# Self-Inconsistency of set theory

Michael Pfender*

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**Abstract**

The consistency formula for **set** theory **T** e. g. Zermelo-Fraenkel set theory **ZF**, can be stated as free-variable predicate in terms of the categorical theory **PR** of primitive recursive functions/maps/predicates. Free-variable p. r. predicates are decidable by **T**, key result. Decidability is built on recursive evaluation of p. r. map codes and *soundness* of that evaluation into theory **T** : internal, *arithmetised* p. r. map code equality is evaluated into map equality of **T**. In particular, the free-variable p. r. consistency predicate of **T** is decidable by **T**. Therefore, by Gödel's second incompleteness theorem, **set** theories **T** turn out to be self-inconsistent.

*michael.pfender@alumni.tu-berlin.de

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# Introduction

The consistency problem for Foundations is open – with a negative bias, since Gödel has derived his two Incompleteness theorems.[1]

The formula which expresses in a gödelian (quantified) arithmetical theory **T** the consistency of **T** can be stated as free-variable primitive recursive predicate in terms of the categorical theory **PR** of primitive recursive ("p. r.") maps/predicates. The latter theory is strongly finitistic with only *bounded* existential quantification in the sense of SKOLEM 1919/1970, p. 153.[2] [3]

Free-variable p. r. predicates are decided by any quantified arithmetical **set** theory **T** which admits Ackermann type double recursion[4]. This *Decidability Theorem* is the key result here. It builds on double recursive *evaluation* of primitive recursive map codes and *soundness* of that evaluation into **T** :

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[1] GÖDEL 1931

[2] "Was ich nun in dieser Abhandlung zu zeigen wünsche ist folgendes: *Faßt man die allgemeinen Sätze der Arithmetik als Funktionalbehauptungen auf, und basiert man sich auf der rekurrierenden Denkweise, so läßt sich diese Wissenschaft in folgerichtiger Weise ohne Anwendung der Russel-Whitehead'schen Begriffe "always" und "sometimes" begründen.*"
[What I now whish to show in this treatise is the following:
If one interprets the general theorems of Arithmetic as function propositions, and bases oneself on the recurring method of thought, so it is possible to found this science in deducible manner without application of the Russel-Whitehead notions "always" and "sometimes"]

[3] The invention of (primitive) recursive Arithmetic is usually attributed to GÖDEL 1931

[4] see PÉTER 1967 as well as EILENBERG/ELGOT 1970

Internal, *arithmetised* primitive recursive[5] map code equality is evaluated into map equality of theory **T**.

In particular the free-variable primitive recursive (!) *consistency predicate* $\mathrm{Con}_{\mathbf{T}}$ of theory **T** – cf. SMORYNSKI 1977 – is decidable by **T**. This decidability gives, by Gödel's second incompleteness theorem, self-inconsistency of theory **T** as a final result.

Theory basis for present *negative* approach to classical foundations is exposition of fully formalised free-variables cartesian categorical theory **PR** of primitive recursion.[6]

For Foundations let us use here categorical language with its absence of formal variables for individuals: categories have only objects and maps as fundamental notions. This circumstance makes coding – *gödelisation* – of categorical theories comparably simple. In cartesian categories, *free variables* (re)enter as names for identic maps, and projections out of (cartesian) products. So free-variables primitive recursive Arithmetic comes back in a conveniently codable way.

It comes in two levels: First as categorical cartesian language **CA** generated over a (proto) natural numbers object $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ – zero and successor functions – and second, built on this, axioms and fundamental theorems making $\mathbb{N}$ into an NNO (*Natural Numbers Object*) in the sense of availability of endo map *iteration* and the axioms of primitive recursion proper: Theory **PR** of primitive recursion.

These explicit axioms, lemmata and theorems are stated for

[5]inserted August 30, 2017

[6] MANIN 2010 "treats, among other things, a categorical approach to the theory of computation, quantum computation, and the P/NP problem."

reference in later sections, in a way to make internalisation – *gödelisation* – comfortable.

**Notation:** The Gentzen type bars below read as ***inferences***

$$\textbf{then}\quad \frac{\text{given } \textbf{\textit{antecedent}}}{\textbf{\textit{postcedent}}}$$

# 1 Cartesian category language

## 1.1 Fundamental object language symbols

$$\{\mathbb{1}, \mathbb{N}, \times, 0, \mathrm{s}, \mathrm{id}, \circ, \Pi, \ell, \mathrm{r}\}$$

$\mathbb{1}$ is the *one-element object,* $\mathbb{N}$ the *natural numbers object* of theories **CA** and **PR** to come, $\times$ the cartesian product of objects and of maps.

0 is the *zero constant* $0 : \mathbb{1} \to \mathbb{N}$, s is the "fundamental" *successor function* $s : \mathbb{N} \to \mathbb{N}$.

Identity id is the family of *identity maps* to all objects, obtained out of $\mathbb{1}$ and $\mathbb{N}$ by *cartesian product* $\times$, $\circ$ is map *composition,* occasionally replaced by concatenation, $\Pi$ symbolises the family of *terminal maps* into object $\mathbb{1}$, $\ell$ and r are left resp. right *projections* out of cartesian product(s) $A \times B$.

Theory **PR** of primitive recursion below will come with an additional symbol $^{\S}$ for endomap *iteration.*

## 1.2 Cartesian category axioms

### Fundamental NNO data

{Obj $\mathbb{1}$, $\mathbb{N}$}

*one-element object* and *natural numbers object*

map $0 : \mathbb{1} \to \mathbb{N}$ *zero constant*

map $\mathrm{s} : \mathbb{N} \to \mathbb{N}$ *successor function*

### Category structure generation

[We use Gentzen's inference bars for description of emerging structure]

$$\text{Ax [id]} \quad \frac{\text{Obj } A}{\text{map id} : A \to A \ \textit{identity map}}$$

Emerging notion '=' of equality of maps is to become an ***equivalence*** *relation:*

$$\text{Ax [}\textit{reflexivity}\text{]} \quad \frac{\text{map } f}{f = f}$$

$$\text{Ax [}\textit{symmetry}\text{]} \quad \frac{\text{map } f, g;\ f = g}{g = f}$$

$$\underline{\text{Ax}}\ [\,\textit{transitivity}\,]\quad \frac{\underline{\text{map}}\ f, g, h;\ \ f = g;\ \ g = h}{f = h}$$

$$\underline{\text{Ax}}\ [\,\circ\,]\quad \frac{f : A \to B;\ \ g : B \to C}{\underline{\text{map}}\ (g\,f) = (g \circ f) = g(f) : A \to C}$$

$(g \circ f) : A \to B \to C$ ***composition***

outmost brackets may be omitted

Emerging composition of maps is to become substitutive and associative:

$$\underline{\text{Ax}}\ [\,\circ\,\text{sub}\,]\quad \frac{f, \tilde{f} : A \to B;\ \ g : B \to C;\ f = \tilde{f}}{g \circ f = g \circ \tilde{f}\ \text{Leibniz' } \textbf{\textit{substitutivity}}}$$

$$\underline{\text{Ax}}\ [\,\text{sub}\,\circ\,]\quad \frac{f : A \to B;\ \ g, \tilde{g} : B \to C;\ g = \tilde{g}}{g \circ f = \tilde{g} \circ f\ \ \textit{second}\ \text{Leibniz}\ \textbf{\textit{substitutivity}}}$$

$$\underline{\text{Ax}}\ [\,\circ\ \text{id}\,]\quad \frac{f : A \to B}{f \circ \text{id} = f \circ \text{id}_A = f}$$

$\text{id} \circ f = \text{id}_B \circ f = f$

***neutrality*** *of identities by composition*

$f : A \to B;$

$\underline{\text{var}}\ a \in A,\ \ a := \mathrm{id}_A$

$$\underline{\text{Lem}}\ [\circ\ \underline{\text{var}}]\quad \frac{f : A \to B; \quad \underline{\text{var}}\ a \in A,\ a := \mathrm{id}_A}{f(a) = f(\mathrm{id}_A) = f \circ \mathrm{id}_A = f}$$

$f(a) = f(\mathrm{id}_A) = f \circ \mathrm{id}_A = f$

***free variable*** *as identity*

$f(a) \in B$ *“dependent variable”*

$$\bullet\ \underline{\text{Ax}}\ [\text{ass}\ \circ]\quad \frac{f : A \to B;\ g : B \to C;\ h : C \to D \quad \underline{\text{var}}\ a \in A}{(h \circ g) \circ f = h \circ (g \circ f) : A \to D}$$

$= h \circ g \circ f = h\,g\,f = h(g(f(a)))$

***associativity*** *of composition*

## Cartesian structure

$$\underline{\text{Ax}}\ [\Pi]\quad \frac{\underline{\text{Obj}}\ A}{\underline{\text{map}}\ \Pi = \Pi_A : A \to \mathbb{1}}$$

***terminal*** *map*

$$\underline{\text{Ax}}\ [!\Pi]\quad \frac{f : A \to \mathbb{1}}{f = \Pi_A}$$

***uniqueness,*** *naturality of family* $\Pi$

DIAGRAM: $$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \Pi} & = & \downarrow{\scriptstyle \Pi} \\ \mathbb{1} & \xrightarrow{\text{id}} & \mathbb{1} \end{array}$$

- $\underline{\text{Ax}}$ [ $\underline{\text{Obj}}$ × ]

$$\frac{\underline{\text{Obj}}\ A, B}{\underline{\text{Obj}}\ (A \times B)}$$

***cartesian product*** *of objects,*

– multidimensional *grid* –

[ Outmost brackets may be omitted ]

- $\underline{\text{Ax}}$ [ $\ell$, r ]

$$\frac{\underline{\text{Obj}}\ A, B \qquad \underline{\text{var}}\ a \in A,\ \underline{\text{var}}\ b \in B}{\begin{array}{l}\underline{\text{map}}\ \ a = \ell = \ell_{A,B} : A \times B \to A \\ \underline{\text{map}}\ \ b = \mathrm{r} = \mathrm{r}_{A,B} : A \times B \to B\end{array}}$$

*left resp. right* ***projection,***

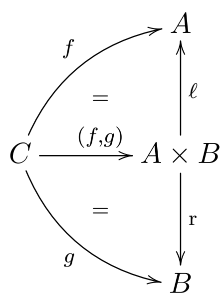
***variables*** *as projections*

• $\underline{\text{Ax}}$ [indu] $\dfrac{\underline{\text{map}}\ f : C \to A,\ \ g : C \to B}{\underline{\text{map}}\ (f, g) : C \to A \times B}$

***induced*** *map into product*

$\ell \circ (f, g) = f,\ \mathrm{r} \circ (f, g) = g$

DIAGRAM




Uniqueness of the induced, a priori expressed by a genuine **inference** of map equations, is obtained by the **equations** hereafter.

$\underline{\text{Ax}}$ [sub( , )] $\dfrac{f, \tilde{f} : C \to A;\ \ g, \tilde{g} : C \to B; \quad f = \tilde{f};\ \ g = \tilde{g}}{(f, g) = (\tilde{f}, \tilde{g})}$

***compatibility*** *of inducing* with '='

Ax [ distr ]

$$\frac{h : D \to C,\ f : C \to A,\ g : C \to B}{(f,g) \circ h = (f \circ h, g \circ h) : D \to (A \times B)}$$

***distributivity*** *of* $\circ$ *over forming*

*the induced map into a product*

[ Lem ]

$$\frac{\underline{\text{var}}\ c \in C,\ c := \mathrm{id}_C}{\ell \circ (f,g)(c) = \ell \circ (f(c), g(c)) = f(c)}$$

$$\mathrm{r} \circ (f,g)(c) = \mathrm{r} \circ (f(c), g(c)) = g(c)$$

q. e. d.

Ax [ retr. pairing ]

$$\frac{h : C \to (A \times B)}{(\ell_{A,B} \circ h, \mathrm{r}_{A,B}\, h) = h}$$

***pairing*** *is* (metamathematically)

***retractive,*** *even* ***isomorphic***

Lem [ !( , ) ]

$$\frac{f : C \to A;\ g : C \to B;\ h : C \to (A \times B);\quad \ell_{A,B} \circ h = f;\ \mathrm{r}_{A,B} \circ h = g}{h = (f,g)}$$

***uniqueness*** *of the induced map*

Proof:

$$h = \mathrm{id}_{A\times B} \circ h$$
$$= (\ell_{A,B} \circ \mathrm{id}_{A,B}, \mathrm{r}_{A,B} \circ \mathrm{id}_{A,B}) \circ h \quad \text{[retr. pairing]}$$
$$= (\ell_{A,B}, \mathrm{r}_{A,B}) \circ h$$
$$= (\ell_{A,B} \circ h, \mathrm{r}_{A,B} \circ h) \quad \text{[distr]}$$
$$= (f, g) : C \to A \times B \quad \text{[sub( , )]}$$

q. e. d.

Lem [ $(\ell, \mathrm{r})$ ]

$$\frac{\underline{\mathrm{Obj}}\ A, B}{(\ell_{A,B}, \mathrm{r}_{A,B}) = \mathrm{id}_{A\times B}}$$

Proof: uniqueness of induced into product $A \times B$ q. e. d.

Def [ $\times$ maps ]

$$\frac{f : A \to A',\ g : B \to B' \qquad \underline{\mathrm{var}}\ a := \ell_{A,B},\ b := \mathrm{r}_{A,B}}{(f \times g) = (f \circ \ell, g \circ \mathrm{r}) : (A \times B) \to (A' \times B')}$$

$$f \times g = (f \times g)(a, b) = (f(a), g(b))$$

***cartesian product*** *of maps*

***Unary*** *cartesian products:*

[ unary $\times$ ]

$$\frac{f : A \to A',\ g : B \to B'}{(A \times g) \ =_{\mathrm{def}}\ (\mathrm{id}_A \times g) : A \times B \to A \times B'}$$

$$(f \times B) \ =_{\mathrm{def}}\ (f \times \mathrm{id}_B) : A \times B \to A' \times B$$

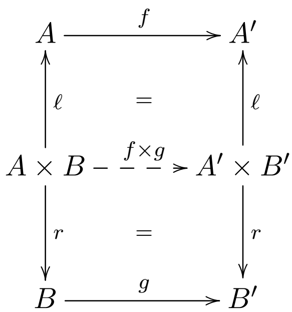
Thm $[\,\mathrm{nat}_{\ell,r}\,]$

$$\frac{\underline{\text{map}}\ f : A \to A',\ \ g : B \to B'}{\ell \circ (f \times g) = f \circ \ell;\ \ \mathrm{r} \circ (f \times g) = g \circ \mathrm{r}}$$

***naturality*** *of projection families* $\ell$ *and* r

Proof: uniqueness of induced map into product $A' \times B'$ q. e. d.

DIAGRAM:



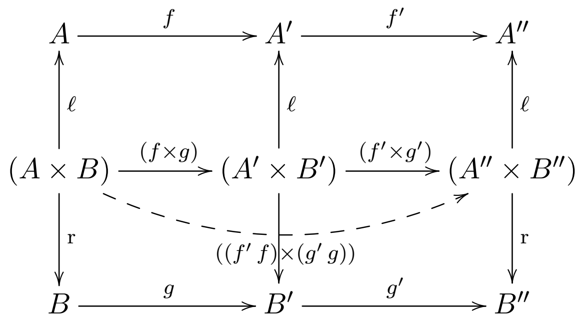

Thm $[\,\times\ \circ\,]$

$$\frac{f : A \to A',\ f' : A' \to A'';\quad g : B \to B',\ g' : B' \to B''}{\mathrm{id}_A \times \mathrm{id}_B = \mathrm{id}_{A\times B} : A \times B \to A \times B \qquad (f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g) : (A \times B) \to (A'' \times B'')}$$

***bifunctoriality*** *of cartesian product*

Proof: uniqueness of induced maps into products within

COMMUTING DIAGRAM




q. e. d.

The **axioms** up to here **define** the theory **CA** of *cartesian categories.* They are satisfied in pariticular by the category of **sets.**[7]

# 2 Primitive Recursion

The cartesian theory **PR** of primitive recursion is **defined** as follows, by

## 2.1 Iteration axioms added

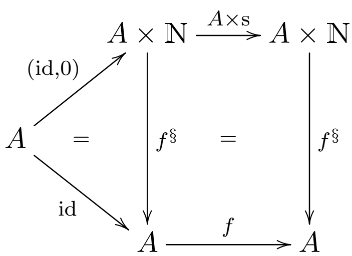
Iteration concept and notation are taken from EILENBERG/ELGOT 1970, case of just one successor map.

[7] corrected August 30, 2017

$$\underline{\text{Ax}}\ [\,\S\,]\quad \frac{f : A \to A,\ \underline{\text{var}}\ a \in A,\ \underline{\text{var}}\ n \in \mathbb{N}}{\begin{array}{l} f^{\S} = f^{\S}(a,n) : A \times \mathbb{N} \to A \\ f^{\S}(a,0) := f^{\S}(\mathrm{id}_A, 0\,\Pi_A) = a = \mathrm{id}_A \\ f^{\S} \circ (A \times \mathrm{s}) = f^{\S}(a, \mathrm{s}\,n) \\ = f \circ f^{\S} = f(f^{\S}(a,n)) : A \times \mathbb{N} \to A \to A \\ f^n(a) := f^{\S}(a,n) \end{array}}$$

***apply iteratively*** *n times endomap f to initial argument a.*

DIAGRAM:



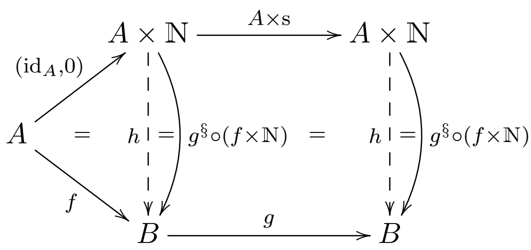

(§)

$$\underline{\text{Ax}}\ [\,\text{FR!}\,]\quad \frac{\begin{array}{l} f : A \to B;\ g : B \to B;\ h : A \times \mathbb{N} \to B; \\ \underline{\text{var}}\ a \in A,\ \underline{\text{var}}\ n \in \mathbb{N}; \\ h(a,0) = f(a); \\ h(a, \mathrm{s}\,n) = g\,h(a,n) \end{array}}{\begin{array}{l} h = g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}})\ \text{i.e.} \\ h(a,n) = g^n(f(a)) \end{array}}$$

Freyd*'s* ***uniqueness*** *of iterated endomap* $g$ ***initialised*** *by a map* $f$.

[ $g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}})$ does the job.]

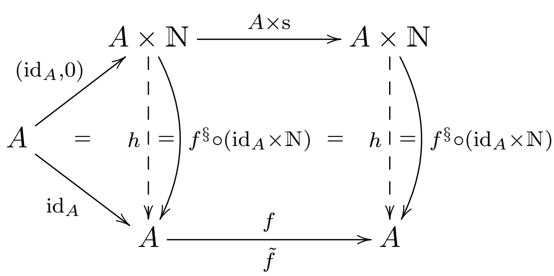
DIAGRAM[8]




$$\underline{\text{Lem}}\ [\S\,]\qquad \frac{f, \tilde{f} : A \to A;\ f = \tilde{f}}{f^{\S} = \tilde{f}^{\S} : A \times \mathbb{N} \to A}$$

***uniqueness*** *of* ***iterated*** *endo* $f$

Proof: Consider the specialised Freyd's uniqueness diagram[9]




$f^{\S} = f^{\S} \circ (\mathrm{id}_A \times \mathbb{N})$ as well as $h = \tilde{f}^{\S}$ are commutative fill ins, whence $\tilde{f}^{\S} = f^{\S}$ q. e. d.

[8] FREYD 1972

[9] changed August 30, 2017

## 2.2 Full schema of primitive recursion

$$
\underline{\text{Thm}}\ [\text{pr}] \quad
\frac{\begin{array}{l} g = g(a) : A \to B \\ h = h((a,n),b) : (A \times \mathbb{N}) \times B \to B \end{array}}
{\begin{array}{l} f = f(a,n) : A \times \mathbb{N} \to B \ \text{ s.t.} \\ \text{(anchor)} \quad f(a,0) = g(a) \text{ and} \\ \text{(step)} \quad f(a, \mathrm{s}\, n) = h((a,n), f(a,n)) \\ f =: \mathrm{pr}[g,h] \\ + \\ \text{(pr!) } \textit{uniqueness of } f \text{ to satisfy} \\ \text{these (anchor) and (step) equations.} \end{array}}
$$

**Interpretation:**

*General primitive recursive map $f = f(a,b)$ initialised by a map $g = g(a)$ and iteratively extended using a* ***step*** *map $h = h((a,n),b)$ which depends on previous value $b$ but (possibly) also on initial argument $a$ as well as from running recursion parameter $n$.*

Schema (pr) without use of free variables:

$$
\text{(pr)}\quad \frac{\begin{array}{l} g : A \to B, \\ h : (A \times \mathbb{N}) \times B \to B \end{array}}{\begin{array}{l} \mathrm{pr}[g,h] := f : A \times \mathbb{N} \to B \\ \quad \text{is given such that} \\ f(\mathrm{id}_A, 0) = g : A \to B \\ f\,(\mathrm{id}_A \times \mathrm{s}) = h\,(\mathrm{id}_{A\times\mathbb{N}}, f) : \\ \quad (A \times \mathbb{N}) \to (A \times \mathbb{N}) \times B \to B \\ (\mathrm{pr!}) : f \ \textit{unique} \end{array}}
$$

Schema (pr) is a consequence of iteration schema (§) and *uniqueness of the initialised iterated,*[10] the latter inference of equations taken above as **axiom** (FR!).

**Remarks:**

- Full schema (pr) of primitive recursion is an **axiom** in the classical theory of primitive recursion, subsystem of any (classical) arithmetical theory **T**.
- Free-Variables Arithmetics of the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$ can be based on the axioms of the cartesian theory **PR** of primitive recursion as defined in the above.
- Goodstein's[11] uniqueness axioms $\mathrm{U}_1$ to $\mathrm{U}_4$ – basic for his *Free-Variables Arithmetics* – are theorems of **PR**.

[10] Romàn 1989 and Pfender/Kröplin/Pape 1994
[11] Goodstein 1971

- In “Begründung der elementären Arithmetik durch die rekurrierende Denkweise ohne die Anwendung scheinbarer Veränderlichen mit unendlichem Ausdehnungsbereich”,[12] SKOLEM 1919 exhibits the strongly finitistic logical kernel of Principia Mathematica **PM,** and forshadows in particular GOODSTEIN 1971.

## 2.3 Predicates

A primitive recursive *predicate* is a primitive recursive map $\chi = \chi(a) : A \to \mathbb{N}$ such that

$$\begin{aligned}
&\mathbf{PR} \vdash\ \chi(a) \leq 1 : A \to \mathbb{N},\ \text{will say} \\
&\mathbf{PR} \vdash\ \operatorname{sgn}\chi = \chi \text{ where} \\
&\operatorname{sgn} : \mathbb{N} \to \mathbb{N} \text{ is p. r. defined by} \\
&\operatorname{sgn}(0) = 0,\ \operatorname{sgn}(\mathrm{s}\, n) = 1 = \mathrm{s}\, 0
\end{aligned}$$

Logically, $0 : \mathbb{1} \to \mathbb{N}$ plays the rôle of false, $1 = s\,0$ that of true.

# 3 Evaluation

From now on we place ourselves in a “*gödelian*” quantified arithmetical **set** theory **T**, with maps defined by Ackermann type *double recursion*[13] for introduction of evaluation below.

**Remark:** This evaluation is $\mu$-recursive, it can be defined by a terminating while loop, see PFENDER 2015/2017.

[12] *Foundation of the elementary Arithmetic by the recurring method of thought without the application of virtual variables with infinite extension domain*

[13] cf. PÉTER 1967 and EILENBERG/ELGOT 1970, Appendix A

## 3.1 Gödelisation

Since the categorical theory **PR** of primitive recursion comes formally without variables and quantification, we can code **PR** maps into an *internal, arithmetised theory* PR $\subset \mathbb{N}$ simply by their LaTeX `utf8` source codes, the Byte strings seen as natural numbers, in particular

- Codes of basic maps

  $\ulcorner \mathrm{ba} \urcorner \in \mathbb{N}$ for

  $\mathrm{ba} \in \mathrm{bas} = \{0, \mathrm{s}, \mathrm{id}, \Pi, \ell, \mathrm{r}\} = \{0, \mathrm{s}, \mathrm{id}_A, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B \ \mathbf{PR} \text{ objects}\}$ :

  $\ulcorner 0 \urcorner = \mathtt{utf8}[0] \in [\mathbb{1}, \mathbb{N}] \subset \mathrm{PR} \subset \mathbb{N}$

  $\ulcorner \mathrm{s} \urcorner = \mathtt{utf8}[\backslash\mathtt{mathrm}\{\mathtt{s}\}] \in [\mathbb{N}, \mathbb{N}] \subset \mathrm{PR} \subset \mathbb{N}$

  $\ulcorner \mathrm{id} \urcorner_A = \ulcorner \mathrm{id}_A \urcorner \in [A, A]$

  $\ulcorner \Pi \urcorner_A = \ulcorner \Pi_A \urcorner \in [A, \mathbb{1}]$

  $\ulcorner \ell \urcorner_{A,B} = \ulcorner \ell_{A,B} \urcorner \in [A \times B, A]$

  $\ulcorner \mathrm{r} \urcorner_{A,B} = \ulcorner \mathrm{r}_{A,B} \urcorner \in [A \times B, B] \subset \mathrm{PR}$

- Coding map composition: With $\odot = \ulcorner \circ \urcorner$

  $$\frac{f : A \to B, \ \ g : B \to C}{\ulcorner (g \circ f) \urcorner = \langle \ulcorner g \urcorner \odot \ulcorner f \urcorner \rangle \in [A, C]}$$

  internal composition:

  $$\frac{\boldsymbol{f} \in [A, B], \ \ \boldsymbol{g} \in [B, C]}{\langle \boldsymbol{g} \odot \boldsymbol{f} \rangle = \ulcorner ( \urcorner \boldsymbol{g} \ulcorner \circ \urcorner \boldsymbol{f} \ulcorner ) \urcorner \in [A, C]}$$

- Coding induced maps: with $\langle ; \rangle = \ulcorner ( , ) \urcorner$

$$\frac{f : C \to A, \;\; g : C \to B}{\ulcorner (f, g) \urcorner = \langle \ulcorner f \urcorner ; \ulcorner g \urcorner \rangle \in [C, A \times B]}$$

internal inducing:

$$\frac{\boldsymbol{f} \in [C, A], \;\; \boldsymbol{g} \in [C, B]}{\langle \boldsymbol{f} ; \boldsymbol{g} \rangle = \ulcorner ( \urcorner \boldsymbol{f} \ulcorner , \urcorner \boldsymbol{g} \ulcorner ) \urcorner \in [C, A \times B]}$$

- Coding map products (redundant): with $\# = \ulcorner \times \urcorner$

$$\frac{f : A \to A', \;\; g : B \to B'}{\ulcorner (f \times g) \urcorner = \langle \ulcorner f \urcorner \# \ulcorner g \urcorner \rangle \in [A \times A', B \times B']}$$

Internal map product:

$$\frac{\boldsymbol{f} \in [A, A'], \;\; \boldsymbol{g} \in [B, B']}{\langle \boldsymbol{f} \# \boldsymbol{g} \rangle = \ulcorner ( \urcorner \boldsymbol{f} \ulcorner \times \urcorner \boldsymbol{g} \ulcorner ) \urcorner \in [A \times B, A' \times B']}$$

- Coding endomap iteration: with $\$ = \ulcorner \S \urcorner$

$$\frac{f : A \to A}{\ulcorner f^{\S} \urcorner = \ulcorner f \urcorner^{\$} \in [A \times \mathbb{N}, A]}$$

internal iteration:

$$\frac{\boldsymbol{f} \in [A, A]}{\boldsymbol{f}^{\$} = \boldsymbol{f}^{\ulcorner \S \urcorner} \in [A \times \mathbb{N}, A]}$$

## 3.2 Evaluation of $\mathrm{PR}$ into set theory

We **define** **T**-recursively an evaluation map

$$\boldsymbol{ev} : \mathrm{PR} \times \mathbb{X} \to \mathbb{X}$$

of **PR** *map code set*

$$\mathrm{PR} = \bigcup_{A,B} [A, B]_{\mathrm{PR}} \subset \mathbb{N}$$

(where $[A, B] \subset \mathbb{N}$ is the **T** set of **PR** codes from $A$ to $B$),

on the *universal arguments-and-values* **T**-*set*

$$\mathbb{X} = \bigcup_{A \,\mathrm{in}\, \mathbf{PR}} A = \bigcup_{A \,\mathrm{in}\, \mathbf{CA}} A$$
$$= \{0\} \cup \mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \cup ((\mathbb{N} \times \mathbb{N}) \times \mathbb{N}) \cup \ldots$$

Objects $\mathbb{1}, \mathbb{N}, A, B, C$ etc. are considered as **PR** objects as well as **T** sets, $\mathbb{X}$ as a **T** set, theory **PR** is a subsystem of **T**.

We **define**

**Basic map/function code evaluation** $\boldsymbol{ev}$ :

$$\boldsymbol{ev}(\ulcorner 0 \urcorner, \ulcorner 0 \urcorner) = \ulcorner 0 \urcorner \in \mathbb{N} \subset \mathbb{X}$$

$$\frac{\underline{\mathrm{Obj}}(A),\ a \in A \subset \mathbb{X}}{\boldsymbol{ev}(\ulcorner \mathrm{id} \urcorner, a) = a}$$

$$\frac{n \in \mathbb{N}}{\boldsymbol{ev}(\ulcorner \mathrm{s} \urcorner, n) = \mathrm{s}\, n \in \mathbb{N} \subset \mathbb{X}}$$

$$\frac{\underline{\mathrm{Obj}}(A),\ a \in A}{\boldsymbol{ev}(\ulcorner \Pi \urcorner, a) = \ulcorner 0 \urcorner \in \mathbb{1} \subset \mathbb{X}}$$

$$\frac{\underline{\mathrm{Obj}}(A),\ \underline{\mathrm{Obj}}(B),\ a \in A,\ b \in B}{\begin{array}{l}\boldsymbol{ev}(\ulcorner \ell \urcorner, (a,b)) = a \\ \boldsymbol{ev}(\ulcorner \mathrm{r} \urcorner, (a,b)) = b\end{array}}$$

**Put together:**

$$\frac{\begin{array}{l}\mathrm{ba} \in \mathrm{bas} = \{\mathrm{id}, 0, s, \Pi, \ell, \mathrm{r}\} \\ = \{\mathrm{id}_A, 0, s, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : \underline{\mathrm{Obj}}\ A, B\} \subset \mathrm{PR} \\ A = \mathrm{Dom}[\mathrm{ba}],\ B = \mathrm{Codom}[\mathrm{ba}],\ a \in A\end{array}}{\begin{array}{l}\boldsymbol{ev}(\ulcorner \mathrm{ba} \urcorner, a) = \mathrm{ba}(a) \in B \subset \mathbb{X} \\ \boldsymbol{ev}(\ulcorner \mathrm{ba} \urcorner, x) = x \text{ for } x \in \mathbb{X} \setminus A \quad \text{(no action)}\end{array}}$$

**Evaluation of composed map codes:**

$$(\mathrm{compos}) \quad \frac{f \in [A,B],\ g \in [B,C],\ a \in A}{\boldsymbol{ev}(g \odot f, a) = \boldsymbol{ev}(g, \boldsymbol{ev}(f, a)) \in C \subset \mathbb{X}}$$

formally (and for **PR** instead of **CA** in fact)

***double recursive*** (within **T**)

$$
\text{(indu)}\quad \frac{f \in [C, A] \subset \mathrm{PR} \subset \mathbb{N},\ g \in [C, B] \subset \mathrm{PR} \qquad c \in C \subset \mathbb{X}}{\begin{array}{l} \boldsymbol{ev}(\langle f; g\rangle, c) = \langle \boldsymbol{ev}(f, c), \boldsymbol{ev}(g, c)\rangle \\ \in (A \times B) \subset \mathbb{X} \\ \textit{\textbf{primitive recursive}} \end{array}}
$$

$$
(\times)\quad \frac{f \in [A, A'] \subset \mathrm{PR},\ g \in [B, B'] \subset \mathrm{PR} \qquad a \in A,\ \ b \in B}{\begin{array}{l} \boldsymbol{ev}(\langle f \# g\rangle, (a; b)) = (\boldsymbol{ev}(f, a), \boldsymbol{ev}(g, b)) \\ \in (A' \times B') \subset \mathbb{X} \\ (\textit{redundant}) \end{array}}
$$

$$
\text{(iter anchor)}\quad \frac{f \in [A, A],\ \ a \in A}{\boldsymbol{ev}(f^{\$}, (a, \ulcorner 0 \urcorner)) = a}
$$

$$
\text{(iter step)}\quad \frac{f \in [A, A],\ \ a \in A,\ \ n \in \mathbb{N}}{\begin{array}{l} \boldsymbol{ev}(f^{\$}, (a, \mathrm{s}\, n) \\ = \boldsymbol{ev}(f, \boldsymbol{ev}(f^{\$}, (a, n)) \in A \\ \textit{\textbf{double recursive}} \\ \text{inner recursion on } n \end{array}}
$$

$$(\text{trash}) \quad \frac{f \in [A,B],\ x \in \mathbb{X} \smallsetminus A}{\boldsymbol{ev}(f,x) = x \text{ (no action)}}$$

## 3.3 Evaluation Theorem

(i) Double recursion above defines a *total* **T**-map

$$\boldsymbol{ev} = \boldsymbol{ev}(f,x) : \mathrm{PR} \times \mathbb{X} \to \mathbb{X}$$

(ii) $\boldsymbol{ev}$ is **characterised** within gödelian Arithmetics **T** by

$$\boldsymbol{ev}(\ulcorner \mathrm{ba} \urcorner, x) = \mathrm{ba}(x)$$
$$\text{for ba} \in \text{bas (basic map constants)}$$
$$\boldsymbol{ev}(g \odot f, a) = \boldsymbol{ev}(g, \boldsymbol{ev}(f,a))$$
$$\boldsymbol{ev}(\langle f; g\rangle, c) = \boldsymbol{ev}(f,c), \boldsymbol{ev}(g,c))$$
$$\boldsymbol{ev}(\langle f \# g\rangle, (a;b)) = (\boldsymbol{ev}(f,a), \boldsymbol{ev}(g,b))$$

as well as

$$\boldsymbol{ev}(f^{\$}, (a, \ulcorner 0 \urcorner)) = a \text{ and}$$
$$\boldsymbol{ev}(f^{\$}, (a, \mathrm{s}\, n) = \boldsymbol{ev}(f, \boldsymbol{ev}(f^{\$}, (a,n))$$
$$a \in A,\ b \in B,\ c \in C,\ n \in \mathbb{N} \text{ all free}$$
$$\boldsymbol{ev}(f,x) = x \text{ for } x \in (\mathbb{X} \smallsetminus \mathrm{Dom}[f])$$

(iii) $\boldsymbol{ev}$ defines within theory **T** a **family**[14]

$$\boldsymbol{ev} = \boldsymbol{ev}_{A,B} = \boldsymbol{ev}_{A,B}(f,a) : [A,B] \times A \to B$$
$$A, B \text{ } \mathbf{PR} \text{ objects, by}$$
$$\boldsymbol{ev}_{A,B}(f,a) = \boldsymbol{ev}(f,a) : [A,B] \times A \to B$$

[14] natural fransformation

(iv) This family $\boldsymbol{ev} = \boldsymbol{ev}_{A,B}$ is (jointly) *objective:*

$$\frac{f : A \to B \ \ \mathbf{PR} \text{ map}, \ a \in A \text{ free}}{\boldsymbol{ev}(\ulcorner f \urcorner, a) = \boldsymbol{ev}_{A,B}(\ulcorner f \urcorner, a) = f(a) \in B}$$

**Proof**

(i) totality of evaluation map – family of maps – defined by an Ackermann type double recursion, is given in **set** theorie(s) **T**.

(ii) the characterisation of evaluation $\boldsymbol{ev}$ within **set** theory **T** follows directly from its double recursive definition.

(iii) splitting $\boldsymbol{ev} : \mathrm{PR} \times \mathbb{X} \to \mathbb{X}$ into family

$$\boldsymbol{ev}_{A,B} = \boldsymbol{ev}_{A,B}(\boldsymbol{f}, a) : [A, B] \times A \to B$$

is immediate.

(iv) objectivity: substitute codes $\ulcorner f \urcorner$, $\ulcorner g \urcorner$ of **PR** maps $f, g$ into code variables $\boldsymbol{f}, \boldsymbol{g} \in \mathrm{PR} \subset \mathbb{N}$ of the evaluation characterisation, beginning with the basic map codes $\ulcorner \mathrm{ba} \urcorner$ **q. e. d.**

# 4 Relative soundness

## 4.1 Internal notion of equality

The objective (notion of) equality of **PR** has an internal-equality enumeration analogon

$$
\begin{aligned}
&\mathrm{eq} = \mathrm{eq}(k) \;=\; \check{\doteq}_k : \mathbb{N} \to \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N} \times \mathbb{N} \\
&k \mapsto (\boldsymbol{f} \,\check{\doteq}_k\, \boldsymbol{g}),\;\; k \in \mathbb{N} \text{ free} \\
&\quad \text{where we write } \boldsymbol{f} \,\check{\doteq}_k\, \boldsymbol{g} \text{ for} \\
&\mathrm{eq}(k) = (\boldsymbol{f}, \boldsymbol{g}) \in \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N} \times \mathbb{N}.
\end{aligned}
$$

This exhaustive enumeration **defines** the internal notion '$\check{\doteq}$' of p. r. equality. It is given by p. r. count (within theory **T**) of *deduction trees* which deduce their root internal equation, say $h \,\check{\doteq}_k\, \tilde{h}$. The deduction trees are made out of internal versions of **PR axioms,** equational or implicational.

**Example:** For $k \in \mathbb{N}$ suitable, a transitivity-of-equality deduction tree has form

$$
\mathrm{dtree}_k \;=\; \Uparrow \frac{\boldsymbol{f} \,\check{\doteq}_k\, \boldsymbol{h}}{\Uparrow \dfrac{\boldsymbol{f} \,\check{\doteq}_i\, \boldsymbol{g}}{\mathrm{dtree}_{ii} \quad \mathrm{dtree}_{ij}} \qquad \Uparrow \dfrac{\boldsymbol{g} \,\check{\doteq}_j\, \boldsymbol{h}}{\mathrm{dtree}_{ji} \quad \mathrm{dtree}_{jj}}}
$$

$i, j < k,\;\; ii, ij < i,\;\; ji, jj < j$ all suitable.

Here the Gentzen type **bar** trees are to be **read** upwards as deduction trees in **set** theory **T**, composed of implications '$\Uparrow$' of **T**.

## 4.2 Evaluation soundness Theorem

With (any) **set** theory $\mathbf{T}$ taken as frame, and for p. r. theory **PR** with its internal notion of equality '$\check{=}$' above, we have for evaluation $\boldsymbol{ev} = [\boldsymbol{ev}_{A,B} : [A, B] \times A \to B]_{A,B}$

(i) PR to $\mathbf{T}$ *evaluation soundness*

$$\mathbf{T} \vdash f \check{=}_k g \implies \boldsymbol{ev}(f, a) = \boldsymbol{ev}(g, a) \qquad (\bullet)$$
$$k \in \mathbb{N},\ f, g \in [A, B],\ a \in A \text{ all free}$$

Substitution of "concrete" **PR** codes into $f$ resp. $g$ entails, by *objectivity* of evaluation $\boldsymbol{ev}$,

(ii) *objective soundness of* $\mathbf{T}$ *relative to* **PR** :

For p. r. maps $f, g : A \to B$

$$\mathbf{T} \vdash \ulcorner f \urcorner \check{=}_k \ulcorner g \urcorner \implies f(a) = g(a)$$
$$k \in \mathbb{N},\ a \in A \text{ both free}$$

(iii) Specialisation to case $f := \chi : A \to \mathbb{N}$ a p. r. *predicate* and to $g := \mathrm{true}_A = \mathrm{s}\,0\,\Pi_A : A \to \mathbb{N}$ gives *logical soundness of* $\mathbf{T}$ *relative to* **PR** :

$$\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall\, a\, \chi(a)$$

***If*** *a p. r. predicate is, within* $\mathbf{T}$, **PR**-*internally* provable, ***then*** *it holds in* $\mathbf{T}$ *for all of its arguments.*

**Proof** of assertion $(\bullet)$ is by primitive recursion on $k$, $\mathrm{dtree}_k$ being the $k$ th deduction tree of internal theory PR and $\mathrm{dtree}_k$ *proving* as its root equation $f \check{=}_k g$. These (argument-free) deduction trees are counted in lexicographical order – with the effect that a branch of such a tree preceeds the tree in that counting order.

**Super Case** of *equational* internal axioms, *flat* deduction trees, in particular

- associativity of (internal) composition:
  $\langle h \odot g \rangle \odot f \doteq h \odot \langle g \odot f \rangle \implies$

  $$\begin{aligned} &\boldsymbol{ev}(\langle h \odot g \rangle \odot f, a) = \boldsymbol{ev}(\langle h \odot g \rangle, \boldsymbol{ev}(f, a)) \\ &= \boldsymbol{ev}(h, \boldsymbol{ev}(g, \boldsymbol{ev}(f, a))) \\ &= \boldsymbol{ev}(h, \boldsymbol{ev}(\langle g \odot f \rangle, a)) = \boldsymbol{ev}(h \odot \langle g \odot f \rangle, a) \end{aligned}$$

  This **proves** assertion (•) in present *associativity-of-composition* case.

- Analogous proof for the other flat – equational – cases, namely *reflexivity of equality, left and right neutrality* of identities, Godement's equations for the induced map:

  $$\ulcorner \ell \urcorner \odot \langle f; g \rangle \doteq f, \ \ \ulcorner \mathrm{r} \urcorner \odot \langle f; g \rangle \doteq g$$

  and definition of cartesian product of maps via induced map, as well as *retractive pairing*

  $$\langle \ulcorner \ell \urcorner \odot h; \ulcorner \mathrm{r} \urcorner \odot h \rangle \doteq h$$

  and distributivity equation

  $$\langle f; g \rangle \odot h \doteq \langle f \odot h; g \odot h \rangle$$

  for composition with an induced.

- proof of (•) for the last equational cases, iteration equations:

– *iteration anchoring, equation*
$f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle \check{=} \ulcorner \mathrm{id} \urcorner$ :

$$\begin{aligned}\mathbf{T} \vdash \boldsymbol{ev}(f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle, a) & \\ = \boldsymbol{ev}(f^{\$}, (\boldsymbol{ev}(\ulcorner \mathrm{id} \urcorner, a), \boldsymbol{ev}(\ulcorner 0 \urcorner, \boldsymbol{ev}(\ulcorner \Pi \urcorner, a))) & \\ = \boldsymbol{ev}(f^{\$}, (a, \boldsymbol{ev}(\ulcorner 0 \urcorner, 0)) & \\ = \boldsymbol{ev}(f^{\$}, (a, 0)) = a = \boldsymbol{ev}(\ulcorner \mathrm{id} \urcorner, a) &\end{aligned}$$

– *iteration step,* case of *genuine iteration equation*

$$f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle \check{=} (f \odot f^{\$})$$

reads:

$$\begin{aligned}&\boldsymbol{ev}(f^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle, (a, n)) \\ &= \boldsymbol{ev}(f^{\$}, \boldsymbol{ev}(\ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner, (a, n))) \\ &= \boldsymbol{ev}(f^{\$}, (a, \mathrm{s}\, n)) \\ &= \boldsymbol{ev}(f, \boldsymbol{ev}(f^{\$}, (a, n))) \\ &= \boldsymbol{ev}(f \odot f^{\$}, (a, n))\end{aligned}$$

[ Internal cartesian map product is defined as an internal induced ]

**Proof** of **PR** to **T** evaluation soundness for the genuine HORN case axioms, of form

$$f \check{=}_i g \wedge \tilde{f} \check{=}_j \tilde{g} \implies h \check{=}_k \tilde{h}, \ i, j < k :$$

**Transitivity-of-equality** case

$$f \check{=}_i g \wedge g \check{=}_j h \implies f \check{=}_k h$$

Evaluate here at argument $a \in A$ and get in fact

$$\begin{aligned}
&\mathbf{T} \vdash f \,\check{\doteq}_k\, h \\
&\qquad \Longrightarrow \ \boldsymbol{ev}(f, a) = \boldsymbol{ev}(g, a) \ \wedge \ \boldsymbol{ev}(g, a) = \boldsymbol{ev}(h, a) \\
&\qquad\quad \text{by hypothesis } f \,\check{\doteq}_i\, g, \ \ g \,\check{\doteq}_j\, h \\
&\qquad \Longrightarrow \ \boldsymbol{ev}(f, a) = \boldsymbol{ev}(h, a) : \\
&\quad \text{transitivity export q. e. d.}
\end{aligned}$$

**Compatibility case** of composition with equality
$g \,\check{\doteq}_i\, g, \ \ f \,\check{\doteq}_j\, \tilde{f} \ \Longrightarrow \ g \odot f \,\check{\doteq}_k\, g \odot \tilde{f}$ :

$$\begin{aligned}
&\boldsymbol{ev}(g \odot f, a) = \boldsymbol{ev}(g, \boldsymbol{ev}(f, a)) = \boldsymbol{ev}(g, \boldsymbol{ev}(\tilde{f}, a)) \\
&= \boldsymbol{ev}(g \odot \tilde{f}, a)
\end{aligned}$$

by hypothesis on $f \,\check{\doteq}\, \tilde{f}$ and by Leibniz' substitutivity in $\mathbf{T}$ q. e. d. in this first compatibility case.

**Case** of compatibility of composition with equality in second factor:
$g \,\check{\doteq}_i\, \tilde{g} \ \Longrightarrow \ g \odot f \,\check{\doteq}_k\, \tilde{g} \odot f$ :

$$\begin{aligned}
&\boldsymbol{ev}(g \odot f, a) = \boldsymbol{ev}(g, \boldsymbol{ev}(f, a)) = \boldsymbol{ev}(\tilde{g}, \boldsymbol{ev}(f, a)) \quad (*) \\
&= \boldsymbol{ev}(\tilde{g} \odot f, a)
\end{aligned}$$

$(*)$ holds by $g \,\check{\doteq}_i\, \tilde{g}$ and induction hypothesis on $i$ : arbitrary argument, here $\boldsymbol{ev}(f, a)$

This proves soundness assertion $(\bullet)$ in this 2nd compatibility case.

**Compatibility case** of internal formation of the induced map with internal equality

$$\boldsymbol{f} \mathrel{\dot{\check{=}}_i} \tilde{\boldsymbol{f}},\ \boldsymbol{g} \mathrel{\dot{\check{=}}_j} \tilde{\boldsymbol{g}} \implies \langle \boldsymbol{f}; \boldsymbol{g}\rangle \mathrel{\dot{\check{=}}_k} \langle \tilde{\boldsymbol{g}}; \tilde{\boldsymbol{f}}\rangle :$$

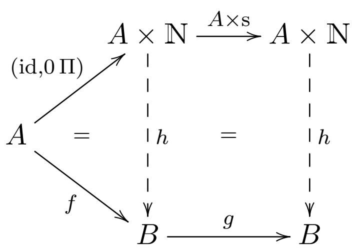
$$\begin{aligned}
&\boldsymbol{ev}(\langle \boldsymbol{f}; \boldsymbol{g}\rangle, c) = (\boldsymbol{ev}(\boldsymbol{f}, c), \boldsymbol{ev}(\boldsymbol{g}, c)) = (\boldsymbol{ev}(\tilde{\boldsymbol{f}}, c), \boldsymbol{ev}(\tilde{\boldsymbol{g}}, c)) \\
&\quad \text{by hypothesis } \boldsymbol{f} \mathrel{\dot{\check{=}}_i} \tilde{\boldsymbol{f}},\ \boldsymbol{g} \mathrel{\dot{\check{=}}_j} \tilde{\boldsymbol{g}} \\
&= \boldsymbol{ev}(\langle \tilde{\boldsymbol{f}}; \tilde{\boldsymbol{g}}\rangle, c)
\end{aligned}$$

Same for compatibility of internal cartesian map product with equality (redundant).

**(Final) case** of Freyd's (internal) uniqueness of the *initialised iterated* is **case**

$$\begin{aligned}
&\boldsymbol{h} \odot \langle \ulcorner \mathrm{id} \urcorner; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle \mathrel{\dot{\check{=}}_i} \boldsymbol{f} \\
&\wedge\ \boldsymbol{h} \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner \rangle \mathrel{\dot{\check{=}}_j} \boldsymbol{g} \odot \boldsymbol{h} \\
&\implies \boldsymbol{h} \mathrel{\dot{\check{=}}_k} \boldsymbol{g}^{\$} \odot \langle \boldsymbol{f} \# \ulcorner \mathrm{id} \urcorner \rangle \qquad\qquad (**)
\end{aligned}$$

internal version of $h$ unique, $h =^{\mathbf{PR}} g^{\S} \circ (f \times \mathrm{id})$ in




**Comment:** $\boldsymbol{h}$ is an internal *comparison candidate* fullfilling the same internal p. r. equations as the *initialised iterated* $\boldsymbol{g}^{\$} \odot \langle \boldsymbol{f} \# \ulcorner \mathrm{id} \urcorner \rangle$. It should be – **is**: soundness – evaluated equal to the latter on $A \times \mathbb{N}$; $\boldsymbol{h}$ corresponds to $h$, $\boldsymbol{f}$ to $f$, $\boldsymbol{g}$ to $g$, and $\boldsymbol{g}^{\$} \odot \langle \boldsymbol{f} \# \ulcorner \mathrm{id} \urcorner \rangle$ to $g^{\S} \circ (f \times \mathrm{id}_{\mathbb{N}})$.

Soundness **proof** in this case, namely case

$$h \odot \langle \ulcorner \mathrm{id} \urcorner, 0\rangle \check{=}_i f \wedge h \odot \langle \ulcorner \mathrm{id} \urcorner \# \mathrm{s}\rangle \check{=}_j g \odot h$$
$$\implies h \check{=}_k g^{\$} \odot \langle \ulcorner \mathrm{id} \urcorner \# f\rangle$$

is the following, by (structural) recursion on $k$ :

$$\begin{aligned} & \boldsymbol{ev}(h, (a, 0)) = \boldsymbol{ev}(f, a) \qquad \text{(hypothesis on } i < k) \\ & = \boldsymbol{ev}(g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner\rangle, (a, 0)) \end{aligned}$$

as well as – *induction on n* –

$$\begin{aligned} & \boldsymbol{ev}(h, (a, \mathrm{s}\, n)) \\ & = \boldsymbol{ev}(h \odot \langle \ulcorner \mathrm{id} \urcorner \# \ulcorner \mathrm{s} \urcorner\rangle, (a, n)) \\ & = \boldsymbol{ev}(g \odot h, (a, n)) \qquad \text{(hypothesis on } j < k) \\ & = \boldsymbol{ev}(g, \boldsymbol{ev}(h, (a, n))) \\ & = \boldsymbol{ev}(g, \boldsymbol{ev}(g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner\rangle, (a, n))) \\ & \quad \text{by induction hypothesis on } n \\ & = \boldsymbol{ev}(g \odot \langle g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner\rangle\rangle, (a, n)) \\ & = \boldsymbol{ev}(g^{\$} \odot \langle f \# \ulcorner \mathrm{id} \urcorner\rangle, (a, \mathrm{s}\, n)) \end{aligned}$$

**q. e. d.**

# 5 Decidability of PR predicates

We consider **PR** predicates $\chi$ for decidability by **set** theorie(s) **T**, without restriction of generality just predicates $\chi = \chi(n) : \mathbb{N} \to \mathbb{N}$ over $\mathbb{N}$.

Basic tool for decision is **PR** to **T** evaluation-soundness of **PR** above, namely

$\chi = \chi(n) : \mathbb{N} \to \mathbb{N}$ **PR** predicate

---

$\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall n\; \boldsymbol{ev}(\ulcorner \chi \urcorner, n) \;=\; \mathrm{true}$

whence by objectivity of evaluation $\boldsymbol{ev}$ :

$\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall n\, \chi(n)$

Within **T** **define** for $\chi : \mathbb{N} \to \mathbb{N}$ a **PR** predicate, a partially defined **predicate decision**

$$\nabla \chi = \begin{cases} \text{false if } \exists k\, \neg \chi(k) \\ \text{true if } \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \\ \bot \; \textit{otherwise} \text{ i. e.} \\ \text{if } \forall k\, \chi(k) \;\wedge\; \forall k\, \neg \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \\ \text{(derivable but not } \textit{provable}\text{)} \end{cases}$$

$$: \mathbb{1} \to \mathbb{2} \cup \{\bot\} = \{\mathrm{false}, \mathrm{true}, \bot\}$$

**well defined** by soundness above of **T** relative to **PR** :

− − − − − − − − − − − − − − − − − − − −

$\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall n\, \chi(n)$

− − − − − − − − − − − − − − − − − − − −

and same as

$$\nabla\chi = \begin{cases} \text{false if } \exists k\, \neg\chi(k) \\ \quad \textit{counterexample against } \chi \textit{ available} \\ \text{true if } \forall k\, \chi(k) \ \wedge\ \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \\ \quad \chi \textit{ derivable and "even" provable} \\ \bot \text{ if } \forall k\, \chi(k) \ \wedge\ \forall k\, \neg\mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \\ \quad \textit{incompleteness at } \chi\textit{:} \\ \quad \chi \textit{ derivable but not (internally) provable} \end{cases}$$

The above gives a *complete* $\mathbf{T}$-*alternative* (for $\nabla\chi$).

(Logical) union of latter two cases of the alternative, gives

### Decidability Theorem

There is the following *complete* $\mathbf{T}$-*alternative for PR predicates* $\chi = \chi(n) : \mathbb{N} \to \mathbb{N}$ :

- *counterexample* $\mathbf{T} \vdash \exists n\, \neg\chi(n)$

  **or else**

- *derivation* $\mathbf{T} \vdash \forall n\, \chi(n)$ **q. e. d.**

**Decision Remark:** This does not mean a priori that *decision algorithm* $\nabla\chi$ *terminates* for all such predicates $\chi$. The theorem says only that $\chi$ is ***decidable*** "by", *within* theory $\mathbf{T}$, that it is ***not independent*** of $\mathbf{T}$.

# 6 Gödel's incompleteness theorems

We visit §2. Gödel's theorems, in Smorynski 1977, Handbook of Mathematical Logic.

First Incompleteness Theorem. *Let* **T** *be a formal theory containing arithmetic. Then there is a sentence* $\varphi$ *which asserts its own unprovability and such that:*

(i) *If* **T** *is consistent,* $\mathbf{T} \nvdash \varphi$

(ii) *If* **T** *is* $\omega$*-consistent,* $\mathbf{T} \nvdash \neg\varphi$

In §3.2.6 Smorynski discusses possible choices of *arithmetic* theory, namely

(a) **PRA** = classical primitive recursive arithmetic.

(b) **PA** = Peano Arithmetic.
Conjecture: $\mathbf{PA} \cong \mathbf{PR} + \exists$

(c) **ZF** = Zermelo-Fraenkel set theory. "This is both a good and a bad example. It is bad because the whole encoding problem is more easily solved in a set theory than in an arithmetical theory. By the same token, it is a good example."

Smorynski's **proof** gives the *First Gödel's incompleteness theorem* for **T** one of the above **set** theories, and from that the following

**Second incompleteness theorem:** Let **T** be one of the quantified arithmetical theory extensions above of **PR**, and **T** consistent. Then

$$\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}}$$

where here $\mathrm{Con}_{\mathbf{T}} = \neg\exists\, k\ \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner\mathrm{false}\urcorner)$ is the sentence asserting the consistency of **T**.

*The consistency formula* $\mathrm{Con}_{\mathbf{T}}$ *of* $\mathbf{T}$ *is not derivable in Metamathematics, even if theory* $\mathbf{T}$ *itself is taken as metamathematical frame,* provided that $\mathbf{T}$ is consistent.

# 7 Inconsistency provability

Predicate $\mathrm{Prov}_{\mathbf{T}}(x, y)$ corresponds to Gödel formula

45. $xBy$, $x$ ist *Beweis* von $y$.

Gödel proves that this formula is *rekursiv,* primitive recursive in contemporary terms.

[ Later Ackermann found "Ackermann recursive" functions growing faster than any "primitive recursive" function,[15] evaluation ***ev*** above is of this type ]

Formula 46. $\exists x\, xBy$ '$y$ ist *beweisbar'* is a priori, formally not primitive recursive, same as for Gödel's "undecidable" formula 17 Gen $r$

But $\mathrm{Con}_{\mathbf{T}} = \neg\exists k\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) = \forall k\, \neg\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner)$ corresponds to the free-variable p. r. predicate $\neg\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to \mathbb{N}$, decidable by Decidability Theorem above.

Gödel's **2nd theorem** infers from this decidability of $\mathrm{Con}_{\mathbf{T}}$

**Inconsistency provability theorem** for **set** theories $\mathbf{T}$ : Such theory $\mathbf{T}$ derives its own inconsistency formula,

$$\mathbf{T} \vdash \neg\, \mathrm{Con}_{\mathbf{T}} \quad \mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) \ \ \textbf{q. e. d.}$$

[15] see Péter 1967 as well as Eilenberg/Elgot 1970

# Proof Résumé

- The consistency formula for "any" theory, in particular for an arithmetical theory, can be stated in terms of a free variable PR predicate: *For any number k (k* $\in \mathbb{N}$ free*), k is not (the enumeration index of) a proof code for (code of)* false.
- **set** theories **T** admit (correctly terminating) *evaluation* of their PR map code sets.
- Such theory **T** is able to decide any p. r. predicate on *counterexample vs. derivability by* **T** : *Decidability Theorem.*
- In particular the consistency formula of such a gödelian theory **T** is **decided** by **T** taken as "metamathematical" frame.
- This result leads to self-inconsistency of **set** theory, as *Corollary* to the second Gödel's *Incompleteness Theorem on non-derivability of theorie's* **T** *consistency formula, provided theory* **T** *is consistent.*

  [ If **T** is inconsistent, then it derives everything, in particular its own inconsistency formula ]

**Note:** Observe that Gödel's "undecidable" formula 17 Gen r is not primitive recursive.

**Remark:** A way out will be given in Pfender 2017:

Choose as "metamathematical" frame self-consistent (ultimate) *iterative descent theory* **R** : p. r. theory **PR** with extension objects of predicates – among these *universal object*

$\mathbb{X}$ of all (internal) numerals and nested numpairs –, and additional **axiom schema** of *non-infinite descent* of all – of the whole spectrum of – complexity controlled iterations: complexity values in any of the polynomial ordinals $\mathbb{N}[\omega_1]\ldots[\omega_{n-1}][\omega]$.

# Discussion

Background for the discussion are the books of Yu. I. Manin 2010 and K. Sigmund 2015.

"Vorbilder des Wiener Kreises sind der Physiker Albert Einstein, der Mathematiker David Hilbert und der Philosoph Bertrand Russell." [16]

Russell had discovered a first contradiction in Frege-Cantor's **set** theory, namely availability of "set" $R = \{x : x \notin x\}$ with paradoxical property $R \in R \iff R \notin R$, and authored with Whitehead 1900 the (typified) Principia Mathematica **PM** in order to exclude this paradoxon from **set** theory.

The first two of Hilbert's famous 10 (later 23) **problems**[17] ask for a provably consistent foundation of Mathematics (and decision of the Continuum Hypothesis **CH**). Hilbert: "Wir wollen wissen, wir werden wissen. ... Niemand wird uns aus dem Paradies vertreiben, in das Cantor uns geführt hat." Hilbert devoted himself to a solution of these first (and second) problems.

In the opinion of the majority of Mathematicians, Gödel has "erledigt" Hilbert's formalistisches **Programm** with the publication of his two **incompleteness** theorems for *Principia*

[16] Sigmund 2015
[17] talk at ICM conference Paris 1900, *Gesammelte Abhandlungen.* Springer 1970

*Mathematica* **PM** *und verwandte Systeme,* such as in particular Zermelo-Fraenkel **set** theory **ZF** and v. Neumann-Gödel-Bernays Foundations **NGB.**

The *anti-idealistic anti-metaphysical Wiener Kreis*, Gödel's intellectual home, celebrated Gödel for his [vermeintliche] Relativierung of the GREEK identity of truth with *provability* in axiomatic Mathematics.[18]

Gödel himself was said to have doubts on his ***assumption*** of $\omega$-consistency, of non-$\omega$-inconsistency. Did he even have doubts on *consistency* of **PM**? As K. Sigmund reports, Gödel became deeply depressive, after his death answers to letters (not given to mail) were found in his desk revealing his platonic convictions.

**Thanks** to J. Sablatnig for valuable comments, critiques, and suggestions.

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[18] cf. MANIN 2010, II 11.7.

# Appendix: Iterative evaluation

We resolve *double recursive* evaluation $\boldsymbol{ev}$ into a **CCI** (*Complexity Controlled Iteration*).

We **construct** it here within framework of (quantified arithmetical) **set** theory **T** with *finite descent* of chains of polynomials out of ordered semiring $\mathbb{N}[\omega]$.

## Evaluation Resolution

Evaluation $\boldsymbol{ev} = \boldsymbol{ev}(f, x) : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}}$ can be resolved into a *Complexity Controlled Iteration* (CCI):

$$\underline{\text{while}}\ \boldsymbol{cx}\, f > 0\ \underline{\text{do}}\ (f, x) := \boldsymbol{e}(f, x)\ \underline{\text{od}}$$

where $\boldsymbol{cx} = \boldsymbol{cx}\, f : \mathrm{PR} \to \mathbb{N}[\omega]$ is a suitable map code *complexity* within the linearily ordered semiring of polynomials with coefficients in $\mathbb{N}$. This complexity descends, eventually down to 0, with each application of *evaluation step* $\boldsymbol{e} = \boldsymbol{e}(f, x) : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR} \times \mathbb{X}_{\underline{\perp}}$ and is to give the evaluation result as value in right component $\mathbb{X}$ upon reaching complexity $0 = \boldsymbol{cx} \ulcorner \mathrm{id} \urcorner$ in left component PR.

## Iterative evaluation of cartesian theory CA

### evaluation step

$$
\begin{aligned}
&\boldsymbol{e} = \boldsymbol{e}\,(f, a) = (\boldsymbol{e}_{\text{map}}(f, a), \boldsymbol{e}_{\text{arg}}(f, a)) : \\
&\text{CA} \times \mathbb{X}_{\underline{\perp}} \longrightarrow \text{CA} \times \mathbb{X}_{\underline{\perp}} \\
&\mathbb{X}_{\underline{\perp}} = \mathbb{X} \cup \{\underline{\perp}\} \subset \mathbb{N},\ \underline{\perp} = \ulcorner \perp \urcorner\ (\textit{trash}) \\
&\mathbb{X} = \cup_A \dot{A} = \{\mathbb{N} : \mathbb{N} \xrightarrow{\mathbb{X}} 2\}\ \text{universal set} \\
&\text{of internal } \textit{numerals} \text{ and (nested) } \textit{numpairs}
\end{aligned}
$$

$\boldsymbol{e}_{\text{arg}}(f, a)$ *is the intermediate argument obtained by one evaluation step applied to the pair* $(f, a)$, *and* $\boldsymbol{e}_{\text{map}}(f, a)$ *is the remaining map code still to be evaluated on intermediate argument* $\boldsymbol{e}_{\text{arg}}(f, a)$, *same then iteratively applied to pair* $(\boldsymbol{e}_{\text{map}}, \boldsymbol{e}_{\text{arg}})$

This evaluation step $\boldsymbol{e}$ is **defined** by recursive case distinction below, *controlled* by $\mathbb{N}$-valued descending **complexity**

$$\boldsymbol{cx} = \boldsymbol{cx}\, f \ \in \mathbb{N}$$

in turn p. r. defined by

$$
\begin{aligned}
&\boldsymbol{cx}\, \ulcorner \text{id} \urcorner := 0 \\
&\boldsymbol{cx}\, \ulcorner \text{ba} \urcorner := 1,\ \text{ba} \in \text{bas} \setminus \{\text{id}\} = \{0, \text{s}, \Pi, \ell, \text{r}\} \\
&\boldsymbol{cx}\, \langle g \odot f \rangle := \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1 \\
&\boldsymbol{cx}\, \langle f; g \rangle := \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1 \\
&\boldsymbol{cx}\, \langle f \# g \rangle := \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1
\end{aligned}
$$

**evaluation step** $\boldsymbol{e} = \boldsymbol{e}(f, a)$ is p. r. defined (and is iteration complexity-controlled) as follows:

- **basic map cases**

$$\boldsymbol{e}\,(\ulcorner \mathrm{id} \urcorner, a) := (\ulcorner \mathrm{id} \urcorner, a),\ \ \boldsymbol{cx}\ \ulcorner \mathrm{id} \urcorner = 0,\ \ \textit{stationary};$$

$$\boldsymbol{e}\,(\ulcorner \mathrm{ba} \urcorner, a) := (\ulcorner \mathrm{id} \urcorner, \boldsymbol{ev}(\ulcorner \mathrm{ba} \urcorner, a))$$

$$\text{with } \boldsymbol{ev}(\ulcorner \mathrm{ba} \urcorner, a) = \nu_B\, \mathrm{ba}\, \nu_A^{-1}\, a$$

$$A = \mathrm{Dom}\, \mathrm{ba},\ \ B = \mathrm{Codom}\, \mathrm{ba}$$

$$\mathrm{ba} \in \mathrm{bas}' = \{0, \mathrm{s}, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B\ \textbf{PR}\ \text{objects}\}$$

$$\nu = \nu_A : A \xrightarrow{\cong} \dot{A}\ \text{(internal)}\ \textit{numeralisation;}$$

$$\boldsymbol{cx}(\ulcorner \mathrm{id} \urcorner) = 0 < \boldsymbol{cx}(\ulcorner \mathrm{ba} \urcorner) = 1,\ \ \mathrm{ba} \in \mathrm{bas}'$$

- **composition cases**
  - identity subcase:

$$\boldsymbol{e}(g \odot \ulcorner \mathrm{id}_A \urcorner, a) := (g, a)$$

$$\boldsymbol{cx}\, g < \boldsymbol{cx}\, g + 0 + 1 = \boldsymbol{cx}\, \langle g \odot \ulcorner \mathrm{id}_A \urcorner \rangle$$

  - For $f \in [A, B],\ \ g \in [B, C],\ \ a \in A,\ \ \boldsymbol{cx}\, f > 0:$

$$\boldsymbol{e}\,(g \odot f, a) = (\boldsymbol{e}_{\mathrm{map}}(g \odot f, a), \boldsymbol{e}_{\mathrm{arg}}(g \odot f, a))$$

$$:= (g \odot \boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a))$$

  **Complexity descent:**

$$\boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}(g \odot f, a)$$

$$= \boldsymbol{cx}\,(g \odot \boldsymbol{e}_{\mathrm{map}}(f, a), a)$$

$$= \boldsymbol{cx}\, \boldsymbol{e}_{\mathrm{map}}(f, a) + \boldsymbol{cx}\, g + 1$$

$$< \boldsymbol{cx}\ f + \boldsymbol{cx}\, g + 1$$

$$= \boldsymbol{cx}\, \langle g \odot f \rangle$$

- **cases of an induced**

– identities subcase:

$$\boldsymbol{e}(\langle \ulcorner \mathrm{id}_C \urcorner ; \ulcorner \mathrm{id}_C \urcorner \rangle, c) := (\ulcorner \mathrm{id}_{C\times C} \urcorner, \langle c; c\rangle)$$
$$\boldsymbol{cx}\ \ulcorner \mathrm{id}_{C\times C} \urcorner = \boldsymbol{c}(\ulcorner \mathrm{id} \urcorner) = 0$$
$$< 1 = \boldsymbol{cx}(\langle \ulcorner \mathrm{id}_C \urcorner ; \ulcorner \mathrm{id}_C \urcorner \rangle)$$

– subcase $f \in [C, A]$, $g \in [C, B]$, not both equal to $\ulcorner \mathrm{id}_C \urcorner$ :

$$\boldsymbol{e}\,(\langle f; g\rangle, c)$$
$$:= (\langle \boldsymbol{e}_{\mathrm{map}}(f, c); \boldsymbol{e}_{\mathrm{map}}(g, c)\rangle, \langle \boldsymbol{e}_{\mathrm{arg}}(f, c); \boldsymbol{e}_{\mathrm{arg}}(g), c\rangle)$$
$$\boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}(\langle f; g\rangle, c)$$
$$= \boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}(f, c) + \boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}(g, c) + 1$$
$$< \boldsymbol{cx}\ f + \boldsymbol{cx}\ g + 1 = \boldsymbol{cx}\ \langle f; g\rangle$$

*since in this case* $\boldsymbol{cx}\ f > 0$ *and/or* $\boldsymbol{cx}\ g > 0$,

*and therefore* $\boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}\,(f, c) < \boldsymbol{cx}\ f$

*and/or* $\boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}(g, c) < \boldsymbol{cx}\ g$

- **cartesian-product cases**

$$\boldsymbol{e}\,(\ulcorner \mathrm{id}_A \urcorner \# \ulcorner \mathrm{id}_B \urcorner, \langle a; b\rangle) := (\ulcorner \mathrm{id}_{A\times B} \urcorner, \langle a; b\rangle)$$
$$\boldsymbol{cx}\ \ulcorner \mathrm{id}_{A\times B} \urcorner = 0$$
$$< 1 = \boldsymbol{cx}\ \ulcorner \mathrm{id}_A \urcorner + \ulcorner \mathrm{id}_B \urcorner + 1 = \boldsymbol{cx}\ \langle \ulcorner \mathrm{id}_A \urcorner \# \ulcorner \mathrm{id}_B \urcorner \rangle$$

For $f \in [A, A']$, $g \in [B, B]$ not both identity codes:

$$\boldsymbol{e}\,(f \# g, \langle a; b\rangle)$$
$$:= (\boldsymbol{e}_{\mathrm{map}}(f, a) \# \boldsymbol{e}_{\mathrm{map}}(g, a), \langle \boldsymbol{e}_{\mathrm{arg}}(f, a); \boldsymbol{e}_{\mathrm{arg}}(g, b)\rangle)$$

*one-step-evaluate both components* $f$ *and* $g$ *in parallel.*

Complexity descent:

$$\begin{aligned}
&\boldsymbol{cx}\,\langle \boldsymbol{e}_{\mathrm{map}}(f,a)\#\boldsymbol{e}_{\mathrm{map}}(g,b)\rangle \\
&= \boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(f,a) + \boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(g,b) + 1 \\
&< \boldsymbol{cx}\,f + \boldsymbol{cx}\,g + 1 = \boldsymbol{cx}\,\langle f\#g\rangle.
\end{aligned}$$

## Evaluation of theory PR

Let $^{\$} = ^{\ulcorner \S \urcorner}$ code the iteration symbol of **PR**

**CA** evaluation **step** $\boldsymbol{e}$ is **extended** by clause:

For endomap code $f \in [A,A] = [A,A]_{\mathbf{PR}}$ and $a \in \dot{A}$

$$\begin{aligned}
&\boldsymbol{e}\,(f^{\$}, \langle a; \ulcorner 0 \urcorner\rangle) := (f^{0}, a) \\
&\boldsymbol{e}\,(f^{\$}, \langle a, \nu(\mathrm{s}\,n)\rangle) := (\langle f \odot f^{n}\rangle, a) \\
&\text{where } f^{0} := \ulcorner \mathrm{id} \urcorner \\
&f^{\mathrm{s}\,n} := \langle f \odot f^{n}\rangle \text{ recursively:}
\end{aligned}$$

***code expansion***

**Complexity extension:**

$$\boldsymbol{cx}\,f^{\$} := (\boldsymbol{cx}\,f + 1)\cdot\omega \;\in\; \mathbb{N}[\omega]$$

$\mathbb{N}[\omega]$ the well-ordered semiring of polynomials in one indeterminate over $\mathbb{N}$, pendant to set theoretic ordinal $\omega^{\omega}$ : Within **set** theory **T**, $\mathbb{N}[\omega]$ has only *finite descending chains.*

In this “acute” iteration case we have

**complexity descent**

$$\boldsymbol{cx}\, f^0 = \boldsymbol{cx}\, \ulcorner\mathrm{id}\urcorner = 0 < (\boldsymbol{cx}\, f + 1)\cdot\omega = \boldsymbol{cx}\, f^{\$}$$

and further inductively

$$\boldsymbol{cx}\, f^{\mathrm{s}\, n} = \boldsymbol{cx}\, \langle f \odot \langle f \ldots\ f\rangle \ldots\rangle$$
$$= \boldsymbol{cx}\, f \cdot \mathrm{s}\, n + n$$
$$< (\boldsymbol{cx}\, f + 1)\cdot(n+1)$$
$$< (\boldsymbol{cx}\, f + 1)\cdot\omega = \boldsymbol{cx}\, f^{\$}$$

**Explication:** $\boldsymbol{cx}$ now takes values within the linearily ordered semiring $\mathbb{N}[\omega] \supset \mathbb{N}$ of polynomials in one indeterminate $\omega$, $\omega$ thought to represent (arbitrarily) big natural numbers. So in fact $\boldsymbol{cx}(f^{\mathrm{s}\, n}) < \boldsymbol{cx}(f^{\$})$ since the former polynomial has lower degree than the latter.

*Linear order* of polynomials $p, q \in \mathbb{N}[\omega]$ is defined hierarchically by first comparison of the *degrees* of $p$ and $q$, second in case of equal degrees by comparison of the pivot coefficients, and third, if the pivot monomials are equal, recursively by comparison of the polynomials $p$ and $q$ with the two pivot monomials deleted.

**Note:** A first approach to evaluate arbitrary constants $c : \mathbb{1} \to A$ of **PR** into numerals/nested numpairs has been given in Lassmann 1981.

## Evaluation Resolution Theorem:

- Evaluation $\boldsymbol{ev}$ of **PR** map code variable $f \in [A, B] = [A, B]_{\mathbf{PR}} \subset \mathrm{PR}$ on (fitting) arguments $a \in \dot{A} \subset \mathbb{X}$ is **totally defined** by the ***complexity controlled iteration***

(CCI)

$$\boldsymbol{ev} = \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{a}) := \begin{cases} \underline{\text{init}} \left\{ (\boldsymbol{h}, x) := (\boldsymbol{f}, \boldsymbol{a}) \right. \\ * \\ \underline{\text{while}}\, [\boldsymbol{cx}\, \boldsymbol{h} > 0] \\ \underline{\text{do}}\ (\boldsymbol{h}, x) := \boldsymbol{e}\,(\boldsymbol{h}, x)\ \underline{\text{od}} \\ * \\ \underline{\text{result}} := x \end{cases}$$

which always **terminates,** (at least) within quantified arithmetical theories **T** with finite descent since there complexity (co)domain $\mathbb{N}[\omega]$ has only *finite descending chains* **whence**

$$\text{(term)}\quad \frac{\boldsymbol{f} \in [A, A] \text{ (endo)map code variable}}{\begin{array}{l}(\exists m \in \mathbb{N})\, \boldsymbol{e}^m(\boldsymbol{f}, a) = (\ulcorner \text{id} \urcorner, \boldsymbol{ev}\,(\boldsymbol{f}, \boldsymbol{a})) \\ [m = m(\boldsymbol{f}, \boldsymbol{a}) = \mu\{\tilde{m} : \boldsymbol{cx}\, \boldsymbol{e}^{\tilde{m}}(\boldsymbol{f}, \boldsymbol{a}) = 0\}] \\ \textbf{so}\quad \boldsymbol{ev}\,(\boldsymbol{f}, \boldsymbol{a}) = \mathrm{r}\, \boldsymbol{e}^m(\boldsymbol{f}, \boldsymbol{a})\end{array}}$$

- $\boldsymbol{ev}$ is **characterised** by the double recursion ("Ackermann")

$$\boldsymbol{ev}(\mathrm{ba}, \boldsymbol{a}) = \nu_B(\mathrm{ba}(\nu_A^{-1}(\boldsymbol{a})))$$
$$\text{for ba} \in \text{bas},\ A = \mathrm{Dom}[\mathrm{ba}],\ B = \mathrm{Codom}[\mathrm{ba}]$$
$$\boldsymbol{ev}(\boldsymbol{g} \odot \boldsymbol{f}, \boldsymbol{a}) = \boldsymbol{ev}(\boldsymbol{g}, \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{a}))$$
$$\boldsymbol{ev}\langle \boldsymbol{f}; \boldsymbol{g}\rangle, \boldsymbol{c}) = \langle \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{c}); \boldsymbol{ev}(\boldsymbol{g}, \boldsymbol{c})\rangle$$
$$\boldsymbol{ev}(\boldsymbol{f} \# \boldsymbol{g}, \langle \boldsymbol{a}; \boldsymbol{b}\rangle) = \langle \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{a}); \boldsymbol{ev}(\boldsymbol{g}, \boldsymbol{b})\rangle$$

as well as

$$\boldsymbol{ev}(\boldsymbol{f}^{\$}, \langle \boldsymbol{a}; \ulcorner 0 \urcorner \rangle) = \boldsymbol{a} \text{ and}$$
$$\boldsymbol{ev}(\boldsymbol{f}^{\$}, \langle \boldsymbol{a}; \nu(\mathrm{s}\, n) \rangle) = \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{ev}(\boldsymbol{f}^{\$}, \langle \boldsymbol{a}; \nu n \rangle)$$

- **define** (natural) *evaluation family*

$$\boldsymbol{ev} = \boldsymbol{ev}_{A,B} = \boldsymbol{ev}_{A,B}(\boldsymbol{f}, a) : [A, B] \times A \to B \text{ by}$$
$$\boldsymbol{ev}_{A,B}(\boldsymbol{f}, a) = \nu_B^{-1}(\boldsymbol{ev}(\boldsymbol{f}, \nu_A(a))$$

This family $\boldsymbol{ev}$ is *objective:*

$$\frac{f : A \to B \ \textbf{PR} \text{ map}}{\boldsymbol{ev}\,(\ulcorner f \urcorner, a) = f(a) : A \to B}$$

*"evaluation is application."*

## Proof of evaluation resolution theorem

by (external) Peano induction on *iteration-index-until-termination* $m = m(\boldsymbol{h}, x) \in \mathbb{N}$, via *case distinction* on **PR** map $\boldsymbol{h}$ and (fitting) $x \in \mathbb{X}$ appearing in the different cases of the asserted conjunction.

- anchor $m = 0, 1 : \boldsymbol{h} = \ulcorner \mathrm{ba} \urcorner,\ \mathrm{ba} \in \mathrm{bas} = \{\mathrm{id}, 0, s, \Pi, \ell, \mathrm{r}\}$ see evaluation definition above.

**cases** $\mu = \mu\{\tilde{m} : \boldsymbol{e}^{\tilde{m}}(\boldsymbol{h}, x) = (\ulcorner \mathrm{id} \urcorner, \boldsymbol{ev}\,(\boldsymbol{h}, x))\} = m + 1 :$

- case $(\boldsymbol{h}, x) = (\boldsymbol{g} \odot \boldsymbol{f}, \boldsymbol{a})$ of an (internally) *composed,* subcase $\boldsymbol{f} = \ulcorner \mathrm{id} \urcorner$ : obvious.

- non-trivial subcase $(h, x) = (g \odot f, a)$, $f \neq \ulcorner \mathrm{id} \urcorner$ :

$$\boldsymbol{ev}(g \odot f, a) = \mathrm{r}\, \boldsymbol{e}^m(g \odot \boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a))$$

*by iterative definition of* $\boldsymbol{ev}$ *in this case,*

$m$ *fold iteration*

$$= \boldsymbol{ev}(g, \boldsymbol{ev}(\boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a)))$$
$$= \boldsymbol{ev}(g, \mathrm{r}\, \boldsymbol{e}^m\, (f, a))$$
$$= \boldsymbol{ev}(g, \boldsymbol{ev}(f, a))$$

*The latter three equations hold (backwards) by induction hypothesis on* $m$

**Objectivity** in this case, substitute $\ulcorner f : A \to B \urcorner$ into $f \in [A, B]$, $\ulcorner g : B \to C \urcorner$ into $g \in [B, C]$ :

$$\boldsymbol{ev}(\ulcorner g \circ f \urcorner, a) = \boldsymbol{ev}(\ulcorner g \urcorner \odot \ulcorner f \urcorner, a)$$
$$= \boldsymbol{ev}(\ulcorner g \urcorner, \boldsymbol{ev}(\ulcorner f \urcorner, a)) \text{ see } \boldsymbol{ev} \text{ just above}$$
$$= \boldsymbol{ev}(\ulcorner g \urcorner, f(a)) = g(f(a))$$

both by hypothesis on $m$

$$= (g \circ f)(a) \ \text{ q. e. d. in this case}$$

- case $(h, x) = (\langle f; g \rangle, c)$ of an (internal) *induced:* Obvious by definition of $\boldsymbol{ev}$ and then of $\boldsymbol{ev}$ on an induced into a product.

- case $(h, x) = (f \# g, \langle a; b \rangle)$ of an (internal) *cartesian product:* Obvious by definition of $\boldsymbol{ev}$ and then of $\boldsymbol{ev}$ on a cartesian product of maps.

- anchor case $(h, x) = (f^\$, \langle a; \ulcorner 0 \urcorner \rangle)$ of an iterated:

$$\boldsymbol{ev}(f^\$, (a, \ulcorner 0 \urcorner)) = a = \boldsymbol{ev}(\ulcorner \mathrm{id} \urcorner, a)$$

- step case $(\boldsymbol{h}, x) = (\boldsymbol{f}^{\$}, \langle a; \nu(\mathrm{s}\, n)\rangle)$ of a genuine (internally) iterated:

$$\begin{aligned}
&\boldsymbol{ev}(\boldsymbol{f}^{\$}, \langle \boldsymbol{a}; \nu(\mathrm{s}\, n)\rangle) \\
&= \boldsymbol{ev}(\boldsymbol{e}(\boldsymbol{f}^{\$}, \langle \boldsymbol{a}; \nu(\mathrm{s}\, n)\rangle) \\
&= \boldsymbol{ev}(\boldsymbol{f}^{s\, n}, \boldsymbol{a}) \ (\textit{definition of evaluation step } \boldsymbol{e}) \\
&= \boldsymbol{ev}(\boldsymbol{f} \odot \boldsymbol{f}^{n}, \boldsymbol{a}) \ (\textit{recursive definition of } \boldsymbol{f}^{s\, n}) \\
&= \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{ev}(\boldsymbol{f}^{n}, \boldsymbol{a})) \ \textit{by induction hypothesis on } m \\
&\qquad \textit{case of a composed map} \\
&= \boldsymbol{ev}(\boldsymbol{f}, \boldsymbol{ev}(\boldsymbol{f}^{\$}, \langle \boldsymbol{a}; \nu n\rangle)
\end{aligned}$$

Proof of **objectivity** in this last case: substitute $\ulcorner f \urcorner$ into $\boldsymbol{f} \in [A, A]$ and get from the above

$$\begin{aligned}
&\boldsymbol{ev}(\ulcorner f^{\S} \urcorner, (a, \mathrm{s}\, n)) \\
&\boldsymbol{ev}(\ulcorner f \urcorner^{\$}, (a, \mathrm{s}\, n)) \\
&= \nu_A^{-1}(\boldsymbol{ev}(\ulcorner f \urcorner^{\$}, \langle \nu_A(a); \nu(\mathrm{s}\, n)\rangle)) \\
&= \nu_A^{-1}(\boldsymbol{ev}(\ulcorner f \urcorner \odot \ulcorner f \urcorner^{\$}, \langle \nu_A(a); \nu n\rangle))) \text{ by the above} \\
&= \nu_A^{-1}(\boldsymbol{ev}(\ulcorner f \circ f^{\S} \urcorner, \langle \nu_A(a); \nu n\rangle))) \\
&= (f \circ f^{\S})(a, n) = f^{\S}(a, \mathrm{s}\, n) \text{ by } \textit{naturality of } \nu
\end{aligned}$$

This shows the theorem in the remaining iteration case **q. e. d.**

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