operatorname{rank}(\begin{bmatrix} Z_{2,1}^H A \end{bmatrix}) = a_\mu.$$

Multiplication of $Z_{2,1}^H A$ with $Z_{2,1}$ from the right does not change the rank and we have $\operatorname{rank}(Z_{2,1}^H A Z_{2,1}) = a_\mu$. In this way we can obtain the complete set of algebraic equations and the symmetry can be preserved. Next, we must get d_μ differential equations which complete these algebraic equations to a strangeness-free differential-algebraic system. Thus, we have to determine a nonsingular matrix T_2 of size $n \times d_\mu$ such that

$$Z_{2,1}^H A T_2 = 0, \quad \operatorname{rank}(E T_2) = d_\mu.$$

It follows that also $\operatorname{rank}(T_2^H E T_2) = d_\mu$ and we can choose $Z_1 = T_2$ in order to preserve symmetry. Altogether, we can construct a pair of Hermitian matrices

$$(\hat{E}, \hat{A}) = \left(\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix}\right), \tag{25}$$

with entries $\hat{E}_{11} = Z_1^H E Z_1$, $\hat{A}_{11} = Z_1^H A Z_1$, $\hat{A}_{22} = Z_{2,1}^H A Z_{2,1}$. Setting $\hat{f}_1 = Z_1^H f$ and $\hat{f}_2 = Z_2^H g_\mu$ and transforming the unknown vector x accordingly, i.e., $\hat{x}_1 = Z_1^H x$, $\hat{x}_2 = Z_{2,1}^H x$ we obtain from the inflated differential-algebraic equation (24) a strangeness-free system

$$\begin{bmatrix} \hat{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix},$$

which is Hermitian and has the same size as the original pair.

Example 20. *Consider the symmetric system*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \tag{26}$$

which has the solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \int_{t_0}^t f_1(\tau) d\tau + c \\ f_3(t) \\ f_2(t) - \dot{f}_3(t) \\ f_4(t) \end{bmatrix},$$

where the integration constant c can be determined by assigning an initial value. The system (26) *has the characteristic quantities $\mu = 1$, $d_\mu = 1$, $a_\mu = 3$, $u_\mu = 0$ and the inflated matrices are given by*

$$\mathcal{M}_\mu = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right], \quad \mathcal{N}_\mu = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We can choose $\begin{bmatrix} Z_{21}^H & Z_{22}^H \end{bmatrix} = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$ *and* $Z_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. *Then we get the new strangeness-free system*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_3 \\ f_4 \\ f_2 - \dot{f}_3 \end{bmatrix},$$

which is again symmetric.

5 Conclusions

We have presented structure preserving condensed forms for pairs of Hermitian matrices and pairs of Hermitian matrix functions associated with linear differential-algebraic equations with constant and variable coefficients, respectively. We have seen that it is only possible to obtain a structure preserving condensed form for pairs of matrix functions if certain additional assumptions hold. In addition, for variable coefficient differential-algebraic systems a structure preserving strangeness-free formulation only exists if the strangeness index $\mu \leq 1$. Further, we have presented a way to derive an equivalent structure preserving strangeness-free formulation using derivative arrays in the case of constant coefficient systems.

References

[1] A. Bunse-Gerstner and V. Mehrmann. The HHDR-algorithm and its application to optimal control problems. *R.A.I.R.O. Autom./Prod. Inform. Industr.*, 23:305–329, 1989.

[2] R. Byers and D. Kressner. Structured condition numbers for invariant subspaces. *SIAM J. Matrix Anal. Appl.*, 2005.

[3] S.L. Campbell. Comment on controlling generalized state-space (descriptor) systems. *Internat. J. Control*, 46:2229–2230, 1987.

[4] H. Fassbender and Kressner D. Structured eigenvalue problems. In *GAMM Mitteilungen, Themenheft Applied and Numerical Linear Algebra, Part II*, 2005. To appear.

[5] F.R. Gantmacher. *The Theory of Matrices.* Chelsea Publishing Company, New York, 1959.

[6] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Computer science and applied mathematics. Academic Press, New York, 1982.

[7] G.H. Golub and C.F. van Loan. *Matrix Computations.* The Johns Hopkins University Press Baltimore and London, 3 edition, 1996.

[8] M. Karow, D. Kressner, and F. Tisseur. Structured eigenvalue condition numbers. Technical Report No. 467, Manchester Centre for Computational Mathematics, Manchester, England, 2005.

[9] P. Kunkel and V. Mehrmann. A new look at pencils of matrix valued functions. *Linear Algebra Appl.*, 212/213:215–248, 1994.

[10] P. Kunkel and V. Mehrmann. Local and global invariants of linear differential-algebraic equations and their relation. *Electr. Trans. Num. Anal.*, 4:138–157, 1996.

[11] P. Kunkel and V. Mehrmann. A new class of discretization methods for the solution of linear differential-algebraic equations with variable coefficients. *SIAM J. Numer. Anal.*, 33(5):1941–1961, 1996.

[12] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations — Analysis and Numerical Solution.* In preparation, Berlin, 2006.

[13] P. Lancaster. *Theory of matrices.* Academic Press, New York, 1969.

[14] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution.* Number 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, 1991.

[15] K. Nomizu. Characteristic roots and vectors of a differentiable family of symmetric matrices. *Linear Mulitilin. Alg.*, 1:159–162, 1973.

[16] W.C. Rheinboldt. On the computation of multi-dimensional solution manifolds of parameterized equations. *Numer. Math.*, 53:165–181, 1988.

[17] R.C. Thompson. The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil. *Linear Algebra Appl.*, 14:135–177, 1976.

[18] R. Zurmühl and S. Falk. *Matrizen und ihre Anwendungen I, 5. Auflage.* Springer-Verlag, Berlin, 1986.

[19] R. Zurmühl and S. Falk. *Matrizen und ihre Anwendungen II, 6. Auflage.* Springer-Verlag, Berlin, 1992.