IMPEDANCE BOUNDARY CONDITIONS FOR ACOUSTIC TIME HARMONIC WAVE PROPAGATION IN VISCOUS GASES

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Abstract. We present Helmholtz or Helmholtz like equations for the approximation of the time-harmonic wave propagation in gases with small viscosity, which are completed with local boundary conditions on rigid walls. We derived approximative models based on the method of multiple scales for the pressure and the velocity separately, both up to order 2. Approximations to the pressure are described by the Helmholtz equations with impedance boundary conditions, which relate its normal derivative to the pressure itself. The boundary conditions from first order on are of Wentzell type and include a second tangential derivative of the pressure proportional to the square root of the viscosity, and take thereby absorption inside the viscosity boundary layer of the underlying velocity into account.

The velocity approximations are described by Helmholtz like equations for the velocity, where the Laplace operator is replaced by $\nabla\,\mathrm{div}$, and the local boundary conditions relate the normal velocity component to its divergence. The velocity approximations are for the so-called far field and do not exhibit a boundary layer. Including a boundary corrector, the so called near field, the velocity approximation is accurate even up to the domain boundary.

The boundary conditions are stable and asymptotically exact, which is justified by a complete mathematical analysis. The results of some numerical experiments are presented to illustrate the theoretical foundation.

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1. Introduction

In this study we are investigating the acoustic equations as a perturbation of the Navier-Stokes equations around a stagnant uniform fluid, with mean density ρ_0 and without heat flux. For gases the (dynamic) viscosity η is very small and leads to *viscosity boundary layers* close to walls. To resolve the boundary layers with (quasi-)uniform meshes, the mesh size has to be of the same order, which leads to very large linear systems to be solved. This is especially the case for the very small boundary layers of acoustic waves. In its turn, the pressure field does not possess a boundary layer, however, this fact cannot be used without some preliminary adjustments as there are no existing physical boundary conditions for pressure.

In an earlier work [11] we derived a complete asymptotic expansion for the problem based on the technique of multiscale expansion in powers of $\sqrt{\eta}$ which takes into account curvature effects. This asymptotic expansion was rigorously justified with optimal error estimates. In this article we propose and justify, based on the asymptotic expansion in [11], (effective) *impedance boundary conditions* for the velocity as well as the pressure for possibly curved boundaries. Similar techniques to derive approximative models have been used for thin sheets [4,8,12] or for conducting bodies [5]. The advantage of using this approach lies in the fact that the solution can be divided into the far field with specified boundary condition, *i.e.*, impedance boundary condition, and a correcting near field, which helps to avoid resolving the boundary layer.

The article is subdivided as follows. In Sec. 2 we define the model problem of the viscous acoustic equations for velocity and pressure and state the impedance boundary conditions for the velocity and for the pressure as well as the stability and modelling error estimates. Sec. 3 is dedicated to the derivation of the impedance boundary condition on the basis of the asymptotic expansion presented in [11]. The well-posedness as well as estimates of the modelling error of the approximative models with the impedance boundary conditions will be shown in Sec. 4. Results of some numerical experiments in Sec. 5 shall emphasize the validity of the theoretical findings.

2. Model problem definition and main results

2.1. Geometry and model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ (Fig. 1(a)). The boundary shall be described by a mapping $\mathbf{x}_{\partial\Omega}(t)$ from an interval $\Gamma \subset \mathbb{R}$. We assume the boundary to be C^∞ such that points close to $\partial\Omega$ can be uniquely written as

$$\mathbf{x}(t,s) = \mathbf{x}_{\partial\Omega}(t) - s\mathbf{n}(t) \tag{2.1}$$

where $\mathbf{n}(t)$ is the outer normalised normal vector and s the distance from the boundary (see Fig. 1(a)). Without loss of generality we can assume $|\mathbf{x}'_{\partial\Omega}(t)| = 1$ for all $t \in \Gamma$. The orthogonal unit vectors in these tangential and normal coordinate directions are $\mathbf{e}_t(t) = -\mathbf{n}^\perp(t)$, where we use the notation $\mathbf{u}^\perp = (u_2, -u_1)^\top$ for a turned vector clockwise by 90°, and $\mathbf{e}_s(t) = -\mathbf{n}(t)$. Furthermore, let $s_0 \in \mathbb{R}$ such that all points with distance smaller than s_0 to the $\partial\Omega$ have a unique closest point on the boundary.

We consider the time-harmonic acoustic velocity $\mathbf{v}$ and acoustic pressure p (the time regime is $\mathrm{e}^{-\mathrm{i}\omega t}$, $\omega \in \mathbb{R}^+$) which are described by the coupled system

$$-\mathrm{i}\omega\rho_0\mathbf{v} + \nabla p - \eta\Delta\mathbf{v} - \eta'\,\nabla\operatorname{div}\mathbf{v} = \mathbf{f}, \qquad \text{in } \Omega, \tag{2.2a}$$

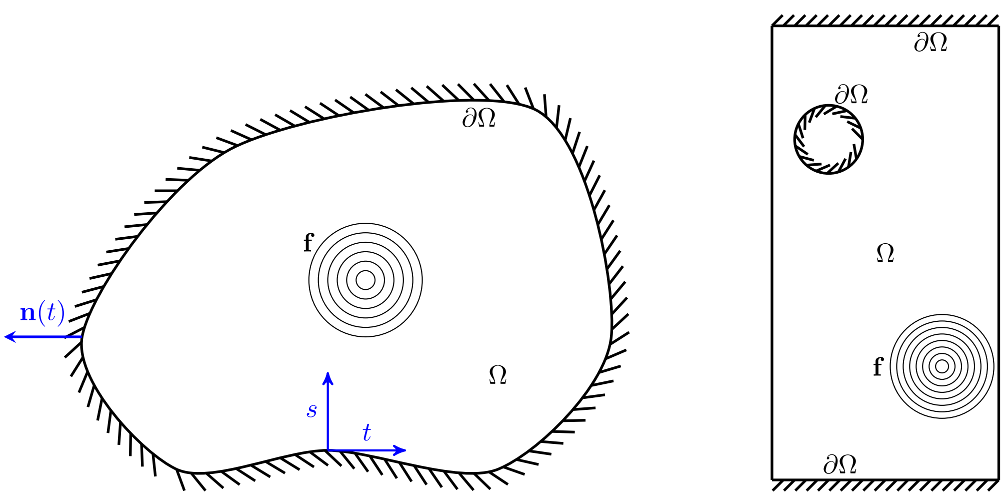
$$-\mathrm{i}\omega p + \rho_0 c^2\,\operatorname{div}\mathbf{v} = 0, \qquad \text{in } \Omega, \tag{2.2b}$$

$$\mathbf{v} = \mathbf{0}, \qquad \text{on } \partial\Omega. \tag{2.2c}$$

In the *momentum equation* (2.2a) with some known source term $\mathbf{f}$ the viscous dissipation in the momentum is not neglected as we consider near wall regions. Here, ρ_0 is the density of the media, c is the speed of sound, $\eta > 0$ is the dynamic viscosity and η' the second (volume) viscosity. Both shall take small values and we call $\gamma' = \eta'/\eta$ their quotient. The system is completed by *no-slip* boundary conditions.

It is well-known that the acoustic velocity field exhibits a boundary layer of thickness $O(\sqrt{\eta})$ starting at the rigid wall, see *e. g.* [11]. In the following we propose definitions of far field velocities, which approximate the acoustic velocity outside the boundary layer, correcting near field velocities and approximative pressure distributions. We propose two alternative definitions,

- a definition of the far field velocity by a PDE including an impedance boundary condition and a-posteriori computation of the pressure (see Sec. 2.2), and



(a) (b)

FIGURE 1. (a) Definition of a general domain with a local coordinate system (t, s) close to the wall; (b) Definition of a torus domain for numerical simulations.

- a definition of the pressure by a PDE including an impedance boundary condition and *a-posteriori* computation of the far field velocity (see Sec. 2.3).

2.2. Impedance boundary conditions for the velocity

In this section we propose approximative models for the far field velocity of order 0, 1 and 2, which especially incorporate impedance boundary conditions. Approximations on the pressure can be computed from the far field velocities. The far field velocity can be used as approximation away from the boundary and has to be corrected by the near field velocity.

Definitions

The approximative model of order 0 is given by

$$\nabla \operatorname{div} \mathbf{v}_{\mathrm{appr},0} + \frac{\omega^2}{c^2}\mathbf{v}_{\mathrm{appr},0} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}, \qquad \text{in } \Omega, \tag{2.3a}$$

$$\mathbf{v}_{\mathrm{appr},0} \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega, \tag{2.3b}$$

that of order 1 by

$$\nabla \operatorname{div} \mathbf{v}_{\mathrm{appr},1} + \frac{\omega^2}{c^2}\mathbf{v}_{\mathrm{appr},1} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}, \qquad \text{in } \Omega, \tag{2.4a}$$

$$\mathbf{v}_{\mathrm{appr},1} \cdot \mathbf{n} - (1+\mathrm{i})\frac{c^2}{\omega^2}\sqrt{\frac{\eta}{2\omega\rho_0}}\,\partial_t^2 \operatorname{div} \mathbf{v}_{\mathrm{appr},1} = \frac{(\mathrm{i}-1)}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\,\partial_t(\mathbf{f}\cdot\mathbf{n}^\perp), \qquad \text{on } \partial\Omega, \tag{2.4b}$$

and

$$\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\right)\nabla \operatorname{div} \mathbf{v}_{\mathrm{appr},2} + \frac{\omega^2}{c^2}\mathbf{v}_{\mathrm{appr},2} = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f} + \frac{\eta}{\rho_0^2 c^2}\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{f}, \qquad \text{in } \Omega, \tag{2.5a}$$

$$\begin{aligned}\mathbf{v}_{\mathrm{appr},2} \cdot \mathbf{n} - \frac{c^2}{\omega^2}\Big((1+\mathrm{i})\sqrt{\frac{\eta}{2\omega\rho_0}}\,\partial_t^2 \operatorname{div} \mathbf{v}_{\mathrm{appr},2} &+ \frac{\mathrm{i}\eta}{2\omega\rho_0}\partial_t(\kappa\partial_t \operatorname{div} \mathbf{v}_{\mathrm{appr},2})\Big) \\ &= \frac{(\mathrm{i}-1)}{\omega\rho_0}\sqrt{\frac{\eta}{2\omega\rho_0}}\,\partial_t(\mathbf{f}\cdot\mathbf{n}^\perp) - \frac{\eta}{2\omega^2\rho_0^2}\partial_t(\kappa\,\mathbf{f}\cdot\mathbf{n}^\perp), \qquad \text{on } \partial\Omega,\end{aligned} \tag{2.5b}$$

defines the approximative model of order 2. Compare also the impedance boundary conditions of 1^{st} order in [1]. The impedance boundary conditions (2.4b) and (2.5b) have similarities with Wentzell's boundary conditions, where, however, the second tangential derivative applies to the Neumann trace $\operatorname{div}\mathbf{v}_{\text{appr},N}$, and not to the Dirichlet trace, which is here $\mathbf{v}_{\text{appr},N}\cdot\mathbf{n}$. For their functional framework and a mixed variational formulation see the proof of stability of an auxilliary system in Lemma 4.1 in Sec. 4.2.

When the far field velocity is computed, we may obtain *a-posteriori* the far field pressure in Ω of order 0, 1 and 2 by

$$p_{\text{appr},N} = -\frac{\mathrm{i}\rho_0 c^2}{\omega}\,\operatorname{div}\mathbf{v}_{\text{appr},N}, \tag{2.6}$$

Close to the wall the far field velocities has to be corrected by a function

$$\mathbf{v}^{BL}_{\text{appr},N} = \sqrt{\frac{2\,\eta}{\omega\rho_0}}\,\mathbf{curl}_{2D}(\phi_{\text{appr},N}\chi), \tag{2.7}$$

where χ is an admissible cut-off function (see Definition 4.6), $\phi_{\text{appr},N}(\mathbf{x}) = \widetilde{\phi}_{\text{appr},N}(t, s\sqrt{\frac{\omega\rho_0}{2\eta}})$ in the so-called χ-neighbourhood of the boundary and

$$\widetilde{\phi}_{\text{appr},N}(t,S) := \frac{1}{2}(1+\mathrm{i})\,\mathrm{e}^{-(1-\mathrm{i})S}\sum_{\ell=0}^{N}\left(\frac{2\,\eta}{\omega\rho_0}\right)^{\frac{\ell}{2}} E_\ell(\mathbf{v}_{\text{appr},N}\cdot\mathbf{n}^\perp)(t,S). \tag{2.8}$$

Here, the operators E_ℓ, which were recursively defined in [11, Lemma A.1] (the parameter $\eta_0 = \omega\rho_0/2$ has to be used in their definition), will be given for $\ell = 0,1,2$ in (3.7).

Analysis

Lemma 2.1 (Stability, existence and uniqueness of $\mathbf{v}_{\text{appr},N}$, $p_{\text{appr},N}$)**.** *Let $\frac{\omega^2}{c^2}$ be distinct from the Neumann eigenvalue of $-\Delta$ and let the source function $\mathbf{f}$ be smooth enough. Then, the system* (2.3)–(2.6) *provides a unique solution* $(\mathbf{v}_{\text{appr},N}, p_{\text{appr},N}) \in (H^1(\Omega))^2\times L^2(\Omega)$ *for $N = 0,1,2$ respectively.*

The proof will be given later in Sec. 4.4, which is based on a stability result for an auxilliary system given in Sec. 4.2 and the asymptotic equivalence of the approximative models and the asymptotic expansion in Sec. 4.3.

Remark 2.2 (Asymptotic equivalence)**.** *We use the notion* asymptotic equivalence *of two functions $u^1_{appr}(\eta)$ and $u^2_{appr}(\eta)$, approximating a function $u(\eta)$ (solution of the original system), if the difference in norm of the two functions is asymptotically at the same order than the difference of one of the two to the solution $u(\eta)$.*

Remark 2.3. *For $N = 2$ the system* (2.5)–(2.6) *provides a unique solution* $(\mathbf{v}_{\text{appr},2}, p_{\text{appr},2}) \in (H^1(\Omega))^2\times L^2(\Omega)$ *for any $\omega > 0$.*

Theorem 2.4 (Modelling error)**.** *Let the assumptions of Lemma 2.1 be satisfied. Then the approximative solution* $(\mathbf{v}_{\text{appr},N}, p_{\text{appr},N}) \in (H^1(\Omega))^2\times L^2(\Omega)$*, of* (2.3)–(2.6) *for $N = 0,1,2$ satisfies*

$$\|\mathbf{v}-\mathbf{v}_{\text{appr},N}-\mathbf{v}^{BL}_{\text{appr},N}\|_{H(\operatorname{div},\Omega)} + \eta^{\frac14}\|\operatorname{curl}_{2D}(\mathbf{v}-\mathbf{v}_{\text{appr},N}-\mathbf{v}^{BL}_{\text{appr},N})\|_{L^2(\Omega)} + \|p-p_{\text{appr},N}\|_{H^1(\Omega)} \le C\,\eta^{\frac{N+1}{2}}, \tag{2.9a}$$

and for any $\delta > 0$

$$\|\mathbf{v}-\mathbf{v}_{\text{appr},N}\|_{(H^1(\Omega\setminus\overline{\Omega}_\delta))^2} \le C_{\delta,N}\,\eta^{\frac{N+1}{2}}, \tag{2.9b}$$

where Ω_δ is the original domain without a δ-neighbourhood of $\partial\Omega$ and where the constants C, $C_{\delta,N} > 0$ do not depend on η.

The proof will be given in Sec. 4.4.

2.3. Impedance boundary conditions for the pressure

In this section we present approximative models for the far field pressureof order 0, 1 and 2, which includes in particular impedance boundary conditions. This is different to the original equations in which no boundary conditions for the pressure, but for both velocity components, are imposed. In analogy to the previous section we propose formulas for velocity approximations. We call the pressure approximations $q_{\text{appr},N}$, $N = 0,1,2$, which we will define by differential equations in this section, which are not necessaryly identical to the pressures $p_{\text{appr},N}$, $N = 0, 1, 2$, which were resulting in a post-processing step from the velocity $\mathbf{v}_{\text{appr},N}$. Furthermore, we denote with $\mathbf{w}_{\text{appr},N}$ the velocity defined a-posteriori from the pressure $q_{\text{appr},N}$.

Definitions

The approximative model of order 0 is given by

$$\Delta q_{\text{appr},0} + \frac{\omega^2}{c^2} q_{\text{appr},0} = \operatorname{div} \mathbf{f}, \tag{2.10a}$$

$$\nabla q_{\text{appr},0} \cdot \mathbf{n} = \mathbf{f} \cdot \mathbf{n}. \tag{2.10b}$$

We define a model of order 1 by

$$\Delta q_{\text{appr},1} + \frac{\omega^2}{c^2} q_{\text{appr},1} = \operatorname{div} \mathbf{f}, \tag{2.11a}$$

$$\nabla q_{\text{appr},1} \cdot \mathbf{n} + (1+\mathrm{i}) \sqrt{\frac{\eta}{2\omega\rho_0}} \partial_t^2 q_{\text{appr},1} = \mathbf{f} \cdot \mathbf{n} - (1+\mathrm{i}) \sqrt{\frac{\eta}{2\omega\rho_0}} \partial_t (\mathbf{f} \cdot \mathbf{n}^\perp), \tag{2.11b}$$

and for order 2 we define

$$\Big(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\Big) \Delta q_{\text{appr},2} + \frac{\omega^2}{c^2} q_{\text{appr},2} = \operatorname{div} \mathbf{f}, \tag{2.12a}$$

$$\nabla q_{\text{appr},2} \cdot \mathbf{n} + (1+\mathrm{i}) \sqrt{\frac{\eta}{2\omega\rho_0}} \partial_t^2 q_{\text{appr},2} + \frac{\mathrm{i}\eta}{2\omega\rho_0} \partial_t (\kappa \partial_t q_{\text{appr},2}) = \tag{2.12b}$$

$$= \Big(1 + \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\Big) \mathbf{f} \cdot \mathbf{n} - (1+\mathrm{i}) \sqrt{\frac{\eta}{2\omega\rho_0}} \partial_t (\mathbf{f} \cdot \mathbf{n}^\perp) - \frac{\mathrm{i}\eta}{2\omega\rho_0} \partial_t (\kappa \mathbf{f} \cdot \mathbf{n}^\perp) - \frac{\mathrm{i}\eta}{\omega\rho_0} \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{f} \cdot \mathbf{n}.$$

When the far field pressure is computed we may obtain *a-posteriori* the far field velocity of order 0, 1 and 2 by

$$\mathbf{w}_{\text{appr},N} = \frac{\mathrm{i}}{\rho_0 \omega} (\mathbf{f} - \nabla q_{\text{appr},N}), \text{ for } N = 0, 1, \qquad \text{in } \Omega, \tag{2.13a}$$

$$\mathbf{w}_{\text{appr},2} = \frac{\mathrm{i}}{\rho_0 \omega} (\mathbf{f} - \nabla q_{\text{appr},2}) - \frac{\eta+\eta'}{\rho_0^2 c^2} \nabla q_{\text{appr},2} + \frac{\eta}{\rho_0^2 \omega^2} \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{f}, \qquad \text{in } \Omega, \tag{2.13b}$$

The near field velocity $\mathbf{w}^{BL}_{\text{appr},N}$ is then given by (2.7) and (2.8) if $\mathbf{v}_{\text{appr},N}$ is replaced by $\mathbf{w}_{\text{appr},N}$.

The impedance boundary conditions (2.11b) and (2.12b) are of Wentzell type. See [2, 10] for the functional framework and variational formulation.

Analysis

Lemma 2.5 (Asymptotic equivalence of the two approximate solutions)**.** *Let the assumptions of Lemma 2.1 be satisfied. Then, the approximate solutions* $(\mathbf{v}_{\text{appr},N}, p_{\text{appr},N})$ *and* $(\mathbf{w}_{\text{appr},N}, q_{\text{appr},N})$ *for* $N = 0, 1$ *are identical, just as the boundary layer correctors* $\mathbf{v}^{BL}_{\text{appr},N}$ *and* $\mathbf{w}^{BL}_{\text{appr},N}$*, and for* $N = 2$ *it holds with a constant* C *independent of* η *that*

$$\|\mathbf{v}_{\text{appr},2} - \mathbf{w}_{\text{appr},2}\|_{H(\operatorname{div},\Omega)} + \eta^{-\frac14} \|\mathbf{v}^{BL}_{\text{appr},2} - \mathbf{w}^{BL}_{\text{appr},2}\|_{H(\operatorname{div},\Omega)}$$
$$+ \eta^{\frac14} \|\operatorname{curl}_{2D}(\mathbf{v}^{BL}_{\text{appr},2} - \mathbf{w}^{BL}_{\text{appr},2})\|_{L^2(\Omega)} + \|p_{\text{appr},2} - q_{\text{appr},2}\|_{H^1(\Omega)} \le C\, \eta^{3/2}.$$

Furthermore, $\operatorname{curl}_{2D}(\mathbf{v}_{\text{appr},2} - \mathbf{w}_{\text{appr},2}) \equiv 0$.

The proof will be given in Sec. 4.5.

As the two approximations are identical for $N = 0, 1$ or have the same asymptotic behaviour for $N = 2$ up to $O(\eta^{3/2})$ we can conclude directly the following stability result and estimate on the modelling error of the approximation solution $(\mathbf{w}_{\text{appr},N}, q_{\text{appr},N})$.

Lemma 2.6 (Stability, existence and uniqueness of $\mathbf{w}_{\text{appr},N}$, $q_{\text{appr},N}$)**.** *Let the assumptions of Lemma 2.1 be satisfied. Then, the system* (2.10)–(2.13) *provides a unique solution* $(\mathbf{w}_{\text{appr},N}, q_{\text{appr},N}) \in (H^1(\Omega))^2 \times H^1(\Omega)$ *for* $N = 0, 1, 2$ *respectively.*

Theorem 2.7 (Modelling error)**.** *Let the assumptions of Lemma 2.1 be satisfied. Then, the approximative solution* $(\mathbf{w}_{\mathrm{appr},N}, q_{\mathrm{appr},N})$ *for* $N = 0, 1, 2$ *satisfies*

$$\|\mathbf{v} - \mathbf{w}_{\mathrm{appr},N} - \mathbf{w}^{BL}_{\mathrm{appr},N}\|_{H(\mathrm{div},\Omega)} + \eta^{\frac{1}{4}} \|\operatorname{curl}_{2D}(\mathbf{v} - \mathbf{w}_{\mathrm{appr},N} - \mathbf{v}^{BL}_{\mathrm{appr},N})\|_{L^2(\Omega)} + \|p - q_{\mathrm{appr},N}\|_{H^1(\Omega)} \le C\,\eta^{\frac{N+1}{2}}, \quad (2.14a)$$

and for any $\delta > 0$

$$\|\mathbf{v} - \mathbf{w}_{\mathrm{appr},N}\|_{(H^1(\Omega\setminus\overline{\Omega}_\delta))^2} \le C_{\delta,N}\,\eta^{\frac{N+1}{2}}, \quad (2.14b)$$

where Ω_δ *is the original domain without a* δ*-neighbourhood of* $\partial\Omega$ *and where the constants* C, $C_{\delta,N} > 0$ *do not depend on* η.

3. Derivation of impedance boundary conditions

3.1. Equations for asymptotically small viscosity

To investigate the solution of (2.2) for small viscosities, we introduce a small dimensionless parameter $\varepsilon \ll 1$, $\varepsilon \in \mathbb{R}^+$ and replace η, η' by $\varepsilon^2\omega\rho_0/2$, $\varepsilon^2\gamma'\omega\rho_0/2$ (corresponding to $\eta_0 = \omega\rho_0/2$, $\eta_0' = \gamma'\,\omega\rho_0/2$ in [11]), respectively. In this way the boundary layer thickness will become proportional to ε. The solution of (2.2), respectively, will be labelled $\mathbf{v}^\varepsilon$ and p^ε due to its dependence on ε , *i.e.*

$$-\mathrm{i}\omega\rho_0\mathbf{v}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2\frac{\omega\rho_0}{2}\Delta\mathbf{v}^\varepsilon - \varepsilon^2\frac{\gamma'\,\omega\rho_0}{2}\nabla\operatorname{div}\mathbf{v}^\varepsilon = \mathbf{f}, \qquad \text{in } \Omega, \quad (3.1a)$$

$$-\mathrm{i}\omega p^\varepsilon + \rho_0 c^2 \operatorname{div}\mathbf{v}^\varepsilon = 0, \qquad \text{in } \Omega, \quad (3.1b)$$

$$\mathbf{v}^\varepsilon = \mathbf{0}, \qquad \text{on } \partial\Omega. \quad (3.1c)$$

Earlier we have proved stability for such a problem for the non-resonant case, which we consider here as well, *i.e.*, for vanishing viscosity and so absorption, the kernel of the system is empty − there is no eigensolution. The eigenvalues of the limit problem coincide with the Neumann eigenvalue of $-\Delta$.

Lemma 3.1 (Stability for the non-resonant case)**.** *For any* $\mathbf{f} \in (H^0(\mathrm{div},\Omega)) \cap H(\mathrm{curl}_{2D},\Omega))'$ *the system* (3.1) *has a unique solution* $(\mathbf{v}^\varepsilon, p^\varepsilon) \in H_0(\mathrm{div},\Omega) \cap H(\mathrm{curl}_{2D},\Omega) \times L^2(\Omega)$. *If* $\frac{\omega^2}{c^2}$ *is not a Neumann eigenvalue of* $-\Delta$, *then there exists a constant* $C > 0$ *independent of* ε, *such that*

$$\|\mathbf{v}^\varepsilon\|_{H(\mathrm{div},\Omega)} + \varepsilon\,\|\operatorname{curl}_{2D}\mathbf{v}^\varepsilon\|_{L^2(\Omega)} + \|p^\varepsilon\|_{L^2(\Omega)} \le C\,\|\mathbf{f}\|_{(H_0(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega))'}, \quad (3.2a)$$

$$\|\nabla p^\varepsilon\|_{L^2(\Omega)} \le C\,\|\mathbf{f}\|_{L^2(\Omega)}. \quad (3.2b)$$

For any $\omega > 0$ *and for* $C^{1,1}$ *boundary* $\partial\Omega$ *it holds*

$$\varepsilon\,\|\mathbf{v}^\varepsilon\|_{(H^1(\Omega))^2} \le C\,\|\mathbf{f}\|_{(H_0(\mathrm{div},\Omega)\cap H(\mathrm{curl}_{2D},\Omega))'}. \quad (3.2c)$$

A proof can be found in [11, Lemma 2.2]. Even so in this work C^∞ was assumed, the proof of (3.2a), (3.2b) does not rely on a higher regularity assumption, see [6].

3.2. Asymptotic expansion

With the above introduced small parameter $\varepsilon = \sqrt{2\eta/(\omega\rho_0)}$ and using the curvilinear coordinates (t, s), which we have introduced in (2.1), close to the boundary the solution of (3.1) inspired by the framework of Vishik and Lyusternik [13] could be written as

$$\mathbf{v} = \sum_{j=0}^{\infty} \varepsilon^j \left(\mathbf{v}^j + \varepsilon\,\mathbf{curl}_{2D}(\phi^j\chi)\right); \qquad p = \sum_{j=0}^{\infty} \varepsilon^j p^j, \quad (3.3)$$

where $\mathbf{v}^j(x, y)$ and $p^j(x, y)$ are terms of the *far field* expansion, the *near field* terms $\phi^j(t, \frac{s}{\varepsilon})$ represent the boundary layer close to the wall, $\mathbf{curl}_{2D} = (\partial_y, -\partial_x)^\top$, and χ is an admissible cut-off function.

The method of multiscale expansion separates the far and near field terms. We restrict ourselves to $j = 0, 1, 2$, as these will be used for the derivation of the impedance boundary conditions where the equations for general $j \in \mathbb{N}$ can be found in [11]. The far field velocity terms $\mathbf{v}^j$ satisfy the PDEs

$$\nabla \operatorname{div} \mathbf{v}^j + \frac{\omega^2}{c^2}\mathbf{v}^j = \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}\cdot\delta_{j=0} + \frac{\mathrm{i}\omega^2}{2c^2}\Delta\mathbf{v}^{j-2} + \frac{\mathrm{i}\gamma'\omega^2}{2c^2}\nabla\operatorname{div}\mathbf{v}^{j-2}, \qquad \text{in } \Omega, \tag{3.4a}$$

$$\mathbf{v}^j\cdot\mathbf{n} = \sum_{\ell=1}^{j}(G_\ell(\partial_t \operatorname{div}\mathbf{v}^{j-\ell}) + H_j(\mathbf{f}), \qquad \text{on } \partial\Omega, \tag{3.4b}$$

where $\mathbf{v}^{-1} = \mathbf{v}^{-2} = \mathbf{0}$, $G_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma)$ and $H_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma)$ are tangential differential operators acting on traces of terms of lower orders or the trace of $\mathbf{f}$ on $\partial\Omega$, respectively. Furthermore, $\delta_{j=0}$ stands for the Kronecker symbol which is 1 if $j = 0$ and 0 otherwise. The operators G_ℓ and H_ℓ up to $\ell = 2$ are given by

$$G_0(v) = 0, \qquad H_0(\mathbf{f}) = 0, \tag{3.5a}$$

$$G_1(v) = (1+\mathrm{i})\frac{c^2}{2\omega^2}\partial_t v, \qquad H_1(\mathbf{f}) = -(1-\mathrm{i})\frac{1}{2\omega\rho_0}\partial_t(\mathbf{f}\cdot\mathbf{n}^\perp), \tag{3.5b}$$

$$G_2(v) = \frac{c^2}{\omega^2}\Big(\frac{\mathrm{i}}{4}\partial_t(\kappa v)\Big), \qquad H_2(\mathbf{f}) = -\frac{1}{4\omega\rho_0}\partial_t(\kappa\,\mathbf{f}\cdot\mathbf{n}^\perp). \tag{3.5c}$$

The near field terms $\mathbf{v}^j_{BL} = \sqrt{\frac{2\eta}{\omega\rho_0}}\,\mathbf{curl}_{2D}(\phi^j\chi)$ for $\phi^j(\mathbf{x}) = \widetilde{\phi}^j(t,S)$ for $S = s\sqrt{\frac{\omega\rho_0}{2\eta}}$ are defined by (*cf.* [11, Lemma A.1])

$$\widetilde{\phi}^j(t,S) = \frac{1}{2}(1+\mathrm{i})\,\mathrm{e}^{-(1-\mathrm{i})S}\sum_{\ell=0}^{j}(E_\ell(\mathbf{v}^{j-\ell}\cdot\mathbf{n}^\perp))(t,S), \tag{3.6}$$

with the operators $E_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma\times[0,\infty))$ for $\ell = 0, 1, 2$

$$E_0(v) = v, \tag{3.7a}$$

$$E_1(v) = \frac{1}{4}(3+\mathrm{i})\,\kappa S v, \tag{3.7b}$$

$$E_2(v) = \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}v + \frac{1}{4}\big(\mathrm{i} + (1+\mathrm{i})S\big)\Big(\frac{3}{4}\kappa^2 v + \partial_t^2 v\Big) + \frac{3}{8}\kappa^2 S^2 v. \tag{3.7c}$$

The far field pressure terms p^j satisfy the PDE

$$\Delta p^j + \frac{\omega^2}{c^2}p^j = \operatorname{div}\mathbf{f}\cdot\delta_{j=0} + \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\Delta p^{j-2}, \qquad \text{in } \Omega, \tag{3.8a}$$

$$\nabla p^j\cdot\mathbf{n} = \sum_{\ell=1}^{j} J_\ell(p^{j-\ell}) + K_j(\mathbf{f}), \qquad \text{on } \partial\Omega, \tag{3.8b}$$

where $p^{-1} = p^{-2} = 0$, and $J_\ell : C^\infty(\Gamma) \to C^\infty(\Gamma)$ and $K_\ell : (C^\infty(\Omega))^2 \to C^\infty(\Gamma)$ are tangential differential operstors acting on traces of terms of lower orders on $\partial\Omega$ or on $\mathbf{f}$, respectively. The operators J_ℓ and K_ℓ up to $\ell = 2$ (*cf.* [11, Sec. 2.4.2]) are given by

$$J_0(v) = 0, \qquad K_0(\mathbf{f}) = \mathbf{f}\cdot\mathbf{n}, \tag{3.9a}$$

$$J_1(v) = -\frac{1}{2}(1+\mathrm{i})\,\partial_t^2 v, \qquad K_1(\mathbf{f}) = -\frac{1}{2}(1+\mathrm{i})\,\partial_t(\mathbf{f}\cdot\mathbf{n}^\perp), \tag{3.9b}$$

$$J_2(v) = -\frac{\mathrm{i}}{4}\partial_t(\kappa\partial_t v), \qquad K_2(\mathbf{f}) = -\frac{\mathrm{i}}{4}\partial_t(\kappa\mathbf{f}\cdot\mathbf{n}^\perp) + \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\mathbf{f}\cdot\mathbf{n} - \frac{\mathrm{i}}{2}\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{f}\cdot\mathbf{n}. \tag{3.9c}$$

3.3. **Derivation of the impedance boundary conditions for velocity and pressure**

Now, we are going to derive the approximative velocity and pressure models for $\mathbf{v}_{\mathrm{appr},N}$ and $p_{\mathrm{appr},N}$ including impedance boundary conditions given in Sec. 2.2. Let $\mathbf{v}^{\varepsilon,N} := \sum_{j=0}^{N} \varepsilon^n \mathbf{v}^j$. Then, by (3.4b) we have

$$
\begin{aligned}
\mathbf{v}^{\varepsilon,N} \cdot \mathbf{n} &= \sum_{j=0}^{N} \varepsilon^j \sum_{\ell=1}^{j} G_\ell(\partial_t \operatorname{div} \mathbf{v}^{j-\ell}) + \sum_{j=0}^{N} \varepsilon^j H_j(\mathbf{f}) \\
&= \sum_{\ell=1}^{N} \varepsilon^\ell \sum_{j=\ell}^{N} \varepsilon^j G_\ell(\partial_t \operatorname{div} \mathbf{v}^{j-\ell}) + \sum_{j=0}^{N} \varepsilon^j H_j(\mathbf{f}) \\
&= \sum_{\ell=1}^{N} \varepsilon^\ell \sum_{j=0}^{N-\ell} \varepsilon^j G_\ell(\partial_t \operatorname{div} \mathbf{v}^{j}) + \sum_{j=0}^{N} \varepsilon^j H_j(\mathbf{f}) \\
&= \sum_{\ell=1}^{N} \varepsilon^\ell G_\ell(\mathbf{v}^{\varepsilon,N}) + \sum_{j=0}^{N} \varepsilon^j H_j(\mathbf{f}) - \varepsilon^{N+1} \sum_{\ell=1}^{N} \sum_{j=0}^{\ell-1} \varepsilon^j G_\ell(\partial_t \operatorname{div} \mathbf{v}^{j+1-N-\ell})
\end{aligned}
$$

Moving all the terms with $\mathbf{v}^{\varepsilon,N}$ from the right hand side to the left hand side and neglecting the terms of order ε^{N+1} on the right hand side and using the equality $\eta = \varepsilon^2 \omega \rho_0 / 2$, we obtain the boundary conditions for $\mathbf{v}_{\mathrm{appr},N}$,

$$
\mathbf{v}_{\mathrm{appr},N} \cdot \mathbf{n} - \sum_{\ell=1}^{N} (\sqrt{2\eta/(\omega\rho_0)})^\ell G_\ell(\partial_t \operatorname{div} \mathbf{v}_{\mathrm{appr},N}) = \sum_{j=0}^{N} (\sqrt{2\eta/(\omega\rho_0)})^j H_j(\mathbf{f}), \tag{3.10}
$$

which is (2.3b), (2.4b) and (2.5b) for $N = 0, 1, 2$.

To obtain the approximative PDEs we are going to simplify (3.4a). Applying curl_{2D} to (3.4a) we obtain

$$
\operatorname{curl}_{2D} \mathbf{v}^j = \frac{\mathrm{i}}{\omega\rho_0} \operatorname{curl}_{2D} \mathbf{f} \cdot \delta_{j=0} - \frac{\mathrm{i}}{2} \operatorname{curl}_{2D} \mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}^{j-2}.
$$

By recursion in j we obtain an expression of $\operatorname{curl}_{2D} \mathbf{v}^j$ in terms of $\mathbf{f}$ only (see (2.11) in [11]). Inserting this expression into (3.4a) we obtain

$$
\nabla \operatorname{div} \mathbf{v}^j + \frac{\omega^2}{c^2} \mathbf{v}^j = \sum_{\ell=1}^{j} L_\ell(\mathbf{v}^{j-\ell}) + M_j(\mathbf{f})
$$

with $L_\ell \equiv 0$ if $\ell \neq 2$, $M_j \equiv 0$ if j is odd and otherwise

$$
L_2 = \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2} \nabla \operatorname{div}, \qquad M_j = \frac{\mathrm{i}\omega}{\rho_0 c^2} \left(-\frac{\mathrm{i}}{2} \mathbf{curl}_{2D} \operatorname{curl}_{2D} \right)^{j/2} \mathbf{f}.
$$

Now, in the same away as above, where G_ℓ and H_j are replaced by L_ℓ and M_j, we find for the approximative velocities (2.3a), (2.3a) and for $N \geq 2$

$$
\left(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2} \right) \nabla \operatorname{div} \mathbf{v}_{\mathrm{appr},N} + \frac{\omega^2}{c^2} \mathbf{v}_{\mathrm{appr},N} = \sum_{j=0}^{N} (\sqrt{2\eta/(\omega\rho_0)})^j M_j(\mathbf{f}),
$$

which is equivalent to (2.5a) for $N = 2$.

Note, that it is possible to keep a term with $\mathbf{curl}_{2D} \operatorname{curl}_{2D} \mathbf{v}_{\mathrm{appr},N}$ on the left hand side and with the gain of the simple source term $\frac{\mathrm{i}\omega}{\rho_0 c^2} \mathbf{f}$ on the right hand side (for any N). However, this PDE needs a further boundary condition, *e. g.*, a prescribed trace of $\operatorname{curl}_{2D} \mathbf{v}_{\mathrm{appr},N}$ in terms of $\mathbf{f}$.

With the same technique as above, but with J_ℓ and K_j instead of G_ℓ and H_j, we get the boundary conditions for the approximative pressure

$$
\nabla q_{\mathrm{appr},N} \cdot \mathbf{n} - \sum_{\ell=1}^{N} (\sqrt{2\eta/(\omega\rho_0)})^\ell J_\ell(q_{\mathrm{appr},N}) = \sum_{j=0}^{N} (\sqrt{2\eta/(\omega\rho_0)})^j K_j(\mathbf{f}),
$$

which is (2.10b), (2.11b) and (2.12b) for $N = 0, 1, 2$.

Similarly, but even simpler than the PDEs for $\mathbf{v}_{\mathrm{appr},N}$ those for $q_{\mathrm{appr},N}$ are derived, leading to (2.10a) and (2.11a) for $N = 0, 1$ and for $N \geq 2$

$$\Big(1 - \frac{\mathrm{i}\omega(\eta + \eta')}{\rho_0 c^2}\Big)\Delta q_{\mathrm{appr},N} + \frac{\omega^2}{c^2} q_{\mathrm{appr},N} = \operatorname{div} \mathbf{f},$$

which is (2.11a) for $N = 2$.

4. Justification of the approximative models

For some frequencies the limit system of vanishing viscosity does not have a unique solution, which has consequences on the stability and error estimates for the systems of order 1 and 2 as well. We discuss the eigensolution for the limit system in Sec. 4.1. In the following stability and error analysis we then exclude those eigenfrequencies. We then in Sec. 4.2 introduce an auxilliary system, which differs from the system of order 1 and 2 only by the source term of the PDE and prove its stability. The approximative solution of the system for the velocity $\mathbf{v}_{\mathrm{appr},N}$ and *a-posteriori* computed pressure $p_{\mathrm{appr},N}$ is asymptotically equivalent to the asymptotic expansion up to order N, which we will show in Sec. 4.3. The stability analysis and error analysis of these systems of order 1 and 2 will be discussed in Sec. 4.4. Finally, in Sec. 4.5 we show that the approximative solution of the system for the pressure $q_{\mathrm{appr},N}$ and *a-posteriori* computed velocity $\mathbf{w}_{\mathrm{appr},N}$ is identical or asymptotically equivalent to approximation $(\mathbf{v}_{\mathrm{appr},N}, p_{\mathrm{appr},N})$, on which the stability and error statement for $(\mathbf{w}_{\mathrm{appr},N}, q_{\mathrm{appr},N})$ is based on.

4.1. Eigensolutions and well-posedness for the limit problem

We concider the approximative model of 0 order for the velocity (2.3)

$$\begin{aligned} -\nabla \operatorname{div} \mathbf{v} - \frac{\omega^2}{c^2}\mathbf{v} &= \tilde{\mathbf{f}}, \quad \text{in } \Omega \\ \mathbf{v} \cdot \mathbf{n} &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

We begin with the case of resonance frequency $\omega = \omega_0$, *i.e.*, $\frac{\omega_0}{c}$ is an eigenvalue, and seek a solution as a decomposition

$$\mathbf{v} = \sum_{i=1}^{N} \alpha_i \boldsymbol{\psi}_i + \mathbf{u}_0$$

where $\boldsymbol{\psi}_i \in E(\omega_0)$, *i.e.* in the eigenfunctions space (finite dimension), and $\mathbf{u}_0$ is perpendicular to all eigenfunctions, *i.e.* $\int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\psi}_i \,\mathrm{d}\mathbf{x} = 0,\ i = 1, \ldots, N$. For all eigenfunctions it holds

$$\begin{aligned} -\nabla \operatorname{div} \boldsymbol{\psi}_i - \frac{\omega^2}{c^2}\boldsymbol{\psi}_i &= 0, \quad \text{in } \Omega \\ \boldsymbol{\psi}_i \cdot \mathbf{n} &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

If the compatibility condition of the source function $\langle \tilde{\mathbf{f}}, \boldsymbol{\psi}_i \rangle_{L^2(\Omega)} = 0,\ i = 1, \ldots, N$ is fulfilled, by Fredholm alternative [9] $\mathbf{u}_0$ exists and is bounded. However, in practise it is rather unlikely that the source is orthogonal to all eigenfunctions and at eigenfrequencies an existence of the limit model is not guaranteed.

Throughout the article we assume that frequency is not an eigenfrequency, $\omega \neq \omega_0$.

4.2. Well-posedness and regularity of an auxilliary system

In this section we analyse a well-posed auxilliary system, which coincides with the approximative velocity systems of order 1 and 2 up to the source term of the PDE. Here, we assume an asymptotically small gradient field as source, which we will need for the proof of Lemma 4.4 on the asymptotic equivalence of the approximative solutions and the asymptotic expansion. The stability of the approximative systems of order 1 and 2, stated in Lemma 2.1, is then a consequence of this asymptotic equivalence and the stability of the terms of the asymptotic expansion, and will be shown in Sec. 4.4.

Lemma 4.1 (Well-posedness of an auxilliary system)**.** *Let* $\frac{\omega^2}{c^2}$ *be distinct from the Neumann eigenvalues of* $-\Delta$*,* $g \in L^2(\partial\Omega)$*,* $h \in L^2(\Omega)$ *with* $h \in H^1(\partial\Omega)$*. Let, furthermore, the constants* $\alpha_1 > 0, \alpha_2 \geq 0, \alpha_3 \in \mathbb{R}$ *and* $\eta_m \in \{\mathbb{R}, \infty\}$ *be constants independent of* $\eta > 0$ *such that* $\sqrt{\eta_m}|\alpha_3|\|\kappa\|_{L^\infty(\partial\Omega)}/\alpha_1 \leq \frac{1}{2}$*,* $\alpha_2\eta_m \leq \frac{1}{3}$*, and* $h = 0$ *if* $\alpha_2 = 0$*. Then, the system*

$$(1 - \mathrm{i}\alpha_2\eta)\nabla \operatorname{div} \mathbf{w} + \frac{\omega^2}{c^2}\mathbf{w} = \eta^{3/4}\,\nabla h, \qquad \textit{in } \Omega, \tag{4.1a}$$

$$\mathbf{w}\cdot\mathbf{n} - (1+\mathrm{i})\sqrt{\eta}\,\alpha_1\partial_t^2 \operatorname{div}\mathbf{w} - \mathrm{i}\eta\,\alpha_3\partial_t(\kappa\partial_t \operatorname{div}\mathbf{w}) = \sqrt{\eta}\,\partial_t g, \qquad \textit{on } \partial\Omega, \tag{4.1b}$$

has for any $\eta \in (0, \eta_m)$ *a unique solution* $\mathbf{w} \in (H^1(\Omega))^2$ *and there exists a constant* $C = C(\eta_m) > 0$ *not depending on* η *such that for all* $\eta \in (0, \eta_m)$

$$\|\mathbf{w}\|_{(H^1(\Omega))^2} \leq C\left(\|g\|_{L^2(\partial\Omega)} + \|h\|_{L^2(\Omega)} + \eta^{\frac{3}{4}}\|\partial_t h\|_{L^2(\partial\Omega)}\right).$$

Proof. Let us write the system (4.1) as a mixed variation formulation and introduce a new function $\lambda = \operatorname{div}\mathbf{w}$ on $\partial\Omega$. We seek $(\mathbf{w}, \lambda) \in (H^1(\Omega))^2 \times H^1(\partial\Omega)$ such that for all $(\mathbf{v}', \lambda') \in (H^1(\Omega))^2 \times H^1(\partial\Omega)$

$$\int_\Omega (1-\mathrm{i}\alpha_2\eta)\operatorname{div}\mathbf{w}\operatorname{div}\mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{w}\cdot\mathbf{v}'\,\mathrm{d}\mathbf{x} - (1-\mathrm{i}\alpha_2\eta)\int_{\partial\Omega}\lambda\,\mathbf{v}'\cdot\mathbf{n}\,\mathrm{d}\sigma(\mathbf{x}) = -\eta^{\frac{3}{4}}\int_\Omega h\,\operatorname{div}\mathbf{v}'\,\mathrm{d}\mathbf{x} + \eta^{\frac{3}{4}}\int_{\partial\Omega} h\,\mathbf{v}'\cdot\mathbf{n}\,\mathrm{d}\sigma(\mathbf{x}), \tag{4.2a}$$

$$\int_{\partial\Omega}\mathbf{w}\cdot\mathbf{n}\,\lambda' + ((1+\mathrm{i})\sqrt{\eta}\,\alpha_1 + \mathrm{i}\eta\,\alpha_3\kappa)\,\partial_t\lambda\,\partial_t\lambda'\,\mathrm{d}\sigma(\mathbf{x}) = \int_{\partial\Omega} -\sqrt{\eta}\,g\,\partial_t\lambda'\,\mathrm{d}\sigma(\mathbf{x}). \tag{4.2b}$$

Subtracting (4.2b) from (4.2a) multiplied with $(1+\mathrm{i}\alpha_2\eta)$ gives

$$\begin{aligned}
&\int_\Omega (1-\mathrm{i}\alpha_2\eta)\operatorname{div}\mathbf{w}\operatorname{div}\mathbf{v}' - \frac{\omega^2}{c^2}\mathbf{w}\cdot\mathbf{v}'\,\mathrm{d}\mathbf{x}\\
&\quad + \int_{\partial\Omega} -(1-\mathrm{i}\alpha_2\eta)\lambda\,\mathbf{v}'\cdot\mathbf{n} - (1+\mathrm{i}\alpha_2\eta)\mathbf{w}\cdot\mathbf{n}\,\lambda' - ((1+\mathrm{i})\sqrt{\eta}\,\alpha_1 + \mathrm{i}\eta\,\alpha_3\kappa)\,(1+\mathrm{i}\alpha_2\eta)\,\partial_t\lambda\,\partial_t\lambda'\,\mathrm{d}\sigma(\mathbf{x})\\
&\qquad = -\eta^{\frac{3}{4}}\int_\Omega h\,\operatorname{div}\mathbf{v}'\,\mathrm{d}\mathbf{x} + \eta^{\frac{3}{4}}\int_{\partial\Omega} h\,\mathbf{v}'\cdot\mathbf{n}\,\mathrm{d}\sigma(\mathbf{x}) + \sqrt{\eta}\,(1+\mathrm{i}\alpha_2\eta)\int_{\partial\Omega} g\,\partial_t\lambda'\mathrm{d}\sigma(\mathbf{x}).
\end{aligned} \tag{4.3}$$

Inserting the test function $\lambda' = \overline{h}$ into (4.2b) and taking the conjugate complex we find

$$\int_{\partial\Omega} h\,\overline{\mathbf{w}}\cdot\mathbf{n}\,\mathrm{d}\sigma(\mathbf{x}) = -\int_{\partial\Omega}((1-\mathrm{i})\sqrt{\eta}\alpha_1 - \mathrm{i}\eta\alpha_3\kappa)\,\partial_t h\,\partial_t\overline{\lambda}\,\mathrm{d}\sigma(\mathbf{x}) - \sqrt{\eta}\int_{\partial\Omega}\partial_t h\,g\,\mathrm{d}\sigma(\mathbf{x}). \tag{4.4}$$

Now, inserting test functions $\mathbf{v}' = \overline{\mathbf{w}}$, $\lambda' = \overline{\lambda}$ into (4.3) and taking the negative imaginary part we obtain

$$\begin{aligned}
&\alpha_2\eta\,\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)} + \sqrt{\eta}\;\alpha_1(1+\alpha_2\eta)\|\partial_t\lambda\|^2_{L^2(\partial\Omega)} + \eta\;\alpha_3\langle\kappa\,\partial_t\lambda, \partial_t\lambda\rangle_{L^2(\partial\Omega)}\\
&\qquad \leq \eta^{\frac{3}{4}}\;\operatorname{Im}\langle h, \operatorname{div}\mathbf{w}\rangle_{L^2(\Omega)} - \eta^{\frac{3}{4}}\;\operatorname{Im}\langle h, \mathbf{w}\cdot\mathbf{n}\rangle_{L^2(\partial\Omega)} + \sqrt{\eta}\,(1+\alpha_2\eta)\;|\,\langle g, \partial_t\lambda\rangle_{L^2(\partial\Omega)}\,|\\
&\qquad \leq \eta^{\frac{3}{4}}\;\operatorname{Im}\langle h, \operatorname{div}\mathbf{w}\rangle_{L^2(\Omega)} + \alpha_1\eta^{\frac{5}{4}}\operatorname{Im}\langle(1-\mathrm{i})\,\partial_t h, \partial_t\lambda\rangle_{L^2(\partial\Omega)} - \alpha_3\eta^{\frac{7}{4}}\operatorname{Im}\langle\mathrm{i}\kappa\partial_t h, \partial_t\lambda\rangle_{L^2(\partial\Omega)}\\
&\qquad\quad + \eta^{\frac{5}{4}}\operatorname{Im}\langle\partial_t h, g\rangle_{L^2(\partial\Omega)} + \tfrac{4}{3}\sqrt{\eta}\,|\,\langle g, \partial_t\lambda\rangle_{L^2(\partial\Omega)}\,|
\end{aligned}$$

where we used (4.4) in the last step and $\alpha_2\eta < \frac{1}{3}$. Now, with $\sqrt{\eta}\,|\alpha_3|\|\kappa\|_{L^\infty(\partial\Omega)} < \frac{1}{2}\alpha_1$ we get

$$\begin{aligned}
\alpha_2\eta\,\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)} + \tfrac{1}{2}\sqrt{\eta}\;\alpha_1\|\partial_t\lambda\|^2_{L^2(\partial\Omega)} &\le \alpha_2\eta\,\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)} + \sqrt{\eta}\;\alpha_1\|\partial_t\lambda\|^2_{L^2(\partial\Omega)} + \eta\;\alpha_3\langle\kappa\,\partial_t\lambda,\partial_t\lambda\rangle_{L^2(\partial\Omega)}\\
&\le \eta^{\frac{3}{4}}\|h\|_{L^2(\Omega)}\|\operatorname{div}\mathbf{w}\|_{L^2(\Omega)} + \eta^{\frac{5}{4}}\left(\sqrt{2}\alpha_1 + \alpha_3\|\kappa\|_{L^\infty(\partial\Omega)}\sqrt{\eta}\right)\|\partial_t h\|_{L^2(\partial\Omega)}\|\partial_t\lambda\|_{L^2(\partial\Omega)}\\
&\quad + \eta^{\frac{5}{4}}\|\partial_t h\|_{L^2(\partial\Omega)}\|g\|_{L^2(\partial\Omega)} + \tfrac{4}{3}\sqrt{\eta}\;\|g\|_{L^2(\partial\Omega)}\|\partial_t\lambda\|_{L^2(\partial\Omega)}\\
&\le \alpha_2\eta\,\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\delta_{\alpha_2>0}}{4\,\alpha_2}\sqrt{\eta}\|h\|^2_{L^2(\Omega)} + 9\,\alpha_1\eta^2\|\partial_t h\|^2_{L^2(\partial\Omega)} + \frac{4}{\alpha_1}\sqrt{\eta}\|g\|^2_{L^2(\partial\Omega)} + \frac{\alpha_1}{4}\sqrt{\eta}\|\partial_t\lambda\|^2_{L^2(\partial\Omega)},
\end{aligned}$$

where we were using sequentially Cauchy–Schwarz inequality and Young's inequality. Here $\delta_{\alpha_2>0}$ stands for the Kronecker symbol which is 1 if $\alpha_2 > 0$ and 0 otherwise. Subtracting $\alpha_2\eta\,\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)} + \frac{\alpha_1}{4}\sqrt{\eta}\|\partial_t\lambda\|^2_{L^2(\partial\Omega)}$ on both sides we find

$$\frac{\alpha_1}{4}\sqrt{\eta}\|\partial_t\lambda\|^2_{L^2(\partial\Omega)} \le \sqrt{\eta}\left(\frac{4}{\alpha_1}\|g\|^2_{L^2(\partial\Omega)} + \frac{\delta_{\alpha_2>0}}{4\,\alpha_2}\|h\|^2_{L^2(\Omega)} + 9\,\alpha_1\eta^{\frac{3}{2}}\|\partial_t h\|^2_{L^2(\partial\Omega)}\right).$$

Finally, division by $\frac{1}{4}\sqrt{\eta}\;\alpha_1$ on both sides and with inequality $d^2 \le a^2 + b^2 + c^2$ with $d, a, b, c \ge 0$, therefore $d \le a + b + c$, we attain the estimate for the second case

$$\|\partial_t\lambda\|_{L^2(\partial\Omega)} \le \frac{4}{\alpha_1}\|g\|_{L^2(\partial\Omega)} + \frac{\delta_{\alpha_2>0}}{\sqrt{\alpha_1\alpha_2}}\|h\|_{L^2(\Omega)} + 6\,\eta^{\frac{3}{4}}\|\partial_t h\|_{L^2(\partial\Omega)}. \tag{4.5}$$

To show uniqueness we assume $g = 0$ and $h = 0$. Then, from the estimate (4.5) $\partial_t\lambda = 0$, *i.e.* λ is a constant and so $\partial_t^2\lambda = 0$. As $\operatorname{div}\mathbf{w} = \lambda$ on $\partial\Omega$, we have in view of (4.1b) $\mathbf{w}\cdot\mathbf{n} = 0$ on $\partial\Omega$. If $\alpha_2 = 0$, then Lemma 4.3 of [11] implies $\mathbf{w} \equiv 0$ in Ω. Otherwise, inserting $\mathbf{v}' = \overline{\mathbf{w}}$ for which $\mathbf{v}'\cdot\mathbf{n} = 0$ into (4.2a) and taking the imaginary part we find that $\operatorname{div}\mathbf{w} \equiv 0$ in Ω. Then, taking the real part we conclude that $\mathbf{w} \equiv 0$ in Ω.

Now, we provide a proof of stability on proof by contradiction. If the statement of the lemma would not hold, then there would exist a sequence $\{(\mathbf{w}_n,\ \lambda_n)\}_{n=1}^\infty$ with $\|(\mathbf{w}_n,\ \lambda_n)\|^2_{(H^1(\Omega))^2\times H^1(\partial\Omega)} := \|\mathbf{w}_n\|^2_{(H^1(\Omega))^2} + \|\lambda_n\|^2_{H^1(\partial\Omega)} = 1$, such that $\mathbf{w}_n,\ \lambda_n$ are solutions of (4.2) or (4.2) with (g_n, h_n) instead of (g, h) where $g_n, h_n \to 0$ with $n \to \infty$. For each bounded sequence there exists a weakly converging subsequence which we call again $\{(\mathbf{w}_n,\ \lambda_n)\}_{n=1}^\infty$. By the Rellich–Kondrachov compactness theorem [9] $\mathbf{w}_n$ strongly converges in $L^2(\Omega)$ and λ_n strongly converges in $L^2(\partial\Omega)$. In the limit $n \to \infty$ we have $\mathbf{w}_n \rightharpoonup \mathbf{w}_\infty \in (H^1(\Omega))^2$, $\lambda_n \rightharpoonup \lambda_\infty \in H^1(\partial\Omega)$ and as it was shown above $\mathbf{w}_\infty = 0$, $\lambda_\infty = 0$. By uniqueness of the limit $(\mathbf{w}_n, \lambda_n)$ converge strongly to 0 in the $L^2(\Omega)\times L^2(\partial\Omega)$–norm, which contradicts our assumption. That proves stability and the solution $(\mathbf{w},\ \lambda)$ is bounded by (g, h), *i.e.*, there exists a constant $C = C(\eta)$ which may depend on η, such that

$$\|(\mathbf{w},\ \lambda)\|_{(H^1(\Omega))^2\times H^1(\partial\Omega)} \le C(\eta)\left(\|g\|_{L^2(\partial\Omega)} + \|h\|_{L^2(\partial\Omega)} + \eta^{\frac{3}{4}}\|\partial_t h\|_{L^2(\partial\Omega)}\right).$$

However, it is easy to see that the solution is even bounded independently of η. For this we consider sequences (g_n, h_n), $\eta_n \to 0$ with $n \to \infty$ as well and a sequence $\{(\mathbf{w}_n,\ \lambda_n)\}_{n=1}^\infty$ with norm 1 as above solution of (4.2) or (4.2) with (g_n, h_n) instead of (g, h). When we call $(\mathbf{w}_\infty, \lambda_\infty)$ its limit again, we have with $\eta_n \to 0$ that $\mathbf{w}_\infty\cdot\mathbf{n} = 0$ on $\partial\Omega$ and so $\mathbf{w}_\infty \equiv 0$ in Ω and so $\lambda_\infty = \operatorname{div}\mathbf{w}_\infty = 0$ on $\partial\Omega$. Again, this is a contradiction. Consequently, there exists a constant $C_m = \max_{0\le\eta\le\eta_m} C(\eta)$ independently of η, such that

$$\|(\mathbf{w},\ \lambda)\|_{(H^1(\Omega))^2\times H^1(\partial\Omega)} \le C_m\left(\|g\|_{L^2(\partial\Omega)} + \|h\|_{L^2(\partial\Omega)} + \eta^{\frac{3}{4}}\|\partial_t h\|_{L^2(\partial\Omega)}\right).$$

This finishes the proof. □

Lemma 4.2 (Higher regularity of an auxilliary system)**.** *Let the assumption of Lemma 4.1 be fulfilled, and let furthermore* $g \in H^{r+1/2}(\partial\Omega)$ *and* $h \in H^1(\Omega)\cap H^r(\widetilde{\Omega})$ *for some* $r \in \mathbb{N}$, $r \ge 1$ *and some* $\widetilde{\Omega} \subseteq \Omega$ *with* $\partial\widetilde{\Omega} \subseteq \partial\Omega$ *and* $h \in H^1(\partial\Omega)$ *. Then, there exist constants* C, C_r *independent of* η *such that for the solution* $\mathbf{w}$ *of* (4.1) *and any* $\widetilde{\Omega}_\subset \subset \widetilde{\Omega}$ *it holds*

$$\|\mathbf{w}\|_{(H^1(\Omega))^2\cap(H^{r+1}(\widetilde{\Omega}_\subset))^2} \le \frac{C_r}{\eta^{r/2}}\left(\|g\|_{H^{r+1/2}(\partial\Omega)} + \|h\|_{H^1(\Omega)\cap H^{r+2}(\widetilde{\Omega})}\right), \tag{4.6a}$$

$$\|\operatorname{div}\mathbf{w}\|_{H^1(\Omega)} \le \frac{C}{\sqrt{\eta}}\left(\|g\|_{H^{3/2}(\partial\Omega)} + \|h\|_{H^1(\Omega)} + \eta^{\frac{3}{4}}\|\partial_t h\|_{L^2(\partial\Omega)}\right). \tag{4.6b}$$

Proof. By Lemma 4.1 the solution $\mathbf{w}$ of (4.1) and, hence, $\operatorname{div}\mathbf{w}$ on $\partial\Omega$ are bounded. In particular, $\mathbf{w}\cdot\mathbf{n} \in H^{1/2}(\partial\Omega)$. Using (4.1b) we find for any $s \in \mathbb{N}_0$, $s \le r-1$

$$\|\operatorname{div}\mathbf{w}\|_{H^{s+5/2}(\partial\Omega)} \le C\left(\frac{1}{\sqrt{\eta}}\|\mathbf{w}\cdot\mathbf{n}\|_{H^{s+1/2}(\partial\Omega)} + \|g\|_{H^{s+3/2}(\partial\Omega)}\right),$$

and so $\operatorname{div}\mathbf{w} \in H^{5/2}(\partial\Omega)$. Note, that $u := \operatorname{div}\mathbf{w}$ solves the boundary value problem

$$\begin{aligned} (1-\mathrm{i}\alpha_2\eta)\Delta u + \frac{\omega^2}{c^2}u &= \eta^{\frac{3}{4}}\Delta h, && \text{in } \Omega,\\ u &= \operatorname{div}\mathbf{w}, && \text{on } \partial\Omega, \end{aligned}$$

and due to the assumption on the frequency we find $\operatorname{div}\mathbf{w} \in H^1(\Omega)$ with

$$\|\operatorname{div}\mathbf{w}\|_{H^1(\Omega)} \le C\left(\eta^{\frac{3}{4}}|h|_{H^1(\Omega)} + \|\operatorname{div}\mathbf{w}\|_{H^{1/2}(\partial\Omega)}\right) \le C\left(\eta^{\frac{3}{4}}|h|_{H^1(\Omega)} + \frac{1}{\sqrt{\eta}}\|\mathbf{w}\cdot\mathbf{n}\|_{H^{1/2}(\partial\Omega)} + \|g\|_{H^{3/2}(\partial\Omega)}\right).$$

which implies (4.6b). Now, by Theorem 4.18 in [7] we have for any $\widetilde{\Omega}_\subset \subset \widetilde{\Omega}$ (and $\widetilde{\Omega}_\subset = \Omega$ if $\widetilde{\Omega} = \Omega$) that

$$\begin{aligned} \|\operatorname{div}\mathbf{w}\|_{H^{s+3}(\widetilde{\Omega}_\subset)} &\le C\left(\|\operatorname{div}\mathbf{w}\|_{H^1(\Omega)} + \|\operatorname{div}\mathbf{w}\|_{H^{s+5/2}(\partial\Omega)} + \eta^{\frac{3}{4}}\|h\|_{H^{s+3}(\widetilde{\Omega})\cap H^1(\Omega)}\right)\\ &\le C\left(\frac{1}{\sqrt{\eta}}\|\mathbf{w}\cdot\mathbf{n}\|_{H^{s+1/2}(\partial\Omega)} + \|g\|_{H^{s+3/2}(\partial\Omega)} + \eta^{\frac{3}{4}}\|h\|_{H^{s+3}(\widetilde{\Omega})\cap H^1(\Omega)}\right). \end{aligned}$$

As, furthermore, $\mathbf{w}$ is given by $\mathbf{w} = \frac{c^2}{\omega^2}\nabla\operatorname{div}\mathbf{w}$ we have

$$\|\mathbf{w}\|_{H^{s+2}(\widetilde{\Omega}_\subset)} \le C\left(\frac{1}{\sqrt{\eta}}\|\mathbf{w}\cdot\mathbf{n}\|_{H^{s+1/2}(\partial\Omega)} + \|g\|_{H^{s+3/2}(\partial\Omega)} + \eta^{\frac{3}{4}}\|h\|_{H^{s+3}(\widetilde{\Omega})\cap H^1(\Omega)}\right).$$

We may choose $\widetilde{\Omega}_\subset$ such that $\partial\widetilde{\Omega}_\subset \subseteq \partial\Omega$ and by the trace theorem we can assert that

$$\|\mathbf{w}\cdot\mathbf{n}\|_{H^{s+3/2}(\partial\Omega)} \le \|\mathbf{w}\|_{H^{s+2}(\widetilde{\Omega}_\subset)} \le C\left(\frac{1}{\sqrt{\eta}}\|\mathbf{w}\cdot\mathbf{n}\|_{H^{s+1/2}(\partial\Omega)} + \|g\|_{H^{s+3/2}(\partial\Omega)} + \eta^{\frac{3}{4}}\|h\|_{H^{s+3}(\widetilde{\Omega})\cap H^1(\Omega)}\right)$$

and so the statement of the lemma follows by induction in s. □

4.3. **Relation of velocity approximations and the related asymptotic expansion**

We start with the proof of asymptotic equivalence of the far field approximation and its asymptotic expansion, followed by the near field approximation.

4.3.1. *Far field approximation*

Lemma 4.3. *Let the assumptions of Lemma 2.1 be fulfilled, in particular, let* $\mathbf{f} \in (L^2(\Omega))^2 \cap (H^m(\widetilde{\Omega}))^2$ *for any* $m \in \mathbb{N}$ *in some neighbourhood* $\widetilde{\Omega} \subset \Omega$ *of* $\partial\Omega$, i.e., $\partial\Omega \subset \partial\widetilde{\Omega}$ *and* $\operatorname{curl}_{2D}\mathbf{f} \in H^m(\Omega)$ *for any* $m \in \mathbb{N}$. *Then, for any* $\widetilde{\Omega}_\subset \subset \widetilde{\Omega}$, *any* $j \in \mathbb{N}_0$ *and any* $m \in \mathbb{N}_0$ *it holds* $\operatorname{div}\mathbf{v}^j \in L^2(\Omega) \cap H^{m+1}(\widetilde{\Omega}_\subset)$.

Proof. By Lemma 2.3 in [11] all terms $\mathbf{v}^j \in (H^1(\Omega))^2$ and by Lemma 4.6 in [11] the terms $\mathbf{v}^j$ have any Sobolev regularity in any subdomain of $\widetilde{\Omega}_\subset$ of $\widetilde{\Omega}$. Using (3.4a) and (2.11) in [11] we find by induction in j

$$\begin{aligned} \nabla\operatorname{div}\mathbf{v}^j &= -\frac{\omega^2}{c^2}\mathbf{v}^j - \frac{\mathrm{i}\omega}{\rho_0 c^2}\mathbf{f}\cdot\delta_{j=0} + \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\operatorname{div}\mathbf{v}^{j-2} - \frac{\mathrm{i}\omega^2}{2c^2}\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{v}^{j-2}\\ &= -\frac{\omega^2}{c^2}\mathbf{v}^j - \delta_{j\text{ is even}}\frac{\mathrm{i}\omega}{\rho_0 c^2}\left(-\frac{\mathrm{i}}{2}\mathbf{curl}_{2D}\operatorname{curl}_{2D}\right)^{j/2}\mathbf{f} + \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla\operatorname{div}\mathbf{v}^{j-2} \in (H^1(\Omega))^2 \cap ((H^{m+1}(\widetilde{\Omega}_\subset)), \end{aligned}$$

and so the statement of the lemma. □

Lemma 4.4. *Let the assumptions of Theorem 2.4 be fulfilled and let* $\mathbf{f} \in (L^2(\Omega))^2 \cap (H^m(\widetilde{\Omega}))^2$ *for any* $m \in \mathbb{N}$ *in some neighbourhood* $\widetilde{\Omega} \subset \Omega$ *of* $\partial\Omega$, i.e., $\partial\Omega \subset \partial\widetilde{\Omega}$ *and* $\operatorname{curl}_{2D} \mathbf{f} \in H^m(\Omega)$. *Then, for any* $m \geq 1$, *any* $\widetilde{\Omega}_\subset \subset \widetilde{\Omega}$ *and* $N = 0, 1, 2$ *there exist constants* η_m *and* $C_{m,N}$ *independent of* η *such that for the solutions* $\mathbf{v}_{\mathrm{appr},N}$ *of the approximative models* (2.3), (2.4), *or* (2.5) *for* $N = 0, 1, 2$, *respectively, for* $p_{\mathrm{appr},N}$ *for* $N = 0, 1, 2$ *given by* (2.6) *and any* $\eta \in (0, \eta_m)$ *it holds*

$$\Big\| \mathbf{v}_{\mathrm{appr},N} - \sum_{j=0}^{N} \left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}} \mathbf{v}^j \Big\|_{(H^1(\Omega))^2 \cap (H^m(\widetilde{\Omega}_\subset))^2} + \Big\| p_{\mathrm{appr},N} - \sum_{j=0}^{N} \left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}} p^j \Big\|_{H^1(\Omega)} \leq C_{m,N} \eta^{\frac{N+1}{2}}. \tag{4.7}$$

Proof. The approximative model for $\mathbf{v}_{\mathrm{appr},0}$ (including boundary condition) for $N = 0$ is identical to that of the first term $\mathbf{v}^0$ of the asymptotic expansion, the difference is zero. As also the relations between $p_{\mathrm{appr},0}$ and $\mathbf{v}_{\mathrm{appr},0}$ given by (2.6) and between p^0 and $\mathbf{v}^0$ given by [11, (2.12)] are the same, the pressure approximation $p_{\mathrm{appr},0}$ is identical to the first term p^0. Hence, the above statement is true for $N = 0$.

For $N = 1, 2$ we write an asymptotic expansion of $\mathbf{v}_{\mathrm{appr},N}$ in the form

$$\mathbf{v}_{\mathrm{appr},N} \approx \mathbf{v}^{N,0} + \sqrt{\tfrac{2\eta}{\omega\rho_0}}\mathbf{v}^{N,1} + \tfrac{2\eta}{\omega\rho_0}\mathbf{v}^{N,2} + \left(\tfrac{2\eta}{\omega\rho_0}\right)^{3/2} \mathbf{v}^{N,3} + \dots, \tag{4.8}$$

where $\mathbf{v}^{N,j} := \mathbf{v}^j$ for $j = 0, 1, \dots, N$, and we call $\delta\mathbf{v}_{\mathrm{appr},N,\ell}$ the difference

$$\delta\mathbf{v}_{\mathrm{appr},N,\ell} = \mathbf{v}_{\mathrm{appr},N} - \sum_{j=0}^{\ell} \left(\tfrac{2\eta}{\omega\rho_0}\right)^{j/2} \mathbf{v}^{N,j}. \tag{4.9}$$

For the definition of $\mathbf{v}^{N,j}$ and estimates on $\delta\mathbf{v}_{\mathrm{appr},N,\ell}$ we distinguish the cases $N = 1$ and $N = 2$, where we denote by C a generic constant independent of η.

(i) $N = 1$. The terms $\mathbf{v}^{1,j}$, $j \geq 2$ satisfy

$$\begin{aligned} \nabla \operatorname{div} \mathbf{v}^{1,j} + \frac{\omega^2}{c^2}\mathbf{v}^{1,j} &= 0, && \text{in } \Omega, \\ \mathbf{v}^{1,j} \cdot \mathbf{n} &= G_1(\partial_t \operatorname{div} \mathbf{v}^{1,j-1}), && \text{on } \partial\Omega. \end{aligned} \tag{4.10}$$

By Lemma 4.3 we have $\operatorname{div} \mathbf{v}^1 \in H^{m+1/2}(\partial\Omega)$ for any $m \in \mathbb{N}$, and so by Lemma 2.3 in [11] $\mathbf{v}^{1,j}$ are well defined and by Lemma 4.6 in [11] it holds $\mathbf{v}^{1,j} \in (H^m(\Omega))^2$ for any $m \in \mathbb{N}$ and any $j \geq 2$. With the equality $\mathbf{v}^{1,1} = \mathbf{v}^1$ this implies $\operatorname{div} \mathbf{v}^{1,j} \in H^{m-3/2}(\partial\Omega)$ for any $m \in \mathbb{N}$ and any $j \geq 1$. The difference $\delta\mathbf{v}_{\mathrm{appr},1,\ell}$ solves by (2.4), (3.4) for $j = 0, 1, \dots, \ell$ and (4.10) the system

$$\nabla \operatorname{div} \delta\mathbf{v}_{\mathrm{appr},1,\ell} + \frac{\omega^2}{c^2}\delta\mathbf{v}_{\mathrm{appr},1,\ell} = 0, \qquad \text{in } \Omega, \tag{4.11a}$$

$$\delta\mathbf{v}_{\mathrm{appr},1,\ell} \cdot \mathbf{n} - \sqrt{\tfrac{2\eta}{\omega\rho_0}}\, G_1(\partial_t \operatorname{div} \delta\mathbf{v}_{\mathrm{appr},1,\ell}) = \left(\tfrac{2\eta}{\omega\rho_0}\right)^{(\ell+1)/2} G_1(\partial_t \operatorname{div} \mathbf{v}^{1,\ell}), \qquad \text{on } \partial\Omega, \tag{4.11b}$$

which coincides with (4.1) for $\alpha_1 = c^2/\omega^2\sqrt{1/(2\omega\rho_0)}$, $\alpha_2 = \alpha_3 = 0$, $g = (1+\mathrm{i})(2\eta/(\omega\rho_0))^{\ell/2}\,\alpha_1\partial_t \operatorname{div} \mathbf{v}^{1,\ell}$ and $h = 0$. So Lemma 4.2 implies

$$\|\delta\mathbf{v}_{\mathrm{appr},1,\ell}\|_{(H^m(\Omega))^2} \leq C\,\eta^{(\ell-m+1)/2}. \tag{4.12}$$

(ii) $N = 2$. The terms $\mathbf{v}^{2,j}$, $j \geq 3$ of the asymptotic expansion of $\mathbf{v}_{\mathrm{appr},2}$ satisfy

$$\begin{aligned} \nabla \operatorname{div} \mathbf{v}^{2,j} + \frac{\omega^2}{c^2}\mathbf{v}^{2,j} &= \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2}\nabla \operatorname{div} \mathbf{v}^{2,j-2}, && \text{in } \Omega \\ \mathbf{v}^{2,j} \cdot \mathbf{n} &= G_1(\partial_t \operatorname{div} \mathbf{v}^{2,j-1}) + G_2(\partial_t \operatorname{div} \mathbf{v}^{2,j-2}), && \text{on } \partial\Omega \end{aligned} \tag{4.13}$$

With the same argumentation as above we achieve regularity for $\operatorname{div} \mathbf{v}^{2,j} \in H^{m-3/2}(\partial\Omega)$ for any $m \in \mathbb{N}$ and any $j \geq 3$. The difference $\delta\mathbf{v}_{\mathrm{appr},2,\ell}$ solves by (2.5), (3.4) for $j = 0, 1, 2$ and (4.13) the system

$$\Big(1 - \frac{\mathrm{i}\omega(\eta+\eta')}{\rho_0 c^2}\Big) \nabla \operatorname{div} \delta\mathbf{v}_{\mathrm{appr},2,\ell-1} + \frac{\omega^2}{c^2} \delta\mathbf{v}_{\mathrm{appr},2,\ell} = \Big(\tfrac{2\eta}{\omega\rho_0}\Big)^{(\ell+1)/2} \frac{\mathrm{i}(1+\gamma')\omega^2}{2c^2} \nabla \operatorname{div}(\mathbf{v}^{2,\ell-1} + \sqrt{\tfrac{2\eta}{\omega\rho_0}}\mathbf{v}^{2,\ell}), \quad \text{in } \Omega, \tag{4.14a}$$

$$\begin{aligned} \delta\mathbf{v}_{\mathrm{appr},2,\ell} \cdot \mathbf{n} - \sqrt{\tfrac{2\eta}{\omega\rho_0}}\, G_1(\partial_t \operatorname{div} \delta\mathbf{v}_{\mathrm{appr},2,\ell}) \qquad & \\ - \tfrac{2\eta}{\omega\rho_0}\, G_2(\partial_t \operatorname{div} \mathbf{v}_{\mathrm{appr},2,\ell}) = \Big(\tfrac{2\eta}{\omega\rho_0}\Big)^{(\ell+1)/2} & \Big(G_1(\partial_t \operatorname{div} \mathbf{v}^{2,\ell}) \\ & + G_2\big(\partial_t \operatorname{div}(\mathbf{v}^{2,\ell-1} + \sqrt{\tfrac{2\eta}{\omega\rho_0}}\mathbf{v}^{2,\ell})\big)\Big), \quad \text{on } \partial\Omega, \end{aligned} \tag{4.14b}$$

which is identical to (4.1) with

$$\begin{aligned} &\alpha_1 = c^2/\omega^2 \sqrt{1/(2\omega\rho_0)}, \qquad \alpha_2 = (1+\gamma')\,\omega/(\rho_0 c^2), \qquad \alpha_3 = c^2/(2\omega^3\rho_0), \\ &g = \mathrm{i}\alpha_3\sqrt{\eta}(2\eta/\omega\rho_0)^{(\ell-1)/2}\kappa\ \partial_t \operatorname{div}(\mathbf{v}^{2,\ell-1} + \sqrt{2\eta/\omega\rho_0}\mathbf{v}^{2,\ell}) + (1+\mathrm{i})\alpha_1(2\eta/\omega\rho_0)^{\ell/2}\partial_t \operatorname{div}\mathbf{v}^{2,\ell}, \\ &h = \mathrm{i}\alpha_2\ \sqrt[4]{\eta}(2\eta/\omega\rho_0)^{(\ell-1)/2} \operatorname{div}(\mathbf{v}^{2,\ell-1} + \sqrt{2\eta/\omega\rho_0}\mathbf{v}^{2,\ell}). \end{aligned}$$

So Lemma 4.2 implies for any $\widetilde{\Omega}_{\subset} \subset \widetilde{\Omega}$

$$\|\delta\mathbf{v}_{\mathrm{appr},2,\ell}\|_{(H^1(\Omega))^2 \cap (H^m(\widetilde{\Omega}_{\subset}))^2} \leq C\,\eta^{(2\ell-2m+1)/4}. \tag{4.15}$$

Now, we continue with both cases $N = 1, 2$. By the triangle inequality and choosing $\ell = m + 2N - 1$ we obtain

$$\begin{aligned} \|\mathbf{v}_{\mathrm{appr},N} - \sum_{j=0}^{N} \Big(\tfrac{2\eta}{\omega\rho_0}\Big)^{j/2} \mathbf{v}_j\|_{(H^1(\Omega))^2\cap(H^m(\widetilde{\Omega}_{\subset}))^2} &= \|\delta\mathbf{v}_{\mathrm{appr},N,N}\|_{(H^1(\Omega))^2\cap(H^m(\widetilde{\Omega}_{\subset}))^2} \\ &= \|\delta\mathbf{v}_{\mathrm{appr},N,m+2N-1} + \sum_{j=N+1}^{m+2N-1} \Big(\tfrac{2\eta}{\omega\rho_0}\Big)^{j/2} \mathbf{v}^{N,j}\|_{(H^1(\Omega))^2\cap(H^m(\widetilde{\Omega}_{\subset}))^2} \\ \leq \|\delta\mathbf{v}_{\mathrm{appr},N,m+2N-1}\|_{(H^1(\Omega))^2\cap(H^m(\widetilde{\Omega}_{\subset}))^2} &+ \sum_{j=N+1}^{m+2N-1} \Big(\tfrac{2\eta}{\omega\rho_0}\Big)^{j/2} \|\mathbf{v}^{N,j}\|_{(H^1(\Omega))^2\cap(H^m(\widetilde{\Omega}_{\subset}))^2} \leq C_m \eta^{\frac{N+1}{2}}, \end{aligned} \tag{4.16}$$

and the statement of the lemma for the velocity follows.

In view of the relation (2.6) between $p_{\mathrm{appr},N}$ and $\mathbf{v}_{\mathrm{appr},N}$ and the relation [11, (2.12)] the proof of the estimates for the pressure for $N = 1, 2$ are equivalent to show that

$$\big|\operatorname{div}\delta\mathbf{v}_{\mathrm{appr},N,N}\big|_{H^1(\Omega)} = \big|\operatorname{div}\big(\mathbf{v}_{\mathrm{appr},N} - \sum_{j=0}^{N}\Big(\frac{2\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\mathbf{v}^j\big)\big|_{H^1(\Omega)} \leq C\eta^{\frac{N+1}{2}}.$$

For $N = 1$ this is a simple consequence of (4.11a) and the estimate (4.16) for $\delta\mathbf{v}_{\mathrm{appr},1,1}$,

$$\big|\operatorname{div}\delta\mathbf{v}_{\mathrm{appr},1,1}\big|_{H^1(\Omega)} = \frac{\omega^2}{c^2}\big\|\operatorname{div}\delta\mathbf{v}_{\mathrm{appr},1,1}\big\|_{L^2(\Omega)} \leq C\,\eta.$$

For $N = 2$ we have in addition to bound $\operatorname{div}\mathbf{v}^{2,1} \equiv \operatorname{div}\mathbf{v}^1$ and $\operatorname{div}\mathbf{v}^{2,2} \equiv \operatorname{div}\mathbf{v}^2$ in the $H^1(\Omega)$-seminorm, which is a consequence of (3.4a) and the assumption on the regularity of $\operatorname{curl}_{2D}\mathbf{f}$. We can conclude with (4.14a) and (4.16) that

$$\big|\operatorname{div}\delta\mathbf{v}_{\mathrm{appr},2,2}\big|_{H^1(\Omega)} \leq \frac{\omega^2}{c^2}\big\|\operatorname{div}\delta\mathbf{v}_{\mathrm{appr},2,2}\big\|_{L^2(\Omega)} + C\eta^{3/2}\big|\operatorname{div}\mathbf{v}^1\big|_{H^1(\Omega)} + C\eta^2\big|\operatorname{div}\mathbf{v}^2\big|_{H^1(\Omega)}. \leq C\,\eta^{3/2}.$$

This finishes the proof. □

4.3.2. *Near field approximation*

For the analysis of the near field approximation we introduce for $J \in \mathbb{N}_0$ the function spaces

$$\Pi^J(\lambda, X) := \Big\{\widetilde{\phi}(t,S) = \mathrm{e}^{-\lambda S}\sum_{j=0}^{J}\widetilde{\phi}_j(t)S^j, \widetilde{\phi}_j \in \mathbb{C}, \|\widetilde{\phi}_j\|_X < \infty, j = 0,\dots,J\Big\},$$

where $\operatorname{Re}\lambda > 0$, which are equipped with a norm defined by

$$\|\widetilde{\phi}\|^2_{\Pi^J(\lambda,X)} := \sum_{j=0}^{J}\frac{2(j+J)!}{(2\operatorname{Re}\lambda)^{2j}}\|\widetilde{\phi}_j\|^2_X. \tag{4.17}$$

It is easy verified that there exists a function $C(\lambda)$ such that

$$\|\partial_S\widetilde{\phi}\|^2_{\Pi^{J-1}(\lambda,X)} \le C(\lambda)\|\widetilde{\phi}\|^2_{\Pi^J(\lambda,X)}. \tag{4.18}$$

Here, we will use function spaces X related to the smoothness in tangential direction. Using Lemma A.1 in [11] the following mapping property of the operators E_ℓ can be easily shown by induction.

Lemma 4.5. *Let $\ell = \mathbb{N}_0$ and $k(\ell) = 2\lfloor\frac{\ell}{2}\rfloor$. Then, for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ and $m \in \mathbb{R}^+$ it holds*

$$e^{-\lambda S}E_\ell : H^m(\Gamma) \to \Pi^\ell(\lambda, H^{m-k(\ell)}(\Gamma)).$$

Definition 4.6 (Admissible cut-off function)**.** *We denote a monotone function $\chi \in C^\infty(\Omega)$ an admissible cut-off function, if there exist constants $0 < s_1 < s_0 < \frac{1}{2}\|\kappa\|^{-1}_{L^\infty(\Gamma)}$ such that $\chi \equiv 0$ outside an s_0-neighbourhood of $\partial\Omega$ and otherwise $\chi(\mathbf{x}) = \widehat{\chi}(s)$, where $\widehat{\chi}(s) = 1$ for $s < s_1$. For an admissible cut-off function χ we call by χ-neighbourhood the $\sup\operatorname{supp}(\widehat{\chi})$-neighbourhood of $\partial\Omega$.*

Lemma 4.7. *Let $\widetilde{\phi} \in \Pi^J(\lambda, H^2(\partial\Omega))$ for some $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, χ an admissible cut-off function, $\phi(\mathbf{x}) = \widetilde{\phi}(t, \frac{s}{\varepsilon})$ in the χ-neighbourhood of the boundary. Then, there exists a constant C_J independent of $\widetilde{\phi}$ such that the function $\mathbf{v}(\mathbf{x}) = \varepsilon\,\mathbf{curl}_{2D}(\phi\,\chi)$ satisfies*

$$\|\mathbf{v}\|_{H(\mathrm{div},\Omega)} \le C_J\sqrt{\varepsilon}\,\|\widetilde{\phi}\|_{\Pi^J(\lambda,H^1(\Gamma))}, \qquad \|\operatorname{curl}_{2D}\mathbf{v}\|_{L^2(\Omega)} \le \frac{C_J}{\sqrt{\varepsilon}}\,\|\widetilde{\phi}\|_{\Pi^J(\lambda,H^2(\Gamma))}.$$

Proof. Due to the presence of the cut-off function it suffices to estimate $\mathbf{v}$ in the χ-neighbourhood of the boundary. Here, we can use the local coordinate system and define $\widehat{\mathbf{v}} = (\widehat{v}_t, \widehat{v}_s)^\top$ such that $\mathbf{v}(\mathbf{x}) = \widehat{v}_t(t,s)\,\mathbf{n}^\perp(t) + 1/(1-s\kappa(t))\,\widehat{v}_s(t,s)\,\mathbf{n}(t)$ for $s < s_0 := \sup\operatorname{supp}(\widehat{\chi})$. Similiarly to Lemma 4.7 in [11] it follows that there exists a constant C such that for an operator $D \in \{\mathbf{e}_1^\top, \mathbf{e}_2^\top, \operatorname{curl}_{2D}, \operatorname{div}\}$,

$$\|D\mathbf{v}\|_{L^2(\Omega)} \le C\,\|D\widehat{\mathbf{v}}\|_{L^2(\Gamma\times(0,s_0))}. \tag{4.19}$$

Using the definition of $\mathbf{v}$ and the representation of $\mathbf{curl}_{2D}$ in local coordinates we find that

$$\widehat{v}_t(t,s) = -\partial_S\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}(s) - \varepsilon\,\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}'(s), \qquad \widehat{v}_s(t,s) = \varepsilon\,\partial_t\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}(s).$$

For the $L^2(\Gamma\times(0,s_0))$-norm of $\operatorname{div}\widehat{v}$ and $\operatorname{curl}_{2D}\widehat{v}$ we need the derivatives

$$\begin{aligned}
\partial_s\widehat{v}_t(t,s) &= -\varepsilon^{-1}\partial_S^2\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}(s) - 2\,\partial_S\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}'(s) - \varepsilon\,\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}''(s),\\
\partial_t\widehat{v}_t(t,s) &= -\partial_t\partial_S\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}(s) - \varepsilon\,\partial_t\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}'(s),\\
\partial_s\widehat{v}_s(t,s) &= \partial_t\partial_S\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}(s) + \varepsilon\partial_t\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}'(s),\\
\partial_t\widehat{v}_s(t,s) &= \varepsilon\,\partial_t^2\widetilde{\phi}(t,\tfrac{s}{\varepsilon})\widehat{\chi}(s).
\end{aligned}$$

Using the Laplace transform of monomials, Youngs inequality and the definition of the norm of $\Pi^J(\lambda,\mathbb{C})$ by (4.17) we find that

$$\int_0^\infty |\widetilde{\phi}(t,S)|^2\,\mathrm{d}S \le \sum_{j=0}^J\sum_{k=0}^J |\widetilde{\phi}_j||\widetilde{\phi}_k|\frac{(j+k)!}{(2\,\mathrm{Re}\lambda)^{j+k}} \le \sum_{j=0}^J \frac{1}{(2\,\mathrm{Re}\lambda)^{2j}}|\widetilde{\phi}_j|^2\sum_{k=0}^J(j+k)!$$
$$\le \sum_{j=0}^J\frac{2(j+J)!}{(2\,\mathrm{Re}\lambda)^{2j}}|\widetilde{\phi}_j|^2 = \|\widetilde{\phi}\|^2_{\Pi^J(\lambda,\mathbb{C})}.$$

Now, using the fact that $\mathrm{d}s = \varepsilon\,\mathrm{d}S$, (4.18) and Lemma 4.8 in [11] to estimate the terms with derivatives of $\widehat{\chi}$ we find that

$$\big\|\widehat{\mathbf{v}}\big\|_{H(\mathrm{div},\Gamma\times(0,s_0))} \le C\sqrt{\varepsilon}\,\|\widetilde{\phi}\|_{\Pi^J(\lambda,H^1(\Gamma))}, \qquad \big\|\operatorname{curl}_{2D}\widehat{\mathbf{v}}\big\|_{L^2(\Gamma\times(0,s_0))} \le \frac{C}{\sqrt{\varepsilon}}\|\widetilde{\phi}\|_{\Pi^J(\lambda,H^2(\Gamma))},$$

and with (4.19) the statement of the lemma follows. □

Lemma 4.8. *Let the assumptions in Lemma 4.4 be fulfilled. Then, for $N=0,1,2$ there exist constants $\eta_{\max}$ and C_N independent of η such that for all $\eta\in(0,\eta_{\max})$*

$$\Big\|\mathbf{v}^{BL}_{\mathrm{appr},N} - \sum_{j=0}^N\Big(\frac{2\,\eta}{\omega\rho}\Big)^{\frac{j}{2}}\mathbf{v}^j_{BL}\Big\|_{H(\mathrm{div},\Omega)} \le C_N\,\eta^{\frac{2N+3}{4}},$$
$$\Big\|\operatorname{curl}_{2D}\big(\mathbf{v}^{BL}_{\mathrm{appr},N} - \sum_{j=0}^N\Big(\frac{2\,\eta}{\omega\rho}\Big)^{\frac{j}{2}}\mathbf{v}^j_{BL}\big)\Big\|_{L^2(\Omega)} \le C_N\,\eta^{\frac{2N+1}{4}}.$$

Proof. We have defined the near field velocity $\mathbf{v}^{BL}_{\mathrm{appr},N}$ in (2.7) in terms of the near field velocity potential $\phi_{\mathrm{appr},N}(\mathbf{x}) = \widetilde{\phi}_{\mathrm{appr},N}(t,S)$ for $S = s\sqrt{\frac{\omega\rho_0}{2\,\eta}}$ and a cut-off function χ. The near field velocity terms $\mathbf{v}^j_{BL}$ of the asymptotic expansion are defined in Sec. 3.2 in a similar way in terms of near field velocity potentials $\phi^j(\mathbf{x}) = \widetilde{\phi}^j(t,S)$.

Hence, we start with the estimation of the near field velocity potentials in $\Gamma\times[0,\infty)$:

$$\widetilde{\phi}_{\mathrm{appr},N} - \sum_{j=0}^N\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\widetilde{\phi}^j = \frac12(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})S}\Big(\sum_{\ell=0}^N\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{\ell}{2}}E_\ell(\mathbf{v}_{\mathrm{appr},N}\cdot\mathbf{n}^\perp) - \sum_{j=0}^N\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\sum_{\ell=0}^j E_\ell(\mathbf{v}^{j-\ell}\cdot\mathbf{n}^\perp)\Big)$$
$$= \frac12(1+\mathrm{i})\mathrm{e}^{-(1-\mathrm{i})S}\Big(\sum_{\ell=0}^N\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{\ell}{2}}E_\ell((\mathbf{v}_{\mathrm{appr},N} - \sum_{j=0}^N\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\mathbf{v}^j)\cdot\mathbf{n}^\perp)$$
$$+\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{N+1}{2}}\sum_{\ell=1}^N E_\ell(\sum_{j=0}^{\ell-1}(\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\mathbf{v}^{N+1-\ell+j})\cdot\mathbf{n}^\perp)\Big).$$

Using Lemma 4.5 we find that for any $m\in\mathbb{R}^+$

$$\Big\|\widetilde{\phi}_{\mathrm{appr},N} - \sum_{j=0}^N\Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\widetilde{\phi}^j\Big\|_{\Pi^N(-(1-\mathrm{i}),H^m(\Gamma))} \le C_{1,N}\Big\|\mathbf{v}_{\mathrm{appr},N}\cdot\mathbf{n}^\perp - \Big(\frac{2\,\eta}{\omega\rho_0}\Big)^{\frac{j}{2}}\mathbf{v}^j\cdot\mathbf{n}^\perp\Big\|_{H^{m+\lfloor N/2\rfloor}(\partial\Omega)}$$
$$+ C_{2,N}\eta^{\frac{N+1}{2}}\sum_{j=0}^N\Big\|\mathbf{v}^j\cdot\mathbf{n}^\perp\Big\|_{H^{m+\lfloor N/2\rfloor}(\partial\Omega)}$$
$$\le C_N\,\eta^{\frac{N+1}{2}}, \tag{4.20}$$

where we have used Lemma 4.4 and the fact that for smooth enough boundary $\partial\Omega$ the terms $\mathbf{v}^j\cdot\mathbf{n}^\perp\in H^m(\partial\Omega)$ for any $m\in\mathbb{R}^+$ (which can be similarly proven than [11, Lemma 2.3]).

Now, using Lemma 4.7 with $\varepsilon = \sqrt{2\eta/(\omega\rho_0)}$ we find the desired statement. □

4.4. **Impedance boundary for velocity: Proof of stability and modelling error**

Now, we are in the position to prove the stability and estimates on the modelling error for the approximative solutions.

Proof of Lemma 2.1. The statement of the lemma is a simple consequence of Lemma 4.4, [11, Lemma 2.3] and the triangle inequality. □

Proof of Theorem 2.4. The estimates on the velocity in (2.9a) and (2.9b) follow from Lemma 2.2 of [11], Lemma 4.4, Lemma 4.8 and the triangle inequality.

Due the the equality

$$p^j = -\frac{\mathrm{i}\rho_0 c^2}{\omega}\operatorname{div}\mathbf{v}^j$$

and a similar definition of $p_{\mathrm{appr},N}$ as multiple of $\operatorname{div}\mathbf{v}_{\mathrm{appr},N}$ in (2.6) the estimate on the pressure in (2.9a) follows by the estimate on the $H^1(\Omega)$-norm of the divergence of the velocities in Lemma 4.4, Lemma 2.2 of [11] and the triangle inequality. □

4.5. **Asymptotic equivalence of the two approximate solutions**

For the proof of the asymptotic equivalence of the two approximative solutions we rely on the following lemma on the asymptotic equivalence of the pressure approximation $q_{\mathrm{appr},N}$ and the related asymptotic expansion of order N. The proof of this lemma uses elements of the proof of Lemma 4.4 and will be left to the reader.

Lemma 4.9. *Let the assumptions of Theorem 2.7 be fulfilled and let $\mathbf{f} \in (L^2(\Omega))^2 \cap (H^m(\widetilde{\Omega}))^2$ for any $m \in \mathbb{N}$ in some neighbourhood $\widetilde{\Omega} \subset \Omega$ of $\partial\Omega$,* i.e., *$\partial\Omega \subset \partial\widetilde{\Omega}$ and $\operatorname{curl}_{2D}\mathbf{f} \in H^m(\Omega)$. Then, for any $m \geq 1$, any $\widetilde{\Omega}_{\subset} \subset \widetilde{\Omega}$ there exist constants $\eta_{\max,m}$ and $C_{m,N}$ independent of η such that for the solutions $q_{\mathrm{appr},N}$ of the approximative models* (2.12) *and any $\eta \in (0, \eta_{\max,m})$ it holds*

$$\Big\| q_{\mathrm{appr},N} - \sum_{j=0}^{N}\left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}} p_j \Big\|_{H^1(\Omega)\cap H^m(\widetilde{\Omega})} \leq C_{m,N}\eta^{\frac{N+1}{2}}.$$

Proof of Lemma 2.5. The proof consists of two parts, first we proof the identity of the approximation for $N = 0$ and $N = 1$, and their equivalence for $N = 2$.

(i) Let $N = 0$. We first take the divergence of (2.3a) and second we evaluate the normal component of (2.3a) on $\partial\Omega$, use (2.3b), and insert $p_{\mathrm{appr},0}$ given by (2.6) in both cases. This gives (2.10) when $q_{\mathrm{appr},0}$ is replaced by $p_{\mathrm{appr},0}$. Hence, the two pressure approximations are identical.

As (2.3a) and (2.6) implies

$$\mathbf{v}_{\mathrm{appr},0} = \frac{\mathrm{i}}{\rho_0\omega}(\mathbf{f} - \nabla p_{\mathrm{appr},0})$$

which exactly the relation between $\mathbf{w}_{\mathrm{appr},0}$ and $q_{\mathrm{appr},0}$ given by (2.13a). So, the two velocity approximations are identical too. As the boundary correctors $\mathbf{v}^{BL}_{\mathrm{appr},0}$ and $\mathbf{w}^{BL}_{\mathrm{appr},0}$ are both defined by (2.7) and (2.8) they are also identical.

For $N = 1$ the identity of the approximations follow in exactly the same way.

(ii) Let $N = 2$. Indeed the pressure approximation $p_{\mathrm{appr},2}$ given by (2.6) satisfy (2.12a) as $q_{\mathrm{appr},2}$, however, it fulfills a different boundary condition than (2.12b). Using Lemma 4.4, Lemma 4.9 and the triangle inequality we find for any $m \geq 1$ that

$$\big\|p_{\mathrm{appr},2} - q_{\mathrm{appr},2}\big\|_{H^1(\Omega)\cap H^m(\widetilde{\Omega})} \leq \big\|p_{\mathrm{appr},2} - \sum_{j=0}^{2}\left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}} p_j\big\|_{H^1(\Omega)\cap H^m(\widetilde{\Omega})} + \big\|q_{\mathrm{appr},2} - \sum_{j=0}^{2}\left(\frac{2\eta}{\omega\rho_0}\right)^{\frac{j}{2}} p_j\big\|_{H^1(\Omega)\cap H^m(\widetilde{\Omega})} \leq C_m\,\eta^{3/2}, \tag{4.21}$$

where C_m are constants independent of η. Then, insertion of (2.6) into (2.5a) implies that the a-posteriori defined velocity is given by

$$\mathbf{v}_{\mathrm{appr},2} = \frac{\mathrm{i}}{\rho_0\omega}(\mathbf{f} - \nabla p_{\mathrm{appr},2}) - \frac{\eta+\eta'}{\rho_0^2 c^2}\nabla p_{\mathrm{appr},2} + \frac{\eta}{\rho_0^2\omega^2}\,\mathbf{curl}_{2D}\operatorname{curl}_{2D}\mathbf{f},$$

which is exactly the relation between $\mathbf{w}_{\mathrm{appr},2}$ and $p_{\mathrm{appr},2}$, see (2.13b). Hence, the difference is given by

$$\mathbf{v}_{\mathrm{appr},2} - \mathbf{w}_{\mathrm{appr},2} = \frac{\mathrm{i}}{\rho_0\omega}\nabla(p_{\mathrm{appr},2} - q_{\mathrm{appr},2}) - \frac{\eta+\eta'}{\rho_0^2 c^2}\nabla(p_{\mathrm{appr},2} - q_{\mathrm{appr},2}), \tag{4.22}$$

and (4.21) implies

$$\|\mathbf{v}_{\mathrm{appr},2} - \mathbf{w}_{\mathrm{appr},2}\|_{(L^2(\Omega))^2\cap(H^{m-1}(\widetilde{\Omega}))^2} \leq C_m\,\eta^{3/2}. \tag{4.23}$$

Applying curl_{2D} to (4.22) the right hand side vanishes and we see that $\mathrm{curl}_{2D}\,\mathbf{v}_{\mathrm{appr},2} \equiv \mathrm{curl}_{2D}\,\mathbf{w}_{\mathrm{appr},2}$.

Now, applying div to (2.13b) and inserting (2.12a) we find that

$$\operatorname{div}\mathbf{w}_{\mathrm{appr},2} = \frac{\mathrm{i}\omega}{\rho_0 c^2} q_{\mathrm{appr},2},$$

which exactly the relation between $\operatorname{div}\mathbf{v}_{\mathrm{appr},2}$ and $p_{\mathrm{appr},2}$ given by (2.6). Hence,

$$\operatorname{div}(\mathbf{v}_{\mathrm{appr},2} - \mathbf{w}_{\mathrm{appr},2}) = -\frac{\mathrm{i}\omega}{\rho_0 c^2}\,(p_{\mathrm{appr},2} - q_{\mathrm{appr},2}),$$

and so (4.21) implies with a constant C independent of η

$$\|\operatorname{div}(\mathbf{v}_{\mathrm{appr},2} - \mathbf{w}_{\mathrm{appr},2})\|_{L^2(\Omega)} = C\,\eta^{3/2}.$$

The estimate on the difference of the boundary layer correctors is similar to the proof of Lemma 4.8, and we give details only if necessary for understanding. As the boundary layer correctors have both the same definition, see (2.7) and (2.8), and using Lemma 4.5 and (4.7) we find that

$$\eta^{-1/4}\|\mathbf{v}^{BL}_{\mathrm{appr},2} - \mathbf{w}^{BL}_{\mathrm{appr},2}\|_{H(\mathrm{div},\Omega)} + \eta^{1/4}\|\,\mathrm{curl}_{2D}(\mathbf{v}^{BL}_{\mathrm{appr},2} - \mathbf{w}^{BL}_{\mathrm{appr},2})\|_{L^2(\Omega)} \leq C\,\|(\mathbf{v}_{\mathrm{appr},2} - \mathbf{w}_{\mathrm{appr},2})\cdot\mathbf{n}^{\perp}\|_{H^3(\partial\Omega)}$$

and using (4.23) we find the desired result.

This finishes the proof. □

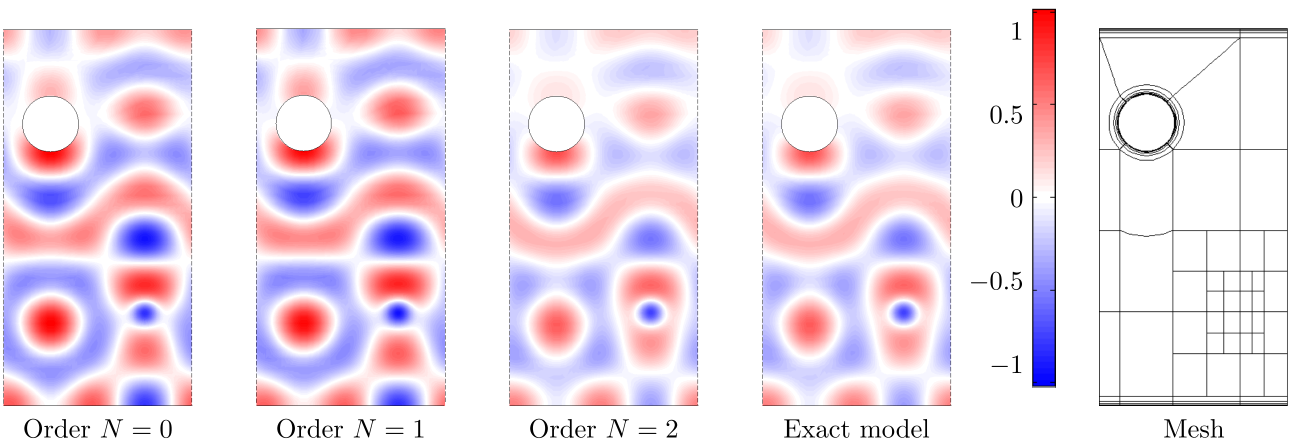


FIGURE 2. Comparison of the real part of the pressure offer the approximate models of order $N = 0, 1, 2$ to the exact pressure ($\eta = 4\cdot 10^{-6}$, $\omega = 15$). The mesh resolving the boundary layers used in the FEM of higher order is shown in the right subfigure.

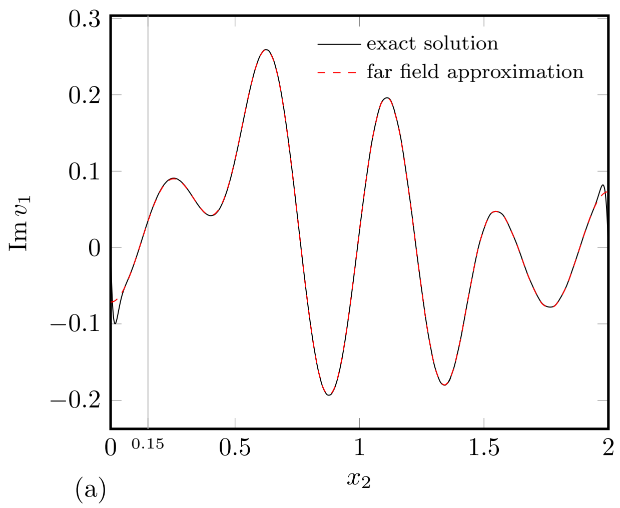
5. Numerical results

For a torus domain with omitted disk, see Fig. 1(b), we have performed numerical simulations for the exact model (2.2) and the approximative pressure models (2.10)–(2.13). We consider the problem in dimensionless quantities. The domain is the rectangle $[0, 1] \times [0, 2]$, where the left and right sides are identified with each other, and the disk of diameter 0.30 is centered at $(0.25, 1.5)$. As source $\mathbf{f}$ we use the gradient of the Gaussian $\exp(-|\mathbf{x}-\mathbf{x}_0|^2/0.005)$ with $\mathbf{x}_0 = (0.75, 0.5)^\top$. Furthermore, we choose for the speed of sound $c = 1$, the (mean) air density as $\rho_0 = 1$ and neglect the second viscosity, $\eta' = 0$.

For the simulation we have used high-order finite elements within the numerical C++ library *Concepts* [3] to push the discretisation error below the modelling error. To resolve the boundary layers in the velocity, we refine the mesh geometrically towards the boundary, see the right picture in Fig. 2. The high gradients of the source term are considered in a further (geometric) mesh refinement towards the point $\mathbf{x}_0$. The far field solution of the approximative models could be computed to a high precision on a rather coarse mesh as no boundary layer has to be resolved. Anyhow, we have computed the far field solution on the mesh illustrated in Fig. 2, which allowed us firstly a straightforward evaluation of norms of the error functions and secondly a representation of the sum of far and near field on the same mesh. We have chosen the polynomial degree to be 11 to obtain low enough discretisation errors such that the modelling errors become visible.

For $\eta = 4 \cdot 10^{-6}$ and $\omega = 15$ we have illustrated the exact pressure and its approximation $q_{\text{appr},0}$, $q_{\text{appr},1}$ and $q_{\text{appr},2}$ of order 0, 1 and 2, respectively, in the first four subfigures of Fig. 2. The colour scaling in all the four subfigures matches to allow for a direct comparison. In this example the approximations of order 0 and order 1 provide a coarse field description, where the pressure amplitude is overestimated. The approximation of order 2, however, predicts the exact quite well. For this example, however, with $\eta = 1.6 \cdot 10^{-3}$ we have illustrated the boundary layer in the tangential velocity component in Fig. 3, both for the exact model and the approximation of order 2. The boundary layer thickness is $d_{\text{BL}} = \sqrt{2\eta/\omega\rho_0} = 1.46 \cdot 10^{-2}$. Here, the approximative far field velocity $w_{\text{appr},2}$ and the respective near field were computed from the pressure approximation $q_{\text{appr},2}$. The representation of the velocity is in a side view for $x_1 = 0$, for which the first component is tangential to the lower boundary at $x_2 = 0$. The approximate solution is the sum of the far field, which does not fulfill a homogeneous Dirichlet boundary condition, and a correcting near field. The far field solution approximates the exact one away from the boundary very well, see Fig. 3(a). In its turn Fig. 3b) shows the near field correction and the behaviour of the solutions close to the wall.

To analyse the modelling error in dependence of the viscosity, and hence ε, we have performed numerical simulations on the simple rectangular torus domain $\Omega = [0, 1] \times [0, 1]$ (*i.e.*, without the hole of the previous problem), for which the left and right sides are again identified with each other. The other parameters are identical to those of the previous problem. The studied frequency $\omega = 15$ is not a Neumann eigenfrequency of $-\Delta$, the closest eigenfrequencies are $\sqrt{20}\,\pi \approx 14.05$ and $5\,\pi \approx 15.71$. We compute the error functions on the subdomain $\Omega_\delta = [0, 1] \times [0.2, 0.8]$, which has a distance of $\delta = 0.2$ to the boundary of Ω. This distance is large enough such that in Ω_δ for the studied viscosities the contribution of the exponentially decaying near fields can



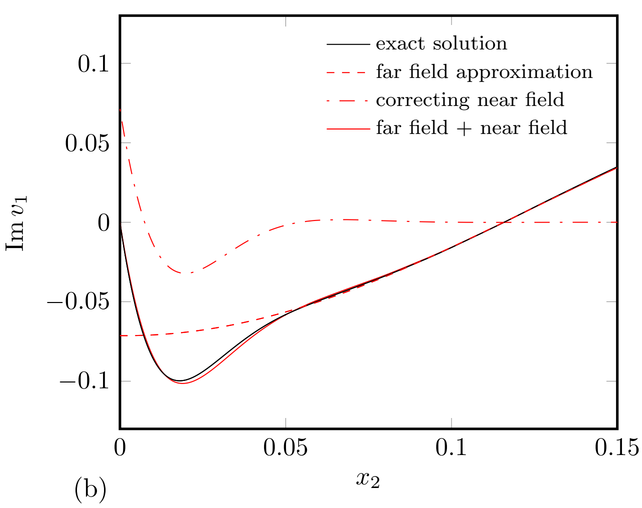


FIGURE 3. Imaginary part of first velocity component in side view for $x_1 = 0$ with $\sqrt{\eta} = 4 \cdot 10^{-2}$, which is at $x_2 = 0$ tangential to the bottom wall. The exact solution v_1 and the approximate (far field) solution $(v_{\text{appr},2})_1$ of order 2, the corresponding near field $(v^{BL}_{\text{appr},2})_1$ and the sum of both are shown, in (a) for the whole line $x_1 = 0$, and in (b) close to the wall.

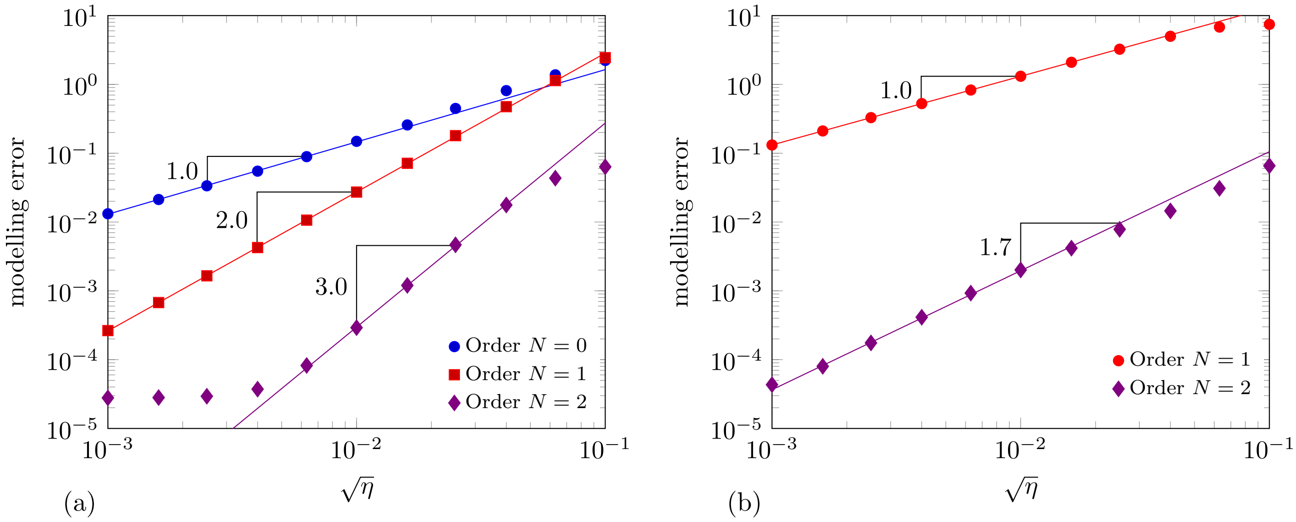


FIGURE 4. The relative modelling error $\|p - q_{\mathrm{appr},N}\|_{H^1(\Omega)}/\|p\|_{H^1(\Omega)} + \|\mathbf{v} - \mathbf{w}_{\mathrm{appr},N}\|_{H(\mathrm{div},\Omega)}/\|\mathbf{v}\|_{H(\mathrm{div},\Omega)}$ for $N = 0, 1, 2$ w.r.t. square root of viscosity for (a) a dimensionless frequency value $\omega = 15$ and (b) an eigenfrequency $\omega = \sqrt{20}\,\pi$.

be neglected. In Fig. 4(a) we have shown the relative modelling error

$$\|p - q_{\mathrm{appr},N}\|_{H^1(\Omega_\delta)}/\|p\|_{H^1(\Omega_\delta)} + \|\mathbf{v} - \mathbf{w}_{\mathrm{appr},N}\|_{H(\mathrm{div},\Omega_\delta)}/\|\mathbf{v}\|_{H(\mathrm{div},\Omega_\delta)}$$

for the approximative solutions of order 0, 1 and 2 in dependence of the (square root of the) viscosity. We observe linear convergence in $\sqrt{\eta}$ for the approximative solution of order 0, quadratic convergence for that of order 1 and convergence of order 3 for the approximative solution of order 2. These results verify that the estimates in Theorem 2.4 are sharp. The error is computed on the above mesh with polynomial degree 14 and included indeed a small discretisation error which becomes visible for small viscosities ($\sqrt{\eta} < 5 \cdot 10^{-3}$) and the approximative model of order 2.

The theoretical estimates are for non-resonant frequencies and the constants may blow up if the frequency tends to a resonant one, *i.e.*, a Neumann eigenfrequency of $-\Delta$. The eigenfrequencies for the studied example are $\omega_{k,m} = \pi\sqrt{k^2 + 4m^2}$, for $k \in \mathbb{N}, m \in \mathbb{N}_0$. In addition we analyse the modelling error in dependence of the viscosity for an eigenfrequency value $\omega_0 = \omega_{2,2} = \omega_{4,1} = \sqrt{20}\,\pi$, see Fig. 4(b). The convergence in this case looses in order, *i.e.*, linear convergence in $\sqrt{\eta}$ for the approximative solution of order 1, convergence of order 1.7 for order 2 and the approximative solution of order 0 explodes and is not represented in the picture.

Furthermore, we analyse the modelling errors of the three approximative solutions in dependence of the frequency for the rectangular domain and $\eta = 1.6 \cdot 10^{-3}$, see Fig. 5. The approximate solution of order 0 and so the modelling error blows up close to the eigenfrequencies. However, the approximate solution of order 1 blows up only close to the eigenfrequency values $\omega_{k,0} = k\pi$ for $k \in \mathbb{N}$. That could be explained by the fact that for $m = 0$ in this example the velocity and so its divergence is constant in x_1 and the additional term in the boundary condition of order 1 disappears. In this case, the order 1 approximation at that frequencies becomes identical to that of order 0. Conversely, the error of the approximate solution of order 2, due to the additional term in the domain, always stays lower than $3 \cdot 10^{-2}$ and, as it was shown earlier, converges w.r.t. viscosity even at the resonance. Yet, in this work we will leave that sentence without a proof and the numerical results are presented for illustration reason only.

Note, that the above simulation corresponds for dimensionful quantities for example to a rectangular domain of size $4\,\mathrm{cm} \times 8\,\mathrm{cm}$, where the hole has a diameter of $1.2\,\mathrm{cm}$, a frequency $\omega = 5.146\,\mathrm{kHz}$, a speed of sound in air $c = 343\,\mathrm{m/s}$, a mean density of air $\rho_0 = 1.2\,\mathrm{kg/m^3}$. Then, a dynamic viscosity of air $\eta = 17.1\,\mathrm{mPa\,s}$ corresponds to a dimensionless viscosity of $1.04 \cdot 10^{-6}$ (dimensionless value of $\sqrt{\eta}$ would be $1.02 \cdot 10^{-3}$), which is close to the lowest viscosity value studied in the above experiments.

6. Conclusion

In this article the acoustic wave propagation in viscous gases inside a bounded two-dimensional domain has been studied as a solution of the compressible linearised Navier-Stokes equation. In frequency domain the

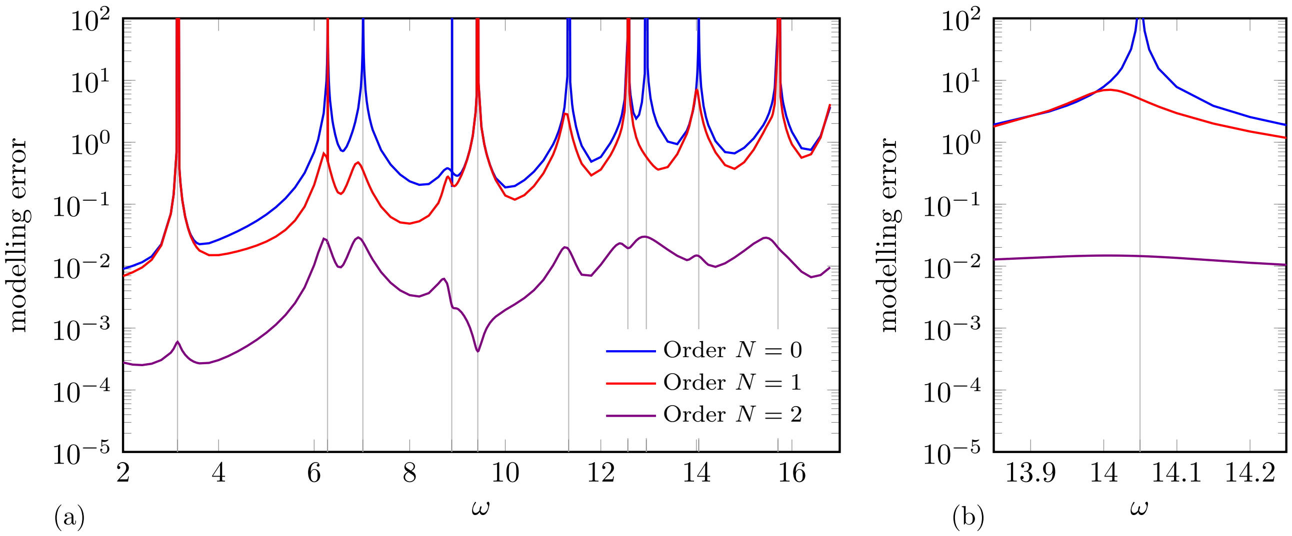


FIGURE 5. The modelling error $\|p - q_{\mathrm{appr},N}\|_{H^1(\Omega)}/\|p\|_{H^1(\Omega)} + \|\mathbf{v} - \mathbf{w}_{\mathrm{appr},N}\|_{H(\mathrm{div},\Omega)}/\|\mathbf{v}\|_{H(\mathrm{div},\Omega)}$ for $N = 0, 1, 2$ w.r.t. dimensionless frequency ω for $\eta = 1.6 \cdot 10^{-3}$.

governing equations are decoupled in equations for the velocity and pressure, where the pressure equation lacks boundary conditions. The velocity exhibits a boundary layer on rigid walls, whose extend scales with the square root of the viscosity and the finite element discretisation requires a heavy mesh refinement in the neighbourhood of the wall. Using the technique of multiscale expansion for small viscosities impedance boundary conditions for velocity and pressure are derived up to second order. The derivation and presented analysis is based on a previous work by the authors [11], where the complete asymptotic expansion of velocity and pressure has been derived. It has be shown that the velocity is represented as a sum of a far field expansion, which does not exhibits a boundary layer, and a correcting near field expansion close to the wall. For the pressure, which does not exhibit a boundary layer, there is only a far field expansion and a near field expansion is absent.

Using boundary conditions for the pressure presented in this work and respective partial differential equations pressure approximations are defined independently of respective velocities. The zero-th order condition is the well-known Neumann boundary condition for rigid walls, and the conditions of first or second order take into account absorption inside the boundary layer. The velocity boundary condition is for a far field approximation, whose finite element discretisation does not need a special mesh refinement close to walls. Here a boundary layer contribution depending on the far field velocity can be added to obtain an overall highly accurate description of the velocity. The derivation of the boundary conditions for either pressure or velocity include curvature effects, where the curvature becomes present in the boundary conditions of order 2.

The approximative models including impedance boundary conditions are justified by a stability and error analysis. The results of the numerical experiments have been provided to illustrate the stability and error estimates. Although, throughout the article the frequency is assumed to be not an eigenfrequency of the limit problem for vanishing viscosity, we show by numerical computations that the second order model provides accurate approximations for all frequencies and the first order model except some of the above mentioned eigenfrequencies. This results give a foundation for future studies for the case of resonances of the limit problem in bounded domains.

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