Generic rank-two perturbations of structured regular matrix pencils

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Abstract

The spectral behavior of classes of structured regular matrix pencils is examined under certain structure-preserving rank-2 perturbations. For T-alternating, palindromic, and skew-symmetric matrix pencils we observe the following effects at each eigenvalue λ under a generic, structure-preserving rank-2 perturbation: 1) The largest two Jordan blocks at λ are destroyed. 2) If hereby the eigenvalue pairing imposed by the structure is violated, also the largest remaining Jordan block at λ will grow in size by one. 3) If λ is a single (double) eigenvalue of the perturbating pencil, one (two) new Jordan blocks of size one will be created at λ.

Key words. Matrix pencil, alternating matrix pencil, palindromic matrix pencil, skew-symmetric matrix pencil, perturbation theory, rank two perturbation, generic perturbation.

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1 Introduction

Rank-1 perturbations of unstructured matrices were studied in [5, 13, 14, 15, 16] and the following result was established: When a matrix is subjected to a generic rank-1 perturbation, its largest Jordan block at each eigenvalue is destroyed.

Then, various classes of matrices that are structured with respect to some indefinite inner product were investigated under structure-preserving rank-1 perturbations in [4, 9, 10, 11, 12]. It was observed that in some cases, not only the largest Jordan block at each eigenvalue was destroyed under perturbation, but that also the second largest Jordan block (i.e., the largest remaining block) would grow in size by one.

Then again, unstructured regular matrix pencils were studied under generic low-rank perturbations in [3]: It was observed that at each eigenvalue of the pencil, not only certain blocks will be destroyed, but also some new blocks of size one will be created. Now, the

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motivation of this paper is to look into similar results for matrix pencils that have a certain symmetry structure and low-rank perturbations that preserve this structure.

We will mainly focus on T-*alternating* matrix pencils $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ (identifying the matrix pair (E, A) with the pencil $\lambda E - A$ whenever convenient), i.e., either E is skew-symmetric and A is symmetric $-$ then (E, A) is called T-*even* $-$ or E is symmetric and A is skew-symmetric $-$ then (E, A) is called T-*odd*. It is well-known that the eigenvalues of T-alternating matrix pencils occur in pairs $(\lambda, -\lambda)$ and that at 0 and ∞ (where this pairing degenerates), the sizes of their Jordan blocks have to fulfill certain conditions (see Theorem 2.5 and also [18, 6]). Some applications that lead to these and other types of structured matrix pencils are presented in [1].

For unstructured matrix pencils (E, A), a rank-1 perturbation will in general perturb both E and A, as such perturbations can, e.g., have the form $(\beta uv^T, \alpha uv^T)$. However, the situation is different for T-alternating matrix pencils: If we consider a T-even rank-1 perturbation $(\Delta E, \Delta A)$, then ΔE must be skew-symmetric and thus have even rank, and at the same time its rank is less than or equal to one, from which we obtain $\Delta E = 0$. Then, ΔA will have rank one and be symmetric, leading to rank-1 perturbations of the form $(0, uu^T)$ and similarly to $(uu^T, 0)$ in the T-odd case.

The generic spectral behavior of T-alternating matrix pencils $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ under structure-preserving rank-1 perturbations of this type was determined in [1, Theorem 3.2] to be as follows. If (E, A) has the partial multiplicities $n_1 \geq \cdots \geq n_m$ at some (possibly infinite) eigenvalue $\widehat{\lambda}$, the partial multiplicities of the perturbed pencil at $\widehat{\lambda}$ are obtained by applying the following steps to the list $(n_1, \ldots, n_m)$:

1) Remove the largest element n_1 from the list.

2) If $n_1 = n_2$ and these two largest blocks are paired, replace n_2 by $n_2 + 1$ in the list.

3) If $\widehat{\lambda}$ is an eigenvalue of the perturbation, add the new entry 1 to the end of the list.

Hereby, as mentioned previously, the situation that identical blocks are paired to one another as in 2) does only occur if $\widehat{\lambda}$ is either 0 or ∞. Further, since the perturbation is equal to $(0, uu^T)$ in the T-even case and $(uu^T, 0)$ in the T-odd case, the condition in 3) is only realized if either (E, A) is T-even and $\widehat{\lambda} = \infty$ or if (E, A) is T-odd and $\widehat{\lambda} = 0$.

Even so, considering T-alternating perturbations where only the symmetric matrix of the pencil is actually perturbed does not suffice to analyze general low-rank perturbations. For example, the T-even rank-2 perturbation $(uv^T - vu^T, 0)$ cannot be decomposed into the sum of T-even rank-1 perturbations. In this paper, we will consider two different classes of T-alternating rank-2 perturbations: First, we regard perturbations of the form

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & -1 \\ -\lambda & -1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & -1 \\ \lambda & 1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix},$$

since they seem to be the more *generic* class of T-alternating rank-2 perturbations (see Section 3).

The other type of T-alternating rank-2 perturbations we will examine has the form

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \beta\lambda - \alpha \\ -\beta\lambda - \alpha & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \beta\lambda - \alpha \\ \beta\lambda + \alpha & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix}.$$

This class of perturbations is important because in practical applications, the matrices E and A from a matrix pencil possibly play very different roles, so that it is realistic to have different perturbations on E and A. Hence, setting one of the parameters α or β to zero, it is evident that perturbations of only E or A are included in the above class of perturbations, and in particular, we can realize purely skew-symmetric rank-2 perturbations of the form $(uv^T - vu^T, 0)$ or $(0, uv^T - vu^T)$.

The next section of this paper will cover preliminary results on low-rank perturbations and structured Kronecker canonical forms. In Section 3, we will then determine the generic spectral behavior of regular, T-alternating matrix pencils under the above structure-preserving rank-2 perturbations. In Section 4, the results from Section 3 are shown to extend to the similarly structured palindromic matrix pencils. Eventually, in Section 5, analogous results are derived for skew-symmetric matrix pencils followed by a conclusion in the final section.

Throughout this paper, for square matrices X and Y (not necessarily of the same dimension), define $X \oplus Y := \operatorname{diag}(X, Y)$ and let $X^{\oplus p} := X \oplus \cdots \oplus X$ (p times). We will denote the jth unit vector in $\mathbb{C}^n$ by $e_{j,n}$, where the second index will be omitted whenever it is clear from the context. Also, we will denote by $J_n(\lambda)$ the $n \times n$ Jordan block corresponding to the eigenvalue λ and denote the $n \times n$ reverse identity matrix by

$$R_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

2 Preliminaries

In this paper, the following notion of genericity will be employed.

Definition 2.1 1) *A set $\mathcal{A} \subseteq \mathbb{C}^n$ is called* algebraic *if there exist finitely many polynomials $p_1(x), \dots, p_k(x)$, such that $a \in \mathcal{A}$ if and only if*

$$p_j(a) = 0 \quad \textit{for} \quad j = 1, \dots, k.$$

2) *An algebraic set $\mathcal{A} \subseteq \mathbb{C}^n$ is called* proper *if $\mathcal{A} \neq \mathbb{C}^n$.*

3) *A set $\Omega \subseteq \mathbb{C}^n$ is called* generic *if $\mathbb{C}^n \setminus \Omega$ is contained in a proper algebraic set.*

Clearly, the intersection of finitely many generic sets is again generic and for an invertible matrix $X \in \mathbb{C}^{n,n}$ the set $X\Omega$ is generic if $\Omega \subseteq \mathbb{C}^n$ is generic. Subsets of $\mathbb{C}^{n,m}$ or $\mathbb{C}^{n,m} \times \mathbb{C}^{n,m}$ are called generic if they can be canonically identified with generic subsets of $\mathbb{C}^{nm}$ or $\mathbb{C}^{2nm}$, respectively.

We continue with a lemma on generic sets that will be essential in the following sections.

Lemma 2.2 *Let $\mathcal{B} \subseteq \mathbb{C}^\ell$ not be contained in any proper algebraic subset of $\mathbb{C}^\ell$. Then, $\mathcal{B} \times \mathbb{C}^k$ is not contained in any proper algebraic subset of $\mathbb{C}^\ell \times \mathbb{C}^k$.*

Proof. First, we observe that the hypothesis that $\mathcal{B}$ is not contained in any proper algebraic subset of $\mathbb{C}^\ell$ is equivalent to the fact that for all nonzero polynomials $p(x)$ in ℓ variables there exists an $x \in \mathcal{B}$ such that $p(x) \neq 0$. Letting now $q(x, y)$ be any nonzero polynomial in $\ell + k$ variables, then the assertion is equivalent to showing that there is an $(x, y) \in \mathcal{B} \times \mathbb{C}^k$ such that $q(x, y) \neq 0$.

Thus, for any such q consider the set

$$\Gamma_q := \left\{ y \in \mathbb{C}^k \mid q(\,\cdot\,, y) \text{ is a nonzero polynomial } \right\}$$

which is not empty (otherwise q would be constantly zero). Now, for any $y \in \Gamma_q$, by hypothesis there exists an $x \in \mathcal{B}$ such that $q(x, y) \neq 0$ but $(x, y) \in \mathcal{B} \times \mathbb{C}^k$. $\square$

2.1 Preliminary results on low-rank perturbations

In this section, we will review some preliminary results on low-rank perturbations of regular matrix pencils. First, let us introduce the following phrase: We will say that a regular matrix pencil has partial multiplicities that are *greater than or equal to* a certain list of multiplicities, e.g., $n_1 \geq \cdots \geq n_k > 0$, at some eigenvalue $\widehat{\lambda}$ if its partial multiplicities at $\widehat{\lambda}$ are given by $n'_1 \geq \cdots \geq n'_m > 0$ with $m \geq k$ and $n'_j \geq n_j$ for $j = 1, \ldots, k$.

Then, the first result that we recap is the following [3, Lemma 2.1]:

Lemma 2.3 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with some eigenvalue $\widehat{\lambda} \in \mathbb{C}$ and let $(\Delta E, \Delta A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ have rank at most k. Then, if the perturbed pencil $(E + \Delta E, A + \Delta A)$ is regular and $k \leq m$, it has partial multiplicities greater than or equal to $(n_{k+1}, \ldots, n_m)$ associated with $\widehat{\lambda}$.*

Hereby, the rank of $(\Delta E, \Delta A)$ means the *normal rank* of this pencil, i.e., the highest rank of the matrix $\lambda\Delta E - \Delta A$ for any $\lambda \in \mathbb{C}$. The next property of low-rank perturbations will frequently be used in the main sections: For all $(E, A), (\Delta E, \Delta A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ we have by [3, Section 1]

$$\begin{aligned} \operatorname{rank}(\lambda E - A) - \operatorname{rank}(\lambda\Delta E - \Delta A) &\leq \operatorname{rank}(\lambda(E + \Delta E) - (A + \Delta A)) \\ &\leq \operatorname{rank}(\lambda E - A) + \operatorname{rank}(\lambda\Delta E - \Delta A) \end{aligned} \tag{2.1}$$

for any $\lambda \in \mathbb{C}$. Therefore, if (E, A) and $(E + \Delta E, A + \Delta A)$ are both regular, the geometric multiplicity of (E, A) at an eigenvalue $\widehat{\lambda}$ cannot change by more than $\operatorname{rank}(\widehat{\lambda}\Delta E - \Delta A)$ under perturbation. Note that only the rank of $\widehat{\lambda}\Delta E - \Delta A$ matters for this estimate and that this number can be zero even for nonzero perturbations.

Lemma 2.4 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and consider a perturbation of the form*

$$(\Delta E, \Delta A) = \begin{bmatrix} u_1 & \ldots & u_k \end{bmatrix} (\delta E, \delta A) \begin{bmatrix} u_1 & \ldots & u_k \end{bmatrix}^T,$$

where $(\delta E, \delta A)$ is an arbitrary but fixed (for the purpose of this lemma) $k \times k$ pencil. Then, the following statements hold:

1) *There exists a generic set* $\Lambda \subseteq (\mathbb{C}^n)^k$*, so that the perturbed pencil* $(E+\Delta E, A+\Delta A)$ *is regular for all* $(u_1,\ldots,u_k) \in \Lambda$.

2) *Let there exist a generic set* $\Lambda' \subseteq (\mathbb{C}^n)^k$ *such that* $(E+\Delta E, A+\Delta A)$ *has at least the algebraic multiplicity* a *at some* $\widehat{\lambda} \in \mathbb{C}$ *for all* $(u_1,\ldots,u_k) \in \Lambda'$*. If* $(E+\Delta E, A+\Delta A)$ *is regular and has the algebraic multiplicity equal to* a *at* $\widehat{\lambda}$ *for one* $(u_1,\ldots,u_k) \in (\mathbb{C}^n)^k$*, this also holds on some generic subset of* $(\mathbb{C}^n)^k$.

Proof. Regarding 1): For fixed $(\delta E, \delta A)$, consider the polynomial

$$\sum_{j=0}^{n} c_j \lambda^j = \det\big(\lambda(E+\Delta E) - A - \Delta A\big),$$

whose coefficients $c_j = c_j(u_1,\ldots,u_k)$ depend polynomially on the entries of $(u_1,\ldots,u_k)$. Hence, since $c_j(0) \neq 0$ holds for at least one j (recall that (E,A) is regular), at least one c_j is not constantly zero as a polynomial in the entries of $(u_1,\ldots,u_k)$. Thus, the set Λ of all $(u_1,\ldots,u_k) \in (\mathbb{C}^n)^k$, such that $c_j(u_1,\ldots,u_k) \neq 0$ for at least one j, is the desired generic set.

Regarding 2): By hypothesis, for all $(u_1,\ldots,u_k) \in \Lambda \cap \Lambda'$, the perturbed pencil is regular and we have

$$\det\big((\lambda+\widehat{\lambda})(E+\Delta E) - A - \Delta A\big) = \lambda^a q(\lambda),$$

for a suitable polynomial $q(\lambda)$, noting that the coefficient $q(0)$ depends polynomially on the entries of $(u_1,\ldots,u_k)$. For continuity reasons, this factorization even holds for all $(u_1,\ldots,u_k) \in (\mathbb{C}^n)^k$. Since there is one particular $(u_1,\ldots,u_k)$ such that $q(0) \neq 0$, by definition $q(0) \neq 0$ is satisfied on some generic set $\Lambda'' \subseteq (\mathbb{C}^n)^k$. Then, clearly, $\Lambda \cap \Lambda' \cap \Lambda''$ is the desired generic set. □

2.2 Structured Kronecker canonical forms

In this section we briefly recap some structured Kronecker canonical forms that will be essential in the main proofs. The following T-even Kronecker form was deduced in [18].

Theorem 2.5 (T-even Kronecker form) *Let* $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ *be a* T*-even matrix pencil. Then, there is a nonsingular matrix* $X \in \mathbb{C}^{n,n}$*, such that*

$$X(E,A)X^T = \mathcal{K}_\mathcal{I} \oplus \mathcal{K}_\mathcal{Z} \oplus \mathcal{K}_\mathcal{F} \oplus \mathcal{K}_\mathcal{S},$$

where

$$\begin{aligned}
\mathcal{K}_\mathcal{I} &= \mathcal{I}_{2\delta_1+1} \oplus \cdots \oplus \mathcal{I}_{2\delta_\ell+1} \oplus \mathcal{I}_{2\epsilon_1} \oplus \cdots \oplus \mathcal{I}_{2\epsilon_m},\\
\mathcal{K}_\mathcal{Z} &= \mathcal{Z}_{2\rho_1+1} \oplus \cdots \oplus \mathcal{Z}_{2\rho_r+1} \oplus \mathcal{Z}_{2\sigma_1} \oplus \cdots \oplus \mathcal{Z}_{2\sigma_s},\\
\mathcal{K}_\mathcal{F} &= \mathcal{F}_{\phi_1} \oplus \cdots \oplus \mathcal{F}_{\phi_t},\\
\mathcal{K}_\mathcal{S} &= \mathcal{S}_{\tau_1} \oplus \cdots \oplus \mathcal{S}_{\tau_u},
\end{aligned}$$

and the blocks are given as follows:

1) $\mathcal{I}_{2\delta_j+1}$ *is one* $(2\delta_j+1)\times(2\delta_j+1)$ *block corresponding to the eigenvalue* ∞*:*

$$\left(\begin{bmatrix} &&&&&&&0\\ &&&&&&\iddots&1\\ &&&&&\iddots&\iddots&\\ &&&0&1&&&\\ &&\iddots&-1&&&&\\ &\iddots&\iddots&&&&&\\ 0&-1&&&&&& \end{bmatrix}, \begin{bmatrix} &&1\\ &\iddots&\\ 1&& \end{bmatrix}\right)\in\mathbb{C}^{(2\delta_j+1)\times(2\delta_j+1)}.$$

2) $\mathcal{I}_{2\epsilon_j}$ *contains two* $2\epsilon_j\times 2\epsilon_j$ *blocks corresponding to the eigenvalue* ∞*:*

$$\left(\left[\begin{array}{cccc|cccc} &&&&&&&0\\ &&&&&&\iddots&1\\ &&&&&\iddots&\iddots&\\ &&&&0&1&&\\ \hline &&&0&&&&\\ &&\iddots&-1&&&&\\ &\iddots&\iddots&&&&&\\ 0&-1&&&&&& \end{array}\right], \left[\begin{array}{ccc|ccc} &&&&&1\\ &&&&\iddots&\\ &&&1&&\\ \hline &&1&&&\\ &\iddots&&&&\\ 1&&&&& \end{array}\right]\right)\in\mathbb{C}^{4\epsilon_j\times 4\epsilon_j}.$$

3) $\mathcal{Z}_{2\rho_j+1}$ *contains two* $(2\rho_j+1)\times(2\rho_j+1)$ *blocks corresponding to the eigenvalue* 0*:*

$$\left(\left[\begin{array}{ccc|ccc} &&&&&1\\ &&&&\iddots&\\ &&&1&&\\ \hline &&-1&&&\\ &\iddots&&&&\\ -1&&&&& \end{array}\right], \left[\begin{array}{cccc|cccc} &&&&&&&0\\ &&&&&&\iddots&1\\ &&&&&\iddots&\iddots&\\ &&&&0&1&&\\ \hline &&&0&&&&\\ &&\iddots&1&&&&\\ &\iddots&\iddots&&&&&\\ 0&1&&&&&& \end{array}\right]\right)\in\mathbb{C}^{(4\rho_j+2)\times(4\rho_j+2)}.$$

4) $\mathcal{Z}_{2\sigma_j}$ *is one* $2\sigma_j\times 2\sigma_j$ *block corresponding to the eigenvalue* 0*:*

$$\left(\begin{bmatrix} &&&&&1\\ &&&&\iddots&\\ &&&1&&\\ &&-1&&&\\ &\iddots&&&&\\ -1&&&&& \end{bmatrix}, \begin{bmatrix} &&&0\\ &&\iddots&1\\ &\iddots&\iddots&\\ 0&1&& \end{bmatrix}\right)\in\mathbb{C}^{2\sigma_j\times 2\sigma_j}.$$

5) $\mathcal{F}_{\phi_j}$ *contains two* $\phi_j \times \phi_j$ *blocks that correspond to the eigenvalues* $\lambda_j, -\lambda_j \in \mathbb{C} \setminus \{0\}$*:*

$$\left(\left[\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array} \right], \left[\begin{array}{ccc|ccc} & & & & & \lambda_j \\ & & & & \iddots & 1 \\ & & & \iddots & \iddots & \\ & & & \lambda_j & 1 & \\ \hline & & \lambda_j & & & \\ & \iddots & 1 & & & \\ \iddots & \iddots & & & & \\ \lambda_j & 1 & & & & \end{array} \right] \right) \in \mathbb{C}^{2\phi_j \times 2\phi_j}.$$

6) $\mathcal{S}_{\tau_j}$ *contains two singular blocks of dimension* $(\tau_j + 1) \times \tau_j$ *and* $\tau_j \times (\tau_j + 1)$*:*

$$\left(\left[\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & 0 \\ & & & 1 & \iddots & \\ & & 0 & 0 & & \\ \hline & -1 & 0 & & & \\ \iddots & \iddots & & & & \\ -1 & 0 & & & & \end{array} \right], \left[\begin{array}{ccc|ccc} & & & & & 0 \\ & & & & \iddots & 1 \\ & & & 0 & \iddots & \\ & & 0 & 1 & & \\ \hline & 0 & 1 & & & \\ \iddots & \iddots & & & & \\ 0 & 1 & & & & \end{array} \right] \right) \in \mathbb{C}^{(2\tau_j+1) \times (2\tau_j+1)}.$$

We note that there exists an analogously structured T-odd Kronecker form that will not be needed in this paper. We refer the reader to [18] for the corresponding theorem.

The following skew-symmetric Kronecker form is also taken from [18].

Theorem 2.6 (Skew-symmetric Kronecker form) *Let* $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ *be a skew-symmetric matrix pencil. Then, there is a nonsingular matrix* $X \in \mathbb{C}^{n,n}$*, such that*

$$X(E, A)X^T = \widehat{\mathcal{K}}_{\mathcal{I}} \oplus \widehat{\mathcal{K}}_{\mathcal{F}} \oplus \widehat{\mathcal{K}}_{\mathcal{S}},$$

where

$$\widehat{\mathcal{K}}_{\mathcal{I}} = \widehat{\mathcal{I}}_{\delta_1} \oplus \cdots \oplus \widehat{\mathcal{I}}_{\delta_\ell}, \qquad \widehat{\mathcal{K}}_{\mathcal{F}} = \widehat{\mathcal{F}}_{\epsilon_1} \oplus \cdots \oplus \widehat{\mathcal{F}}_{\epsilon_m}, \qquad \widehat{\mathcal{K}}_{\mathcal{S}} = \widehat{\mathcal{S}}_{\tau_1} \oplus \cdots \oplus \widehat{\mathcal{S}}_{\tau_u},$$

and the blocks are given as follows:

1) $\widehat{\mathcal{I}}_{\delta_j}$ *contains two* $\delta_j \times \delta_j$ *blocks corresponding to the eigenvalue* ∞*:*

$$\left(\left[\begin{array}{ccc|ccc} & & & & & 0 \\ & & & & \iddots & 1 \\ & & & \iddots & \iddots & \\ & & & 0 & 1 & \\ \hline & & 0 & & & \\ & \iddots & -1 & & & \\ \iddots & \iddots & & & & \\ 0 & -1 & & & & \end{array} \right], \left[\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array} \right] \right) \in \mathbb{C}^{2\delta_j \times 2\delta_j}.$$

2) *$\widehat{\mathcal{F}}_{\epsilon_j}$ contains two $\epsilon_j \times \epsilon_j$ blocks corresponding to the eigenvalue $\lambda_j \in \mathbb{C}$:*

$$\left(\left[\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array}\right], \left[\begin{array}{cccc|cccc} & & & & & & & \lambda_j \\ & & & & & & \iddots & 1 \\ & & & & & \iddots & \iddots & \\ & & & & \lambda_j & 1 & & \\ \hline & & & -\lambda_j & & & & \\ & & \iddots & -1 & & & & \\ & \iddots & \iddots & & & & & \\ -\lambda_j & -1 & & & & & & \end{array}\right]\right) \in \mathbb{C}^{2\epsilon_j \times 2\epsilon_j}.$$

3) *$\widehat{\mathcal{S}}_{\tau_j}$ contains two singular blocks of dimension $(\tau_j + 1) \times \tau_j$ and $\tau_j \times (\tau_j + 1)$:*

$$\left(\left[\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & 0 \\ & & & 1 & \iddots & \\ & & 0 & 0 & & \\ \hline & -1 & 0 & & & \\ \iddots & \iddots & & & & \\ -1 & 0 & & & & \end{array}\right], \left[\begin{array}{ccc|ccc} & & & & & 0 \\ & & & & \iddots & 1 \\ & & & 0 & \iddots & \\ & & 0 & 1 & & \\ \hline & 0 & -1 & & & \\ \iddots & \iddots & & & & \\ 0 & -1 & & & & \end{array}\right]\right) \in \mathbb{C}^{(2\tau_j+1)\times(2\tau_j+1)}.$$

3 T-alternating low-rank perturbations

Let us now turn to rank-2 perturbations of T-alternating matrix pencils, i.e., the normal rank of the perturbation $(\Delta E, \Delta A)$ is prescribed to be two. First, we aim to derive a *generic* T-even Kronecker form of T-even rank-2 perturbations assuming the dimension n to be greater than two. Clearly, if $(\Delta E, \Delta A)$ is a T-even matrix pencil with normal rank two, then both ΔE and ΔA have rank two, i.e., the pencil will have the form $(uv^T - vu^T, xy^T + yx^T)$ for certain $u, v, x, y \in \mathbb{C}^n$.

Then, assuming the generic condition that u and v are linearly independent (otherwise there were $\Delta E = 0$), there must exist an invertible $S \in \mathbb{C}^{n,n}$ so that $S^T[u, v] = [e_1, e_2]$ since this is a transformation to reduced row echelon form, i.e.,

$$S^T(\Delta E, \Delta A)S = (e_1 e_2^T - e_2 e_1^T, \widetilde{x}\widetilde{y}^T + \widetilde{y}\widetilde{x}^T)$$

setting $\widetilde{x} := S^T x$ and $\widetilde{y} := S^T y$. Now, it is a generic assumption that the third entry of $\widetilde{y}$ is nonzero, i.e., there exists an invertible $T \in \mathbb{C}^{n,n}$ so that $T^T\widetilde{y} = e_3$ and also $T^T[e_1, e_2] = [e_1, e_2]$, so that

$$T^T S^T(\Delta E, \Delta A)ST = (e_1 e_2^T - e_2 e_1^T, \widehat{x} e_3^T + e_3 \widehat{x}^T)$$

setting $\widehat{x} := T^T\widetilde{x}$. Clearly, if the normal rank of this matrix pencil shall be equal to two, $\widehat{x}$ must have the form $[x_1, x_2, 0, \ldots, 0]^T$. But now, whenever the generic condition $x_1 \neq 0$ is satisfied, multiplying the third row and column by $1/x_1$ and then adding a suitable multiple of the first row and column onto the second, we obtain the matrix pencil

$$(e_1 e_2^T - e_2 e_1^T, e_1 e_3^T + e_3 e_1^T),$$

whose T-even Kronecker form is given by $\mathcal{S}_1 \oplus \mathcal{S}_0^{\oplus n-3}$ in terms of the blocks defined in Theorem 2.5. Since similar arguments hold in the T-odd case, a T-alternating matrix pencil with normal rank two can generically be displayed in the form

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & -1 \\ -\lambda & -1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ \lambda & -1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \tag{3.1}$$

in the T-even or T-odd case, respectively. This observation seems related to [2, Theorem 3.2] describing the generic Kronecker structure of matrix pencils with fixed rank. In particular, the generic Kronecker structure of a matrix pencil with rank one is given by one singular block corresponding to either a left or a right minimal index one. Since the singular blocks of T-even matrix pencils come in pairs, the simplest nontrivial singular structure that is allowed for T-even matrix pencils is the block $\mathcal{S}_1$, and by the above observation this singular structure is also generic if we prescribe the normal rank to be equal to two.

On the other hand, one can consider T-alternating rank-k perturbations of the form

$$\begin{bmatrix} \widetilde{u}_1 & \dots & \widetilde{u}_k \end{bmatrix} \big(\delta E, \delta A\big) \begin{bmatrix} \widetilde{u}_1 & \dots & \widetilde{u}_k \end{bmatrix}^T,$$

assuming that $(\delta E, \delta A)$ is a generic T-alternating $k \times k$ pencil and that $\widetilde{u}_1, \dots, \widetilde{u}_k \in \mathbb{C}^n$ are generic vectors. Hereby, the set of T-alternating $k \times k$ pencils forms a vector space of dimension k^2, which is why we consider a subset of it to be generic if it can canonically be identified with a generic subset of $\mathbb{C}^{k^2}$.

In [1, Section 3], it was shown that under these conditions, the above rank-k perturbation is generically the sum of both rank-1 perturbations of the form $(0, uu^T)$ or $(uu^T, 0)$ treated in [1] and rank-2 perturbations of the form

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda\beta - \alpha \\ -\lambda\beta - \alpha & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda\beta - \alpha \\ \lambda\beta + \alpha & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix}, \tag{3.2}$$

in the T-even or T-odd case, respectively. In particular, considering this type of perturbations is useful because setting the parameter α to zero allows us to only perturb the skew-symmetric matrix of a T-alternating pencil (perturbations of only the symmetric matrix were already analyzed in [1]).

Let us now consider an example of perturbations as in (3.1) that also illustrates the main idea of the proof of Theorem 3.4.

Example 3.1 Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular T-even matrix pencil that has the partial multiplicities $(6, 5, 5, 4)$ at the eigenvalue 0 and let $(\Delta E, \Delta A)$ be a generic T-even rank-2 perturbation as in (3.1).

From Lemma 2.3 follows that $(E + \Delta E, A + \Delta A)$ has partial multiplicities greater than or equal to $(5, 4)$ at 0, but there cannot occur an odd number of blocks of size 5 at 0 by Theorem 2.5. Hence, the algebraic multiplicity of $(E + \Delta E, A + \Delta A)$ at 0 cannot fall below 10, and in fact (for details see the proof of Theorem 3.4) it is generically equal to 10. Therefore, the generic partial multiplicities can be either $(6, 4)$ or $(5, 5)$.

In order to decide between the possible partial multiplicities $(6,4)$ and $(5,5)$ at 0, we consider a further T-even rank-1 perturbation $(0, xx^T)$ of $(E+\Delta E, A+\Delta A)$: By Lemma 2.3, for any x so that $(E+\Delta E, A+\Delta A+xx^T)$ is regular, its partial multiplicities at 0 are given by

$$\begin{aligned}
&(6,5,5,4) \xrightarrow{\text{rank-2}} (6,4) \xrightarrow{\text{rank-1}} \geq (4) \quad \text{or} \\
&(6,5,5,4) \xrightarrow{\text{rank-2}} (5,5) \xrightarrow{\text{rank-1}} \geq (5),
\end{aligned}$$

where $\geq (k)$ stands for 'greater than or equal to k.' Then again, a perturbation of the form $(\Delta E, \Delta A + xx^T)$ is a T-even rank-3 perturbation, that we will show in Subsection 3.1 to generically produce the following partial multiplicities at 0:

$$(6,5,5,4) \xrightarrow{\qquad\qquad \text{rank-3} \qquad\qquad} (4).$$

It is now intuitive (for details see again the proof of Theorem 3.4) that this leads to a contradiction if $(6,4)$ are not the generic partial multiplicities of $(E+\Delta E, A+\Delta A)$ at 0.
□

This can be interpreted as follows: The multiplicity sequences $(6,4)$ and $(5,5)$ do not differ in geometric or algebraic multiplicity – even so, the situation $(6,4)$ is 'more sensitive' to future perturbation, which is why it might generically be created.

Example 3.2 Let $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be a regular T-even matrix pencil that has the partial multiplicities $(4,1,1)$ at the eigenvalue 0 and let $(\Delta E, \Delta A)$ be a generic T-even rank-2 perturbation as in (3.1). As in Example 3.1, we obtain that $(E+\Delta E, A+\Delta A)$ has multiplicities greater than or equal to (1) at 0, but again, there cannot occur an odd number of blocks of size 1 at 0. Thus, the possible partial multiplicities are (2) and $(1,1)$, whereby the difference to Example 3.1 is that $(1,1)$ includes one new block being created instead of an existing one growing in size.

Now, from Examples 3.1 and 3.2 we conclude that to get the full picture on T-even rank-2 perturbations, we need some information on T-even rank-3 perturbations, which is why we dedicate the next subsection to studying them.

3.1 T-even rank-3 perturbations

Let us now prove the rank-3 perturbation result that will be essential for our main theorem.

Theorem 3.3 *Let $(E,A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and T-even with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with some eigenvalue $\widehat{\lambda}$. Also, consider a T-even rank-3 perturbation of the form $(\Delta E, \Delta A + xx^T)$, where $(\Delta E, \Delta A)$ is a T-even rank-2 perturbation.*

1) *If $\widehat{\lambda} = 0$, n_1 is even, and $n_2 = n_3$ is odd, the following statements hold:*

(a) *If* $(\Delta E, \Delta A)$ *has the form* (3.2)*, then for each* $(\alpha, \beta) \in (\mathbb{C} \times \mathbb{C}) \setminus \{0\}$ *there is a generic set* $\Omega \subseteq (\mathbb{C}^n)^3$ *such that for all* $(u, v, x) \in \Omega$*, the perturbed pencil* $(E + \Delta E, A + \Delta A + xx^T)$ *is regular and has the partial multiplicities* $(n_4, \ldots, n_m, 1, 1)$ *if* $\alpha = 0$ *and* $(n_4, \ldots, n_m)$ *otherwise at* 0*.*

(b) *If* $(\Delta E, \Delta A)$ *has the form* (3.1)*, then there is a generic set* $\Omega' \subseteq (\mathbb{C}^n)^4$ *such that for all* $(u, v, w, x) \in \Omega'$*, the perturbed pencil* $(E + \Delta E, A + \Delta A + xx^T)$ *is regular and has the partial multiplicities* $(n_4, \ldots, n_m)$ *at* 0*.*

2) *If* $\widehat{\lambda} = \infty$*,* n_1 *is odd, and* $n_2 = n_3$ *is even, the following statements hold:*

(a) *If* $(\Delta E, \Delta A)$ *has the form* (3.2)*, then for each* $(\alpha, \beta) \in (\mathbb{C} \times \mathbb{C}) \setminus \{0\}$ *there is a generic set* $\Omega \subseteq (\mathbb{C}^n)^3$ *such that for all* $(u, v, x) \in \Omega$*, the perturbed pencil* $(E + \Delta E, A + \Delta A + xx^T)$ *is regular and has the partial multiplicities* $(n_4, \ldots, n_m, 1, 1, 1)$ *if* $\beta = 0$ *and* $(n_4, \ldots, n_m, 1)$ *otherwise at* ∞*.*

(b) *If* $(\Delta E, \Delta A)$ *has the form* (3.1)*, then there is a generic set* $\Omega' \subseteq (\mathbb{C}^n)^4$ *such that for all* $(u, v, w, x) \in \Omega'$*, the perturbed pencil* $(E + \Delta E, A + \Delta A + xx^T)$ *is regular and has the partial multiplicities* $(n_4, \ldots, n_m, 1)$ *at* ∞*.*

Proof. We consider the proof of 1) since 2) is shown by analogous arguments. First, by Lemma 2.3 and (2.1) it is clear that if the perturbed pencil $(E + \Delta E, A + \Delta A + xx^T)$ is regular, it has partial multiplicities greater than or equal to the above given partial multiplicities at $\widehat{\lambda}$ in each case.

Thus, by Lemma 2.4 it is sufficient to show that there exist particular (u, v, x) or (u, v, w, x) in the case $(1a)$ or $(1b)$, respectively, so that $(E + \Delta E, A + \Delta A + xx^T)$ has the algebraic multiplicity $n_4 + \cdots + n_m + 2$ if $\alpha = 0$ in case $(1a)$ and $n_4 + \cdots + n_m$ otherwise. To construct these particular perturbations, let us in the following assume that (E, A) is already in T-even Kronecker form as in Theorem 2.5, where the $\widehat{\lambda}$ blocks come first and are ordered decreasingly with respect to their size.

Concerning $(1a)$, let us regard the specific perturbation defined by $u = e_{n_1+1}$, $v = e_{n_1+n_2+1}$, and $x = e_1$, since then the perturbed part of $(E + \Delta E, A + \Delta A + xx^T)$ is given by

$$\left[\lambda \begin{bmatrix} 0 & R_{n_1/2} \\ -R_{n_1/2} & 0 \end{bmatrix} - R_{n_1} J_{n_1}(0) - e_1 e_1^T \right] \oplus \begin{bmatrix} 0 & -R_{n_2} J_{n_2}(-\lambda) + (\beta\lambda - \alpha) e_1 e_1^T \\ -R_{n_2} J_{n_2}(\lambda) - (\beta\lambda + \alpha) e_1 e_1^T & 0 \end{bmatrix} \tag{3.3}$$

having the determinant $\big(\lambda^{n_1} - (-1)^{n_1/2}\big)\big(\lambda^{n_2} + \beta\lambda + \alpha\big)\big(\lambda^{n_2} + \beta\lambda - \alpha\big)$.

On the other hand, in the case $(1b)$ we consider the particular perturbation with $u = 0$, $v = e_{n_1+1}$, $w = e_{n_1+n_2+1}$ and $x = e_1$, as then the perturbed part of $(E + \Delta E, A + \Delta A + xx^T)$ also has the form (3.3) setting $\beta = 0$ and $\alpha = 1$. Clearly, since the blocks not included in (3.3) are unchanged by these particular perturbations, we obtain in the case that $(\Delta E, \Delta A)$ has the form (3.2) and $\alpha = 0$, that the perturbed pencil has the partial multiplicities $(n_4, \ldots, n_m, 1, 1)$ at 0 and otherwise that its multiplicities at 0 are given by $(n_4, \ldots, n_m)$. $\square$

3.2 T-alternating rank-2 perturbations

Now, we are in a position to prove our main theorem on T-alternating rank-2 perturbations. Since it will in the following be crucial, we recall that $\widehat{\lambda}$ is an eigenvalue of the singular perturbating pencil $(\Delta E, \Delta A)$ if the rank of $\widehat{\lambda}\Delta E - \Delta A$ is less than the normal rank of $(\Delta E, \Delta A)$ (which is two in (3.1) and (3.2)). In particular, the perturbation (3.1) has no eigenvalues and (3.2) has only the eigenvalues α/β and $-\alpha/\beta$.

Theorem 3.4 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and T-alternating with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda}$ and consider a structure-preserving rank-2 perturbation $(\Delta E, \Delta A)$. Then, the following statements hold:*

1) *If $(\Delta E, \Delta A)$ has the form (3.2), then for each $(\alpha, \beta) \in (\mathbb{C}\times\mathbb{C})\setminus\{0\}$ there is a generic set $\Omega \subseteq (\mathbb{C}^n)^2$ such that for all $(u, v) \in \Omega$, the perturbed pencil $(E + \Delta E, A + \Delta A)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ as in Table 3.1.*

2) *If $(\Delta E, \Delta A)$ has the form (3.1), then there is a generic set $\Omega' \subseteq (\mathbb{C}^n)^3$ such that for all $(u, v, w) \in \Omega'$, the perturbed pencil $(E + \Delta E, A + \Delta A)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ as in Table 3.1.*

Table 3.1: Rank-2 perturbations of T-alternating matrix pencils.

$(\Delta E, \Delta A)$	eigenvalue $\widehat{\lambda}$	$n_1 + n_2$	multiplicities
$\widehat{\lambda}$ no eigenvalue of $(\Delta E, \Delta A)$	$\widehat{\lambda} \in \{0, \infty\}$	even	$(n_3, n_4, \dots, n_m)$
		odd	$(n_3 + 1, n_4, \dots, n_m)$
	$\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$		$(n_3, n_4, \dots, n_m)$
$\widehat{\lambda}$ eigenvalue of $(\Delta E, \Delta A)$	$\widehat{\lambda} \in \{0, \infty\}$	even	$(n_3, n_4, \dots, n_m, 1, 1)$
		odd	$(n_3 + 1, n_4, \dots, n_m, 1, 1)$
	$\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$		$(n_3, n_4, \dots, n_m, 1)$

Proof. It is sufficient to prove this theorem if (E, A) is T-even, since otherwise we can consider the reverse pencil (A, E). The proof will in the following be given distinguishing by $\widehat{\lambda}$: We will first consider the case $\widehat{\lambda} \in \{0, \infty\}$ and then the case $\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$.

In the remainder of this proof, let us always assume that (E, A) is already in T-even Kronecker form as in Theorem 2.5, where the $\widehat{\lambda}$ blocks come first and are ordered decreasingly with respect to their size.

Case $\widehat{\lambda} \in \{0, \infty\}$: We will tackle this proof assuming $\widehat{\lambda} = 0$, since the other case is almost identical. In view of the Lemmas 2.3 and 2.4, the perturbed pencil $(E + \Delta E, A + \Delta A)$ is generically regular and has partial multiplicities greater than or equal to $(n_3, \dots, n_m)$ at 0. If, in addition, 0 is an eigenvalue of $(\Delta E, \Delta A)$, it must be a double eigenvalue and we even obtain that these partial multiplicities are greater than or equal to $(n_3, \dots, n_m, 1, 1)$ because of (2.1). We proceed considering the following two subcases.

Subase $n_1 + n_2$ *even*: This case is realized if either n_1, n_2 are even or n_1, n_2 are odd. In the latter case, as odd-sized 0 blocks occur an even number of times, we obtain $n_1 = n_2$.

Let us first consider the case that $(\Delta E, \Delta A)$ has the form (3.2). We regard the particular perturbation with $u = e_1$ and $v = e_{n_1+1}$, since then the perturbed blocks of $(E + \Delta E, A + \Delta A)$ are given by

$$\begin{bmatrix} \lambda \left[\begin{smallmatrix} 0 & R_{n_1/2} \\ -R_{n_1/2} & 0 \end{smallmatrix}\right] - R_{n_1} J_{n_1}(0) & (\beta\lambda - \alpha) e_1 e_1^T \\ -(\beta\lambda + \alpha) e_1 e_1^T & \lambda \left[\begin{smallmatrix} 0 & R_{n_2/2} \\ -R_{n_2/2} & 0 \end{smallmatrix}\right] - R_{n_2} J_{n_2}(0) \end{bmatrix} \tag{3.4}$$

or

$$\begin{bmatrix} 0 & -R_{n_1} J_{n_1}(-\lambda) + (\beta\lambda - \alpha) e_1 e_1^T \\ -R_{n_1} J_{n_1}(\lambda) - (\beta\lambda + \alpha) e_1 e_1^T & 0 \end{bmatrix}, \tag{3.5}$$

depending on n_1, n_2 being both even or $n_1 = n_2$ being odd, respectively.

On the other hand, if $(\Delta E, \Delta A)$ has the form (3.1), we consider a perturbation given by $u = 0$, $v = e_1$, and $w = e_{n_1+1}$. Again, the perturbed blocks of $(E + \Delta E, A + \Delta A)$ have the form (3.4) or (3.5), respectively, whereby $\beta = 0$ and $\alpha = 1$.

As in both cases, no other blocks are affected by these particular perturbations, we compute the algebraic multiplicity of $(E+\Delta E, A+\Delta A)$ at 0 to be equal to $n_3 + \cdots + n_m + 2$ if 0 is an eigenvalue of $(\Delta E, \Delta A)$ and equal to $n_3 + \cdots + n_m$ otherwise. Therefore, by Lemma 2.4, $(E + \Delta E, A + \Delta A)$ is generically regular and has these algebraic multiplicities − and hence the partial multiplicities in the first and fourth row of Table 3.1 − at 0.

Subcase $n_1 + n_2$ *odd*: As odd-sized 0 blocks occur an even number of times, this case can only be realized if n_1 is even and n_2 is odd; then also $n_2 = n_3$ is obtained by the same argument.

In this case, we observe that neither the partial multiplicity sequence $(n_3, \dots, n_m)$ nor $(n_3, \dots, n_m, 1, 1)$ can occur at 0 in a T-even pencil as n_3 is odd and they include an odd number of chains of length n_3. Thus, the algebraic multiplicity of $(E + \Delta E, A + \Delta A)$ at 0 generically has to be at least $n_3 + \cdots + n_m + 3$ if 0 is an eigenvalue of $(\Delta E, \Delta A)$ and at least $n_3 + \cdots + n_m + 1$ otherwise.

To show that this minimum algebraic multiplicity is generically attained, consider the following argument. If $(\Delta E, \Delta A)$ has the form (3.2), regard the particular perturbation with $u = e_1$ and $v = e_{n_1+1} + e_{n_1+2} + e_{n_1+n_2+1}$; then the perturbed blocks of $(E+\Delta E, A+\Delta A)$ are given by

$$\begin{bmatrix} \lambda \left[\begin{smallmatrix} 0 & R_{n_1/2} \\ -R_{n_1/2} & 0 \end{smallmatrix}\right] - R_{n_1} J_{n_1}(0) & e_1 (e_1 + e_2)^T (\beta\lambda - \alpha) & e_1 e_1^T (\beta\lambda - \alpha) \\ (e_1 + e_2) e_1^T (-\beta\lambda - \alpha) & 0 & -R_{n_2} J_{n_2}(-\lambda) \\ e_1 e_1^T (-\beta\lambda - \alpha) & -R_{n_2} J_{n_2}(\lambda) & 0 \end{bmatrix}. \tag{3.6}$$

The determinant of this pencil is computed in the appendix (setting the dummy elements to $x = \beta\lambda - \alpha$ and $y = -\beta\lambda - \alpha$) to be given by

$$\lambda^{n_1 + 2n_2} + 2(-1)^{n_1/2} (\beta^2 \lambda^2 - \alpha^2) \lambda^{n_2 + 1}.$$

Then again, if $(\Delta E, \Delta A)$ has the form (3.1), consider a perturbation given by $u = 0$, $v = e_1$, and $w = e_{n_1+1} + e_{n_1+2} + e_{n_1+n_2+1}$. Then, the perturbed blocks of $(E+\Delta E, A+\Delta A)$ also have the form (3.6) with $\beta = 0$ and $\alpha = 1$. Thus, in both cases, applying Lemma 2.4 yields that $(E + \Delta E, A + \Delta A)$ is generically regular and has the algebraic multiplicity $n_3 + \cdots + n_m + 3$ if 0 is an eigenvalue of $(\Delta E, \Delta A)$ and $n_3 + \cdots + n_m + 1$ otherwise.

Now, in order to determine the generic partial multiplicities of $(E + \Delta E, A + \Delta A)$ at 0, let us group together Jordan blocks of the same size, i.e., let

$$(n_1, n_2, n_3, \ldots, n_m) = (s_1, \underbrace{s_2, \ldots, s_2}_{t_2}, \ldots, \underbrace{s_\nu, \ldots, s_\nu}_{t_\nu}),$$

then we have $s_1 = n_1$ with $t_1 = 1$ and $s_2 = n_2 = n_3$ where $t_2 \geq 2$ is even. Now, the partial multiplicities of the perturbed pencil at 0 are greater than or equal to $(n_3, \ldots, n_m, 1, 1)$ or $(n_3, \ldots, n_m)$, i.e.,

$$(\underbrace{s_2, \ldots, s_2}_{t_2-1}, \underbrace{s_3, \ldots, s_3}_{t_3}, \ldots, \underbrace{s_\nu \ldots, s_\nu}_{t_\nu}, 1, 1) \quad \text{or} \quad (\underbrace{s_2, \ldots, s_2}_{t_2-1}, \underbrace{s_3, \ldots, s_3}_{t_3}, \ldots, \underbrace{s_\nu, \ldots, s_\nu}_{t_\nu}),$$

respectively, whereby either exactly one of these blocks will be larger by one or exactly one more block of size one will exist. But to have an even number of Jordan chains of length s_2 at 0 in the perturbed pencil, this can only be realized by either

$$(s_2+1, \underbrace{s_2, ..., s_2}_{t_2-2}, \underbrace{s_3, ..., s_3}_{t_3}, ..., \underbrace{s_\nu, ..., s_\nu}_{t_\nu}, 1, 1) \quad \text{or} \quad (s_2+1, \underbrace{s_2, ..., s_2}_{t_2-2}, \underbrace{s_3, ..., s_3}_{t_3}, ..., \underbrace{s_\nu, ..., s_\nu}_{t_\nu}) \tag{3.7}$$

if 0 is an eigenvalue of $(\Delta E, \Delta A)$ or not, respectively; or for $\nu \geq 3$ and $s_3 = s_2 - 1$ by:

$$(\underbrace{s_2, ..., s_2}_{t_2}, \underbrace{s_3, ..., s_3}_{t_3-1}, ..., \underbrace{s_\nu, ..., s_\nu}_{t_\nu}, 1, 1) \quad \text{or} \quad (\underbrace{s_2, ..., s_2}_{t_2}, \underbrace{s_3, ..., s_3}_{t_3-1}, ..., \underbrace{s_\nu, ..., s_\nu}_{t_\nu}) \tag{3.8}$$

if 0 is an eigenvalue of $(\Delta E, \Delta A)$ or not, respectively; or for $\nu = 2$ and $s_2 = 1$ by:

$$(\underbrace{s_2, \ldots, s_2}_{t_2}) \tag{3.9}$$

if 0 is not an eigenvalue of $(\Delta E, \Delta A)$. (If 0 is an eigenvalue of $(\Delta E, \Delta A)$, the geometric multiplicity at 0 is fixed under perturbation by (2.1), i.e., no additional block of size one can be there.) Illustrating these possibilities, we note that in Example 3.1 we chose (3.7) over (3.8), whereas in Example 3.2 we had to decide between (3.7) and (3.9).

Then, aiming to prove that the partial multiplicities in (3.7) are generically realized in $(E + \Delta E, A + \Delta A)$ at 0, let us assume the opposite: First, in the case that $(\Delta E, \Delta A)$ is as in (3.2), let there exist some (E, A) so that $(E + \Delta E, A + \Delta A)$ is regular and has the partial multiplicities from (3.8) or (3.9) at 0 for all $(u, v) \in \mathcal{B}$, where $\mathcal{B}$ is not contained in any proper algebraic subset of $(\mathbb{C}^n)^2$. Then, we apply a T-even rank-1 perturbation $(0, xx^T)$ to $(E + \Delta E, A + \Delta A)$. By Lemma 2.3 (or equivalently, by [1, Theorem 2.7]), for all $(u,v,x) \in \mathcal{B} \times \mathbb{C}^n$ that are such that the pencil $(E+\Delta E, A+\Delta A + xx^T)$ is regular, it has partial multiplicities at 0 that are greater than or equal to

$$(\underbrace{s_2,\ldots,s_2}_{t_2-1},\underbrace{s_3,\ldots,s_3}_{t_3-1},\ldots,\underbrace{s_\nu,\ldots,s_\nu}_{t_\nu},1,1) \quad \text{or} \quad (\underbrace{s_2,\ldots,s_2}_{t_2-1},\underbrace{s_3,\ldots,s_3}_{t_3-1},\ldots,\underbrace{s_\nu,\ldots,s_\nu}_{t_\nu})$$

resulting from (3.8) if $\alpha = 0$ or $\alpha \neq 0$, respectively, or greater than or equal to

$$(\underbrace{s_2,\ldots,s_2}_{t_{2-1}})$$

resulting from (3.9). On the other hand, Theorem 3.3 ($1a$) states that $(E+\Delta E, A+\Delta A + xx^T)$ is regular and has the partial multiplicities at 0 given by

$$(\underbrace{s_2,\ldots,s_2}_{t_2-2},\underbrace{s_3,\ldots,s_3}_{t_3},\ldots,\underbrace{s_\nu,\ldots,s_\nu}_{t_\nu},1,1) \quad \text{or} \quad (\underbrace{s_2,\ldots,s_2}_{t_2-2},\underbrace{s_3,\ldots,s_3}_{t_3},\ldots,\underbrace{s_\nu,\ldots,s_\nu}_{t_\nu})$$

if $\alpha = 0$ or $\alpha \neq 0$, respectively, for all $(u,v,x) \in \widehat{\Omega}$, where $\widehat{\Omega}$ is a generic subset of $(\mathbb{C}^n)^3$ (that includes the case $\nu = 2$ and $s_2 = 1$ of (3.9)). Then, a contradiction is obtained as by Lemma 2.2 the set $\mathcal{B} \times \mathbb{C}^n$ is not contained in any proper algebraic subset of $(\mathbb{C}^n)^3$ and thus, clearly, $(\mathcal{B} \times \mathbb{C}^n) \cap \widehat{\Omega}$ is not empty.

In the second case that $(\Delta E, \Delta A)$ is as in (3.1), a contradiction is obtained by similar arguments using Theorem 3.3 ($1b$). Therefore, there exist generic sets $\Omega \subseteq (\mathbb{C}^n)^2$ and $\Omega' \subseteq (\mathbb{C}^n)^3$ such that $(E+\Delta E, A+\Delta A)$ is regular and has the partial multiplicities (3.7) – i.e., the ones in the second and fifth row of Table 3.1 – at 0 for all $(u,v) \in \Omega$ or $(u,v,w) \in \Omega'$, respectively.

Case $\widehat{\lambda} \in \mathbb{C} \setminus \{0\}$: Resulting from the Lemmas 2.3 and 2.4 and equation (2.1), the perturbed pencil $(E+\Delta E, A+\Delta A)$ is generically regular and has partial multiplicities greater than or equal to the ones from the third and sixth row of Table 3.1 at $\widehat{\lambda}$.

Thus, it remains to show that the respective partial multiplicities of $(E+\Delta E, A+\Delta A)$ generically cannot exceed $(n_3,\ldots,n_m,1)$ or $(n_3,\ldots,n_m)$ depending on $\widehat{\lambda}$ being an eigenvalue of $(\Delta E, \Delta A)$ or not, respectively, using Lemma 2.4. Let us first consider the case that $(\Delta E, \Delta A)$ has the form (3.2). Since the diagonal block of (E,A) including the largest blocks at $\widehat{\lambda}$ is given by

$$(P,J) = \left(\begin{bmatrix} 0 & R_{n_1} & & \\ -R_{n_1} & 0 & & \\ & & 0 & R_{n_2} \\ & & -R_{n_2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & R_{n_1}J_{n_1}(\widehat{\lambda}) & & \\ R_{n_1}J_{n_1}(\widehat{\lambda}) & 0 & & \\ & & 0 & R_{n_2}J_{n_2}(\widehat{\lambda}) \\ & & R_{n_2}J_{n_2}(\widehat{\lambda}) & 0 \end{bmatrix} \right),$$

we consider the particular perturbation with $u = e_1 + e_{2n_1+n_2+1}$ and $v = e_{n_1+1} + e_{2n_1+1}$. Then, the first two blocks of the perturbed pencil $\lambda(E+\Delta E) - A - \Delta A$, that we left-multiply with P^T are given by

$$\begin{bmatrix} J_{n_1}(\mu_+) + (\beta\lambda+\alpha)e_{n_1}e_1^T & 0 & 0 & (\beta\lambda+\alpha)e_{n_1}e_1^T \\ 0 & -J_{n_1}(-\mu_-) + (\beta\lambda-\alpha)e_{n_1}e_1^T & (\beta\lambda-\alpha)e_{n_1}e_1^T & 0 \\ 0 & -(\beta\lambda-\alpha)e_{n_2}e_1^T & J_{n_1}(\mu_+) - (\beta\lambda-\alpha)e_{n_2}e_1^T & 0 \\ -(\beta\lambda+\alpha)e_{n_2}e_1^T & 0 & 0 & -J_{n_1}(-\mu_-) - (\beta\lambda+\alpha)e_{n_2}e_1^T \end{bmatrix},$$

using the notation $\mu_+ := \lambda + \widehat{\lambda}$ and $\mu_- := \lambda - \widehat{\lambda}$. The determinant of the above pencil is computed in the appendix (setting $\nu_- := \beta\lambda - \alpha$ and $\nu_+ := \beta\lambda + \alpha$) to be given by

$$\begin{aligned} &\Big[(\lambda+\widehat{\lambda})^{n_1}(\lambda-\widehat{\lambda})^{n_2} - (\lambda+\widehat{\lambda})^{n_1}(\beta\lambda+\alpha) - (-1)^{n_1}(\lambda-\widehat{\lambda})^{n_2}(\beta\lambda+\alpha)\Big] \\ &\cdot \Big[(\lambda-\widehat{\lambda})^{n_1}(\lambda+\widehat{\lambda})^{n_2} + (-1)^{n_2}(\lambda-\widehat{\lambda})^{n_1}(\beta\lambda-\alpha) + (\lambda+\widehat{\lambda})^{n_2}(\beta\lambda-\alpha)\Big]. \end{aligned}$$

Thus, as $\det P^T = 1$ holds, in the first block-part of the perturbed pencil the eigenvalue $\widehat{\lambda}$ (and also $-\widehat{\lambda}$) does not occur if $\widehat{\lambda}\beta \neq \pm\alpha$ and only occurs with algebraic multiplicity 1 if $\widehat{\lambda}\beta = \pm\alpha$.

If, on the other hand, $(\Delta E, \Delta A)$ has the form (3.1), we consider a perturbation with $u = 0$, $v = e_1 + e_{2n_1+n_2+1}$ and $w = e_{n_1+1} + e_{2n_1+1}$; then analogous arguments show that $\widehat{\lambda}$ is not an eigenvalue of the first block-part of the perturbed pencil. Therefore, in both cases we obtain by Lemma 2.4 that the perturbed pencil $(E+\Delta E, A+\Delta A)$ is generically regular and has the partial multiplicities from the third and sixth row of Table 3.1 associated with $\widehat{\lambda}$. □

4 T-palindromic rank-2 perturbations

In this section, let us consider *palindromic* matrix pencils. A matrix pencil $P(\lambda)$ is called palindromic if it is either T-*palindromic*, i.e., $P(\lambda) = \lambda B + B^T$ for some $B \in \mathbb{C}^{n,n}$ or if it is T-*anti-palindromic*, i.e., $P(\lambda) = \lambda B - B^T$ for some $B \in \mathbb{C}^{n,n}$.

In order to investigate the impact of structure-preserving rank-2 perturbations on palindromic matrix pencils, we aim to use the results on T-alternating rank-2 perturbations obtained in Section 3. To that end, recall that the Cayley transformations with pole at $+1$ and -1 are given by

$$\mathcal{C}_{+1}(P)(\mu) = (1-\mu)P\left(\frac{1+\mu}{1-\mu}\right) \quad \text{and} \quad \mathcal{C}_{-1}(P)(\mu) = (1+\mu)P\left(\frac{\mu-1}{1+\mu}\right)$$

and that the structure of $P(\lambda)$ corresponds to that of its Cayley transform as in the following table that is extracted from [7].

Clearly, T-alternating and palindromic matrix pencils are closely related by these Cayley transformations and the reader is referred to [8] for a collection of properties and invariants of this type of transformations (in the more general setting of Möbius transformations of matrix polynomials). In particular, we will derive analogous classes of palindromic rank-2 perturbations by applying $\mathcal{C}_{-1}$ to the T-alternating rank-2 perturbations from Section 3.

$P(\lambda)$	$\mathcal{C}_{-1}(P)(\mu)$	$\mathcal{C}_{+1}(P)(\mu)$
T-palindromic	T-odd	T-even
T-anti-palindromic	T-even	T-odd
T-even	T-palindromic	T-anti-palindromic
T-odd	T-anti-palindromic	T-palindromic

Concerning T-alternating rank-2 perturbations as in (3.1), applying $\mathcal{C}_{-1}$ yields

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda-1 \\ 0 & 0 & -\lambda-1 \\ -\lambda+1 & -\lambda-1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda-1 \\ 0 & 0 & -\lambda-1 \\ \lambda-1 & \lambda+1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \tag{4.1}$$

in the T-palindromic or T-anti-palindromic case, respectively. We note that the 'middle pencil' of these palindromic perturbations can be transformed to a *palindromic Kronecker form* derived in [17], by replacing $[u, v, w]$ with some $[\widetilde{u}, \widetilde{v}, \widetilde{w}]$. However, since $[u, v, w]$ and $[\widetilde{u}, \widetilde{v}, \widetilde{w}]$ can then be transformed into one another by multiplication with an invertible matrix, it is equivalent to consider *generic* perturbations of the one or the other form, and as our proof is based on reusing results from Section 3, we will for simplicity regard the above form (4.1).

Similarly, palindromic analogues to (3.2) are given by (applying $\mathcal{C}_{-1}$)

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda\gamma+1 \\ \lambda+\gamma & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda\gamma+1 \\ -\lambda-\gamma & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix}, \tag{4.2}$$

in the T-palindromic or T-anti-palindromic case, respectively (setting $-1 = \alpha + \beta$ and $\gamma = \beta - \alpha$). In [1, Section 4], an argument was presented that shows that these types of perturbations stem from a certain type of generic palindromic rank-k perturbation. In particular, perturbations of this type include the important special case $\gamma = 0$, i.e., the matrix B standing for the palindromic matrix pencil $\lambda B \pm B^T$ is subjected to a generic rank-1 perturbation of the form $B + uv^T$.

The generic change in Jordan structure of palindromic pencils under these types of structure-preserving rank-2 perturbations is described in the following theorem, where the symbol $\mathbb{C}_\infty$ stands for $\mathbb{C} \cup \{\infty\}$.

Theorem 4.1 *Let $P(\lambda) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and palindromic with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda}$ and consider a structure-preserving rank-2 perturbation $Q(\lambda)$. Then, the following statements hold:*

1) *If $Q(\lambda)$ has the form (4.2), then for each $\gamma \in \mathbb{C}$ there is a generic set $\Omega \subseteq (\mathbb{C}^n)^2$ such that for all $(u, v) \in \Omega$, the perturbed pencil $P(\lambda) + Q(\lambda)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ as in Table 4.1.*

2) *If $Q(\lambda)$ has the form (4.1), then there is a generic set $\Omega' \subseteq (\mathbb{C}^n)^3$ such that for all $(u,v,w) \in \Omega'$, the perturbed pencil $P(\lambda)+Q(\lambda)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ as in Table 4.1.*

Table 4.1: Rank-2 perturbations of palindromic matrix pencils.

$Q(\lambda)$	eigenvalue $\widehat{\lambda}$	n_1+n_2	multiplicities
$\widehat{\lambda}$ no eigenvalue of $Q(\lambda)$	$\widehat{\lambda} \in \{1,-1\}$	even	$(n_3, n_4, \ldots, n_m)$
		odd	$(n_3+1, n_4, \ldots, n_m)$
	$\widehat{\lambda} \in \mathbb{C}_\infty \setminus \{1,-1\}$		$(n_3, n_4, \ldots, n_m)$
$\widehat{\lambda}$ eigenvalue of $Q(\lambda)$	$\widehat{\lambda} \in \{1,-1\}$	even	$(n_3, n_4, \ldots, n_m, 1, 1)$
		odd	$(n_3+1, n_4, \ldots, n_m, 1, 1)$
	$\widehat{\lambda} \in \mathbb{C}_\infty \setminus \{1,-1\}$		$(n_3, n_4, \ldots, n_m, 1)$

Proof. We restrict ourselves to the case that $P(\lambda)$ is T-palindromic; otherwise an analogous proof is obtained. Thus, for any perturbation $Q(\lambda)$, applying $\mathcal{C}_{+1}$ yields

$$\mathcal{C}_{+1}(P+Q)(\mu) = \mathcal{C}_{+1}(P)(\mu) + \mathcal{C}_{+1}(Q)(\mu),$$

as $\mathcal{C}_{+1}$ is a linear transformation on the vector space $\mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$. Also, since $Q(\lambda)$ is T-palindromic, $\mathcal{C}_{+1}(P)(\mu)$ and $\mathcal{C}_{+1}(Q)(\mu)$ are T-even and $\mathcal{C}_{+1}(P)(\mu)$ is regular with partial multiplicities $(n_1, \ldots, n_m)$ associated with the transformed eigenvalue $\widehat{\mu} = (\widehat{\lambda}-1)/(\widehat{\lambda}+1)$ by [8, Theorem 5.3] (see also [19]). Further, if $Q(\lambda)$ is as in (4.2), we compute than

$$\mathcal{C}_{+1}(Q)(\mu) = \begin{bmatrix} u & v \end{bmatrix} \left(\mu \begin{bmatrix} 0 & \gamma-1 \\ 1-\gamma & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma+1 \\ \gamma+1 & 0 \end{bmatrix} \right) \begin{bmatrix} u^T \\ v^T \end{bmatrix},$$

is a T-even rank-2 perturbation of $\mathcal{C}_{+1}(P)(\mu)$ that has the form (3.2). Analogously, if $Q(\lambda)$ is as in (4.1), then

$$\mathcal{C}_{+1}(Q)(\mu) = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\mu \\ 0 & 0 & -2 \\ -2\mu & -2 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix},$$

is a T-even rank-2 perturbation of $\mathcal{C}_{+1}(P)(\mu)$ of the form (3.1).

Thus, by Theorem 3.4, there exist generic sets $\Omega \subseteq (\mathbb{C}^n)^2$ and $\Omega' \subseteq (\mathbb{C}^n)^3$ such that for all $(u,v) \in \Omega$ or $(u,v,w) \in \Omega'$, respectively, the perturbed pencil $\mathcal{C}_{+1}(P)(\mu) + \mathcal{C}_{+1}(Q)(\mu)$ is regular and has the partial multiplicities at $\widehat{\mu}$ given by Table 3.1, where $(\Delta E, \Delta A)$ is replaced by $\mathcal{C}_{+1}(Q)(\mu)$ and $\widehat{\lambda}$ is replaced by $\widehat{\mu}$.

Now, applying the inverse transformation $\mathcal{C}_{-1}$, we obtain that for all $(u,v) \in \Omega$ or $(u,v,w) \in \Omega'$, respectively, the perturbed pencil $P(\lambda)+Q(\lambda)$ is regular and has the partial multiplicities at $\widehat{\lambda} = (1+\widehat{\mu})/(1-\widehat{\mu})$ given by Table 4.1 (using again [8, Theorem 5.3]). $\square$

5 Skew-symmetric rank-2 perturbations

In this section we will consider skew-symmetric matrix pencils $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$, i.e., both E and A are skew-symmetric. Since for each $\lambda \in \mathbb{C}$ the matrix $\lambda E - A$ is skew-symmetric, it follows that n is even if we assume that (E, A) is regular. Also, by Theorem 2.6 skew-symmetric matrix pencils have each Jordan block appearing twice: if (E, A) has the partial multiplicities $n_1 \geq n_2 \geq \cdots \geq n_m > 0$ at some eigenvalue, then m is even with $n_{2j-1} = n_{2j}$ for $j = 1, 2, \ldots, m/2$, but there is no eigenvalue pairing for skew-symmetric matrix pencils as for T-alternating ones.

Similar considerations as in the third section of this paper (which we do not elaborate here for the sake of brevity) lead to the following two classes of skew-symmetric rank-2 perturbations. First, there are skew-symmetric rank-2 perturbations of the form

$$\lambda \Delta E - \Delta A = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & -1 \\ -\lambda & 1 & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}, \tag{5.1}$$

and second, there are rank-2 perturbations of the form

$$\lambda \Delta E - \Delta A = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 0 & \lambda\beta - \alpha \\ -\lambda\beta + \alpha & 0 \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} \tag{5.2}$$

for $\alpha, \beta \in \mathbb{C}$. The following theorem characterizes the generic change in Jordan structure of regular skew-symmetric matrix pencils under these types of rank-2 perturbations.

Theorem 5.1 *Let $(E, A) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n}$ be regular and skew-symmetric with the partial multiplicities $n_1 \geq \cdots \geq n_m > 0$ associated with an eigenvalue $\widehat{\lambda} \in \mathbb{C}$ and consider a skew-symmetric rank-2 perturbation $(\Delta E, \Delta A)$. Then, the following statements hold:*

1) *If $(\Delta E, \Delta A)$ has the form (5.2), then for each $(\alpha, \beta) \in (\mathbb{C} \times \mathbb{C}) \setminus \{0\}$ there is a generic set $\Omega \subseteq (\mathbb{C}^n)^2$ such that for all $(u, v) \in \Omega$, the perturbed pencil $(E + \Delta E, A + \Delta A)$ is regular and has the partial multiplicities at $\widehat{\lambda}$ given by $(n_3, \ldots, n_m, 1, 1)$ if $\beta\widehat{\lambda} = \alpha$ and $(n_3, \ldots, n_m)$ otherwise.*

2) *If $(\Delta E, \Delta A)$ has the form (5.1), then there is a generic set $\Omega' \subseteq (\mathbb{C}^n)^3$ such that for all $(u, v, w) \in \Omega'$, the perturbed pencil $(E + \Delta E, A + \Delta A)$ is regular and has the partial multiplicities $(n_3, \ldots, n_m)$ at $\widehat{\lambda}$.*

Proof. By the Lemmas 2.3 and 2.4 and inequalities (2.1), in each of the cases from above, the perturbed pencil is generically regular and has partial multiplicities greater than or equal to the ones stated in the assertion. Thus, in view of Lemma 2.4, it is sufficient to give a particular perturbation that creates these partial multiplicities in each of the cases. Thus, let us in the following assume that (E, A) is in skew-symmetric Kronecker form as in Theorem 2.6 and that the blocks corresponding to $\widehat{\lambda}$ come first and are ordered decreasingly with respect to their size.

If $(\Delta E, \Delta A)$ has the form (5.2), consider a perturbation with $u = e_1$ and $v = e_{n_1+1}$ since then the perturbed part of $(E + \Delta E, A + \Delta A)$ is given by

$$\begin{bmatrix} 0 & -R_{n_1} J_{n_1}(\widehat{\lambda} - \lambda) + (\beta\lambda - \alpha) e_1 e_1^T \\ R_{n_1} J_{n_1}(\widehat{\lambda} - \lambda) - (\beta\lambda - \alpha) e_1 e_1^T & 0 \end{bmatrix}.$$

On the other hand, if $(\Delta E, \Delta A)$ is as in (5.1), then we let $u = 0$, $v = e_1$, and $w = e_{n_1+1}$ to also obtain that the perturbed part of $(\Delta E, \Delta A)$ is given by the above pencil setting $\beta = 0$ and $\alpha = 1$. In both cases, for this particular perturbation, the perturbed pencil at $\widehat{\lambda}$ clearly has the partial multiplicities $(n_3, \ldots, n_m, 1, 1)$ if $\widehat{\lambda}$ is an eigenvalue of $(\Delta E, \Delta A)$ and $(n_3, \ldots, n_m)$ otherwise, which implies the assertion. $\square$

Remark 5.2 An analogous result for the infinite eigenvalue of (E, A) is obtained by applying the above theorem to the reverse pencil (A, E).

6 Conclusion

We have investigated regular T-alternating matrix pencils under two classes of structure-preserving rank-2 perturbations. The difference to T-alternating rank-1 perturbations studied in [1] is that now both matrices of the pencil are subjected to perturbation, so that the perturbation is not forced to have the eigenvalue 0 or ∞, but instead a pair of complex (possibly infinite) eigenvalues $(\gamma, -\gamma)$.

Underlying all the different cases that were considered, we find the following principles governing T-alternating rank-2 perturbations: At each eigenvalue $\widehat{\lambda}$ of (E, A), the Jordan structure of the perturbed pencil $(E + \Delta E, A + \Delta A)$ is that of (E, A) except for the following changes:

1) The largest two Jordan blocks corresponding to $\widehat{\lambda}$ are destroyed.

2) If the largest Jordan block at $\widehat{\lambda}$ is unpaired and the second largest block is paired to an identical one, this largest remaining Jordan block will grow in size by one.

3) If $\widehat{\lambda}$ is a single (or double) eigenvalue of the perturbation $(\Delta E, \Delta A)$, i.e., $\pm\widehat{\lambda} = \gamma$, one (or two, respectively) new Jordan block(s) of size one will be created at $\widehat{\lambda}$.

Using Cayley transformations, we saw that parallel results hold for palindromic matrix pencils. Further, skew-symmetric matrix pencils were investigated under structure-preserving rank-2 perturbations, as a nontrivial skew-symmetric perturbation will at least have rank two. The result was that at each eigenvalue $\widehat{\lambda}$ of the skew-symmetric pencil, the pair consisting of the largest two Jordan blocks is destroyed under perturbation and that two new blocks of size one are created if $\widehat{\lambda}$ is an eigenvalue of the perturbation.

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Appendix (1)

Letting x and y be dummy elements, we aim to compute the determinant of the following matrix pencil. Let us assume that n_1 is even, n_2 is odd, and that in the top-left $n_1 \times n_1$ block there are $n_1/2$ instances of each $-\lambda$ and λ on the anti-diagonal.

$$\det T(\lambda) = \left|\begin{array}{ccc|ccc|ccc}
 & & \lambda & x & x & & x & & \\
 & \iddots & -1 & & & & & & \\
 \iddots & \iddots & & & & & & & \\
 -\lambda & -1 & & & & & & & \\ \hline
 y & & & & & & & & \lambda \\
 y & & & & & & & \iddots & -1 \\
 & & & & & & \iddots & \iddots & \\
 & & & & & & \lambda & -1 & \\ \hline
 y & & & & & -\lambda & & & \\
 & & & & \iddots & -1 & & & \\
 & & & \iddots & \iddots & & & & \\
 & & & -\lambda & -1 & & & &
\end{array}\right|
\begin{array}{l}
 \\ n_1 \\ \\ \\ \\ \\ n_2. \\ \\ \\ \\ n_2 \\ \\
\end{array}$$

We observe that an odd number of row permutations gives

$$-\det T(\lambda) = \left|\begin{array}{ccc|ccc|ccc}
 & & \lambda & x & x & & x & & \\
 & \iddots & -1 & & & & & & \\
 \iddots & \iddots & & & & & & & \\
 -\lambda & -1 & & & & & & & \\ \hline
 & & & -\lambda & -1 & & & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & \ddots & -1 & & & \\
 y & & & & & -\lambda & & & \\ \hline
 & & & & & & \lambda & -1 & \\
 & & & & & & & \ddots & \ddots \\
 y & & & & & & & \ddots & -1 \\
 y & & & & & & & & \lambda
\end{array}\right|
\begin{array}{l}
 \\ n_1 \\ \\ \\ \\ \\ n_2. \\ \\ \\ \\ n_2 \\ \\
\end{array}$$

We make a Laplace expansion with respect to the last row

$$-\det T(\lambda) = -y \det T_1(\lambda) + \lambda \det T_2(\lambda),$$

where

$$T_1(\lambda) = \begin{array}{cl} \left[\begin{array}{ccc|cccc|cccc} & & \lambda & x & x & & & x & & & \\ & \cdot^{\cdot^{\cdot}} & -1 & & & & & & & & \\ -\lambda & \cdot^{\cdot^{\cdot}} & & & & & & & & & \\ -1 & & & & & & & & & & \\ \hline & & & -\lambda & -1 & & & & & & \\ & & & & \ddots & \ddots & & & & & \\ & & & & & \ddots & -1 & & & & \\ & & & & & & -\lambda & & & & \\ \hline & & & & & & & \lambda & -1 & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & \lambda & -1 \end{array}\right] & \begin{array}{l} n_1 \\ \\ n_2 \\ \\ n_2 - 1 \end{array} \\ \begin{array}{ccc} n_1 - 1 & \qquad\qquad n_2 \qquad\qquad & n_2 \end{array} & \end{array}$$

and

$$T_2(\lambda) = \begin{array}{cl} \left[\begin{array}{ccc|cccc|cccc} & & \lambda & x & x & & & x & & & \\ & \cdot^{\cdot^{\cdot}} & -1 & & & & & & & & \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & & & & & & \\ -\lambda & -1 & & & & & & & & & \\ \hline & & & -\lambda & -1 & & & & & & \\ & & & & \ddots & \ddots & & & & & \\ & & & & & \ddots & -1 & & & & \\ y & & & & & & -\lambda & & & & \\ \hline & & & & & & & \lambda & -1 & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & \ddots & -1 \\ y & & & & & & & & & & \lambda \end{array}\right] & \begin{array}{l} n_1 \\ \\ n_2 \\ \\ n_2 - 1 \end{array} \\ \begin{array}{ccc} n_1 & \qquad\qquad n_2 \qquad\qquad & n_2 - 1 \end{array} & \end{array} .$$

Another Laplace expansion with respect to the last row yields

$$\det T_2(\lambda) = y \det T_3(\lambda) + \lambda \det T_4(\lambda)$$

where

$$T_3(\lambda) = \begin{array}{cc} \left[\begin{array}{ccc|cccc|cccc} & & \lambda & x & x & & & x & & & \\ & \iddots & -1 & & & & & & & & \\ -\lambda & \iddots & & & & & & & & & \\ -1 & & & & & & & & & & \\ \hline & & & -\lambda & -1 & & & & & & \\ & & & & \ddots & \ddots & & & & & \\ & & & & & \ddots & -1 & & & & \\ & & & & & & -\lambda & & & & \\ \hline & & & & & & & \lambda & -1 & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & \lambda & -1 \end{array}\right] & \begin{array}{l} n_1 \\ \\ n_2 \\ \\ n_2-2 \end{array} \\ \begin{array}{ccc} n_1-1 & \qquad n_2 \qquad & n_2-1 \end{array} & \end{array}$$

and

$$T_4(\lambda) = \begin{array}{cc} \left[\begin{array}{cccc|cccc|cccc} & & & \lambda & x & x & & & x & & & \\ & & \iddots & -1 & & & & & & & & \\ & \iddots & \iddots & & & & & & & & & \\ -\lambda & -1 & & & & & & & & & & \\ \hline & & & & -\lambda & -1 & & & & & & \\ & & & & & \ddots & \ddots & & & & & \\ & & & & & & \ddots & -1 & & & & \\ y & & & & & & & -\lambda & & & & \\ \hline & & & & & & & & \lambda & -1 & & \\ & & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & & \ddots & -1 \\ & & & & & & & & & & & \lambda \end{array}\right] & \begin{array}{l} n_1 \\ \\ n_2 \\ \\ n_2-2 \end{array} \\ \begin{array}{ccc} n_1 & \qquad n_2 \qquad & n_2-2 \end{array} & \end{array} .$$

We go on to compute

$$\begin{aligned} \det T_4(\lambda) &= \lambda^{n_2-2}(-1)^{n_1/2} \begin{array}{cc} \left|\begin{array}{cccc|cccc} -\lambda & -1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & -1 & & & & \\ & & & \lambda & x & x & & \\ \hline & & & & -\lambda & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ y & & & & & & & -\lambda \end{array}\right| & \begin{array}{l} n_1 \\ \\ n_2 \end{array} \\ \begin{array}{cc} n_1 & \qquad n_2 \end{array} & \end{array} \\ &= \lambda^{n_2-2}\Big[-\lambda^{n_1+n_2} + (-1)^{n_1/2} y \det T_5(\lambda)\Big], \end{aligned}$$

where

$$\det T_5(\lambda) = \begin{array}{c} \left|\begin{array}{cccc|cccc} -1 & & & & & & & \\ -\lambda & \ddots & & & & & & \\ & \ddots & -1 & & & & & \\ & & \lambda & & x & x & & \\ \hline & & & & -\lambda & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -\lambda & -1 \end{array}\right| \begin{array}{l} \\ n_1 \\ \\ \\ \\ n_2 - 1 \\ \\ \end{array} \\ \begin{array}{cc} \quad n_1 - 1 \qquad\qquad & \qquad n_2 \qquad \end{array} \end{array} = \det \begin{bmatrix} x & x \\ -\lambda & -1 \end{bmatrix} = x(\lambda - 1).$$

We obtain

$$\begin{aligned} \det T_4(\lambda) &= \lambda^{n_2-2}\big[-\lambda^{n_1+n_2} + (-1)^{n_1/2} y \det T_5(\lambda)\big] \\ &= \lambda^{n_2-2}\big[-\lambda^{n_1+n_2} + (-1)^{n_1/2} xy(\lambda - 1)\big] \\ &= -\lambda^{n_1+2n_2-2} + (-1)^{n_1/2}\lambda^{n_2-2} xy\big(\lambda - 1\big). \end{aligned}$$

Inserting this into the formula for $\det T(\lambda)$ yields

$$\begin{aligned} \det T(\lambda) &= y \det T_1(\lambda) - \lambda y \det T_3(\lambda) - \lambda^2 \det T_4(\lambda) \\ &= y \det T_1(\lambda) - \lambda y \det T_3(\lambda) + \lambda^{n_1+2n_2} + (-1)^{n_1/2}\lambda^{n_2} xy\big(1 - \lambda\big). \end{aligned}$$

We finally compute

$$\det T_1(\lambda) = x \begin{array}{c} \left|\begin{array}{cccc|cccc} & & \lambda & -1 & & & & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & & & \\ -\lambda & \cdot^{\cdot^{\cdot}} & & & & & & \\ -1 & & & & & & & \\ \hline & & & & -\lambda & -1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & -1 \\ & & & & & & & -\lambda \end{array}\right| \begin{array}{l} \\ n_1 - 1 \\ \\ \\ \\ \\ n_2 \\ \\ \end{array} \\ \begin{array}{cc} \quad n_1 - 1 \qquad\qquad & \qquad n_2 \qquad \end{array} \end{array} = -(-1)^{n_1/2} x \lambda^{n_2},$$

and since $\det T_3(\lambda) = -\det T_1(\lambda)$ holds it is:

$$\begin{aligned} \det T(\lambda) &= y \det T_1(\lambda) - \lambda y \det T_3(\lambda) + \lambda^{n_1+2n_2} + (-1)^{n_1/2}\lambda^{n_2} xy\big(1 - \lambda\big) \\ &= \lambda^{n_1+2n_2} - 2(-1)^{n_1/2} xy \lambda^{n_2+1}. \end{aligned}$$

Appendix (2)

The following matrix is denoted by T and we want to compute its determinant:

$$
\begin{array}{c}
\left[\begin{array}{c|c|c|c}
\begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \nu_+ & & & \mu_+ \end{matrix} & & & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_+ & & & \end{matrix} \\
\hline
& \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_- & & & \end{matrix} & \\
\hline
& \begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_- & & & \end{matrix} & \begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \\
\hline
\begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_+ & & & \end{matrix} & & & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -\nu_+ & & & \mu_- \end{matrix}
\end{array}\right] . \\
\begin{array}{cccc} \qquad n_1 \qquad & \qquad n_1 \qquad & \qquad n_2 \qquad & \qquad n_2 \qquad \end{array}
\end{array}
$$

Laplace expansion with respect to the first column gives

$$\det T = \mu_+ \det T_1 + (-1)^{n_1+1} \nu_+ \det T_2 + \nu_+ \det T_3,$$

where

$$
\det T_1 = \begin{array}{c}
\left|\begin{array}{c|c|c|c}
\begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu_+ \end{matrix} & & & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_+ & & & \end{matrix} \\
\hline
& \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_- & & & \end{matrix} & \\
\hline
& \begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_- & & & \end{matrix} & \begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \\
\hline
& & & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -\nu_+ & & & \mu_- \end{matrix}
\end{array}\right| \\
\begin{array}{cccc} \qquad n_1 - 1 \qquad & \qquad n_1 \qquad & \qquad n_2 \qquad & \qquad n_2 \qquad \end{array}
\end{array}
$$

$$= \mu_+^{n_1-1}\left(\mu_-^{n_2}-\nu_+\right)\det T_{mid},$$

denoting by T_{mid} the middle $(n_1+n_2)\times(n_1+n_2)$ block of T. Moreover, we have

$$\det T_2 = \left|\begin{array}{c|c|c|c}
\begin{matrix} 1 & & & \\ \mu_+ & \ddots & & \\ & \ddots & \ddots & \\ & & \mu_+ & 1 \end{matrix} & & & \\ \hline
& \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_- & & & \end{matrix} & \\ \hline
& \begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_- & & & \end{matrix} & \begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \\ \hline
& & & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -\nu_+ & & & \mu_- \end{matrix}
\end{array}\right|$$

$$\begin{array}{cccc} \quad n_1-1 \quad & \quad n_1 \quad & \quad n_2 \quad & \quad n_2 \quad \end{array}$$

$$= \left(\mu_-^{n_2}-\nu_+\right)\det T_{mid}$$

and

$$\det T_3 = \left|\begin{array}{c|c|c|c}
\begin{matrix} 1 & & \\ \mu_+ & \ddots & \\ & \ddots & 1 \\ & & \mu_+ \end{matrix} & & & \begin{matrix} & & \\ & & \\ & & \\ \nu_+ & & \end{matrix} \\ \hline
& \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \nu_- & & & \mu_- \end{matrix} & \begin{matrix} & & & \\ & & & \\ & & & \\ \nu_- & & & \end{matrix} & \\ \hline
& \begin{matrix} & & & \\ & & & \\ & & & \\ -\nu_- & & & \end{matrix} & \begin{matrix} \mu_+ & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\nu_- & & & \mu_+ \end{matrix} & \\ \hline
& & & \begin{matrix} \mu_- & -1 & & \\ & \ddots & \ddots & \\ & & \mu_- & -1 \end{matrix}
\end{array}\right|$$

$$\begin{array}{cccc} \quad n_1-1 \quad & \quad n_1 \quad & \quad n_2 \quad & \quad n_2 \quad \end{array}$$

$$= (-1)^{n_1+1}\nu_+\det T_{mid}.$$

Putting these computations together, we obtain

$$\begin{aligned}\det T &= \Big[\mu_+^{n_1}(\mu_-^{n_2}-\nu_+)+(-1)^{n_1+1}\nu_+(\mu_-^{n_2}-\nu_+)+\nu_+^2(-1)^{n_1+1}\Big]\det T_{mid}\\ &= \Big[\mu_+^{n_1}\mu_-^{n_2}-\mu_+^{n_1}\nu_+-(-1)^{n_1}\mu_-^{n_2}\nu_+\Big]\det T_{mid}.\end{aligned}$$

We continue with computing

$$\det T_{mid} = \begin{array}{c}\left|\begin{array}{cccc|cccc} \mu_- & -1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & -1 & & & & \\ \nu_- & & & \mu_- & \nu_- & & & \\ \hline & & & & \mu_+ & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ -\nu_- & & & & -\nu_- & & & \mu_+ \end{array}\right| \\ \qquad n_1 \qquad\qquad\qquad\qquad n_2 \end{array}.$$

A Laplace expansion with respect to the first column yields

$$\det T_{mid} = \mu_- \det T_4 + (-1)^{n_1+1}\nu_- \det T_5 + (-1)^{n_1+n_2}\nu_- \det T_6,$$

where

$$\det T_4 = \begin{array}{c}\left|\begin{array}{cccc|cccc} \mu_- & -1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & -1 & & & & \\ & & & \mu_- & \nu_- & & & \\ \hline & & & & \mu_+ & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & -\nu_- & & & \mu_+ \end{array}\right| \\ \qquad n_1-1 \qquad\qquad\qquad\qquad n_2 \end{array} = \mu_-^{n_1-1}\Big[\mu_+^{n_2}+(-1)^{n_2}\nu_-\Big]$$

and

$$\det T_5 = \begin{array}{c}\left|\begin{array}{cccc|cccc} -1 & & & & & & & \\ \mu_- & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \mu_- & -1 & & & & \\ \hline & & & & \mu_+ & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & -\nu_- & & & \mu_+ \end{array}\right| \\ \qquad n_1-1 \qquad\qquad\qquad\qquad n_2 \end{array} = (-1)^{n_1-1}\Big[\mu_+^{n_2}+(-1)^{n_2}\nu_-\Big]$$

as well as

$$\det T_6 = \left|\begin{array}{cccc|cccc} -1 & & & & & & & \\ \mu_- & \ddots & & & & & & \\ & \ddots & -1 & & & & & \\ & & \mu_- & & \nu_- & & & \\ \hline & & & & \mu_+ & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \mu_+ & 1 \end{array}\right| = (-1)^{n_1-1}\nu_-.$$
$$\qquad\qquad n_1 - 1 \qquad\qquad\qquad n_2$$

Hence, we obtain

$$\begin{aligned} \det T_{mid} &= \mu_-^{n_1}(\mu_+^{n_2} + (-1)^{n_2}\nu_-) + \nu_-(\mu_+^{n_2} + (-1)^{n_2}\nu_-) + (-1)^{n_2+1}\nu_-^2 \\ &= \mu_-^{n_1}\mu_+^{n_2} + (-1)^{n_2}\mu_-^{n_1}\nu_- + \mu_+^{n_2}\nu_-, \end{aligned}$$

which altogether yields

$$\det T = \Big[\mu_+^{n_1}\mu_-^{n_2} - \mu_+^{n_1}\nu_+ - (-1)^{n_1}\mu_-^{n_2}\nu_+\Big]\Big[\mu_-^{n_1}\mu_+^{n_2} + (-1)^{n_2}\mu_-^{n_1}\nu_- + \mu_+^{n_2}\nu_-\Big].$$