# Global attractors of sixth order PDEs describing the faceting of growing surfaces

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**Abstract**

A spatially two-dimensional sixth order PDE describing the evolution of a growing crystalline surface $h(x, y, t)$ that undergoes faceting is considered with periodic boundary conditions, such as its reduced one-dimensional version. These equation are expressed in terms of the slopes $u_1 = h_x$ and $u_2 = h_y$ to establish the existence of global, connected attractors for both of the equations. Since unique solutions are guaranteed for initial conditions in $\dot{H}^2_{per}$, we consider the solution operator $S(t) : \dot{H}^2_{per} \to \dot{H}^2_{per}$, to gain the results. We prove the necessary continuity, dissipation and compactness properties.

## 1 Introduction

We study the long time dynamics of the following system arising in modeling of the self assembly of nanostructures,

$$\begin{array}{ll} h_t = \frac{\delta}{2}|\nabla h|^2 + \Delta(\Delta^2 h - \Delta \mathrm{div}\, D_F W(\nabla h)) & \text{in } \Omega^2 \times \mathbb{R}_+, \\ h(x,0) = h_0(x) & \text{for } x \in \Omega^2. \end{array} \quad (1)$$

Here, $h : \Omega^2 \times \mathbb{R}_+ \to \mathbb{R}$ is the height of a growing crystalline surface undergoing faceting during growth, $\delta > 0$ is related to the deposition strength, $W : \mathbb{R}^d \to \mathbb{R}$ is the anisotropy function derived from the surface energy anisotropy and $\Omega^2 \subset \mathbb{R}^d$ is the spatial domain with $d = 1$ or $d = 2$. Dependent on the dimension we consider two specific cases for $W$ given below, see (2), (3). We restrict our attention to special geometries and work with $\Omega^2 = (0, L)^d$, $d = 1, 2$ and furthermore we assume periodic boundary conditions for $h$.

We show existence of global attractors for above system and we shall also explain, why this result seems best we can hope for. The PDE was introduced by Savina *et al.*, see [14]. It has been derived by invoking Mullins' surface diffusion formula [11], a normally impinging flux of adatoms to the surface and a strongly anisotropic surface energy formula. The reduced evolution equation is obtained

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by carrying out a long-wave approximation. The choice of periodic boundary conditions is realistic as the patterns of the nanostructures statistically repeat throughout the domain, which is much larger than the length-scales of interest. Numerical simulations imposing these kind of boundary conditions show good agreement with experimentally observed behavior of crystalline materials undergoing faceting and coarsening [14, 5]. We also notice that the analysis on periodic domains is easy to transfer for the numerical analysis of simulation schemes based on trigonometric interpolation. Such collocation methods are applied frequently to such problems.

We consider the following anisotropy functions $W$. If $d = 1$, then we take

$$W(F) = \frac{1}{4}(F^2 - 1)^2 \tag{2}$$

yielding a double well potential and in the case $d = 2$ we deal with

$$W(F) \equiv W(F_1, F_2) = \frac{\alpha}{12}(F_1^4 + F_2^4) + \frac{\beta}{2}F_1^2 F_2^2 - \frac{1}{2}(F_1^2 + F_2^2) + A, \tag{3}$$

where $\alpha, \beta > 0$ are anisotropy coefficients.

Formula (3) gives a quadruple well that is responsible for the faceting of the growing surface in shape of pyramids with four preferred orientations and hence preferred slopes. A constant $A$ may be chosen such that $W$ is always nonnegative.

In two related works, [7, Theorem 1.1] and [8, Theorem 2.1], we proved the existence of global in time weak solutions to (1) with periodic boundary data. There was no size restrictions on the data.

In [7] and [8] we showed only exponential bounds on the growth of solutions which is not particularly suitable for studying long time behavior. We will find the remedy here and we will show existence of a global attractor of (1) for $d = 1, 2$. The destabilizing term does not give us much hope to establish convergence to an equilibrium state. However, if we had a Liapunov functional, then we could hope to use methods based on Łojasiewicz inequality to show convergence of solutions to a steady state, see [12].

Our plan is to study first the one-dimensional problem, so that we can develop ideas that are used later also in the more complex case. It turns out that the trick applied in [14] and [8] works very nicely. Namely, after differentiating (1) with respect to $x$ we obtain a *slope equation* for the new unknown quantity $u = h_x$, see (4). One advantage is that we obtain a new conserved quantity, $\int_0^L u\,dx = 0$. This will imply that the semigroup generated by $\Delta^3$ has an exponential decay. Another advantage is, the resulting equation is similar to the convective Cahn-Hilliard equation, which has already been analyzed to some extent. Equation (1) may be interpreted as a convective Cahn-Hilliard (CCH) type equation of higher order, hence we call it the HCCH equation. Note that it is a gradient system perturbed by a destabilizing Kardar-Parisi-Zhang type term $|\nabla h|^2$.

Here, we use ideas from the theory of infinite dimensional dynamical systems [13, 4] combined with the available results on convective Cahn-Hilliard equation, e.g. [9, 3, 1]. Eden and Kalantarov noticed, see [1], that the structure of the lower order convective Cahn-Hilliard equation permits to deduce bounds implying existence of an absorbing set. The same method can be applied here.

We first prove the existence of an absorbing set in the $H^1$ topology, then we extend the result to $H^2$. Showing its compactness in $H^2$ requires further improvement of the regularity of weak solutions. Once we achieve that we can conclude existence of a global attractor, see [2, Theorem 1].

We notice that, if we take gradient of (1) with respect to the spatial variables in the two dimensional case, then the resulting system, see (8), has the structure which permits to carry the calculations we did for the one-dimensional problem. Thus, we establish existence of the global attractor for the corresponding slope systems, $u = \nabla h$. Finally, we deduce from this existence of a global attractor of the original equation (1), see Theorems 1, 2, 3.

We proceed as follows. In the next section we recall the notions of weak solutions and the necessary facts from [7, 8], in addition we state the main results. In Section 3 we prove the existence of absorbing balls in $H^1$ for the one-dimensional problem (4). This is done with the help of ideas taken from [1]. We also show in this section the necessary auxiliary facts. In Section 4, we study system, which is the result of gradient of (1), we call it the slope system. Its advantage is we can use exactly the same method, as in one-dimension to show existence of an absorbing ball in $H^1$. Next section is devoted to the proof of higher order regularity and compactness in $H^2$ of the absorbing balls, we use the parameter variation formula for this purpose. This is done both cases of the space dimension.

Finally we discuss the results and future plans in Section 6.

# 2 Preliminaries and main statements

## 2.1 Properties of solutions and main statements

As we mentioned in the introduction, we treat cases $d = 1$ and $d = 2$ differently. For the one-dimensional problem we switch to a new variable, the slope $u = h_x$, i.e. we differentiate (1) with respect to $x$. The resulting problem is

$$\begin{array}{ll} u_t - \frac{\delta}{2}(u^2)_x - (u_{xx} - f(u))_{xxxx} = 0, & \text{in } (x,t) \in (0,L) \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{for } x \in (0,L), \end{array} \tag{4}$$

where

$$f(u) = W'(u) = u^3 - u.$$

In [8] we adopted a natural definition of a weak solution of the one-dimensional problem (4). We say that a function

$$u \in L^2(0,T;\dot{H}^3_{per}(\Omega)) \cap L^4(0,T;\dot{L}^4(\Omega)) \cap C^0([0,T],\dot{L}^2(\Omega)) \text{ with } u_t \in L^2(0,T;H^{-3}(\Omega))$$

is a *weak solution* to (4) provided that it fulfills (see [8]),

$$\int_{\Omega_T} u_t \varphi dxdt + \delta \int_{\Omega_T} g(u)\varphi_x dxdt + \int_{\Omega_T} u_{xxx}\varphi_{xxx} dxdt - \int_{\Omega_T} f'(u)u_x\varphi_{xxx} dxdt = 0,$$
$$\text{for all} \quad \varphi \in L^2(0,T,\dot{H}^3_{per}(\Omega))\,. \tag{5}$$

We showed the existence of such solutions in [8, Theorem 2.1] for initial condition $u_0 \in \dot{H}^1_{per}$. The symbol $\dot{H}^k_{per}$ denotes the Sobolev spaces of periodic functions with zero mean.

For the two-dimensional problem (1) and (3) we established a similar result in [7, Theorem 1.1], i.e. we have shown the existence of a weak solution to (1) with periodic boundary conditions, understood as a function $h \in C([0,T), H^3_{per})$ with $h(\cdot, 0) = h_0$ and $h_t \in L_\infty((0,T), H^{-3})$, such that $h$ satisfies (1) in the distributional sense. We assumed that $h_0 \in H^2_{per}$. Here, $\Omega^2$ is a two-dimensional flat torus, that may be scaled to represent $(0,L)^2$ for arbitrary $L$ and periodic boundary conditions.

We showed global in time existence and uniqueness (see [8, Theorem 4.2] and [7, Theorem 1.1]) for more regular weak solution. Namely, for any $T > 0$ and $d = 1$ we used

$$u \in L^2(0,T; \dot{H}^4_{per}(\Omega) \cap L^\infty(0,T; W^{1,\infty}(\Omega)), \tag{6}$$

while for $d = 2$ we needed,

$$h \in L^2(0,T; H^5_{per}(\Omega)) \cap L^\infty(0,T; H^2_{per}(\Omega)), \quad h_t \in L^2(0,T; H^{-1}_{per}(\Omega)). \tag{7}$$

This kind of smoothness is implied by better regularity of the initial data, i.e. $u_0 \in \dot{H}^2_{per}$ and respectively $h_0 \in H^3_{per}$, (see [8, Theorem 3.3] and [7, Theorem 1.2]). The solutions depend continuously on the initial conditions.

Furthermore, analysis of the two-dimensional problem gets simplified after we transform equation (1) to a system for the slopes, $u = (u_1, u_2) = (h_x, h_y)$ and we study

$$\begin{array}{ll} u_t = \frac{\delta}{2}\nabla|u|^2 + \Delta^3 u - \nabla\Delta \mathrm{div}\, D_u W(u_1,u_2) & \text{in } \Omega^2 \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \text{for } x \in \Omega^2. \end{array} \tag{8}$$

The derivation of this equation and specification of $W$ will be carried out in Section 5.

The following theorems are our main results proved in this paper. Here, we denote by $S(t)u_0$ the unique solution $u(t)$ to (4) if $d = 1$, respectively (8) if $d = 2$, with $u_0 \in (H^2_{per})^d$. This way of writing exposes the family of operators $S(t) : (\dot{H}^2_{per})^d \to (\dot{H}^2_{per})^d$. Their continuity follows from results in [7, 8]. The uniqueness theorems imply that the family $\{S(t)\}_{t\geq 0}$ has the semigroup property.

**Theorem 1 (1D Attractor in $\dot{H}^2_{per}(\Omega)$)** Let us consider $\Omega = (0,L)$ with $L > 0$ arbitrary. The semigroup $S(t) : \dot{H}^2_{per}(\Omega) \to \dot{H}^2_{per}(\Omega), u_0 \mapsto S(t)u_0 = u(t)$ generated by the HCCH equation (4) with periodic boundary conditions has a global attractor.

**Theorem 2 (2D Attractor in $(\dot{H}^2_{per}(\Omega^2))^2$)** Let us consider $\Omega^2 = (0,L)^2$ with $L > 0$ arbitrary. The semigroup $S(t) : (\dot{H}^2_{per}(\Omega^2))^2 \to (\dot{H}^2_{per}(\Omega^2))^2, u_0 \mapsto S(t)u_0 = u(t)$ generated by equation (8) with periodic boundary conditions has a global attractor.

Once we show these results we may address the question of behavior of the solutions once they are transformed back to the actual shape $h$ of the original problem. We notice that one can easily recover a continuous function $f$ from its derivative and its mean. Thus, the above results imply.

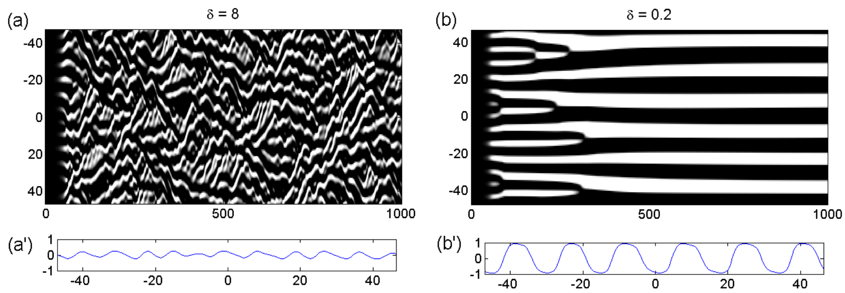


Figure 1: Simulation of the evolution of the one-dimensional HCCH equation (4). Initial condition: small random perturbation of the zero state. (a) Time-space plot of the evolution for $\delta = 8$, brighter areas correspond to larger values; chaotic evolution; (a') shows the solution at one point of time; The time-space plot in (b) and one of the solutions at a late time point in (b') show regular coarsening similar as in the Cahn-Hilliard equation, here for $\delta = 0.2$.

**Theorem 3** The semigroup generated by equation (1) has a global attractor in $H^3_{per}$ for $d = 1$ and $d = 2$.

We note that we have numerical evidence of the existence of such an attractor in the one-dimensional setting. Figure 1 shows similar pictures of the evolution as in [8] for two values of $\delta$. For its large value $\delta$ a strange attractor seems to be existent, see the time-space plot in (a) and one particular solution in (a'). For smaller values we numerically expect stationary solutions as in (b) and (b)', or traveling wave (time-periodic) solutions. Note that once the structures form, the solutions stay in an $\dot{H}^2_{per}$ ball as the theory predicts. One might hope to be able to prove that at least for small initial data the $L^\infty$ norm of $u$ stays roughly below 1, independently of the value for $\delta$. Our analytical result however gives us information of different nature. We can take bigger initial conditions and still the absorption is in the same ball. This property is indicated in Figure 2 where another typical evolution of equation (4) is shown together with the decrease of the norm of the discrete solution and the three phase spaces $(u, u_x), (u, u_{xx})$ and $(u_x, u_{xx})$. However, these runs calculated with a pseudospectral method discussed elsewhere [8], are particular examples with special initial conditions, fixed domain length and deposition parameter. The theory establishes a general result. We like to remark, that more simulation results, also for the two-dimensional setting, can be found in [14].

## 2.2 Tools of dynamical systems

We will use the methods of the infinite dimensional dynamical systems, see the books by Hale, [4], Temam, [16] or Robinson, [13]. However, we will use the theorem guaranteeing existence of a global attractor as stated in [2]. From now on we shall assume that $S(t) : H \to H$ is a semigroup, here $H$ is a Hilbert space. Following [2], we recall the necessary notions.

Let us suppose $C_1, C_2 \subset H$, by dist $(C_1, C_2)$ we denote their Hausdorff semi-distance,

$$\text{dist}\,(C_1, C_2) = \sup_{x\in C_1} \inf_{y\in C_2} d(x, y).$$

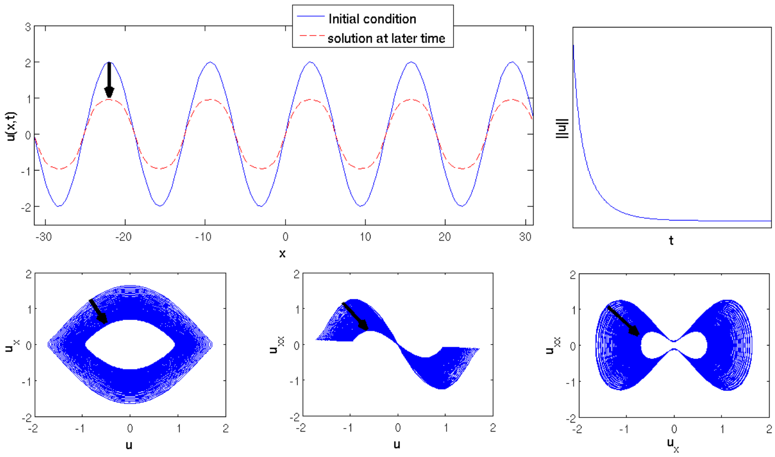

Figure 2: Top: Initial condition $u_0(x) = 2\sin(x/2)$ and shape of $u$ after evolution of the HCCH equation to near stationary state on a $20\pi$ long domain. The figure on the right indicates a decrease of the $L^2$-norm of $u$. Bottom row: Phase spaces $(u, u_x), (u, u_{xx}), (u_x, u_{xx})$ for the same initial condition. The lines indicate the solutions at different times, all shrinking in these plains.

A non-empty set $K \subset H$ is invariant, if

$$S(t)K = K, \quad t \geq 0,$$

it attracts $B \subset H$ if

$$\lim_{t\to\infty} \text{dist}\,(S(t)B, K) = 0.$$

A set $K \subset H$ is called an *absorbing set* if for any bounded $B \subset H$ there is time $t_{K,B} \geq 0$ such that

$$S(t)B \subset K \quad \text{for } t \geq t_{K,B}.$$

Here, we note that any absorbing set attracts bounded sets.

For a bounded set $B \subset H$, we define its $\omega$-limit set by formula

$$\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}.$$

A global attractor for $S(\cdot)$ is a maximal compact invariant set.

**Theorem 4** *(see [2, Theorem 1])* Let us suppose that $S(\cdot)$ has a compact attracting set $K$. Then there is a global attractor for $S(\cdot)$ and $\mathcal{A} = \omega(K)$.

This result will imply our Theorems 1 and 2 once we show its assumptions are fulfilled. For this purpose we need to show the following properties of the solution operator $S(t) : (\dot{H}^2_{per})^d \to (\dot{H}^2_{per})^d$, $d = 1, 2$.

- $S$ is a $C^0$-semigroup operator on $\dot{H}^2_{per}$. This is to be established for the one-dimensional case. This property was proved in [7] for (1) in 2-D.

- Existence of a compact attracting set $K$. This will be achieved in two steps. First, we will show existence of an absorbing set in $H^1$. Next, by application of a different method, the existence of an absorbing set in $H^2$ and its compactness will be proved.

Note that $\dot{H}^2_{per}$ is the correct choice of space for the slope systems (4) and (8), because we could not work with solution operators acting on lower order spaces due to the lack of uniqueness.

We also we have to show that the Galerkin method applied to (4) yields a strongly continuous semigroup $S(t) : \dot{H}^2_{per} \to \dot{H}^2_{per}$.

### 2.3 The inverse Laplacian

We present an important tool that will be useful throughout the document.

**Lemma 5** The operator $(-\Delta)^{-1} : \dot{H}^k_{per}(\Omega) \to \dot{H}^{k+2}_{per}(\Omega)$, where $\Omega = (0, L)^d$, $d = 1, 2$ given by (9) is well defined and continuous.

Let us define the operator $\mathcal{L} : \dot{H}^k_{per}(\Omega) \to \dot{H}^{k+2}_{per}(\Omega)$ as an multiplicator acting on the Fourier series of $u$ by the formula

$$(\mathcal{L}u)\hat{}(\xi) = \begin{cases} \left(\frac{L}{2\pi}\right)^2 \frac{\hat{u}(\xi)}{|\xi|^2} & \xi \neq 0, \\ 0 & \xi = 0, \end{cases}$$

where $\xi \in \mathbb{Z}^d$, $d = 1, 2$ and $|\cdot|$ is an Euclidean norm. We have

$$\begin{aligned} \|\mathcal{L}u\|^2_{H_{k+2}} &= \sum_{\xi \in \mathbb{Z}^d, \xi \neq 0} \left(\frac{L}{2\pi}\right)^2 \left(1 + |\xi|^2\right)^{k+2} \frac{|\hat{u}(\xi)|^2}{|\xi|^4} \\ &\leq C(L) \sum_{\xi \in \mathbb{Z}^d, \xi \neq 0} \left(1 + |\xi|^2\right)^k |\hat{u}(\xi)|^2 = C(L)\|u\|^2_{H_k}. \end{aligned}$$

Also $\int_\Omega \mathcal{L}u = |\Omega|(\mathcal{L}u)\hat{}(0) = 0$. Note that

$$(-\Delta u)\hat{}(\xi) = \left(\frac{2\pi}{L}\right)^2 \hat{u}(\xi)|\xi|^2. \tag{9}$$

Since for $u \in \dot{H}^k_{per}(\Omega)$, we have $\hat{u}(0) = 0$, we obtain

$$(\mathcal{L} \circ (-\Delta)u)\hat{}(\xi) = \begin{cases} \left(\frac{L}{2\pi}\right)^2 \frac{1}{|\xi|^2} \left(\frac{2\pi}{L}\right)^2 \hat{u}(\xi)|\xi|^2 & \xi \neq 0 \\ 0 & \xi = 0 \end{cases}$$

Thus, $(\mathcal{L} \circ (-\Delta)u)\hat{}(\xi) = \hat{u}(\xi)$ and therefore $\mathcal{L} \circ (-\Delta) = Id$. □

The above argument justifies the natural definition of $(-\Delta)^s : \dot{H}^t_{per} \to \dot{H}^{t+s}_{per}$ by the following formula,

$$((-\Delta)^s u)\hat{}(\xi) := |\xi|^s \hat{u}(\xi).$$

# 3 The one-dimensional problem

In the following subsections we prove, by using Gronwall estimates, that there exists an absorbing ball in $H^1$. Throughout the calculations, we denote by $C$ a constant that may change from estimate to estimate, but does not depend on the initial condition. This quantity may rely on the domain length and the deposition related parameter, $L$ and $\delta$, respectively. Numbers whose actual value is needed for balances with other estimates are denoted by $C_j$, where $j$ is an integer index, and these numbers are fixed.

In the second part of this section we will show that the semigroup $S(t) : \dot{H}^2_{per} \to \dot{H}^2_{per}$ is indeed strongly continuous. This will be done by a series of a priori estimates of Galerkin approximations and passing to the limit.

## 3.1 Absorbing ball in $H^1$

Consider the HCCH equation (4) with periodic boundary conditions on a domain $\Omega = (0, L)$ and initial condition $u(x, 0) = u_0(x)$. We extend the analysis from [8] by showing that the solutions are in fact absorbed into a ball whose radius does not depend on the initial value's norm. To prove this result we will need to combine several estimates that we want to formulate as separate statements. Subsequently, we write the $L^2$-norm as $\| \cdot \| = \| \cdot \|_{L^2(0,L)}$ and the $L^2$ scalar product by $(\cdot, \cdot)$. Other norms are equipped with a corresponding subscript.

**Lemma 6** Weak solutions to equation (4) with $u_0 \in \dot{H}^2_{per}$ fulfill

$$\frac{d}{dt}\left[\int_\Omega W(u)dx + \frac{1}{2}\|u_x\|^2\right] + \frac{1}{2}\|(-\Delta)^{-1}u_t\|^2 \le C_1\|u\|^4_{L^4} \tag{10}$$

**Proof** Application of the integral operator $(-\Delta)^{-2} : \dot{L}^2 \to \dot{H}^4_{per}$ to both sides of equation (4) yields

$$(-\Delta)^{-2}u_t - \delta(-\Delta)^{-2}[g(u)_x] + u^3 - u - u_{xx} = 0\,, \tag{11}$$

with $g(u) = u^2/2$. The regularity guaranteed by (6) and equation (4) imply that we may test (11) with $u_t$. Next, integration by parts, rearranging and using the Cauchy inequality with $\epsilon$ yield,

$$\frac{d}{dt}\left[\frac{1}{4}\|u\|^4_{L^4} - \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_x\|^2\right] + \|(-\Delta)^{-1}u_t\|^2 \le \frac{\delta}{2\epsilon_0}\|(-\Delta)^{-1}[g(u)_x]\|^2 + \frac{\delta\epsilon_0}{2}\|(-\Delta)^{-1}u_t\|^2\,.$$

Since $\|(-\Delta)^{-1}[g(u)_x]\| \le C\|g(u)\|$, with an $L$ dependent constant, we get

$$\frac{d}{dt}\left[\frac{1}{4}\|u\|^4_{L^4} - \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_x\|^2\right] + (1 - \frac{\delta\epsilon_0}{2})\|(-\Delta)^{-1}u_t\|^2 \le \frac{\delta C}{4\epsilon_0}\|u^2\|^2\,. \tag{12}$$

Choosing $\epsilon_0 = 1/\delta$, we obtain

$$\frac{d}{dt}\left[\frac{1}{4}\|u\|^4_{L^4} - \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_x\|^2\right] + \frac{1}{2}\|(-\Delta)^{-1}u_t\|^2 \le C_1\|u\|^4_{L^4}\,, \tag{13}$$

with a constant $C_1 = C_1(L, \delta)$. In fact, by noting that

$$0 \leq W(u) := \frac{1}{4}(u^2 - 1)^2 = \frac{1}{4}u^4 - \frac{1}{2}u^2 + \frac{1}{4}\,,$$

we gain estimate (10). □

The result shown above can be used for proving the existence of absorbing sets in $\dot{H}^1$, therefore one needs to take care of the right hand side in (10). This is done in the following lemma.

**Lemma 7** The inequality

$$\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{-1}u\|^2 + \frac{1}{2}\|u\|_{L^4}^4 + \|u_x\|^2 \leq C_2\,. \tag{14}$$

is true for all weak solutions of (4) with $u_0 \in \dot{H}^2_{per}$.

**Proof** We use $u$ as a test function in the same, transformed equation (11),

$$((-\Delta)^{-1}u_t, (-\Delta)^{-1}u) - \delta((-\Delta)^{-1}[g(u)_x], (-\Delta)^{-1}u) + \|u_x\|^2 + \|u\|_{L^4}^4 - \|u\|^2 = 0\,.$$

This permitted again by (6). Putting the convective term on the right hand side, we estimate it as

$$\begin{aligned} \delta((-\Delta)^{-1}[g(u)_x], (-\Delta)^{-1}u) &\leq \delta\|(-\Delta)^{-1}[g(u)_x]\|\|(-\Delta)^{-1}u\| \\ &\leq C\|u^2\|\|u\| \leq C_3 + \frac{1}{4}\|u\|_{L^4}^4. \end{aligned}$$

Furthermore, by using $\|u\|^2 \leq \frac{1}{4}\|u\|_{L^4}^4 + \|1\| = \frac{1}{4}\|u\|_{L^4}^4 + C_4$ we finally derived (14) with $C_2 = C_3 + C_4$, where both $C_3$ and $C_4$ depend upon $L$. □

Now we are able to prove the existence of the first absorbing set.

**Theorem 8 (Absorbing balls in $\dot{H}^1_{per}(\Omega)$)** The semigroup $S(t) : \dot{H}^2_{per}(\Omega) \to \dot{H}^2_{per}(\Omega)$, $u_0 \mapsto S(t)u_0 = u(t)$ generated by equation (4) with periodic boundary conditions (i.e. the unique weak solutions) has an $H^1$ absorbing ball $\mathcal{B} = \{u \in \dot{H}^1_{per}(\Omega) : \|u\|_{\dot{H}^1_{per}} \leq \rho\}$, i.e. for a set $B \subset \dot{H}^2_{per}(\Omega)$ bounded in the $\dot{H}^1_{per}$ topology there is $t_B \geq$ such that $S(t)u_0 = u(t) \in \mathcal{B}$ for $u_0 \in B$ and $t \geq t_B$.

**Proof** We define the 'energy'

$$\mathcal{E}_1(t) := \int_\Omega W(u)dx + \frac{1}{2}\|u_x\|^2 + 2C_1\|(-\Delta)^{-1}u\|^2, \tag{15}$$

Then, by adding $4C_1$ times estimate (14) to (10), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_1(t) + \epsilon\mathcal{E}_1(t) - \epsilon\left(\int_\Omega W(u)dx + \frac{1}{2}\|u_x\|^2 + 2C_1\|(-\Delta)^{-1}u\|^2\right) \\ + 2C_1\|u\|_{L^4}^4 + 4C_1\|u_x\|^2 \leq C_1\|u\|_{L^4}^4 + 4C_1C_2\,. \end{aligned}$$

Here we added and subtracted a small fraction of $\mathcal{E}_1$ ($\epsilon > 0$). A rearrangement yields

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_1(t) + \epsilon\mathcal{E}_1(t) + C_1\|u\|_{L^4}^4 + (4C_1 - \epsilon/2)\|u_x\|^2 \\ \leq 4C_1C_2 + \epsilon\left(\int_\Omega W(u)dx + 2C_1\|(-\Delta)^{-1}u\|^2\right).\end{aligned}$$

The $H^1$ term does not make any trouble, as $\epsilon$ can be chosen arbitrarily small. Furthermore, as

$$\int_\Omega W(u)dx = \int_\Omega \left(\frac{1}{4}u^4 - \frac{1}{2}u^2 + \frac{1}{4}\right)dx \leq \frac{1}{4}\int_\Omega u^4 dx + \frac{L}{4},$$

and

$$\|(-\Delta)^{-1}u\|^2 \leq C_5\left(\int_\Omega u^4 dx + 1\right)$$

we can estimate the right hand side and put the terms back on the left hand side to balance them with the $L^4$ term. In this way we derive

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_1(t) + \epsilon\mathcal{E}_1(t) + (C_1 - \epsilon(1/4 + 2C_1C_5))\|u\|_{L^4}^4 + (4C_1 - \epsilon/2)\|u_x\|^2 \\ \leq 4C_1C_2 + \epsilon\left(\frac{L}{4} + 2C_1C_5\right) = C_6 \,. \qquad (16)\end{aligned}$$

Choosing $\epsilon$ sufficiently small yields

$$\frac{d}{dt}\mathcal{E}_1(t) + \epsilon\mathcal{E}_1(t) \leq C_6$$

and Gronwall leads to

$$\mathcal{E}_1(t) \leq \left(\mathcal{E}_1(0) - \frac{C_6}{\epsilon}\right)e^{-\epsilon t} + \frac{C_6}{\epsilon}. \qquad (17)$$

□

**Remark** We now know that $\|u\|^2 \leq C, \|u_x\|^2 \leq C$ and $\|u\|_{L^4}^4 \leq C$, for a constant $C$ independent of the initial condition that is undershot after a transient time. As we are in one dimension the result leads to a uniform $L^\infty$ bound on $u$. Furthermore, it was neither necessary to impose any restrictions to the deposition related parameter $\delta$ nor to the domain length $L$ to achieve the result.

By the same method we can establish the existence of an absorbing set in the $H^2$ topology, but the argument is more involved. Possibly, we may show its compactness. However, this is of no use in the two dimensional case. This is why we will use a more general tool capable of handling both dimensional cases simultaneously. However, the starting point is the specific estimate like (17).

## 3.2 Strong continuity of $S(\cdot)$

Equation (4) generates in one dimension a strongly continuous semigroup. We show this by applying the Galerkin method to equation (4). This gives a strongly continuous semigroup on $H^2$.

**Proposition 9** Let us suppose that $u_0 \in \dot{H}^2_{per}$, then $S(t)u_0 \equiv u(t)$ converges to $u_0$ in the $\dot{H}^2_{per}$ topology, as $t \to 0^+$, where $S(t)$ is the semigroup operator to (4).

**Proof** We will use the observation that if we have $u \in L^2(0,T;\dot{H}^3_{per})$ and $u_t \in L^2(0,T;\dot{H}^{-3}_{per})$, then $u \in C^0([0,T],\dot{L}^2)$. By the same token, $u_{xx} \in L^2(0,T;\dot{H}^3_{per})$ and $u_{txx} \in L^2(0,T;\dot{H}^{-3}_{per})$ will imply that

$$u_{xx} \in C^0([0,T],\dot{L}^2).$$

The fact $u \in L^2(0,T;\dot{H}^5_{per})$ or equivalently $u_t \in L^2(0,T;\dot{H}^{-1}_{per})$ is the content of Lemma 11. $\square$

We need the Gagliardo-Nirenberg inequality that holds on bounded domains $\Omega \subset \mathbb{R}^n$, for $n \leq 3$. It states

$$\|D^j u\|_{L^p} \leq c_1 \|D^m u\|^a_{L^r} \|u\|^{1-a}_{L^q} + c_2 \|u\|_{L^q}, \tag{18}$$

where

$$j/m \leq a \leq 1 \quad \text{and} \quad 1/p = j/n + a(1/r - m/n) + (1-a)/q\,,$$

and where $c_1$ and $c_2$ are positive constants. For $p = \infty$ the fraction $1/p$ is interpreted as 0.

**Lemma 10** Let us suppose that $u_0 \in B \subset \dot{H}^2_{per}$, where $B$ is a bounded subset of $\dot{H}^1_{per}$. Then, weak solutions to equation (4) with $u_0 \in B$ for $t \geq t_B$ fulfill

$$\|(u^3)_{xxx}\|^2 \leq C(\|u_{xxxx}\|^2 + \|u_{xxx}\|^2 + 1)\,. \tag{19}$$

**Proof** We note that

$$\int_\Omega (u^3)^2_{xxx} dx \leq C \left( \int_\Omega u_x^6 + u^2 u_x^2 u_{xx}^2 + u^4 u_{xxx}^2 dx \right) \tag{20}$$

and we estimate each of the three terms separately. Using Gagliardo-Nirenberg's inequality with $n = 1, j = 1, p = 6, m = 4, a = 1/3, r = 2$ and $q = 2$ we deduce

$$\int_\Omega u_x^6 dx \leq C\|u_{xxxx}\|^2 \|u\|^4 + C\|u\|^6 \leq C(\|u_{xxxx}\|^2 + 1)\,. \tag{21}$$

Now, the inequality $a^2 b^2 \leq a^6/3 + 2b^3/3$ for positive $a$ and $b$ implies

$$\int_\Omega u^2 u_x^2 u_{xx}^2 dx \leq C \int_\Omega \frac{1}{3} u_x^6 + \frac{2}{3}|u_{xx}|^3 dx \tag{22}$$

The first term can be estimated as before, for the latter we again apply Gagliardo-Nirenberg's inequality. We set $j = 2, p = q = 3, m = 4, r = 2, n = 1, a = 12/23$ so that

$$\int_\Omega |u_{xx}|^3 dx \leq C\|u_{xxxx}\|^{\frac{36}{23}} \|u\|^{\frac{33}{23}}_{L^3} + \|u\|^3_{L^3}\,.$$

Finally, we use $ab \leq a^p/p + b^q/q$ for conjugate numbers $p = 23/18$ and $q = 23/5$. This yields the overall estimate

$$\int_\Omega |u_{xx}|^3 dx \leq C \left( \frac{18}{23} \|u_{xxxx}\|^2 + \frac{5}{23} \|u\|^{33/5}_{L^3} + \|u\|^3_{L^3} \right) \leq C(\|u_{xxxx}\|^2 + 1)\,.$$

The last term in (20) is just bounded by $C\|u_{xxx}\|^2$, so that we derived (19). $\square$

**Lemma 11** Let us suppose that $u$ is a weak solution to (4) with initial condition $u_0$ in $\dot{H}^2_{per}$. Then, $u \in L^2(0,T;\dot{H}^5_{per})$.

**Proof** It is sufficient to show that $u_t \in L^2(0,T;H^{-1})$ as one can also see from (4) that

$$u_t \in L^2(0,T;\dot{H}^{-1}_{per}) \Leftrightarrow u \in L^2(0,T;\dot{H}^5_{per}).$$

We act at the level of Galerkin approximation $u^N$. We apply the integral operator $(-\Delta)^{-1} : \dot{L}^2 \to \dot{H}^2_{per}$ to both sides of equation (4) to derive

$$(-\Delta)^{-1}u^N_t - \delta(-\Delta)^{-1}[g(u^N)_x](u^N_{xx} + u^N - (u^N)^3)_{xx} = 0\,, \tag{23}$$

We test this equation by $u^N_t$ and estimate

$$\begin{aligned}
&\|(-\Delta)^{-1/2}u^N_t\|^2 + \frac{1}{2}\frac{d}{dt}\|u^N_{xx}\|^2 \qquad (24)\\
&\quad\le \frac{\delta}{2}((-\Delta)^{-1/2}((u^N)^2)_x, (-\Delta)^{-1/2}u^N_t) + (f(u^N)_{xx}, u^N_t)\\
&\quad\le C\|(u^N)^2\|\|(-\Delta)^{-1/2}u^N_t\| + ((-\Delta)^{1/2}f(u^N)_{xx}, (-\Delta)^{-1/2}u^N_t)\\
&\quad\le C\|u^N\|^4_{L^4} + \frac{1}{4}\|(-\Delta)^{-1/2}u^N_t\|^2 + C(\|((u^N)^3)_{xxx}\| + \|u^N_{xxx}\|)\|(-\Delta)^{-1/2}u^N_t\|\\
&\quad\le C + \frac{1}{2}\|(-\Delta)^{-1/2}u^N_t\|^2 + C(\|((u^N)^3)_{xxx}\|^2 + \|u^N_{xxx}\|^2). \qquad (25)
\end{aligned}$$

Collecting the $\|(-\Delta)^{-1/2}u^N_t\|^2$ term on the left hand side yields

$$\frac{1}{2}\|(-\Delta)^{-1/2}u^N_t\|^2 + \frac{1}{2}\frac{d}{dt}\|u^N_{xx}\|^2 \le C\left(1 + \|((u^N)^3)_{xxx}\|^2 + \|u^N_{xxx}\|^2\right).$$

After the application of estimate (19) and integrating with respect to $t$ we get,

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|(-\Delta)^{-1/2}u^N_t\|^2 dt + \frac{1}{2}\|u^N_{xx}\|^2(T) \qquad (26)\\
\le&\frac{1}{2}\|u^N_{xx}\|^2(0) + C\int_0^T \left(1 + \|u^N_{xxxx}\|^2 + \|u^N_{xxx}\|^2\right) dt.
\end{aligned}$$

Since the right hand side is bounded uniformly in $N$ due to the existence result established for the HCCH equation, we can pass to the limit and conclude that indeed that our claim holds.

□

# 4 The slope system in the two dimensional setting

For the purpose of analysis of the two-dimensional spatial domain, we rewrite equation (1) as a system of slope equations. The surface with height $h$ over the reference plane depends on the domain size, it grows due to coarsening that leads to an increase of the average size of the evolving structures. The slopes have a more dissipative character as the anisotropy of the surface energy forces the slopes to stay at a certain level that is independent of the domain size.

We write $u_1 = h_x, u_2 = h_y$ and note that the function in (3) is now used with $u_1$ and $u_2$ as arguments. In the evolution equation we need to calculate the gradient of $W$ with respect to its arguments $u_1$ and $u_2$, we denote it here by $D_u$,

$$D_u W = \begin{pmatrix} \frac{\alpha}{3}u_1^3 + \beta u_1 u_2^2 - u_1 \\ \frac{\alpha}{3}u_2^3 + \beta u_2 u_1^2 - u_2 \end{pmatrix}$$

and further note that $\operatorname{div} D_u W$ yields the second order linear and nonlinear terms. The fourth order term in the same potential stems from a corner regularization in the extended surface energy $\tilde{W} = W + (\Delta h)^2/2$.

Now we transform equation (1) to a slope equation, using the same notation as introduced above. For this purpose we apply a gradient to (1). If we set $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} h_x \\ h_y \end{pmatrix} = \nabla h$, then we will arrive at

$$u_t = \frac{\delta}{2}\nabla|u|^2 + \Delta^3 u - \nabla\Delta \operatorname{div} D_u W(u_1, u_2). \tag{27}$$

It is obvious from (27) that any solution to this equation is a gradient.

The advantage of (27) is that it has the same structure as in the one-dimensional setting. Due to this fact and the prescription of periodic boundary conditions we have

$$\int_\Omega u\, dx = 0 \tag{28}$$

and hence again work with Sobolev spaces with zero mean. We establish the first absorption property:

**Lemma 12** There is a constant $C > 0$ such that for initial conditions in a bounded $(\dot{H}^2_{per})^2$ set, $u_0 \in B \subset (\dot{H}^2_{per})^2$ after sufficient time of evolution $\tilde{t}_B$ the following uniform bounds are given

$$\|\nabla u\|^2 + \|W(u_1, u_2)\|^2 \leq C\,.$$

**Proof** We consider now $(-\Delta)^{-2}$ to be the inverse operator of the bi-Laplacian $\Delta^2 : \dot{H}^4_{per} \subset \dot{L}^2 \to \dot{L}^2$ and apply it to our new transformed system,

$$\begin{aligned} (-\Delta)^{-2}u_t &= \tfrac{\delta}{2}(-\Delta)^{-2}\nabla|u|^2 + \Delta u - \Delta^{-1}\nabla \operatorname{div} D_u W(u_1, u_2) \\ &= \tfrac{\delta}{2}(-\Delta)^{-2}\nabla|u|^2 + \Delta u - \nabla\Delta^{-1} \operatorname{div} D_u W(u_1, u_2). \end{aligned} \tag{29}$$

The last equality is based on the observation that for any vector field $X \in \dot{H}^1_{per}(\Omega^2; \mathbb{R}^2)$ we have

$$\Delta^{-1}\nabla \operatorname{div} X = \nabla\Delta^{-1} \operatorname{div} X. \tag{30}$$

This becomes obvious, after application of the Fourier transform to both side to (30),

$$-|\xi|^{-1}\xi(\operatorname{div} X)^\wedge = \xi(-|\xi|^{-1})(\operatorname{div} X)^\wedge.$$

As before, we can test (29) by $u_t$. However, this time we integrate over a two-dimensional domain, and as we deal with a system, we add the two components together, where we write shortly $\|(-\Delta)^{-1}u_t\|^2 = \|(-\Delta)^{-1}(u_1)_t\|^2 + \|(-\Delta)^{-1}(u_2)_t\|^2$ and keep this notation for all norms with arguments that are two-dimensional vectors, i.e. $\|u\|_{L^p} = \|\sqrt{(u_1)^2 + (u_2)^2}\|_{L^p}$.

We arrive at

$$
\begin{aligned}
\|(-\Delta)^{-1}u_t\|^2 &= \frac{\delta}{2}((-\Delta)^{-1}\nabla|u|^2, (-\Delta)^{-1}u_t) - \int_{\Omega^2} \nabla u \cdot \nabla u_t dxdy \\
&\quad - \int_{\Omega^2} \nabla\Delta^{-1}\mathrm{div}\, D_u W(u_1,u_2) u_t dxdy. \qquad (31)
\end{aligned}
$$

A series of integration by parts based on $u_t = \nabla h_t$ yields,

$$
\begin{aligned}
&\int_{\Omega^2} \nabla\Delta^{-1}\mathrm{div}\, D_u W(u_1,u_2)u_t dxdy = \int_{\Omega^2} \nabla\Delta^{-1}\mathrm{div}\, D_u W(u_1,u_2)\nabla h_t dxdy \\
= &\; -\int_{\Omega^2} \Delta\Delta^{-1}\mathrm{div}\, D_u W(u_1,u_2) h_t dxdy = -\int_{\Omega^2} \mathrm{div}\, D_u W(u_1,u_2) h_t dxdy \\
= &\; \int_{\Omega^2} D_u W(u_1,u_2)\nabla h_t dxdy \int_{\Omega^2} D_u W(u_1,u_2) u_t dxdy = \frac{d}{dt}\int_{\Omega^2} W(u_1,u_2)\, dxdy.
\end{aligned}
$$

Using the identity in (31) we derive

$$
\begin{aligned}
\frac{d}{dt}[\int_{\Omega^2} \frac{1}{2}|\nabla u|^2 dxdy + \int_{\Omega^2} W(u_1,u_2)dxdy] + \frac{1}{2}\|(-\Delta)^{-1}u_t\|^2 &= \frac{\delta^2}{8}\|(-\Delta)^{-1}\nabla|u|^2\|^2 \\
&\le D_1\|u\|_{L^4}^4, \quad (32)
\end{aligned}
$$

corresponding to (10) in the one-dimensional setting.

Testing (29) by $u$ and adding the components yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{-1}u\|^2 \\
&= \frac{\delta}{2}((-\Delta)^{-1}\nabla|u|^2, (-\Delta)^{-1}u) - \int_{\Omega^2}|\nabla u|^2 dxdy - \int_{\Omega^2} \frac{\alpha}{3}(u_1^4 + u_2^4) + 2\beta u_1^2 u_2^2 - u_1^2 - u_2^2 dxdy \\
&\le \frac{\delta}{2}\|(-\Delta)^{-1}\nabla|u|^2\|\|(-\Delta)^{-1}u\| - \|\nabla u\|^2 - \frac{\alpha}{6}\|u\|_{L^4}^4 - 2\beta\|u_1u_2\|^2 + \|u\|^2.
\end{aligned}
$$

Similarly, as before use the identity

$$
\int_{\Omega^2} \nabla\Delta^{-1}\mathrm{div}\, D_u W(u_1,u_2)u\, dxdy = \int_{\Omega^2} D_u W(u_1,u_2)u\, dxdy,
$$

which can be derived by the same argument as above.

We estimate the two terms with the wrong sign as

$$
\frac{\delta}{2}\|(-\Delta)^{-1}\nabla|u|^2\|\|(-\Delta)^{-1}u\| + \|u\|^2 \le C\|u^2\|\|u\| + \|u\|^2 \le C + \frac{\alpha}{18}\|u\|^4\,,
$$

where $C$ depends on the domain parameter $L$ as we applied Young's inequality to $u^2 \cdot 1$. Overall we have derived

$$
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{-1}u\|^2 + \|\nabla u\|^2 + \frac{2\alpha}{9}\|u\|_{L^4}^4 + 2\beta\|u_1u_2\|^2 \le D_2 \qquad (33)
$$

Multiplication of (32) by $\alpha/(9D_1)$ and adding to the above estimate yields

$$
\begin{aligned}
&\frac{d}{dt}[\frac{1}{2}\|(-\Delta)^{-1}u\|^2 + \frac{\alpha}{9D_1}(\|\nabla u\|^2 + \|W(u_1,u_2)\|^2)] \\
&+ \frac{\alpha}{18D_1}(\|(-\Delta)^{-1}u_t\|^2) + \|\nabla u\|^2 + \frac{\alpha}{9}\|u\|_{L^4}^4 + 2\beta\|u_1u_2\|^2 \le D_2
\end{aligned}
$$

We define the energy

$$\mathcal{E}_{1,2D}(t) = \frac{1}{2}\|(-\Delta)^{-1}u\|^2 + \frac{\alpha}{9D_1}(\|\nabla u\|^2 + \|W(u_1,u_2)\|^2) \tag{34}$$

and proceed in exact analogy as in the proof of Theorem 8. Hence, once again Gronwall yields the existence of absorbing sets in $H^1$. □

# 5 The global attractor

## 5.1 The integral representation of solutions

The energy estimates become more tedious in two dimensions. Therefore, we choose a different approach to prove the higher oder absorption.

Let us consider the 2-d system, (27), for $u = (u_1, u_2)$. We note that we have the formula

$$u(t) = e^{\Delta^3(t-t_0)}u_0 + \int_{t_0}^t e^{\Delta^3(t-s)}(\delta\nabla|u|^2 + \nabla\Delta\mathrm{div}\, D_uW(u))\,ds. \tag{35}$$

that has been used for the surface equation for $h$ in [7]. Here the exponential operator is defined by

$$e^{\Delta^3 t}f = \left(e^{-|\cdot|^6 t}\hat{f}(\cdot)\right)^{\vee},$$

where the right hand side is the inverse Fourier transform, while $\hat{f}$ denotes the Fourier coefficients. For more details we refer to the cited work.

It turns out that we can derive the same formula for solutions of the one dimensional problem (4). Indeed, since we have a unique weak solution, we may apply the Fourier transform to both sides of (4). The knowledge of the parameter variation formula for ODE's yields,

$$u(t) = e^{\Delta^3(t-t_0)}u_0 + \int_{t_0}^t e^{\Delta^3(t-s)}(\delta(|u|^2)_x + (D_uW(u))_{xxxx})\,ds. \tag{36}$$

Let us come back to the two-dimensional case. The inspection of the proof of [7, eq. (10)] reveals that in the case considered here we obtain a better estimate, due to the fact that $u$ has zero mean. Namely, after setting

$$v(s) = (\delta\nabla|u|^2 + \nabla\Delta\mathrm{div}\, D_uW(u))(s)$$

we can prove

**Lemma 13** If $\epsilon > 0$ and $\sup_{s\in[t_0,t]}\|v(\cdot,s)\|_{H^{p-6(1-\epsilon)}} < \infty$, then

$$J_p := \|\int_{t_0}^t e^{\Delta^3(t-s)}v(s)\,ds\|_{H^p} \le C(\epsilon,\lambda_0)\sup_{s\in[t_0,t]}\|v(\cdot,s)\|_{H^{p-6(1-\epsilon)}}, \tag{37}$$

where $\lambda_0 = L^6/2$.

We will present a sketch of argument. We work with the Fourier variables (see also [7] for the details). Because of (28) there is no zeroth mode in the Fourier variables, hence

$$|\xi|^6 - \frac{L^6}{2} \geq \lambda_0 > 0. \tag{38}$$

Thus, there is a positive constant $C_p > 0$ such that for all $0 \neq \xi \in (L\mathbb{Z})^d$, $d = 1, 2$, we have

$$(1 + |\xi|^2)^3 \leq C_p(|\xi|^6 - \frac{L^6}{2}). \tag{39}$$

This and the identity $e^{-|\xi|^6(t-s)} = e^{-\lambda_0(t-s)}e^{-(|\xi|^6-\lambda_0(t-s)}$ imply that

$$\begin{aligned} & e^{-|\xi|^6(t-s)}(t-s)^{1-\epsilon}(1+|\xi|^2)^{3(1-\epsilon)} \\ & \quad \leq Ce^{\lambda_0(t-s)}e^{-(t-s)(|\xi|^6-\lambda_0)}(t-s)^{1-\epsilon}(1+|\xi|^2)^{3(1-\epsilon)} \\ & \quad \leq Ce^{\lambda_0(t-s)}e^{-(t-s)(|\xi|^6-\lambda_0)}(t-s)^{1-\epsilon}(|\xi|^6-\lambda_0)^{1-\epsilon} \\ & \quad \leq \tilde{C}(\epsilon)e^{-\lambda_0(t-s)}. \end{aligned} \tag{40}$$

Here the last inequality follows from fast exponential decay $e^{-y}y^{1-\epsilon} \leq \tilde{C}$ that is true for any positive $y$. We used $y = (t-s)(|\xi|^6 - \lambda_0)$. Thus,

$$\begin{aligned} J_p & \leq \tilde{C}(\epsilon)\int_{t_0}^{t} \frac{e^{-\frac{1}{2}(t-s)}}{(t-s)^{1-\epsilon}}\|v(s)\|_{H^{p-6(1-\epsilon)}}\, ds \\ & = \tilde{C}(\epsilon)\left(\int_{t_0}^{t-1} + \int_{t-1}^{t}\right)\frac{e^{-\lambda_0(t-s)}}{(t-s)^{1-\epsilon}} \sup_{s\in[t_0,t]} \|v(s)\|_{H^{p-6(1-\epsilon)}} \\ & \quad + C(\epsilon, \lambda_0) \sup_{s\in[t_0+1,t]} \|v(s)\|_{H^{p-6(1-\epsilon)}}. \end{aligned}$$

and hence (37). We notice that the dimensionality of the problem does not intervene here.

We may now establish new results based on (37).

**Lemma 14** There is a constant $C > 0$ such that for any $u_0 \in B$, where $B \subset \dot{H}^2_{per}$ is bounded, there exist a time instant $t_B > 0$ such that

$$\sup_{t\geq t_B} \|u\|_\infty \leq C.$$

*Proof.* It is sufficient to show the following bound for any $\alpha > 0$

$$\|u(t)\|_{H^{1+\alpha}} \leq K < \infty$$

for all $t \geq t_B$.

We note that indeed

$$\|e^{\Delta t}u_0\|_{L^2} \leq Ce^{-\lambda_0 t}\|u_0\|_{L^2}.$$

and

$$\|e^{\Delta^3 t}u_0\|_{H^p} = \|e^{-|\xi|^6}(1+|\xi|^2)^{s/2}\hat{u}_0\|_{L^2}.$$

Proceeding as in the proof of (37), we conclude that

$$\|e^{\Delta^3 t}u_0\|_{H^s} \le Ct^{-s/6}e^{-\lambda_0}\|u_0\|_{L^2}\,.$$

Hence it follows from (37) for $p = 1+\alpha$, where $\alpha > 0$,

$$\begin{aligned}\|u(t)\|_{H^{1+\alpha}} \le Ce^{-\lambda_0 t}\|u_0\|_{H^{1+\alpha}} &+ C\sup_t \|\nabla\Delta \mathrm{div}\, D_u W(u)\|_{H^{1+\alpha-6(1-\epsilon)}} \\ &+ C\sup_t \|\nabla|u|^2\|_{H^{1+\alpha-6(1-\epsilon)}}\,,\end{aligned}$$

where $D_u W(u)$ is the cubic nonlinearity.

We notice that

$$\begin{aligned}\|\nabla\Delta \mathrm{div}\, D_u W(u)\|_{H^{1+\alpha-6(1-\epsilon)}} &= \|D_u W(u)\|_{H^{-1+\alpha+6\epsilon}} \le C\|D_u W(u)\|_{L^2} \\ &\le C\|u\|^3_{L^6} \le C\|\nabla u\|^3 \le C.\end{aligned}$$

Furthermore, we can estimate the other nonlinear term by

$$\begin{aligned}\|\nabla|u|^2\|_{H^{1+\alpha-6(1-\epsilon)}} &= \||u|^2\|_{H^{-4+\alpha+6\epsilon}} \\ &\le \||u|^2\| \le \|u\|^2_{L^4} \le C\|W(u_1,u_2)\| \le C.\end{aligned}$$

The last estimate is a consequence of (34). The uniformity of the constants of the last two estimates also comes from the uniform absorption of bounded sets of the energy (34). □

## 5.2 Compactness of absorbing balls

Using Lemma 13 we can not only show the existence of absorbing sets in $H^2$ but also their compactness. Therefore we make the following key observation.

**Proposition 15** There exists some $\alpha > 0$ and a constant $C(\alpha) > 0$, such that for any initial condition $u_0 \in B$, where $B \subset (\dot{H}^2_{per})^d$, $d = 1, 2$, is a bounded set, for all times bigger than $t'_B \ge t_B$

$$\sup_{t\ge t'_B} \|u(t)\|_{H^{2+\alpha}(\Omega^2;\mathbb{R}^d)} \le C(\alpha).$$

*Proof.* We use again formula (37), this time we take $p = 2+\alpha$. We get

$$\begin{aligned}\|u(t)\|_{H^{2+\alpha}} \quad \le \quad & \|(-\Delta)^{\alpha/2}e^{\Delta^3 t}\Delta u_0\| \\ & + C(\epsilon)\sup_t \|\Delta^2 D_u W(u)\|_{H^{2+\alpha-6(1-\epsilon)}}\end{aligned}$$

We recall that $\|(-\Delta)^{\alpha}e^{\Delta^3(t-t_0)}u_0\| \le C(t-t_0)^{-\alpha/3}\|u_0\|$. Hence,

$$\|u(t)\|_{H^{2+\alpha}} \le C(t-t_0)^{-\alpha/6}\|u_0\|_{H^1} + C(\epsilon)\sup_t \|D_u W(u)\|_{H^{\alpha+6\epsilon}}.$$

We also observe that

$$\|D_u W(u)\|_{H^1} \le C\|\nabla u^3\|_{L^2} \le C\|u\|^2_\infty \|\nabla u\|_{L^2} \le K \quad \text{for } t \ge t_B.$$

Collecting these estimates we conclude that our claim holds for $t \ge t_B + 1$. □

We may complete the *proofs of Theorems 1 and 2* in one stroke. Proposition 15 yields compactness of an absorbing ball in $H^2$ topology. On the other hand we have already established the strong continuity of the semigroup $S(t)$. Thus, an application of Theorem 4 finishes the proof. $\square$

Now, we prove the final assertion. We transfer the above results to the problem expressed in terms of the shape $h$ in (1).

**Proof of Theorem 3:** Exactly as in the slope system it is sufficient to show existence of a compact absorbing set in the $H^3$ topology. First, we notice that it is easy to reconstruct a function $h : \Omega = (0, L)^d \to \mathbb{R}$, when it is given its derivative $u = \nabla h$ and the mean $m = \int_\Omega h$. Indeed, we have the following formulas,

$$h(x) = \frac{1}{L}(m - \int_0^L \int_0^x u(s)\, dsdx) + \int_0^x u(s)\, ds, \tag{41}$$

in case $d = 1$ and

$$h(x, y) = \frac{1}{L^2}(m - \int_0^L \int_0^x u(s)\, dsdx) + \int_0^x u_1(s, 0)\, ds + \int_0^y u_2(x, s)\, ds, \tag{42}$$

if $d = 2$.

These two formulas and Theorems 1 and 2 imply existence of compact absorbing sets, hence existence of a global attractor in $H^3$. $\square$

# 6 Conclusions and outlook

We have established the existence of global attractors in $\dot{H}^2_{per}$ for the slope equations (4) and (8). This enable us to show existence of global attractors in $H^3_{per}$ for (1) in the 1+1D and 1+2D settings. On the way, we showed that solutions to (4) and (8) enjoy further regularity. For the one-dimensional case we succeed in deriving proper uniform estimates by repeated application of Gronwall inequality. As we needed uniform constants for the estimates, the work may seem somewhat tedious at certain points, e.g. during the application of Gagliardo-Nirenberg's inequality. Because of its repeated application this approach is not feasible the two-dimensional setting. Instead we reconsidered the constant variation formula from our previous work [7] to improve the regularity result. It turns out, once this approach is understood, the semigroup ansatz seems more elegant for this problem.

We are content with the results obtained for the presented equations. They coincide with the observations made with the help of a pseudospectral numerical method in the previous work [8], though we were not yet able to show or negate the existence of stationary or traveling wave solutions, which have been discussed in this publication. As we are not able to find a Lyapunov function, we were not in the position to use approaches based on the Łojasiewicz-Simon inequality (e.g. [10, 15]).

We do not know much about the $\omega$-limit set, but as Figure 1 has already indicated, we expect to have time-periodic or stationary solutions for smaller values of $\delta$ and a strange attractor for increased values of the deposition rate dependent parameter. Note that once the structures form, the solutions in this figure stay in an $\dot{H}^2_{per}$ ball as predicted. The numerical simulations suggest that at least for small initial data the $L^\infty$ norm of $u$ stays roughly below 1, independently of the value for $\delta$.

We plan to extend the analysis to heteroepitaxial quantum dot growth systems, where a nonlocal elastic term and a wetting nonlinearity are added to the partial differential equation, which makes the analytical treatment more challenging. In particular we will work on the PDEs discussed by Korzec et al. [5, 6].


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