Port-Hamiltonian descriptor systems

Christopher Beattie* and Volker Mehrmann† and Hongguo Xu‡ and Hans Zwart §

May 29, 2017

Abstract

The modeling framework of port-Hamiltonian systems is systematically extended to constrained dynamical systems (descriptor systems, differential-algebraic equations). A new algebraically and geometrically defined system structure is derived. It is shown that this structure is invariant under equivalence transformations, and that it is adequate also for the modeling of high-index descriptor systems. The regularization procedure for descriptor systems to make them suitable for simulation and control is modified to deal with the port-Hamiltonian structure. The relevance of the new structure is demonstrated with several examples.

Keywords: port-Hamiltonian system, descriptor system, differential-algebraic equation, passivity, stability, system transformation, differentiation-index, strangeness-index, skew-adjoint operator.
AMS subject classification.: 93A30, 93B17, 93B11.

1 Introduction

Modeling packages such as MODELICA (`https://www.modelica.org/`), MATLAB/SIMULINK (`http://www.mathworks.com`) or SIMPACK [42] have come to provide excellent capabilities for the automated generation of models describing dynamical systems originating in different physical domains that may include mechanical, mechatronic, fluidic, thermic, hydraulic, pneumatic, elastic, plastic, or electric components [1, 16, 20, 40, 41]. Due to the explicit incorporation of constraints, the resulting systems comprise *differential-algebraic equations (DAEs)*, also referred to as *descriptor systems* in the system theory context. Descriptor systems may contain hidden constraints, consistency requirements for initial conditions, and unexpected regularity requirements. Therefore, these models usually require further regularization to be suitable for numerical simulation and control, see [11, 27, 30]. Our main focus will be on linear-time varying descriptor systems, as they may arise from the linearization of

nonlinear DAE systems along a (non-stationary) reference trajectory, see [10]. These have the form

$$
\begin{aligned}
E(t)\dot{x} &= A(t)x + B(t)u, \\
y &= C(t)x + D(t)u,
\end{aligned}
\tag{1}
$$

together with an initial condition $x(t_0) = x_0$. The coefficient matrices $E, A \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $C \in C^0(\mathbb{I}, \mathbb{R}^{m,n})$, and $D \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$, where we denote by $C^j(\mathbb{I}, \mathcal{X})$ $j \in \{0, 1, 2, 3, \ldots\}$ the set of j-times continuously differentiable functions from a compact time interval $\mathbb{I} = [t_0, t_f] \subseteq \mathbb{R}$ to $\mathcal{X} = \mathbb{R}^n$. If it is otherwise clear from the context, the argument t of the coefficient functions is suppressed.

An important development in recent years has been to employ *energy based modeling* via bond graphs [4, 12]. This has been implemented recently in 20-SIM (http://www.20sim.com/), for example. The resulting systems have a *port-Hamiltonian (pH) structure*, see e. g. [18, 24, 33, 37, 36], that encodes underlying physical principles such as conservation laws directly into the structure of the system model. The standard form for *pH systems* appears as

$$
\begin{aligned}
\dot{x} &= (J - R)\,\nabla_x \mathcal{H}(x) + (B - P)u, \\
y &= (B + P)^T \nabla_x \mathcal{H}(x) + (S + N)u,
\end{aligned}
\tag{2}
$$

where the function $\mathcal{H}(x)$ is the *Hamiltonian* which describes the distribution of internal energy among energy storage elements of the system, $J = -J^T \in \mathbb{R}^{n,n}$ is the *structure matrix* describing energy flux among energy storage elements within the system; $R = R^T \in \mathbb{R}^{n,n}$ is the *dissipation matrix* describing energy dissipation/loss in the system; $B \pm P \in \mathbb{R}^{n,m}$ are *port* matrices, describing the manner in which energy enters and exits the system, and $S + N$, with $S = S^T \in \mathbb{R}^{m,m}$ and $N = -N^T \in \mathbb{R}^{m,m}$, describes the direct *feed-through* from input to output. It is necessary that

$$
W = \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0,
\tag{3}
$$

where we write $W > 0$ (or $W \geq 0$) to assert that a real symmetric matrix W is positive definite (or positive semi-definite). Port-Hamiltonian systems generalize *Hamiltonian systems*, in the sense that the *conservation of energy* for Hamiltonian systems is replaced by the *dissipation inequality*:

$$
\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{t_1} y(t)^T u(t)\ dt.
\tag{4}
$$

In the language of system theory, (4) shows that the dynamical system described in (2) is a *passive* system [8]. Furthermore, $\mathcal{H}(x)$ defines a Lyapunov function for the unforced system, so pH systems are implicitly Lyapunov stable [21]. Inequality (4) is an immediate consequence of (3) and holds even when the coefficient matrices J, R, B, P, S, and N depend on x or explicitly on time t, see [31], or when they are defined as linear operators acting on infinite dimensional spaces [24, 39].

The physical properties of pH systems are encoded in the algebraic structure of the coefficient matrices and in geometric structures associated with the flow of the differential equation. This leads to a remarkably robust modeling paradigm that greatly facilitates the combination and manipulation of pH systems. Note in particular that the family of pH systems is closed under *power-conserving interconnection* (see [25]); model reduction of pH systems via

Galerkin projection yields (smaller) pH systems [2, 19, 35]; and conversely, pH systems are easily extendable in the sense that new state variables can be included while preserving the structure of (2), and so, the range of application of the model can be increased while ensuring that the basic conservation principle (4) remains in force.

When state constraints are included in a pH system, the resulting system is a *port-Hamiltonian descriptor system (differential-algebraic equation)* (pHDAE). pHDAE systems arise also in singularly perturbed pH systems when small parameters are set to zero, see [38]. Significantly, there is no systematic way that has yet emerged to describe this problem class consistently, in a way that reflects both the pH structure and the DAE structure accurately. The first main topic of this paper is to propose such a systematic approach. This is a challenging task, in particular when constraints of the DAE are 'hidden', which is often signaled with the terminology 'high-index DAE' [5, 27, 30]. Such DAEs are not well-suited for numerical simulation and control and so, either a reformulation or a regularization of the model must first be carried out, [11, 27]. We will briefly summarize the fundamentals of this technique in Section 4.

It is sometimes stated in the literature, see e. g. [38], that port-Hamiltonian DAEs are of *differentiation-index at most one*, i. e., that they do not contain hidden constraints arising from derivatives. In contrast, we will show that higher-index pHDAEs are actually very common and so a regularization procedure is necessary. Unfortunately, the usual regularization strategies do not preserve a given pHDAE structure of the model and so, how one should go about this task while respecting pHDAE structure is the second main topic of the paper.

The paper is organized as follows. In Section 2 we give a definition of *port-Hamiltonian differential-algebraic systems* and demonstrate that this is a relevant class for many applications. The main properties of this new class of pHDAE systems (such as stability and dissipativity) are discussed in Section 3. Section 4 extends the definition to the nonlinear case. The analysis of 'index at most one' pHDAEs is discussed in Section 5 while the structured regularization procedure is discussed in Section 6.

2   Linear port-Hamiltonian Differential-Algebraic Equations

In this section we introduce a new definition of systems of port-Hamiltonian descriptor systems (pHDAEs). Our new definition is slightly different from the concepts discussed in [38] and is based on the concept of skew-adjoint differential-algebraic operators, see [29] for the corresponding self-adjoint case.

Definition 1 *A (differential-algebraic) operator*

$$\mathcal{L} := \mathcal{E}\frac{d}{dt} - \mathcal{A} : \Omega \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n)$$

with coefficient functions $\mathcal{E} \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $\mathcal{A} \in C(\mathbb{I}, \mathbb{R}^{n,n})$ is called skew-adjoint, *if $\mathcal{E}^T(t) = \mathcal{E}(t)$ and $\dot{\mathcal{E}}(t) = -(\mathcal{A}(t) + \mathcal{A}^T(t))$ for all $t \in \mathbb{I}$.*

This definition is motivated by the following observation: starting with vector functions $x_1(t), x_2(t)$ that are absolutely continuous on the interval $\mathbb{I} = (t_0, t_f)$ each with square integrable derivative and $x_i(t_0) = x_i(t_f) = 0$ for $i = 1, 2$, and then denoting the usual L_2 inner

product as $\langle x_1, x_2 \rangle = \int_{t_0}^{t_f} x_2^T x_1 \, dt$, we have

$$
\begin{aligned}
\langle x_1, \mathcal{L}(x_2) \rangle &= \langle x_1, \mathcal{E}\dot{x}_2 - \mathcal{A}x_2 \rangle = \langle x_1, \frac{d}{dt}(\mathcal{E}x_2) - \mathcal{A}x_2 - \dot{\mathcal{E}}x_2 \rangle \\
&= x_2^T \mathcal{E} x_1 \big|_{t_0}^{t_f} - \langle \mathcal{E}^T \dot{x}_1, x_2 \rangle - \langle (\mathcal{A}^T + \dot{\mathcal{E}}^T) x_1, x_2 \rangle \\
&= \langle -\mathcal{E}^T \dot{x}_1 - (\mathcal{A}^T + \dot{\mathcal{E}}^T) x_1, x_2 \rangle = \langle -\mathcal{E}\dot{x}_1 + \mathcal{A}x_1, x_2 \rangle.
\end{aligned}
$$

So formally, the adjoint operator $\mathcal{L}^*$ satisfies $\mathcal{L}^* = -\mathcal{L}$. Note the boundary terms arising in partial integration will vanish under a wide variety of conditions replacing the requirement of zero end conditions on $x_1(t)$ and $x_2(t)$.

Skew-adjoint operators stay skew-adjoint under time-varying congruence transformations.

Lemma 2 *Consider a skew-adjoint differential-algebraic operator*

$$
\mathcal{L} := \mathcal{E}\frac{d}{dt} - \mathcal{A} : \Omega \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n)
$$

with coefficient functions $\mathcal{E} \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ and $\mathcal{A} \in C(\mathbb{I}, \mathbb{R}^{n,n})$. Then for every $\mathcal{V} \in C^1(\mathbb{I}, \mathbb{R}^{n,r})$, the operator $\mathcal{L}_\mathcal{V}$ defined by

$$
\mathcal{L}_\mathcal{V}(x) := \mathcal{V}^T \mathcal{E} \mathcal{V} \dot{x} - (\mathcal{V}^T \mathcal{A} \mathcal{V} - \mathcal{V}^T \mathcal{E} \dot{\mathcal{V}}) x
$$

is again skew-adjoint, i. e., $\mathcal{L}_\mathcal{V}^* = -\mathcal{L}_\mathcal{V}$.

Proof. Since $\mathcal{V}^T \mathcal{E} \mathcal{V} = (\mathcal{V}^T \mathcal{E} \mathcal{V})^T$, it remains to consider the coefficient of x. Using $\mathcal{E}^T = \mathcal{E}$ and $\dot{\mathcal{E}} = -(\mathcal{A} + \mathcal{A}^T)$, we have

$$
\begin{aligned}
\frac{d}{dt}(\mathcal{V}^T \mathcal{E} \mathcal{V}) &= \dot{\mathcal{V}}^T \mathcal{E} \mathcal{V} + \mathcal{V}^T \dot{\mathcal{E}} \mathcal{V} + \mathcal{V}^T \mathcal{E} \dot{\mathcal{V}} \\
&= \dot{\mathcal{V}}^T \mathcal{E} \mathcal{V} - \mathcal{V}^T (\mathcal{A} + \mathcal{A}^T) \mathcal{V} + \mathcal{V}^T \mathcal{E} \dot{\mathcal{V}} \\
&= -(\mathcal{V}^T \mathcal{A} \mathcal{V} - \mathcal{V}^T \mathcal{E} \dot{\mathcal{V}}) - (\mathcal{V}^T \mathcal{A} \mathcal{V} - \mathcal{V}^T \mathcal{E} \dot{\mathcal{V}})^T.
\end{aligned}
$$

□

It should be noted that for any $t \in \mathbb{I}$ and $x \in C(\mathbb{I}, \mathbb{R}^n)$ we have $\mathcal{L}_\mathcal{V}(x(t)) = \mathcal{V}^T(t)\mathcal{L}(\mathcal{V}(t)x(t))$.

Remark 3 Note that in Lemma 2 we do not need that the transformation matrix V is invertible. This implies, in particular, that with a projection matrix

$$
V = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}
$$

the projected system is still skew-adjoint.

Using the definition of skew-adjoint differential-algebraic operators we now present a definition of pHDAEs.

Definition 4 *A linear variable coefficient descriptor system of the form*

$$
\begin{aligned}
E\dot{x} &= [(J-R)Q - EK]\,x + (B-P)u, \\
y &= (B+P)^T Q x + (S+N)u,
\end{aligned} \tag{5}
$$

with $E, Q \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $J, R, K \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B, P \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $S = S^T, N = -N^T \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$, is called port-Hamiltonian descriptor system *(port-Hamiltonian differential-algebraic system) (pHDAE) if the following properties are satisfied:*

i) the differential algebraic operator

$$\mathcal{L} := Q^T E \frac{d}{dt} - (Q^T JQ - Q^T EK) : D \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n) \qquad (6)$$

is skew-adjoint, *i. e. we have that* $Q^T E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *and for all* $t \in \mathbb{I}$,

$$
\begin{aligned}
Q^T(t)E(t) &= E^T(t)Q(t), \; and \\
\frac{d}{dt}(Q^T(t)E(t)) &= Q^T(t)[E(t)K(t) - J(t)Q(t)] + [E(t)K(t) - J(t)Q(t)]^T Q(t);
\end{aligned}
$$

ii) the matrix function $Q^T E$ is bounded from below by a constant symmetric matrix H_0, i. e., $Q^T(t)E(t) - H_0 \geq 0$ for all $t \in \mathbb{I}$;

ii) the matrix function

$$W := \begin{bmatrix} Q^T R Q & Q^T P \\ P^T Q & S \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{n+m,n+m}) \qquad (7)$$

is positive semidefinite, i. e., $W(t) = W^T(t) \geq 0$ for all $t \in \mathbb{I}$.

The associated Hamiltonian *is defined as*

$$\mathcal{H}(x) := \frac{1}{2} x^T Q^T E x : C^1(\mathbb{I}, \mathbb{R}^n) \to \mathbb{R}. \qquad (8)$$

Besides the matrix function E in front of the derivative and the different definition of the Hamiltonian, which gives the option of having singular matrices E and Q, a major difference to the definition of standard pH systems is the extra additive term $-EKx$ on the right hand side of (5), which is needed to accommodate time-varying changes of basis. Note further that in this definition no further properties of the differential-algebraic operator are assumed, in particular it is not assumed that it has a certain index as a differential-algebraic equation.

The assumption that the matrix function $Q^T E$ is bounded by a constant matrix H_0 from below implies that the Hamiltonian $\mathcal{H}$ is bounded from below by a constant. This constant is irrelevant when the derivative of $\mathcal{H}$ is considered, but it guarantees that the Hamiltonian can be interpreted as energy in a real physical system.

Example 5 Consider the model of a simple RLC network, see e. g. [13, 17], given by a linear constant coefficient DAE

$$
\underbrace{\begin{bmatrix} G_c C G_c^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=E}
\begin{bmatrix} \dot{V} \\ \dot{I}_l \\ \dot{I}_v \end{bmatrix}
=
\underbrace{\begin{bmatrix} -G_r R_r^{-1} G_r^T & -G_l & -G_v \\ G_l^T & 0 & 0 \\ G_v^T & 0 & 0 \end{bmatrix}}_{:=(J-R)I}
\begin{bmatrix} V \\ I_l \\ I_v \end{bmatrix}, \qquad (9)
$$

with real symmetric constant matrices $L > 0$, $C > 0$, $R_r > 0$ describing inductances, capacitances, and resistances, respectively that are present in the network. Here, G_v is of full column rank, and the subscripts r, c, l, and v refer to edge quantities corresponding to the resistors, capacitors, inductors, and voltage sources, while V, I denote the voltage and current, respectively, on or across the branches of the given RLC network. This model has a pHDAE

structure with vanishing B, P, S, N, K, the matrix Q is the identity, $E = E^T$, $J = -J^T$, $Q^T R Q = R \geq 0$, and

$$\mathcal{H} = \begin{bmatrix} V \\ I_l \\ I_v \end{bmatrix}^T E \begin{bmatrix} V \\ I_l \\ I_v \end{bmatrix} = \begin{bmatrix} V \\ I_l \end{bmatrix}^T \begin{bmatrix} G_c C G_c^T & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} V \\ I_l \end{bmatrix}.$$

Example 6 In [14, 15] the propagation of pressure waves on the acoustic time scale in a network of gas pipelines is considered and an infinite-dimensional pHDAE is derived. A structure preserving mixed finite element space discretization leads to a block-structured pHDAE system

$$\begin{aligned} E\dot{x} &= (J - R)Qx + Bu, \\ y &= B^T Qx, \\ x(t_0) &= x^0, \end{aligned} \tag{10}$$

with $Q = I$, $P = 0$, $S + N = 0$,

$$E = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & -\tilde{G} & 0 \\ \tilde{G}^T & 0 & \tilde{N}^T \\ 0 & -\tilde{N} & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{D} & 0 \\ 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 \\ \tilde{B}_2 \\ 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where the vector valued functions $x_1 : \mathbb{R} \to \mathbb{R}^{n_1}, x_2 : \mathbb{R} \to \mathbb{R}^{n_2}$ represent the discretized pressure and flux, respectively, and $x_3 : \mathbb{R} \to \mathbb{R}^{n_3}$ represents the Lagrange multiplier for satisfying the space-discretized constraints. The coefficients $M_1 = M_1^T$, $M_2 = M_2^T$, and $\tilde{D} = \tilde{D}^T$ are positive definite, and the matrices $\tilde{N}$ and $\begin{bmatrix} \tilde{G}^T & \tilde{N}^T \end{bmatrix}^T$ have full row rank. The Hamiltonian is given by $\mathcal{H}(x) = x^T E^T Q x = x_1^T M_1 x_1 + x_2^T M_2 x_2$.

Definition 4 brings the pH modeling framework and the DAE framework together in a structured way. It should be noted, however, that in a DAE we may have hidden constraints that arise from differentiations, which are not explicitly formulated and the formulation of the DAE that is used in simulation and control is not unique. One can for example add derivatives of constraints which leads to an over-determined system, then one can add dummy variables or Lagrange multipliers to make the number of variables equal to the number of equation or one can remove some of the dynamical equations to achieve this goal, see [5, 16, 27, 30] for detailed discussions on this topic. To rewrite these different formulations in the pHDAE formulation is not always obvious. Let us demonstrate this with an example from multi-body dynamics.

Example 7 A benchmark example for a nonlinear DAE system is the model of a two-dimensional three-link mobile manipulator, see [6, 22], which is modeled as

$$\begin{aligned} \tilde{M}(\Theta)\ddot{\Theta} + \tilde{C}(\Theta, \dot{\Theta}) + \tilde{G}(\Theta) &= \tilde{B}_1 \tilde{u} + \Psi^T \lambda, \\ \psi(\Theta) &= 0, \end{aligned} \tag{11}$$

where $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 & \Theta_3 \end{bmatrix}^T$ is the vector of joint displacements, $\tilde{u}$ is vector of control torques at the joints, $\tilde{M}$ is mass matrix, $\tilde{C}$ is the vector of centrifugal and Coriolis forces, and $\tilde{G}$ is the

gravity vector. The term $\Psi^T\lambda$ with $\Psi = \frac{\partial \psi}{\partial \Theta}$ is the generalized constraint force with Lagrange multiplier λ associated with the constraint

$$\psi(\Theta) = \left[\begin{array}{c} l_1\cos(\Theta_1) + l_2\cos(\Theta_1 + \Theta_2) + l_3\cos(\Theta_1 + \Theta_2 + \Theta_3)l_3 - l \\ \Theta_1 + \Theta_2 + \Theta_3 \end{array} \right] = 0.$$

Besides the explicit constraint this system contains the first and second time derivative of ψ as hidden algebraic constraints, see e. g. [16, 27]. There are several regularization procedures that one can employ to make the system better suited for numerical simulation and control. One possibility is to replace the original constraint by its time derivative $\Psi(\Theta)\dot{\Theta} = 0$. In this case the model equation can easily be written in a pHDAE formulation. Using Cartesian coordinates for positions p, scaling the constraint equation by -1, and linearizing around a non-stationary reference solution yields a linear time-varying DAE of the form

$$\begin{aligned} \tilde{M}\delta\ddot{p} &= -\tilde{D}\delta\dot{p} - \tilde{S}\delta p + \tilde{G}^T\delta\lambda + \tilde{B}_1\delta u, \\ 0 &= -\tilde{G}\delta\dot{p}, \end{aligned} \tag{12}$$

with pointwise symmetric positive definite matrix functions $\tilde{M}, \tilde{S}$ and pointwise symmetric and positive semidefinite $\tilde{D}$. Adding a tracking output of the form $y = \tilde{B}_1^T\delta\dot{p}$, see e. g. [23], and transforming to first order by introducing

$$x = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] := \left[\begin{array}{c} \delta\dot{p} \\ \delta p \\ \delta\lambda \end{array} \right], \quad u = \delta u,$$

one obtains a linear time-varying pHDAE system $E\dot{x} = (J - R)Qx + Bu, y = B^TQx$, with

$$\begin{aligned} E &:= \left[\begin{array}{ccc} \tilde{M} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{array} \right], \ R := \left[\begin{array}{ccc} \tilde{D} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \ Q := \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & \tilde{S} & 0 \\ 0 & 0 & I \end{array} \right], \\ J &:= \left[\begin{array}{ccc} 0 & -I & \tilde{G}^T \\ I & 0 & 0 \\ -\tilde{G} & 0 & 0 \end{array} \right], \ B := \left[\begin{array}{c} \tilde{B}_1 \\ 0 \\ 0 \end{array} \right], \ P = 0, \ S + N = 0. \end{aligned}$$

The Hamiltonian in this case is given by $\mathcal{H}(x) = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]^T \left[\begin{array}{cc} \tilde{M} & 0 \\ 0 & \tilde{S} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]$.

Since the Lagrange multipliers in the multibody framework can be interpreted as external forces, it is also possible to incorporate them in the input $(B - P)u$ to achieve a pHDAE formulation as in Definition 4, but also other formulations are possible, e. g. we can keep the original algebraic constraint as well and use an extra Lagrange multiplier for the first time derivative.

Remark 8 *A special case of* (5) *takes the following form:*

$$\begin{aligned} E\dot{x} &= (J - R)x + (B - P)u, \\ y &= (B + P)^Tx + (S + N)u, \end{aligned} \tag{13}$$

where $E = E^T \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $J = -J^T, R = R^T, K \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B, P \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $S = S^T, N = -N^T \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$ as before but now we require,

i) the differential algebraic operator

$$\mathcal{L} := E\frac{d}{dt} - J : D \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n) \tag{14}$$

is skew-adjoint, *so that we have for all* $t \in \mathbb{I}$,

$$\frac{d}{dt}E(t) \;\; = \;\; -\left[J(t) + J(t)^T\right];$$

ii) $E(t)$ *is positive semidefinite:* $E(t) \geq 0$ *for all* $t \in \mathbb{I}$; *and*

ii) $W(t) := \begin{bmatrix} R(t) & P(t) \\ P^T(t) & S(t) \end{bmatrix} \geq 0$ *for all* $t \in \mathbb{I}$.

The effective Hamiltonian *is now*

$$\mathcal{H}(x) := \frac{1}{2}x^T E x : C^1(\mathbb{I}, \mathbb{R}^n) \to \mathbb{R}. \tag{15}$$

Notice that in this model description we have merged the roles of Q *and* E. *This is always possible when* Q *is pointwise invertible, see Section 3 but this formulation may not be possible when* Q *is singular.*

3   Properties of pHDAE systems

To analyze the properties of pHDAE systems, we first derive the conservation of energy and the dissipation inequality.

Theorem 9 *A linear time-varying pHDAE system has the following properties:*

i) If $W \equiv 0$ *in (7) then* $\frac{d}{dt}\mathcal{H} = u^T y$.

ii) The system satisfies the dissipation inequality (4).

Proof. By Definition 4 we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} \;\; &= \;\; \frac{1}{2}\left[\dot{x}^T(Q^T E)x + x^T\frac{d}{dt}(Q^T E)x + x^T(Q^T E)\dot{x}\right] \\
&= \;\; \frac{1}{2}x^T\frac{d}{dt}(Q^T E)x + x^T Q^T(E\dot{x}) \\
&= \;\; \frac{1}{2}x^T\frac{d}{dt}(Q^T E)x + x^T Q^T([JQ - RQ - EK]x + Bu - Pu) \\
&= \;\; \frac{1}{2}x^T\frac{d}{dt}(Q^T E)x + x^T Q^T JQx - x^T Q^T RQx - x^T Q^T EKx + x^T Q^T Pu + u^T B^T Qx \\
&= \;\; \frac{1}{2}x^T\frac{d}{dt}(Q^T E)x + x^T Q^T JQx - x^T Q^T RQx - x^T Q^T EKx - x^T Q^T Pu \\
&\qquad + u^T(y - P^T Qx - Su - Nu) \\
&= \;\; u^T y + \frac{1}{2}x^T\frac{d}{dt}(Q^T E)x + x^T Q^T JQx - x^T Q^T RQx - x^T Q^T EKx \\
&\qquad - x^T Q^T Pu - u^T P^T Qx - u^T Su \\
&= \;\; u^T y + \frac{1}{2}\left(x^T\frac{d}{dt}(Q^T E)x + x^T[Q^T(JQ - EK) + (JQ - EK)^T Q]x\right) \\
&\qquad - \begin{bmatrix} x \\ u \end{bmatrix}^T W \begin{bmatrix} x \\ u \end{bmatrix}.
\end{aligned}
$$

From the skew-adjointness of $\mathcal{L}$ we have that

$$\frac{d}{dt}\mathcal{H} = u^T y - \begin{bmatrix} x \\ u \end{bmatrix}^T W \begin{bmatrix} x \\ u \end{bmatrix}.$$

Part i) then follows immediately from the assumption $W \equiv 0$, while in Part ii) the fact that $W(t) \geq 0$ for all $t \in \mathbb{I}$ implies that for any $t_1 \geq t_0$,

$$\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} \frac{d}{dt}\mathcal{H}\,dt \leq \int_{t_0}^{t_1} y^T u\,dt.$$

□

An important feature of pHDAE systems is that a change of basis and a scaling with an invertible matrix function preserves the pHDAE structure and the Hamiltonian.

Theorem 10 *Consider a pHDAE system of the form (5) with Hamiltonian (8). Let $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ be pointwise invertible in $\mathbb{I}$. Then the transformed DAE*

$$\begin{aligned} \tilde{E}\dot{\tilde{x}} &= [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}]\tilde{x} + (\tilde{B} - \tilde{P})u, \\ y &= (\tilde{B} + \tilde{P})^T \tilde{Q}\tilde{x} + (S + N)u \end{aligned}$$

with

$$\begin{aligned} \tilde{E} &= U^T E V, \ \tilde{Q} = U^{-1} Q V, \ \tilde{J} = U^T J U, \\ \tilde{R} &= U^T R U, \ \tilde{B} = U^T B, \ \tilde{P} = U^T P, \\ \tilde{K} &= V^{-1} K V + V^{-1}\dot{V}, \ x = V\tilde{x} \end{aligned}$$

is still a pHDAE system with the same Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{Q}^T \tilde{E}\tilde{x} = \mathcal{H}(x)$.

Proof. The transformed DAE system is obtained from the original DAE system by setting $x = V\tilde{x}$ in (5), by pre-multiplying with U^T and by inserting UU^{-1} in front of Q. The transformed operator corresponding to $\mathcal{L}$ in (14) is

$$\mathcal{L}_V := \tilde{Q}^T \tilde{E}\frac{d}{dt} - \tilde{Q}^T(\tilde{J}\tilde{Q} - \tilde{E}\tilde{K}).$$

Because

$$\tilde{Q}^T \tilde{E} = V^T Q^T E V, \quad \tilde{Q}^T \tilde{J}\tilde{Q} = V^T Q^T J Q V, \quad \tilde{Q}^T \tilde{E} V^{-1}\dot{V} = V^T Q^T E\dot{V},$$

by Lemma 2, $\mathcal{L}_V$ is skew-adjoint since $\mathcal{L}$ defined in (14) is skew-adjoint. Hence,

$$\begin{aligned} \tilde{Q}^T \tilde{E} &= \tilde{E}^T \tilde{Q}, \\ \frac{d}{dt}(\tilde{Q}^T \tilde{E}) &= -\tilde{Q}^T(\tilde{J}\tilde{Q} - \tilde{E}\tilde{K}) - (\tilde{J}\tilde{Q} - \tilde{E}\tilde{K})^T \tilde{Q}. \end{aligned}$$

It is straightforward to show that

$$\frac{d}{dt}\tilde{\mathcal{H}}(\tilde{x}) = y^T u - \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}^T \tilde{W} \begin{bmatrix} \tilde{x} \\ u \end{bmatrix},$$

where

$$\tilde{W} = \begin{bmatrix} \tilde{Q}^T \tilde{R} \tilde{Q} & \tilde{Q}^T \tilde{P} \\ \tilde{P}^T \tilde{Q} & S \end{bmatrix} = \begin{bmatrix} V^T Q^T RQV & V^T Q^T P \\ P^T QV & S \end{bmatrix}$$

$$= \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}^T W \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix},$$

and W is defined in (7). Because $W(t)$ is positive semidefinite for all $t \in \mathbb{I}$, so is $\tilde{W}(t)$. Therefore, for any $t_1 \geq t_0$,

$$\tilde{\mathcal{H}}(\tilde{x}(t_1)) - \tilde{\mathcal{H}}(\tilde{x}(t_0)) \leq \int_{t_0}^{t_1} y^T(t) u(t) dt,$$

which establishes the dissipation inequality. Since

$$\tilde{Q}^T(t) \tilde{E}(t) - V^T(t) H_0 V(t) = V^T(t)(Q^T(t) E(t) - H_0) V(t) \geq 0 \text{ for all } t \in \mathbb{I},$$

and since V is continuous, and thus $V^T H_0 V$ is bounded on $\mathbb{I}$, it follows that there exist a constant symmetric matrix $\tilde{H}_0$ such that $\tilde{Q}^T(t) \tilde{E}(t) \geq \tilde{H}_0$ for all $t \in \mathbb{I}$. □

An important point to note is that the Hamiltonian stays invariant under time-varying changes of basis and the operator $\mathcal{L}_V$, the Hamiltonian $\tilde{\mathcal{H}}(\tilde{x})$, and the matrix function $\tilde{W}$ are independent of the choice of the matrix function U.

As we have already pointed out, our definition of pHDAE systems has the extra term $-EKx$ on the right hand side which is needed to incorporate time-varying changes of basis. Even if $K = 0$ in the original system, after the transformation given in Theorem 10 the extra term $-\tilde{E}\tilde{K}$ with $\tilde{K} = V^{-1}\dot{V}$ will appear. Note that if an orthogonal change of basis is carried out in a system with $K = 0$ then the resulting $\tilde{K} = V^{-1}\dot{V}$ is skew-symmetric. Furthermore, even if $K \neq 0$, this term can be removed via a change of basis transformation which does not change the Hamiltonian.

Lemma 11 *Consider a pHDAE system*

$$\begin{aligned} \tilde{E}\dot{\tilde{x}} &= [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}]\tilde{x} + (\tilde{B} - \tilde{P})u, \\ y &= (\tilde{B} + \tilde{P})^T \tilde{Q}\tilde{x} + (S + N)u \end{aligned}$$

with Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{Q}^T \tilde{E}\tilde{x}$, where $\tilde{K} \in C(\mathbb{I}, \mathbb{R}^{n,n})$. If $V_{\tilde{K}} \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ is a pointwise invertible solution of the matrix differential equation $\dot{V} = \tilde{K}V$ with initial condition $V(t_0) = I$, then defining

$$\begin{aligned} E &= \tilde{E}V_K^{-1}, Q = \tilde{Q}V_k^{-1}, \\ J &= \tilde{J}, R = \tilde{R}, B = \tilde{B}, \\ P &= \tilde{P}, \tilde{x} = V_K^{-1}x, \end{aligned}$$

the system

$$\begin{aligned} E\dot{x} &= (J - R)Qx + (B - P)u, \\ y &= (B + P)^T Qx + (S + N)u \end{aligned}$$

is again pHDAE with the same Hamiltonian $\mathcal{H}(x) = \tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}x^T Q^T Ex$.

Proof. For a given matrix function $\tilde{K}$, the system $\dot{V}_K = \tilde{K}V_K$ always has a solution V_K that is pointwise invertible. The remainder of the proof follows by reversing the proof of Theorem 10 with $U = I$. □

Remark 12 Note that if K is real and skew-symmetric, then the matrix function V_K in Lemma 11 can be chosen to be pointwise real orthogonal.

Following Theorem 10, if E is pointwise invertible, then the original system can be transformed into the one with new $\hat{E}$ being the identity, so into a standard port-Hamiltonian system; and whenever Q is pointwise invertible, then the original system can be transformed into the one with new $\hat{Q}$ being the identity. Which of these formulations is preferable will depend on the sensitivity (conditioning) of these transformations. In the context of numerical simulation and control methods, these transformations should be avoided if they are ill-conditioned.

4 Nonlinear DAEs and pHDAEs

In this section we briefly recall the theory of nonlinear DAE systems and then extend these results to pHDAEs. Consider a general descriptor system of the form

$$
\begin{aligned}
F(t, x, \dot{x}, u) &= 0, \\
x(t_0) &= x^0 \\
y &= G(t, x, u).
\end{aligned}
\tag{16}
$$

Assume that $F \in C^0(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}} \times \mathbb{D}_u, \mathbb{R}^n)$ and $G \in C^0(\mathbb{I} \times \mathbb{D}_x \times \mathbb{D}_u, \mathbb{R}^m)$ are sufficiently smooth, and that $\mathbb{D}_x, \mathbb{D}_{\dot{x}} \subseteq \mathbb{R}^n$, and $\mathbb{D}_u$ are open sets. Note that (in order to deal with pHDAEs) in contrast to the more general case in [11], we assume square systems with an equal number of equations and variables and with an equal number of inputs and outputs.

For the analysis and the regularization procedure we make use of the *behavior approach* [34], which introduces a *descriptor vector* $v = [x^T, u^T]^T$. We could also include the output vector y in v, but in the context of pHDAEs it is preferable to keep the output equation separate. The behavior formulation has the form

$$
\mathcal{F}(t, v, \dot{v}) = 0,
\tag{17}
$$

with $\mathcal{F} \in C^0(\mathbb{I} \times \mathbb{D}_v \times \mathbb{D}_{\dot{v}}, \mathbb{R}^n)$ together with a set of initial conditions $c(v(t_0)) = v^0$ which results from the original initial condition. Note that although no initial condition is given for u in the context of the regularization procedure discussed in [11] such conditions may arise, so we formally state a condition for $v(t_0)$.

To regularize DAEs for numerical simulation and control, see [9, 11, 27], one uses the behavior system (17) and some or all of its derivatives to produce an equivalent system with the same solution set (all variables keep their physical interpretation), but where all explicit and hidden constraints are available. The approach of [11] (adapted for the analysis of pHDAEs) uses the state equation of (17) to form a derivative array, see [9],

$$
\mathcal{F}_\mu(t, v, \dot{v}, \dots, v^{(\mu+1)}) = 0,
\tag{18}
$$

which stacks the equation and its time derivatives up to level μ into one large system. We denote partial derivatives of $\mathcal{F}_\mu$ with respect to selected variables ζ from $(t, v, \dot{v}, \dots, v^{(\mu+1)})$

by $\mathcal{F}_{\mu;\zeta}$, and the solution set of the nonlinear algebraic equation associated with the derivative array $\mathcal{F}_\mu$ for some integer μ (considering the variables as well as their derivatives as algebraic variables) by $\mathbb{L}_\mu = \{v_\mu \in \mathbb{I} \times \mathbb{R}^{n+m} \times \ldots \times \mathbb{R}^{n+m} \mid \mathcal{F}_\mu(v_\mu) = 0\}$.

The main assumption for the analysis is that the DAE satisfies the following hypothesis, which in the linear case can be proved as a Theorem, see [27].

Hypothesis 13 *Consider the system of nonlinear DAEs (17). There exist integers μ, r, a, d, and ν such that $\mathbb{L}_\mu$ is not empty and such that for every $v_\mu^0 = (t_0, v_0, \dot{v}_0, \ldots, v_0^{(\mu+1)}) \in \mathbb{L}_\mu$ there exists a neighborhood in which the following properties hold.*

1. *The set $\mathbb{L}_\mu \subseteq \mathbb{R}^{(\mu+2)(n+m)+1}$ forms a manifold of dimension $(\mu+2)(n+m)+1-r$.*

2. *We have $\operatorname{rank} \mathcal{F}_{\mu;v,\dot{v},\ldots,v^{(\mu+1)}} = r$ on $\mathbb{L}_\mu$.*

3. *We have $\operatorname{corank} \mathcal{F}_{\mu;v,\dot{v},\ldots,v^{(\mu+1)}} - \operatorname{corank} \mathcal{F}_{\mu-1;v,\dot{v},\ldots,v^{(\mu)}} = \nu$ on $\mathbb{L}_\mu$, where the corank is the dimension of the corange and the convention is used that corank of $\mathcal{F}_{-1;v}$ is 0.*

4. *We have $\operatorname{rank} \mathcal{F}_{\mu;\dot{v},\ldots,v^{(\mu+1)}} = r - a$ on $\mathbb{L}_\mu$ such that there exist smooth full rank matrix functions Z_2 and T_2 of size $(\mu+1)n \times a$ and $(n+m) \times (n+m-a)$, respectively, satisfying $Z_2^T \mathcal{F}_{\mu;\dot{v},\ldots,v^{(\mu+1)}} = 0$, $\operatorname{rank} Z_2^T \mathcal{F}_{\mu;v} = a$, and $Z_2^T \mathcal{F}_{\mu;v} T_2 = 0$ on $\mathbb{L}_\mu$.*

5. *We have $\operatorname{rank} \mathcal{F}_{\dot{v}} T_2 = d = n - a - \nu$ on $\mathbb{L}_\mu$ such that there exists a smooth full rank matrix function Z_1 of size $n \times d$ satisfying $\operatorname{rank} Z_1^T \mathcal{F}_{\dot{v}} T_2 = d$.*

The smallest μ for which Hypothesis 13 holds is called the *strangeness-index* of (17), see [27]. It generalizes the concept of *differentiation-index* [5] to over- and under-determined systems but in contrast to the differentiation-index, ordinary differential equations and purely algebraic equations have $\mu = 0$ and for other systems the differentiation-index (if defined) is $\mu + 1$, see [27]. The quantity ν gives the number of trivial equations $0 = 0$ in the system. Of course, these equations can be simply removed and so for our further analysis we assume that $\nu = 0$.

If Hypothesis 13 holds then, locally (via the implicit function theorem) there exists, see [26, 27], a system (in the same variables)

$$
\begin{aligned}
\hat{\mathcal{F}}_1(t, v, \dot{v}) &= 0, \\
\hat{\mathcal{F}}_2(t, v) &= 0,
\end{aligned}
\tag{19}
$$

in which the first d equations $\hat{\mathcal{F}}_1 = Z_1^T \mathcal{F}$ form a (linear) projection of the original set of equations representing the dynamics of the system, while the second set $\hat{\mathcal{F}}_2(t, v) = 0$ of a equations contains all explicit and hidden constraints and can be used to parameterize the solution manifold and to characterize when an initial condition is consistent. Adding again the output equation and writing (19) in the original variables we obtain the system

$$
\begin{aligned}
\hat{\mathcal{F}}_1(t, x, \dot{x}, u) &= 0, \\
\hat{\mathcal{F}}_2(t, x, u) &= 0, \\
x(t_0) &= x^0 \\
y &= G(t, x, u),
\end{aligned}
\tag{20}
$$

It should be noted that although formally also derivatives of u have been used to form the derivative array, no derivatives of u appear in the regularized system (20). This has been

shown in various contexts [7, 27, 28] and is due to the fact that only derivatives of equations where $F_{\mu;u} \equiv 0$ are needed to generate (20). This means, in particular, that the equations in $\hat{\mathcal{F}}_2(t, x, u) = 0$ can be partitioned further into equations that arise from the original system, which include those algebraic equations in the original system which are explicit constraints (in the behavior sense) so that the system can be made to be of differentiation index at most one by a feedback (the part that is impulse controllable or controllable at infinity), and implicit hidden constraints arising from differentiations of equations for which $F_{\mu,u} \equiv 0$ in the derivative array (the parts that are not impulse controllable).

Using these observations, the regularized system can be (locally in the nonlinear case) written as

$$
\begin{aligned}
\hat{E}_1 \dot{x} &= \hat{A}_1 x + B_1 u, \\
0 &= \hat{A}_2 x + B_2 u, \\
0 &= \hat{A}_3 x, \\
x(t_0) &= x^0, \\
y &= C x + D u,
\end{aligned}
\tag{21}
$$

where the third equation that is representing all the hidden algebraic constraints of differentiation index larger that one. Performing an appropriate (local) change of basis one can identify some (transformed variables) which vanish and the remaining system consisting of the first two equations is of index at most one in the behavior sense, see [7, 27, 28] for details. For the first two equations in (20) and (21) one can always find an initial feedback $u = k(x) + \tilde{u}$ so that the resulting system is strangeness-free (of differentiation-index one) as a system with input $\tilde{u} = 0$, see [3, 11] for a detailed analysis and regularization procedures. In the following we assume that this reinterpretation has been done, so that the $n \times n$ matrix (functions)

$$
\begin{bmatrix} \hat{E}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{bmatrix}, \quad \begin{bmatrix} (\hat{\mathcal{F}}_1)_{\dot{x}}(t, x, \dot{x}, u) \\ (\hat{\mathcal{F}}_2)_x(t, x, u) \\ (\hat{\mathcal{F}}_3)_x(t, x) \end{bmatrix},
\tag{22}
$$

respectively, are locally invertible, see [27]. Furthermore there exists a (local) partitioning of the variables so that the strangeness-free formulation takes the form

$$
\begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} & \hat{E}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \end{bmatrix} u
\tag{23}
$$

with the property that $\hat{A}_{33}$ is invertible and the reduced system obtained by solving for x_3 is strangeness-free (of differentiation index at most one) when setting $u = 0$.

The described regularization procedure holds for general DAEs but it does not reflect an available port-Hamiltonian structure. We will now modify this approach for nonlinear systems with a pHDAE structure, which (based on the linear time-varying formulation) we define as follows.

Definition 14 *Consider a general DAE model in the form (16) and a Hamiltonian $\mathcal{H}(x)$ with the property that for a given input $u(t)$ and associated trajectory $x(t)$ the Hessian $Y_{loc}(t) = \mathcal{H}_{xx}(x(t))$ can be expressed locally as $E_{loc}(t)^T Q_{loc}(t)$, where $E_{loc}(t) = F_{\dot{x}}(t)$, $F_x(t) = (J_{loc}(t) -$*

$R_{loc}(t))Q_{loc}(t) - E_{loc}(t)K_{loc}(t)$, $F_u(t) = B_{loc}(t) - P_{loc}(t)$, $G_x(t) = (B_{loc}(t) + P_{loc}(t))^T Q_{loc}(t)$, $G_u(t) = S_{loc}(t) + N_{loc}(t)$, with $E_{loc}(t), A_{loc}(t), Q_{loc}(t), R_{loc}(t) = R_{loc}^T(t), K_{loc}(t) \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B_{loc}(t), P_{loc}(t) \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $S_{loc}(t) = S_{loc}^T(t), N_{loc}(t) = -N_{loc}^T(t) \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$. *Then the system is called pHDAE system if the following properties are satisfied:*

  i) *the (local) differential-algebraic operator*

$$\mathcal{L}_{loc} := Q_{loc}^T(t)E_{loc}(t)\frac{d}{dt} - Q_{loc}^T(t)(J_{loc}(t)Q_{loc}(t) - E_{loc}(t)K_{loc}(t)) \tag{24}$$

  is skew-adjoint,

  ii) *locally there exists a constant matrix $\mathcal{H}_0$ such that Hessian $\mathcal{H}_{xx}(t) - \mathcal{H}_0$ is positive semidefinite;*

  ii) *locally*

$$W_{loc}(t) = \begin{bmatrix} Q_{loc}(t)^T R_{loc}(t)Q_{loc}(t) & Q_{loc}(t)^T P_{loc}(t) \\ P_{loc}^T(t)Q_{loc}(t) & S_{loc}(t) \end{bmatrix} \geq 0 \text{ for all } t \in \mathbb{I}. \tag{25}$$

Clearly standard pH systems of the form (2) and linear time-varying systems as in Definition 4 directly fit in this framework. The same holds for multibody systems as in Example 7 with the derivative of the constraint

$$\begin{aligned} \tilde{M}(\Theta)\ddot{\Theta} + \tilde{C}(\Theta, \dot{\Theta}) + \tilde{G}(\Theta) &= \tilde{u} + \Psi^T\lambda, \\ \Psi(\Theta)\dot{\Theta} &= 0. \end{aligned} \tag{26}$$

Remark 15 There is a lot of choice in the local matrices Q_{loc} and E_{loc} when factoring the Hessian. In some cases we can just choose Q_{loc} to be the identity (see Remark 8), so that $E_{loc} = E_{loc}^T$ defines the Hessian. In other cases one chooses the block-diagonal matrix $E_{loc} = \text{diag}(I_d, 0)$ and obtains a semi-explicit formulation of the pHDAE. However, in general, this freedom should be chosen to make the system robust to perturbations for simulation and control methods.

There are multiple reasons why constraints may arise in pH systems. A typical example arises as a limiting situation in a singularly perturbed problem which has pH structure. Typical examples are mechanical multibody systems where small masses are ignored.

Example 16 Finite element modeling of the acoustic field in the interior of a car, see e. g. [32], leads (after several simplifications) to a large scale constant coefficient differential-algebraic equation system of the form

$$M\ddot{p} + D\dot{p} + Kp = B_1 u,$$

where p is the coefficient vector associated with the pressure in the air and the displacements of the structure, $B_1 u$ is an external force, M is a symmetric positive semidefinite mass matrix, D is a symmetric positive semidefinite matrix, and K is a symmetric positive definite stiffness matrix. Here M is only semidefinite since small masses were set to zero, so M is a perturbation

of a positive definite matrix. Performing a first order formulation we obtain the state equation of a pHDAE system $E\dot{z} = (J - R)Qz + Bu$, where

$$E \quad := \quad \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \ J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \ R := \begin{bmatrix} D & \\ & 0 \end{bmatrix}, \ z := \begin{bmatrix} \dot{p} \\ p \end{bmatrix},$$

$$Q \quad := \quad \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}, \ B := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \ P := 0,$$

and the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(z^T E^T Q z) = \frac{1}{2}(\dot{p}^T M \dot{p} + p^T K p).$$

Note that this model is nonlinear originally, but the simplifications carried out in the modeling process, e. g. linearization and omitting nonlinear terms with small coefficients leads to a linear model.

The other class of examples are systems such as as Example 7, where the dynamics is constrained to a manifold. If the system like in this example has hidden constraints, then the formulation as pHDAE system is not unique because different formulations of the equations and the constraints can be be made. We will come back to this question in Section 6.

As mentioned in the introduction, it is sometimes claimed that port-Hamiltonian DAEs are of differentiation-index at most one (i. e., satisfy Hypothesis 13 with $\mu = 0$). If this would be the case then in the derivative array with $\mu = 0$ the matrix $\mathcal{F}_{\dot{x}}$ locally has constant rank d and if Z_2^T is a maximal rank matrix such that (locally) $Z_2^T \mathcal{F}_{\dot{x}} = 0$ and Z_1 is such that it completes Z_2 to an invertible matrix $Z = [Z_1, Z_2]$, then the matrix $\bar{E} := \begin{bmatrix} Z_1^T \mathcal{F}_{\dot{x}} \\ Z_2^T \mathcal{F}_x \end{bmatrix}$ is locally invertible.

Let us check this for some of the examples. In Example 5 we have $Z_2^T = \begin{bmatrix} 0 & 0 & I \end{bmatrix}$ and obtain

$$\bar{E} = \begin{bmatrix} G_c C G_c^T & 0 & 0 \\ 0 & L & 0 \\ -G_v^T & 0 & 0 \end{bmatrix}$$

which is clearly not invertible, except if the last row and column is empty. The same matrix Z_2 can be used in Example 6 and yields

$$\bar{E} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & N & 0 \end{bmatrix}$$

which is also not invertible except if the last row and column is empty. Actually due to the special structure it can be shown that both systems have $\mu = 1$, i. e., differentiation-index two, when the input is chosen to be 0. The analysis of Example 7 with the original constraint $\Psi(\Theta)$ has $\mu = 2$ (differentiation-index three) and the formulation as pHDAE is not straightforward, but using as constraint its derivative yields $\mu = 1$ (differentiation-index two) if $\tilde{G}\tilde{G}^T$ is invertible. This replacement corresponds to an index reduction. How to carry out such a regularization for pHDAEs will be discussed in Section 6. But let us first (in the next section) analyze in detail the case of differentiation-index one pHDAEs.

5 PHDAEs of differentiation-index at most one

In this section we characterize pHDAE systems of differentiation-index at most one ($\mu = 0$) and we first study linear time-varying pHDAE systems. In this case Hypothesis 13 implies that the matrix function $E(t)$ has constant rank. Then, see e. g. Theorem 3.9 in [27], there exist pointwise orthogonal matrix functions $\tilde{U}$ and V such that

$$\tilde{U}^T E V = \left[\begin{array}{cc} E_{11} & 0 \\ 0 & 0 \end{array} \right],$$

where E_{11} is pointwise invertible. Because $Q^T E$ is real symmetric, in

$$\tilde{U}^T Q V = \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right],$$

one has $Q_{11}^T E_{11} = E_{11}^T Q_{11}$ and also $Q_{12} = 0$. Partition in the same way

$$\begin{aligned}
\tilde{U}^T J \tilde{U} &= \left[\begin{array}{cc} J_{11} & J_{12} \\ J_{21} & J_{22} \end{array} \right], \ \tilde{U}^T R \tilde{U} = \left[\begin{array}{cc} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{array} \right] \ \tilde{x} = V^T x = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right], \\
\tilde{U}^T \left[\begin{array}{cc} B & P \end{array} \right] &= \left[\begin{array}{cc} B_1 & P_1 \\ B_2 & P_2 \end{array} \right], \ \tilde{K} = K + V^T \dot{V} = \left[\begin{array}{cc} K_{11} & K_{12} \\ K_{21} & K_{22} \end{array} \right], \\
\tilde{U}^T (J - R) \tilde{U} &= \left[\begin{array}{cc} J_{11} & J_{12} \\ J_{21} & J_{22} \end{array} \right] - \left[\begin{array}{cc} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{array} \right] =: \left[\begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array} \right],
\end{aligned}$$

so that the transformed pHDAE system has the form

$$\begin{aligned}
\left[\begin{array}{cc} E_{11} & 0 \\ 0 & 0 \end{array} \right] \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right] &= \left(\left[\begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array} \right] \left[\begin{array}{cc} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{array} \right] - \left[\begin{array}{cc} E_{11} K_{11} & E_{11} K_{12} \\ 0 & 0 \end{array} \right] \right) \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \\
&\quad + \left[\begin{array}{c} B_1 - P_1 \\ B_2 - P_2 \end{array} \right] u, \\
y &= \left[\begin{array}{cc} (B_1 + P_1)^T & (B_2 + P_2)^T \end{array} \right] \left[\begin{array}{cc} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + (S + N) u.
\end{aligned}$$

Since the system has differentiation-index at most one, the block $L_{22} Q_{22}$ either does not occur (in this case we have an implicitly defined standard pH system) or it must be pointwise invertible, see [27], i. e., both L_{22} and Q_{22} are pointwise invertible. Let $U = \tilde{U} T$, where

$$T := \left[\begin{array}{cc} I & 0 \\ T_{21} & I \end{array} \right], \qquad T_{21} = -L_{22}^{-T} (L_{12} - E_{11} K_{12} Q_{22}^{-1})^T.$$

Then a transformation of the original pHDAE with U and V yields the pHDAE system

$$\begin{aligned}
\tilde{E} \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right] &= [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + (\tilde{B} - \tilde{P}) u, \\
y &= (\tilde{B} + \tilde{P})^T \tilde{Q} \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + (\tilde{S} + \tilde{N}) u, \tag{27}
\end{aligned}$$

where

$$\tilde{E} = T^T \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{K} = K, \tilde{S} = S, \quad \tilde{N} = N,$$

$$\tilde{Q} = T^{-1} \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} - T_{21}Q_{11} & Q_{22} \end{bmatrix} =: \begin{bmatrix} Q_{11} & 0 \\ \tilde{Q}_{21} & Q_{22} \end{bmatrix},$$

$$\tilde{J} = T^T \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} T = \begin{bmatrix} J_{11} + T_{21}^T J_{21} + J_{12}T_{21} + T_{21}^T J_{22}T_{21} & J_{12} + T_{21}^T J_{22} \\ J_{21} + J_{22}T_{21} & J_{22} \end{bmatrix}$$

$$=: \begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & J_{22} \end{bmatrix},$$

$$\tilde{R} = T^T \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} T = \begin{bmatrix} R_{11} + T_{21}^T R_{12}^T + J_{12}R_{12} + T_{21}^T R_{22}T_{21} & R_{12} + T_{21}^T R_{22} \\ R_{12}^T + R_{22}T_{21} & R_{22} \end{bmatrix}$$

$$=: \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^T & R_{22} \end{bmatrix},$$

$$\tilde{B} = T^T \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 + T_{21}^T B_2 \\ B_2 \end{bmatrix} =: \begin{bmatrix} \tilde{B}_1 \\ B_2 \end{bmatrix}, \quad \tilde{P} = T^T P = \begin{bmatrix} P_1 + T_{21}^T P_2 \\ P_2 \end{bmatrix} =: \begin{bmatrix} \tilde{P}_1 \\ P_2 \end{bmatrix}.$$

Following Theorem 10 this transformation will not change the Hamiltonian, and based on the construction of T,

$$\begin{aligned} (\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K} &= T((J - R)Q - EK) \\ &= \begin{bmatrix} (\tilde{J}_{11} - \tilde{R}_{11})Q_{11} - E_{11}(K_{11} - K_{12}Q_{22}^{-1}\tilde{Q}_{21}) & 0 \\ (\tilde{J}_{21} - \tilde{R}_{12}^T)Q_{11} + (J_{22} - R_{22})\tilde{Q}_{21} & (J_{22} - R_{22})Q_{22} \end{bmatrix}. \end{aligned}$$

Note that these transformations should not be performed in a numerical integration or control design technique, since the inversion of the matrices Q_{22} and L_{22} may be highly ill-conditioned. However, from an analytic point of view have the following theorem.

Theorem 17 *Suppose that the pHDAE system (5) is of differentiation-index at most one (i. e. satisfies Hypothesis 13 with $\mu = 0$) for $\nu = 0$, and that $E(t)$ has constant rank. Let U, V and $\tilde{E}, \tilde{Q}, \tilde{J}, \tilde{R}, \tilde{B}, \tilde{P}$ be as in (27) and let $V^T x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$. Then system (5) can be reduced to the implicit pHDAE system (for the state x_1)*

$$\begin{aligned} \hat{E}\dot{x}_1 &= [(\hat{J} - \hat{R})\hat{Q} - \hat{E}\hat{K}]x_1 + (\hat{B} - \hat{P})u \\ y &= (\hat{B} + \hat{P})^T \hat{Q}x_1 + (\hat{S} + \hat{N})u \end{aligned} \tag{28}$$

with Hamiltonian $\hat{\mathcal{H}}(x_1(t)) = \frac{1}{2}x_1^T \hat{Q}^T \hat{E}x_1 = \mathcal{H}(x)$, and coefficients

$$\hat{E} = E_{11}, \quad \hat{Q} = Q_{11}, \quad \hat{J} = \tilde{J}_{11}, \quad \hat{R} = \tilde{R}_{11}, \quad \hat{K} = K_{11} - K_{12}Q_{22}^{-1}\tilde{Q}_{21},$$

$$\hat{B} = \tilde{B}_1 - \frac{1}{2}(\tilde{J}_{21}^T - \tilde{R}_{12})L_{22}^{-T}(B_2 + P_2), \quad \hat{P} = \tilde{P}_1 - \frac{1}{2}(\tilde{J}_{21}^T - \tilde{R}_{12})L_{22}^{-T}(B_2 + P_2),$$

$$\hat{S} = S - \frac{1}{2}[(B_2 + P_2)^T L_{22}^{-1}(B_2 - P_2) + (B_2 - P_2)^T L_{22}^{-T}(B_2 + P_2)],$$

$$\hat{N} = N - \frac{1}{2}[(B_2 + P_2)^T L_{22}^{-1}(B_2 - P_2) - (B_2 - P_2)^T L_{22}^{-T}(B_2 + P_2)],$$

together with the explicit algebraic constraint

$$L_{22}Q_{22}x_2 = -[(\tilde{J}_{21} - \tilde{R}_{12}^T)Q_{11} + L_{22}\tilde{Q}_{21}]x_1 - (B_2 - P_2)u, \tag{29}$$

for the state x_2, which also gives a consistency constraint for the initial condition

$$L_{22}(t_0)Q_{22}(t_0)x_2(t_0) = -[(L_{21}(t_0))Q_{11}(t_0) + L_{22}(t_0)\tilde{Q}_{21}(t_0)]x_1(t_0) - (B_2(t_0) - P_2(t_0))u(t_0).$$

Proof. Equations (28) and (29) follow directly from (27). The output equation is obtained directly by substituting (29). It remains to prove that (28) is port-Hamiltonian. From

$$\frac{d}{dt}\tilde{Q}^T\tilde{E} = -\tilde{Q}^T(\tilde{J} + \tilde{J}^T)\tilde{Q} + \tilde{Q}^T\tilde{E}\tilde{K} + \tilde{K}^T\tilde{E}^T\tilde{Q},$$

and since Q_{22} is invertible, one obtains

$$\begin{aligned}
0 &= J_{22} + J_{22}^T, \\
0 &= -Q_{11}^T(\tilde{J}_{12} + \tilde{J}_{21}^T) - \tilde{Q}_{21}^T(J_{22} + J_{22}^T) + Q_{11}^T E_{11} K_{12} Q_{22}^{-1}, \\
\frac{d}{dt}Q_{11}^T E_{11} &= -Q_{11}^T(\tilde{J}_{11} + \tilde{J}_{11}^T)Q_{11} - Q_{11}^T(\tilde{J}_{12} + \tilde{J}_{21}^T)\tilde{Q}_{21} \\
&\quad -\tilde{Q}_{21}^T(\tilde{J}_{12} + \tilde{J}_{21}^T)^T Q_{11} - \tilde{Q}_{21}^T(J_{22} + J_{22}^T)\tilde{Q}_{21} + Q_{11}^T E_{11} K_{11} + K_{11}^T E_{11}^T Q_{11},
\end{aligned}$$

which leads to

$$J_{22} + J_{22}^T = 0, \tag{30}$$
$$Q_{11}^T(\tilde{J}_{12} + \tilde{J}_{21}^T) = Q_{11}^T E_{11} K_{12} Q_{22}^{-1}, \tag{31}$$

and

$$\frac{d}{dt}Q_{11}^T E_{11} = -Q_{11}^T(\tilde{J}_{11} + \tilde{J}_{11}^T)Q_{11} + Q_{11}^T E_{11}(K_{11} - K_{12}Q_{22}^{-1}\tilde{Q}_{21}) + (K_{11} - K_{12}Q_{22}^{-1}\tilde{Q}_{21})^T E_{11}^T Q_{11},$$

and this last equation is just

$$\frac{d}{dt}\hat{Q}^T\hat{E} = -\hat{Q}^T(\hat{J} + \hat{J}^T)\hat{Q} + \hat{Q}^T\hat{E}\hat{K} + \hat{K}^T\hat{E}^T\hat{Q}.$$

Since $\hat{Q}^T\hat{E} = \hat{E}^T\hat{Q}$, the operator $\hat{Q}^T\hat{E}\frac{d}{dt} - \hat{Q}^T(\hat{J}\hat{Q} - \hat{E}\hat{K})$ is skew-adjoint.

The invariance of the Hamiltonian follows directly, since

$$\hat{\mathcal{H}}(x_1) = \frac{1}{2}x_1^T\hat{Q}^T\hat{E}x_1 = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \tilde{Q}^T\tilde{E}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}x^T Q^T E x = \mathcal{H}(x).$$

It remains to prove the dissipation inequality. We have that

$$\frac{d}{dt}\hat{\mathcal{H}}(x_1) = \frac{d}{dt}\mathcal{H}(x) = y^T u - \begin{bmatrix} x \\ u \end{bmatrix}^T W \begin{bmatrix} x \\ u \end{bmatrix} = y^T u - \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}^T \tilde{W} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}$$

where

$$\tilde{W} = \begin{bmatrix} Q_{11} & 0 & 0 \\ \tilde{Q}_{21} & Q_{22} & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & \tilde{P}_1 \\ \tilde{R}_{12}^T & R_{22} & P_2 \\ \tilde{P}_1^T & P_2^T & S \end{bmatrix} \begin{bmatrix} Q_{11} & 0 & 0 \\ \tilde{Q}_{21} & Q_{22} & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Eliminating x_2 by using (29), we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}^T \tilde{W} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} x_1 \\ u \end{bmatrix}^T \hat{W} \begin{bmatrix} x_1 \\ u \end{bmatrix},$$

where $\hat{W} = X^T \tilde{W} X$ with

$$X = \begin{bmatrix} I & 0 \\ -Q_{22}^{-1}(L_{22}^{-1}(\tilde{J}_{21} - \tilde{R}_{12}^T)Q_{11} + \tilde{Q}_{21}) & -Q_{22}^{-1}L_{22}^{-1}(B_2 - P_2) \\ 0 & I \end{bmatrix},$$

Note that $L_{22} = J_{22} - R_{22}$ and by (30) we have $J_{22} = -J_{22}^T$, and thus $R_{22} = -\frac{1}{2}(L_{22} + L_{22}^T)$. Also, from (31) and the formulas of $\tilde{J}_{21}, \tilde{R}_{12}, T_{21}$, it follows that $Q_{11}^T(\tilde{J}_{21}^T + \tilde{R}_{12}) = 0$. Then, by straightforward calculations we obtain

$$\hat{W} = \begin{bmatrix} Q_{11}^T \hat{R} Q_{11} & Q_{11}^T \hat{P} \\ \hat{P}^T Q_{11} & \hat{S} \end{bmatrix}.$$

Hence

$$\frac{d}{dt}\hat{\mathcal{H}} = y^T u - \begin{bmatrix} x_1 \\ u \end{bmatrix}^T \hat{W} \begin{bmatrix} x_1 \\ u \end{bmatrix}.$$

Since W is symmetric positive semidefinite, so is $\hat{W}$, and hence the reduced system in x_1 is still port-Hamiltonian with Hamiltonian $\hat{\mathcal{H}}(x_1)$. $\quad \square$

Note that for the numerical integration or in the control context, as for general DAEs, it is sufficient to carry out the transformation with $\tilde{U}$ pointwise from the left and the insertion of $I = \tilde{U}\tilde{U}^T$ before Q. In this way a differentiation of a computed transformation matrix can be avoided and the pHDAE structure is preserved nonetheless.

Remark 18 For nonlinear pHDAE systems with differentiation index at most one ($\mu = 0$), the corresponding local result follows directly via the implicit function theorem and application of Theorem 17 to the linearization as in Definition 14.

6 Regularization of higher index pHDAE systems

In this section we discuss how to modify the regularization procedure discussed for general DAEs in Section 4 to preserve the pHDAE structure. Let us first consider the linear time-varying case (5) and set $L = J - R$. Suppose that the state equation with $u = 0$ already satisfies Hypothesis 13, i. e., as discussed in Section 4, no reinterpretation of variables or initial feedbacks are necessary. It has been shown in [7] that the extra constraint equations (hidden constraints) that arise from derivatives are uncontrollable, because otherwise the index reduction could have been done via feedback. This means that these extra constraint equations are of the form $\hat{A}_3 x = 0$ which corresponds to $\hat{F}_3(t, x) = 0$ in the nonlinear case. We add just these constraint equations to our original pHDAE and obtain an overdetermined strangeness-free system. Note again that under our assumptions the explicit algebraic constraints are included in the first two equations in (20), resp. (21).

Let us make the weak assumption that $E(t)$ has constant rank. This is a restriction that however holds in all examples that we have encountered so far, and it can be removed

by considering the system in a piecewise fashion, see [27]. Then there exist real orthogonal matrix functions $U, V_1 \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that

$$U_1^T E V_1 = \begin{bmatrix} \tilde{E}_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

with pointwise invertible $\tilde{E}_{11}$. Perform a transformation of the pHDAE (5) as in Theorem 10 and also form $\hat{A}_3 V = \begin{bmatrix} \hat{A}_{31} & \hat{A}_{32} \end{bmatrix}$ partitioned accordingly. By the property that $\hat{A}_3$ contains all the high index constraints it follows that $\hat{A}_{23}$ has full row rank for all $t \in \mathbb{I}$, and hence there exists a real orthogonal matrix function V_2 such that $\hat{A}_{32} V_2 = \begin{bmatrix} 0 & A_{33} \end{bmatrix}$ with A_{33} pointwise invertible. Performing a change of variables of the pHDAE with

$$V := V_1 \begin{bmatrix} I & 0 \\ 0 & V_2 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\hat{A}_{31} A_{33}^{-1} & 0 & I \end{bmatrix}$$

we obtain a pHDAE of the form

$$
\begin{bmatrix} \tilde{E}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \tilde{L} \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$
$$
- \begin{bmatrix} \tilde{E}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$
$$
+ \begin{bmatrix} \tilde{B}_1 - \tilde{P}_1 \\ \tilde{B}_2 - \tilde{P}_2 \\ \tilde{B}_3 - \tilde{P}_3 \end{bmatrix} u, \tag{32}
$$
$$
y = \begin{bmatrix} (\tilde{B}_1 + \tilde{P}_1)^T & (\tilde{B}_2 + \tilde{P}_2)^T & (\tilde{B}_3 + \tilde{P}_3)^T \end{bmatrix} \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$
$$
+ (S + N)u,
$$

where $\tilde{K} = (V^T K V + \dot{V})$, $\tilde{L} = LV$, together with the constraint $0 = A_{33} x_3$, i. e. $x_3 = 0$.

Assuming further that the matrix function

$$\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \\ \tilde{Q}_{31} & \tilde{Q}_{32} \end{bmatrix}$$

has constant rank, there exists a pointwise real orthogonal matrix function U_2 such that

$$U_2^T \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & 0 & Q_{33} \end{bmatrix}$$

20

Transforming the pHDAE (34) with U_2^T we get a pHDAE of the form

$$
\begin{bmatrix} E_{11} & 0 & 0 \\ E_{21} & 0 & 0 \\ E_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

$$
- \begin{bmatrix} \tilde{E}_{11} & 0 & 0 \\ \tilde{E}_{21} & 0 & 0 \\ \tilde{E}_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \tilde{B}_1 - \tilde{P}_1 \\ \tilde{B}_2 - \tilde{P}_2 \\ \tilde{B}_3 - \tilde{P}_3 \end{bmatrix} u, \tag{33}
$$

$$
y = \begin{bmatrix} (\tilde{B}_1 + \tilde{P}_1)^T & (\tilde{B}_2 + \tilde{P}_2)^T & (\tilde{B}_3 + \tilde{P}_3)^T \end{bmatrix} \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

$$
+ (S + N)u,
$$

together with the constraint $0 = x_3$.

By Theorem 10, system (34) is still a pHDAE system, and the solution of the overdetermined system (34) together with $x_3 = 0$ is the same as that of (34) and the Hamiltonian is unchanged. Since the resulting system is still port-Hamiltonian, using that $x_3 = 0$, we have that the subsystem given by the first two block rows together with output equation is an index at most one phDAE which has the form

$$
\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

$$
- \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 - \tilde{P}_1 \\ \tilde{B}_2 - \tilde{P}_2 \end{bmatrix} u, \tag{34}
$$

$$
y = \begin{bmatrix} (\tilde{B}_1 + \tilde{P}_1)^T & (\tilde{B}_2 + \tilde{P}_2)^T \end{bmatrix} \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

$$
+ (S + N)u,
$$

To this system we can apply the results of the previous section and obtain that the system can be further reduced to an implicit standard pH system.

Example 19 Consider again the semidiscretized Example 6. It has been shown in [15] that for a (permuted) singular value decomposition (SVD) of N^T

$$
N^\top = U_N^\top \begin{bmatrix} 0 \\ \Sigma \end{bmatrix} V_N,
$$

with real orthogonal matrices U_N, V_N and a nonsingular diagonal matrix $\Sigma \in \mathbb{R}^{n_3, n_3}$. Scaling the second row of (10) with U_N and setting $x_2 = V_N \begin{bmatrix} x_{2,2}^\top & x_{2,3}^\top \end{bmatrix}^\top$, as well as $x_2^0 = V_N \begin{bmatrix} x_{2,2}^{0\top} & x_{2,3}^{0\top} \end{bmatrix}^\top$ we obtain a transformed system

$$
\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_{2,2} & M_{2,3} & 0 \\ 0 & M_{2,3}^\top & M_{3,3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_{2,2} \\ x_{2,3} \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & G_{1,2} & G_{1,3} & 0 \\ -G_{1,2}^\top & D_{2,2} & D_{2,3} & 0 \\ -G_{1,3}^\top & D_{2,3}^\top & D_{3,3} & -\Sigma \\ 0 & 0 & \Sigma & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_{2,2} \\ x_{2,3} \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ B_{2,2} \\ B_{3,2} \\ 0 \end{bmatrix} u. \tag{35}
$$

It follows immediately that $x_{2,3} = 0$, which is the uncontrollable (index two) constraint in the DAE that in particular the initial condition $x_{2,3}^0$ has to satisfy. The vectors $x_1, x_{2,2}$ are solutions of the implicit ordinary pH system

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_{2,2} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_{2,2} \end{bmatrix} + \begin{bmatrix} 0 & G_{1,2} \\ -G_{1,2}^\top & D_{2,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ B_{2,2} \end{bmatrix} u, \tag{36}$$

with initial conditions $x_1(0) = x_1^0$, $x_{2,2}(0) = x_{2,2}^0$, so they are well-defined continuously differentiable functions for any piecewise continuous u and any choice of the initial conditions.

Finally we get the component x_3 (the Lagrange multiplier) via

$$x_3 = \Sigma^{-1}(M_{2,3}^\top \frac{d}{dt} x_{2,2} - G_{1,3}^\top x_1 + D_{2,3}^\top x_{2,2} - B_{3,2} u), \tag{37}$$

and this is the implicit index one constraint in the DAE. Since both type of (the explicit and the hidden) constraints have to be satisfied for the initial condition, it means that the transformed initial condition also has to satisfy the consistency condition

$$x_3(0) = \Sigma^{-1}(M_{2,3}^\top \frac{d}{dt} x_{2,2}(0) - G_{1,3}^\top x_1(0) + D_{2,3}^\top x_{2,2}(0) - B_{3,2} u(0)) \tag{38}$$

Condition (38) leads to a relationship between the input u and the state at $t = 0$, which is a constraint that has to be satisfied to have a classical solution. Furthermore, we see immediately that to obtain a continuous x_3 the function $B_{3,2} u$ has to be continuous and u has to be such that $B_{3,2} u$ leads to a continuous $M_{2,3}^\top \frac{d}{dt} x_{2,2}$. The implicit ordinary pH system (36) describes the dynamics of the system, while the other two equations describe the constraints.

Remark 20 For nonlinear pHDAE systems satisfying Hypothesis 13 with $\mu > 0$, the corresponding local result follows directly via linearization and the implicit function theorem.

Conclusion

A new definition of port-Hamiltonian descriptor systems has been derived. It has been shown that this formulation is valid also for DAEs of differentiation-index larger than one, and it has been demonstrated that under some (local) constant rank assumption any such pHDAE can be reformulated as an implicitly defined standard PH system plus an algebraic constraint that describes the manifold where the dynamics of the system takes place and that also describes the consistent initial conditions. As for standard DAEs the reformulated system is well suited for numerical integration and control, since all constraints are available.

Acknowledgments

References

[1] J. Bals, G. Hofer, A. Pfeiffer, and C. Schallert. Virtual iron bird – A multidisciplinary modelling and simulation platform for new aircraft system architectures. In *Deutscher Luft- und Raumfahrtkongress*, Friedrichshafen, Germany, 2005.

[2] C. Beattie and S. Gugercin. Structure-preserving model reduction for nonlinear port-Hamiltonian systems. In *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), 2011*, pages 6564–6569. IEEE, 2011.

[3] A. Binder, V. Mehrmann, A. Miedlar, and P. Schulze. A `Matlab` toolbox for the regularization of descriptor systems arising from generalized realization procedures. Preprint 24–2015, Institut für Mathematik, TU Berlin, 2015.

[4] P. C. Breedveld. *Modeling and Simulation of Dynamic Systems using Bond Graphs*, pages 128–173. EOLSS Publishers Co. Ltd./UNESCO, Oxford, UK, 2008.

[5] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations*. SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[6] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols. Feedback design for regularizing descriptor systems. *Linear Algebra Appl.*, 299:119–151, 1999.

[7] R. Byers, P. Kunkel, and V. Mehrmann. Regularization of linear descriptor systems with variable coefficients. *SIAM J. Control Optim.*, 35(1):117–133, 1997.

[8] C. I. Byrnes, A. Isidori, and J. C. Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Autom. Control*, 36:1228–1240, 1991.

[9] S. L. Campbell. A general form for solvable linear time varying singular systems of differential equations. 18:1101–1115, 1987.

[10] S. L. Campbell. Linearization of DAEs along trajectories. *Z. Angew. Math. Phys.*, 46:70–84, 1995.

[11] S. L. Campbell, P. Kunkel, and V. Mehrmann. Regularization of linear and nonlinear descriptor systems. In L. T. Biegler, S. L. Campbell, and V. Mehrmann, editors, *Control and Optimization with Differential-Algebraic Constraints*, Advances in Control and Design, chapter 2, pages 17–36. 2012.

[12] F. Couenne, C. Jallut, B. M. Maschke, M. Tayakout, and P. C. Breedveld. Bond graph for dynamic modelling in chemical engineering. *Chemical engineering and processing, Elsevier, Amsterdam*, 47:1994–2003, 2008.

[13] L. Dai. *Singular Control Systems*, volume 118 of *Lecture Notes in Control and Inform. Sci.* Springer-Verlag, Berlin/Heidelberg, 1989.

[14] H. Egger and T. Kugler. Damped wave systems on networks: Exponential stability and uniform approximations. Technical report, 2016. http://arxiv.org/abs/1605.03066.

[15] H. Egger, T. Kugler, B. Liljegren-Sailer, N. Marheineke, and V. Mehrmann. Structure preserving model reduction for damped wave propagation in transport networks. Preprint 06-2017, Institut für Mathematik, TU Berlin, 2017. http://arxiv.org/abs/1704.03206.

[16] E. Eich-Soellner and C. Führer. *Numerical Methods in Multibody Dynamics*. Teubner, Stuttgart, 1998.

[17] R. W Freund. The sprim algorithm for structure-preserving order reduction of general rcl circuits. In *Model reduction for circuit simulation*, pages 25–52. Springer, 2011.

[18] G. Golo, A. J. van der Schaft, P. C. Breedveld, and B. M. Maschke. Hamiltonian formulation of bond graphs. In A. Rantzer R. Johansson, editor, *Nonlinear and Hybrid Systems in Automotive Control*, pages 351–372. Springer, Heidelberg, 2003.

[19] S. Gugercin, R. V. Polyuga, C. Beattie, and A. J. van der Schaft. Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems. *Automatica*, 48:1963–1974, 2012.

[20] M. Hiller and K. Hirsch. Multibody system dynamics and mechatronics. *Z. Angew. Math. Mech.*, 86(2):87–109, 2006.

[21] D. Hinrichsen and A. J. Pritchard. *Mathematical Systems Theory I. Modelling, State Space Analysis, Stability and Robustness*. Springer-Verlag, New York, NY, 2005.

[22] M. Hou. A three–link planar manipulator model. Sicherheitstechnische Regelungs- und Meßtechnik, Bergische Universität–GH Wuppertal, Germany, May 1994.

[23] M. Hou and P. C. Müller. *LQ* and tracking control of descriptor systems with application to constrained manipulator. Technical report, Sicherheitstechnische Regelungs- und Meßtechnik, Universität Wuppertal, Gauß-Straße 20, D-5600 Wuppertal 1, Germany, 1994.

[24] B. Jacob and H. Zwart. *Linear port-Hamiltonian systems on infinite-dimensional spaces*. Operator Theory: Advances and Applications, 223. Birkhäuser/Springer Basel AG, Basel CH, 2012.

[25] C. Kleijn. *20-sim 4C 2.1 Reference manual*. Controlab Products B.V., 2013.

[26] P. Kunkel and V. Mehrmann. Analysis of over- and underdetermined nonlinear differential-algebraic systems with application to nonlinear control problems. *Math. Control Signals Systems*, 14:233–256, 2001.

[27] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.

[28] P. Kunkel, V. Mehrmann, and W. Rath. Analysis and numerical solution of control problems in descriptor form. *Math. Control Signals Systems*, 14:29–61, 2001.

[29] P. Kunkel, V. Mehrmann, and L. Scholz. Self-adjoint differential-algebraic equations. *Math. Control Signals Syst.*, 26:47–76, 2014.

[30] R. Lamour, R. März, and C. Tischendorf. *Differential-algebraic equations: a projector based analysis*. Springer Science & Business Media, 2013.

[31] B. M. Maschke, A. J. van der Schaft, and P. C. Breedveld. An intrinsic Hamiltonian formulation of network dynamics: non-standard poisson structures and gyrators. *J. Franklin Inst.*, 329:923–966, 1992.

[32] V. Mehrmann and C. Schröder. Nonlinear eigenvalue and frequency response problems in industrial practice. *J. Math. Ind.*, 1:18, 2011.

[33] R. Ortega, A. J. van der Schaft, Y. Mareels, and B. M. Maschke. Putting energy back in control. *Control Syst. Mag.*, 21:18–33, 2001.

[34] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioural Approach*. Springer-Verlag, New York, NY, 1998.

[35] R. V. Polyuga and A. J. van der Schaft. Structure preserving model reduction of port-Hamiltonian systems by moment matching at infinity. *Automatica*, 46:665–672, 2010.

[36] A. J. van der Schaft. Port-Hamiltonian systems: an introductory survey. In J. L. Verona M. Sanz-Sole and J. Verdura, editors, *Proc. of the International Congress of Mathematicians, vol. III, Invited Lectures*, pages 1339–1365, Madrid, Spain.

[37] A. J. van der Schaft. Port-Hamiltonian systems: network modeling and control of nonlinear physical systems. In *Advanced Dynamics and Control of Structures and Machines*, CISM Courses and Lectures, Vol. 444. Springer Verlag, New York, N.Y., 2004.

[38] A. J. van der Schaft. Port-Hamiltonian differential-algebraic systems. In *Surveys in Differential-Algebraic Equations I*, pages 173–226. Springer-Verlag, 2013.

[39] A. J. van der Schaft and B. M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *J. Geom. Phys.*, 42:166–194, 2002.

[40] W. Schiehlen. *Advanced multibody system dynamics*. Kluwer Academic Publishers, Stuttgart, Germany, 1993.

[41] K. Schlacher and A. Kugi. Automatic control of mechatronic systems. *Int. J. Appl. Math. Comput. Sci.*, 11(1):131–164, 2001.

[42] W. Trautenberg. Simpack 8.9. Manual, INTEC GmbH, Angelrieder Feld 13, 82234 Wessling.