LOW RANK PERTURBATIONS OF QUATERNION MATRICES

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Dedicated to the memory of Leiba Rodman, whose work inspired us greatly.

Abstract. Low rank perturbations of right eigenvalues of quaternion matrices are considered. For real and complex matrices it is well known that under a generic rank-k perturbation the k largest Jordan blocks of a given eigenvalue will disappear while additional smaller Jordan blocks will remain. In this paper, it is shown that the same is true for real eigenvalues of quaternion matrices, but for complex nonreal eigenvalues the situation is different: not only the largest k, but the largest $2k$ Jordan blocks of a given eigenvalue will disappear under generic quaternion perturbations of rank k. Special emphasis is also given to Hermitian and skew-Hermitian quaternion matrices and generic low rank perturbations that are structure-preserving.

1. Introduction. In this paper we will consider an $n \times n$ matrix A with entries from the skew-field $\mathbb{H}$ of the quaternions. Recall from [20] that a number $\lambda \in \mathbb{H}$ is called a right eigenvalue if there is a vector $x \in \mathbb{H}^n \setminus \{0\}$ such that $Ax = x\lambda$. Since for every $\alpha \in \mathbb{H}$ we have $A(x\alpha) = (x\alpha)(\alpha^{-1}\lambda\alpha)$, we see that together with λ also every similar number $\alpha^{-1}\lambda\alpha$ is a right eigenvalue. Restricting oneself to one representative of each equivalence class of similar right eigenvalues, one can assume without loss of generality that the right eigenvalues are in fact complex numbers with nonnegative imaginary part. This concept then allows the computation of a Jordan canonical form for the matrix A, to be precise: there exists an invertible quaternion matrix S such that

$$S^{-1}AS = \mathcal{J}_{m_1}(\lambda_1) \oplus \cdots \oplus \mathcal{J}_{m_p}(\lambda_p),$$

with $\lambda_1, \ldots, \lambda_p \in \mathbb{C}$ having nonnegative imaginary part for $j = 1, \ldots, p$. Here, $\lambda_1, \ldots, \lambda_p$ are not necessarily pairwise distinct and $\mathcal{J}_m(\lambda)$ stands for the upper triangular complex Jordan block of size $m \times m$ associated with the eigenvalue $\lambda \in \mathbb{C}$.

The question we will consider is the following: what happens to the Jordan canonical form of A when we apply a generic additive perturbation of rank k, i.e., when we consider the matrix $A + UV^T$ for some (U, V) from a generic set $\Omega \subseteq \mathbb{H}^n \times \mathbb{H}^n \cong \mathbb{H}^{2n}$. For the complex case this problem was studied in [12], and later in [19, 21, 22]. An alternative treatment for the complex case was given in [14], and also the real case has been studied, see [16]. Furthermore, also the case of complex matrix pencils has been studied in [9, 8] for the regular case and in [6] for the singular case, while the case of regular matrix polynomials was treated in [7]. The related questions for matrices with a symmetry structure have been addressed in a series of papers starting with [14] and continued in [15, 16, 17, 18] and [13, 11] for many different classes of structures and the case of structure-preserving rank-one perturbation. A generalization to the case of structure-preserving rank-k perturbations was then given in [5]. Also, structure-preserving low-rank perturbations of regular matrix pencils with symmetry structures have been considered, see [1, 2, 3, 4] for special perturbations of rank one or two and [10] for the general case.

To be precise about the nature of the term "generic", we introduce the isomorphism $\chi : \mathbb{H}^n \to \mathbb{R}^{4n}$ as a particular standard representation of $\mathbb{H}^n$ seen as a real

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vector space as follows: If $(1, \mathsf{i}, \mathsf{j}, \mathsf{k})$ is the canonical basis for $\mathbb{H}$ over $\mathbb{R}$, we define

$$\chi(u) = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

for a vector $u = u_0 + u_1\mathsf{i} + u_2\mathsf{j} + u_3\mathsf{k} \in \mathbb{H}^n$, with $u_i \in \mathbb{R}^n$, $i = 0, 1, 2, 3$. Then a set $\Omega \subseteq \mathbb{H}^n$ is said to be *generic* (or more precisely *generic with respect to the real components*) if the set $\chi(\Omega)$ is a generic set in $\mathbb{R}^{4n}$, i.e., its complement is contained in a proper algebraic subset of $\mathbb{R}^{4n}$. (Recall that a subset $\mathcal{A} \subseteq \mathbb{R}^{4n}$ is called *algebraic* if it is the set of common zeros of finitely many real polynomials in $4n$ variables, and it is called *proper* if it is not the full space $\mathbb{R}^{4n}$.) Similarly, a set $\Omega \subseteq \mathbb{C}^n$ is said to be *generic* (or more precisely *generic with respect to the real and imaginary parts*) if it is generic when viewed as the canonical subspace of $\mathbb{R}^{2n}$. It is easy to see that if $\Omega \subseteq \mathbb{F}^n$ is generic and $S \in \mathbb{F}^{n,n}$ is invertible, then the sets $S\Omega$ and ΩS are generic as well, where $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$.

A general result on rank-k perturbations of complex matrices says that the geometric multiplicity of a fixed eigenvalue can change at most by k if any (not necessarily being generic) rank-k perturbation is applied, see, e.g., [19] or [14], and the question arises if this remains true for quaternion matrices. As a first example, consider the quaternion matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -\mathsf{k} \\ \mathsf{k} & 1 \end{bmatrix}, \quad \text{and} \quad A + B = \begin{bmatrix} 2 & -\mathsf{k} \\ \mathsf{k} & 2 \end{bmatrix}.$$

Then A is actually a real matrix with eigenvalue 1 with algebraic and geometric multiplicity two and B is a quaternion matrix of rank one. Then one easily checks that

$$(A + B)\begin{bmatrix} 1 \\ \mathsf{k} \end{bmatrix} = \begin{bmatrix} 1 \\ \mathsf{k} \end{bmatrix} 3, \quad \text{and} \quad (A + B)\begin{bmatrix} 1 \\ -\mathsf{k} \end{bmatrix} = \begin{bmatrix} 1 \\ -\mathsf{k} \end{bmatrix} 1,$$

so that the eigenvalues of $A + B$ are 3 and 1. This shows that for this example the geometric multiplicity of the eigenvalue 1 does change by only one from two to one as the reader may have expected.

Surprisingly, however, it needs no longer be the case for quaternion matrices that a rank-k perturbation can change the geometric multiplicity by at most k. To see this, consider the example

$$C := \begin{bmatrix} \mathsf{i} & 0 \\ 0 & \mathsf{i} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -\mathsf{k} \\ \mathsf{k} & 1 \end{bmatrix},$$

where B is the same rank-one matrix as above. Then setting

$$S := \begin{bmatrix} -1 + 2\mathsf{j} & 1 - \mathsf{k} \\ 2 + \mathsf{j} & -\mathsf{i} - \mathsf{j} - \mathsf{k} \end{bmatrix}$$

a straight forward calculation shows that

$$\begin{bmatrix} 1 + \mathsf{i} & -\mathsf{k} \\ \mathsf{k} & 1 + \mathsf{i} \end{bmatrix} S = \begin{bmatrix} -1 + 2\mathsf{j} & 2\mathsf{j} - \mathsf{k} \\ 2 + \mathsf{j} & 2 - \mathsf{i} - \mathsf{k} \end{bmatrix} = S \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

or equivalently $S^{-1}(C+B)S = \mathcal{J}_2(1)$, so the perturbed matrix $C+B$ has the eigenvalue 1 with geometric multiplicity one and algebraic multiplicity two which means that the geometric multiplicity of the eigenvalue i of the original matrix C has changed by 2 from two to zero. Although we will see below that the perturbation B does not show the generic behaviour as it generates a Jordan block of size two of the newly created eigenvalue 1, this example shows that the effect of quaternion rank one perturbations of quaternion matrices may be significantly different from the analogous effect observed for complex rank-one perturbations of complex matrices.

The observant reader may suspect at this moment that the different behavior of the matrices A and C under a perturbation with B may be caused by the given symmetry-structure and its preservation or non-preservation, respectively. Indeed, the matrices A and B are Hermitian while C is skew-Hermitian, so the perturbation with B is structure-preserving for A, but not for C. However, the surprising effect in the second example remains true even for perturbations preserving the skew-Hermitian structure as our third example with the matrices

$$C = \begin{bmatrix} \mathsf{i} & 0 \\ 0 & \mathsf{i} \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} \mathsf{j} & -1 \\ 1 & \mathsf{j} \end{bmatrix}$$

will show. Here, D is skew-Hermitian and has rank one. When we consider

$$C + D = \begin{bmatrix} \mathsf{i}+\mathsf{j} & -1 \\ 1 & \mathsf{i}+\mathsf{j} \end{bmatrix},$$

then a straightforward computation shows that

$$\begin{aligned}
(C+D)\begin{bmatrix} 1+\sqrt{2}+\mathsf{k} \\ -(1+\sqrt{2})\mathsf{i}-\mathsf{j} \end{bmatrix} &= \begin{bmatrix} 1+\sqrt{2}+\mathsf{k} \\ -(1+\sqrt{2})\mathsf{i}-\mathsf{j} \end{bmatrix}(1+\sqrt{2})\mathsf{i}, \\
(C+D)\begin{bmatrix} -(1+\sqrt{2})\mathsf{i}-\mathsf{j} \\ 1+\sqrt{2}+\mathsf{k} \end{bmatrix} &= \begin{bmatrix} -(1+\sqrt{2})\mathsf{i}-\mathsf{j} \\ 1+\sqrt{2}+\mathsf{k} \end{bmatrix}(\sqrt{2}-1)\mathsf{i}.
\end{aligned}$$

This shows that the geometric multiplicity of the eigenvalue i of C drops from two to zero even under a structure-preserving rank-one perturbation.

In this paper we will show that the different behavior observed in the examples above is due to the nature of the occurring eigenvalues. Indeed, real eigenvalues behave differently than complex eigenvalues: if a generic rank-k perturbation is applied to a square quaternion matrix, then the largest k Jordan blocks associated with any real eigenvalue will disappear from the Jordan canonical form while additional smaller Jordan blocks will remain. For a given complex eigenvalue, however, it will now be the corresponding largest $2k$ Jordan blocks that disappear while again additional smaller ones will remain. This effect can be observed for both generic rank-one perturbations of general quaternion matrices as well as for generic structure-preserving rank-one perturbations of Hermitian or skew-Hermitian quaternion matrices.

The remainder of the paper is organized as follows. In the next section, we present one of the main tools in our investigations by reviewing the well-known connection between quaternion matrices and a subclass of complex matrices with a special symmetry structure. This class will be denoted by $\mathcal{Q}_{n,n}$, where the symbol $\mathcal{Q}$ has been chosen as a reminder of the quaternions. In Section 3, we generalize some results on low-rank perturbations from the literature so that they can be applied to structure-preserving low rank perturbations in $\mathcal{Q}_{n,n}$. In Section 4, we then discuss the changes in the Jordan structure under generic structure-preserving rank-k perturbations within

$\mathcal{Q}_{n,n}$ and translate this result in Section 5 to quaternion matrices. In Section 6 and 7, we investigate structure-preserving quaternion rank-k perturbations of Hermitian and skew-Hermitian, respectively, and show that the same behavior as under generic perturbations that ignore the structure can be observed.

2. Reduction to a structured matrix problem. It is well known that the map $\omega : \mathbb{H} \to \omega(\mathbb{H}) \subseteq \mathbb{C}^{2,2}$ with

$$\omega(\alpha_1 + \mathrm{i}\alpha_2 + \mathrm{j}\alpha_3 + \mathrm{k}\alpha_4) \mapsto \begin{bmatrix} \alpha_1 + \mathrm{i}\alpha_2 & \alpha_3 + \mathrm{i}\alpha_4 \\ -\alpha_3 + \mathrm{i}\alpha_4 & \alpha_1 - \mathrm{i}\alpha_2 \end{bmatrix}$$

for $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, is a skew-field isomorphism. Its extension (also denoted by ω) to matrices will be an important tool in this paper: given a quaternion matrix $A \in \mathbb{H}^{n,m}$, we can write $A = A_1 + A_2\mathrm{j}$, where A_1 and A_2 are complex matrices. Then

$$\omega(A) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$$

and by [20, Theorem 5.7.1], the Jordan form of the quaternion matrix A is given by

$$\mathcal{J}_{m_1}(\lambda_1) \oplus \cdots \oplus \mathcal{J}_{m_p}(\lambda_p),$$

if and only if the Jordan form of $\omega(A)$ is given by

$$\begin{bmatrix} \mathcal{J}_{m_1}(\lambda_1) & 0 \\ 0 & \mathcal{J}_{m_1}(\overline{\lambda_1}) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{m_p}(\lambda_p) & 0 \\ 0 & \mathcal{J}_{m_p}(\overline{\lambda_p}) \end{bmatrix}. \tag{2.1}$$

Note that the eigenvalues in (2.1) are allowed to be real. In particular, it follows that each real eigenvalue has even algebraic and geometric multiplicity, and all partial multiplicities occur an even number of times. In the following it will be useful to use a slight variant of (2.1) that is itself in the range of ω. Applying a block permutation the matrix in (2.1) is easily seen to be similar to

$$\begin{bmatrix} \mathcal{J} & 0 \\ 0 & \overline{\mathcal{J}} \end{bmatrix} \quad \text{with} \quad \mathcal{J} = \mathcal{J}_{m_1}(\lambda_1) \oplus \cdots \oplus \mathcal{J}_{m_p}(\lambda_p).$$

The map ω mapping quaternion matrices to complex matrices has the properties that it is linear (with respect to real scalars), multiplicative, and respects the transpose operation [20, Section 3.4]. In particular, we have

$$\omega(A + UV^T) = \omega(A) + \omega(U)\omega(V)^T \tag{2.2}$$

for $A \in \mathbb{H}^{n,n}$ and $U, V \in \mathbb{H}^{n,k}$. Thus, to study the effect of rank-k perturbations on the quaternion matrix A we can study the effect of structure-preserving perturbations of $\omega(A)$. However, if $U = U_1 + U_2\mathrm{j}$ and $V = V_1 + V_2\mathrm{j}$ with $U_i, V_i \in \mathbb{C}^{n,k}$ have rank k, then it follows by [20, Proposition 3.2.5(e)] and the properties of ω that

$$\omega(U) = \begin{bmatrix} U_1 & U_2 \\ -\overline{U_2} & \overline{U_1} \end{bmatrix} \quad \text{and} \quad \omega(V) = \begin{bmatrix} V_1 & V_2 \\ -\overline{V_2} & \overline{V_1} \end{bmatrix}$$

have rank $2k$. Thus, rank-k perturbations of quaternion matrices lead to rank-$2k$ perturbations of complex matrices that are structured as in the range of ω.

In the following it will be useful to use an alternative characterization of this particular class of structured complex matrices. For this, we will introduce the following notation.

DEFINITION 2.1. *Let $n, k \in \mathbb{N}$ and*

$$J := J_{2n} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

Then we define

$$\mathcal{Q}_{n,k} = \left\{ X \in \mathbb{C}^{2n,2k} \,\middle|\, J_{2n}X = \overline{X}J_{2k} \right\}.$$

It is straightforward to check that a $2n \times 2k$ matrix X is in the range of ω if and only if $X \in \mathcal{Q}_{n,k}$. In the following, we will sometimes switch between the sets $\mathbb{H}^{n,m}$ and $\mathcal{Q}_{n,m}$, and in order to make it easier for the reader to keep track in which set we currently are, we adopt the convention to use "hatted" symbols for matrices in $\mathcal{Q}_{n,m}$. Thus, if $A \in \mathbb{H}^{n,m}$ then we denote $\widehat{A} = \omega(A)$ and similarly, if $\widehat{B} \in \mathcal{Q}_{n,m}$, then $B = \omega^{-1}(\widehat{B}) \in \mathbb{H}^{n,m}$.

The next proposition shows that the formula in (2.2) gives a parametrization of matrices in $\mathcal{Q}_{n,n}$ that have rank $2k$ so that we are able to identify and describe generic sets of such matrices.

PROPOSITION 2.2. *Let $\widehat{B} \in \mathcal{Q}_{n,n}$ be a matrix of rank $2k$. Then there exist two matrices $\widehat{U}, \widehat{V} \in \mathcal{Q}_{n,k}$ of full rank $2k$ such that $\widehat{B} = \widehat{U}\widehat{V}^T$.*

Proof. Since $\widehat{B}$ is in the range of ω, there exists a quaternion matrix $B \in \mathbb{H}^{n,n}$ such that $\widehat{B} = \omega(B)$, and by the properties of ω B must have rank k. But then there exists matrices $U, V \in \mathbb{H}^{n,k}$ of rank k such that $B = UV^T$ by [20, Proposition 3.2.5(e)], and hence we obtain $\widehat{B} = \widehat{U}\widehat{V}^T$ with $\widehat{U} = \omega(U), \widehat{V} = \omega(V) \in \mathcal{Q}_{n,k}$ having rank $2k$. □

As a side-note, observe that matrices in the class $\mathcal{Q}_{n,n}$ can never have odd rank, so the smallest rank perturbation of matrices in that class is a perturbation of rank two.

3. Localization results. In this section, we establish a result that allows us to determine the behavior of a possibly structured complex matrix under generic structure-preserving low-rank perturbations by studying the effect of perturbations that locally perturb an arbitrary, but fixed eigenvalue of the matrix. The main theorem is a generalization of [5, Theorem 2.6] and in fact contains that result as a special case. A key ingredient for its proof is the following lemma which is a generalization of [17, Lemma 8.1]. We highlight that although the lines of the proofs of the results in this section follow the lines of the proofs of the previously obtained results, they are not immediate. Therefore, a careful revision of each single step in the proof is necessary to obtain the full generality in the main theorem presented here. In this way, the result will not only be applicable in the remainder of this paper, but also for any class of structured matrices and corresponding structure-preserving low-rank perturbations that can be parameterized by polynomial functions.

The next lemma is needed for the proof of the main result and states that newly created eigenvalues of perturbed matrices will generically have multiplicities that are as small as possible.

LEMMA 3.1. *Let $A \in \mathbb{C}^{n,n}$ have the pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ with algebraic multiplicities $a_1, \ldots, a_m$, and let $\varepsilon > 0$ be such that the discs*

$$D_j := \{\mu \in \mathbb{C} \,|\, |\lambda_j - \mu| < \varepsilon^{2/n}\}, \quad j = 1, \ldots, m$$

are pairwise disjoint. Furthermore, let $B : \mathbb{R}^m \to \mathbb{C}^{n,n}$ be an analytic function with $B(0) = A$ such that the following conditions are satisfied:

1) *For all $u \in \mathbb{R}^m$, the algebraic multiplicity of any eigenvalue of $B(u)$ is always a multiple of $\ell \in \mathbb{N} \setminus \{0\}$.*
2) *There exists a generic set $\Omega \subseteq \mathbb{R}^m$ such that for all $u \in \Omega$ the matrix $B(u)$ has the eigenvalues $\lambda_1, \ldots, \lambda_m$ with algebraic multiplicities $\widetilde{a}_1, \ldots, \widetilde{a}_m$, where $\widetilde{a}_j \leq a_j$ for $j = 1, \ldots, m$. (Here, we allow $\widetilde{a}_j = 0$ in the case that λ_j no longer is an eigenvalue of $B(u)$.)*
3) *For each $j = 1, \ldots, m$ there exists $u_j \in \mathbb{R}^m$ with $\|u_j\| < \varepsilon$ such that the matrix $B(u)$ has exactly $(a_j - \widetilde{a}_j)/\ell$ pairwise distinct eigenvalues in D_j different from λ_j and each with algebraic multiplicity exactly ℓ.*

Then there exists a generic set $\Omega_0 \subseteq \mathbb{R}^m$ such that for all $u \in \Omega_0$ the eigenvalues of $B(u)$ that are different from those of A have algebraic multiplicity exactly ℓ.

Proof. First observe that there exists a constant K only depending on A such that for any $u \in \mathbb{R}^m$ with $\|u\| < K \cdot \min\{1, \varepsilon\}$ the matrix $B(u)$ has exactly a_j eigenvalues in the disc D_j. Indeed, this follows from the continuity of B and well-known results on matching distance of eigenvalues of nearby matrices, see, e.g., [23, Section IV.1] and references therein. In the following, we denote $\varepsilon' = K \cdot \min\{1, \varepsilon\}$.

Next, we fix λ_j and denote by $\chi(\lambda_j, u)$ the characteristic polynomial (in the independent variable t) of the restriction of $B(u)$ to the spectral invariant subspace corresponding to the eigenvalues of $B(u)$ within D_j. Then the coefficients of $\chi(\lambda_j, u)$ are analytic functions of the components of u, see, e.g., [14, Lemma 2.5] for more details.

Let $q(\lambda_j, u)$ be the number of distinct eigenvalues of $B(u)$ in the disk D_j. Furthermore, denote by $S(p_1, p_2)$ the Sylvester resultant matrix of the two polynomials $p_1(t)$, $p_2(t)$ and recall that $S(p_1, p_2)$ is a square matrix of size $\deg(p_1) + \deg(p_2)$ and that the rank deficiency of $S(p_1, p_2)$ coincides with the degree of the greatest common divisor of the polynomials $p_1(t)$ and $p_2(t)$. We have

$$q(\lambda_j, u) = \frac{1}{\ell}\left(\text{rank } S\left(\chi(\lambda_j, u), \frac{\partial^\ell \chi(\lambda_j, u)}{(\partial t)^\ell}\right) - a_j\right) + 1.$$

The entries of $S\big(\chi(\lambda_j, u), \frac{\partial^\ell \chi(\lambda_j, u)}{(\partial t)^\ell}\big)$ are scalar multiples (which are independent of u) of the coefficients of $\chi(\lambda_j, u)$, and therefore the set $Q(\lambda_j)$ of all $u \in \mathbb{R}^m$, $\|u\| < \varepsilon'$, for which $q(\lambda_j, u)$ is maximal is the complement of the set of zeros of an analytic function of the entries of u. (In fact, this analytic function can be chosen to be the product of minors of that order that is equal to the maximal value of $q(\lambda_j, u)$.) In particular, $Q(\lambda_j)$ is open and dense in

$$\{u \in \mathbb{R}^m \,|\, \|u\| < \varepsilon'\}.$$

By hypothesis, there exists $u_j \in \mathbb{R}^m$ such that $B(u_j)$ has exactly $\frac{1}{\ell}(a_j - \widetilde{a}_j)$ eigenvalues with algebraic multiplicity exactly ℓ in D_j different from λ_j. If u_j happens not to be in Ω, then we may slightly perturb u_j to obtain a new $u_j' \in \Omega$ such that $B(u_j')$ has the eigenvalues $\lambda_1, \ldots, \lambda_m$ with algebraic multiplicities $\widetilde{a}_1, \ldots, \widetilde{a}_m$ and $\frac{1}{\ell}(a_j - \widetilde{a}_j)$

eigenvalues with algebraic multiplicity exactly ℓ in D_j different from λ_j. Such choice of u_j' is possible because Ω is generic, the property of eigenvalues having algebraic multiplicity exactly ℓ persists under small perturbations of $B(u_j)$ by assumption 1), and the total number of eigenvalues of $B(u)$ within D_j, counted with multiplicities, is equal to a_j, for every $u \in \mathbb{R}^m$ with $\|u\| < \varepsilon'$. Since Ω is open, clearly there exists $\delta > 0$ such that for all $u \in \mathbb{R}^m$ with $\|u - u_j\| < \delta$ the matrix $B(u_j)$ has the eigenvalues $\lambda_1, \ldots, \lambda_m$ with algebraic multiplicities $\widetilde{a}_1, \ldots, \widetilde{a}_m$ and $\frac{1}{\ell}(a_j - \widetilde{a}_j)$ eigenvalues with algebraic multiplicity exactly ℓ in D_j different from λ_j. Since the set of all such vectors u is open in $\mathbb{R}^m$, it follows from the properties of the set $Q(\lambda_j)$ established above that in fact we have

$$q(\lambda_j, u) = \frac{1}{\ell}(a_j - \widetilde{a}_j), \quad \text{for all} \quad u \in \mathbb{R}^m, \quad \|u - u_j\| < \delta.$$

So for the open set

$$\Omega_j := Q(\lambda_j) \cap \Omega$$

which is dense in $\{u \in \mathbb{R}^m \mid \|u\| < \varepsilon'\}$, we have that all eigenvalues of $B(u)$ within D_j different from λ_j have algebraic multiplicity exactly ℓ. Now let

$$\Omega' = \bigcap_{j=1}^{m} \Omega_j \subseteq \Omega.$$

Note that Ω' is nonempty as the intersection of finitely many sets that are open dense in $\{u \in \mathbb{R}^m \mid \|u\| < \varepsilon'\}$.

Finally, let $\chi(u)$ denote the characteristic polynomial (in the independent variable t) of $B(u)$. Then the number of distinct roots of $\chi(u)$ is given by

$$\operatorname{rank} S\left(\chi(u), \frac{\partial \chi(u)}{\partial t}\right) - n + 1$$

and therefore, the set of all $u \in \Omega$ on which the number of distinct roots of $\chi(u)$ is maximal, is a generic set. Since Ω' constructed above is nonempty, this maximal number is equal to $\sum_{j=1}^{m} \frac{1}{\ell}(a_j - \widetilde{a}_j)$, i.e., generically all eigenvalues of $B(u)$ that are different from $\lambda_1, \ldots, \lambda_m$ have algebraic multiplicity exactly ℓ. $\square$

Next, we consider the analogue of Theorem 2.6 of [5] which describes the possible changes in the Jordan structure of a fixed eigenvalue λ of a matrix from $\mathcal{Q}_{n,k}$ under low rank perturbations, and also presents conditions when a generic behavior can be observed.

THEOREM 3.2. *Let $A \in \mathbb{C}^{n,n}$ and let the Jordan canonical form of A be given by*

$$\mathcal{J}_{n_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \tilde{\mathcal{J}},$$

with $n_1 \geq \cdots \geq n_m$ and $\lambda \notin \sigma(\tilde{\mathcal{J}})$. Furthermore, let $P \in \mathbb{R}^{n,n}[t_1, \ldots, t_r]$ be a matrix whose entries are polynomials in the independent indeterminate variables $t_1, \ldots, t_r$. Assume that for all $u = (u_1, \ldots, u_r) \in \mathbb{R}^r$ we have

(i) $\operatorname{rank} P(u) \leq \kappa$;

(ii) *the algebraic multiplicity of any eigenvalue of $A + P(u)$ is always a multiple of $\ell \in \mathbb{N} \setminus \{0\}$.*

Then the following statements hold:

1. *For each* $(u_1, \ldots, u_r) \in \mathbb{R}^r$ *there exist integers* $\eta_1 \geq \cdots \geq \eta_\ell$ *such that*
   (a) *the Jordan canonical form of* $A + P(u_1, \ldots, u_r)$ *is given by*
$$\mathcal{J}_{\eta_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{\eta_\ell}(\lambda) \oplus \check{\mathcal{J}},$$
   where $\lambda \notin \sigma(\check{\mathcal{J}})$,
   (b) $(\eta_1, \ldots, \eta_\ell)$ *dominates* $(n_{\kappa+1}, \ldots, n_m)$; *that is, we have* $l \geq m - \kappa$ *and* $\eta_j \geq n_{j+\kappa}$ *for* $j = 1, \ldots, m - \kappa$.
2. *Assume that for all* $u = (u_1, \ldots, u_r) \in \mathbb{R}$ *the algebraic multiplicity* a_u *of* λ *as an eigenvalue of* $A + P(u)$ *satisfies* $a_u \geq a$ *for some* $a \in \mathbb{N}$. *If there exists* $u_0 \in \mathbb{R}^r$ *such that* $a_{u_0} = a$, *then the set*
$$\Omega = \{u \in \mathbb{R}^r \mid a_u = a\}$$
   is a generic set.
3. *Assume that for any* $\varepsilon > 0$ *there exists* $u_0 \in \mathbb{R}^r$ *with* $\|u_0\| < \varepsilon$ *such that the Jordan form of* $A + P(u_0)$ *is described by*
   (a) $\mathcal{J}_{n_{\kappa+1}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \check{\mathcal{J}}, \qquad \lambda \notin \sigma(\check{\mathcal{J}})$,
   (b) *all eigenvalues that are not eigenvalues of* A *have multiplicity* ℓ *precisely.*
   Then there exists a generic set $\Omega \subseteq \mathbb{R}^r$ *such that the Jordan canonical form of* $A + P(u)$ *is described by* (a) *and* (b) *for all* $u \in \Omega$.

Proof. Part (1) is a direct consequence of [9, Lemma 2.1] using the fact that the rank of $P(u)$ is at most κ for any $u \in \mathbb{R}^r$.

For part (2), let $Y(u) = (A + P(u) - \lambda I_n)^n$. Then the hypothesis tells us that $\operatorname{rank} Y(u_0) = n - a$ for some $u_0 \in \mathbb{R}^r$. Thus we can apply [14, Lemma 2.1] (or [5, Lemma 2.2]) to see that the set
$$\Omega := \{u \in \mathbb{R}^r \mid \operatorname{rank} Y(u) \geq n - a\}$$
is a generic set. Note that the condition $\operatorname{rank} Y(u) \geq n-a$ is equivalent to $a_u \leq a$, and since the reverse inequality $a_u \geq a$ holds by assumption it is equivalent to $a_u = a_0$. Hence, Ω is the desired generic set.

Concerning part (3) observe that by part (1) of the theorem, the list of partial multiplicities of $A + P(u)$ corresponding to the eigenvalue λ dominates the list $(n_{\kappa+1}, \ldots, n_m)$. Hence, the algebraic multiplicity a_u of $A + P(u)$ at λ is at least $a := n_{\kappa+1} + \cdots + n_m$. By the hypothesis there exists a particular $u_0 \in \mathbb{R}^r$ such that $a_{u_0} = n_{\kappa+1} + \cdots + n_m = a$. Then by part (2) the set Ω_1 of all $u \in \mathbb{R}^r$ with $a_u = a$ is generic. Since the only list of partial multiplicities that both dominates $(n_{\kappa+1}, \ldots, n_m)$ and has $a_u = a_0$ is the list $(n_{\kappa+1}, \ldots, n_m)$ itself, this shows that the Jordan form described in part (a) is attained by all matrices in Ω_1. Moreover, since P is analytic and u_0 can be chosen arbitrarily small with $A + P(u_0)$ satisfying the condition in (b), it follows by Lemma 3.1 that the set Ω_2 of all $u \in \mathbb{R}^r$ satisfying (b) is also generic. Then the set $\Omega = \Omega_1 \cap \Omega_2$ is the desired generic set. □

4. Even rank perturbations within $\mathcal{Q}_{n,n}$. We are now ready to state the first main result of this paper, which basically says that for each eigenvalue of a matrix $\widehat{A}$ in $\mathcal{Q}_{n,n}$ under generic perturbations with matrices of rank $2k$ in $\mathcal{Q}_{n,n}$ the largest $2k$ partial multiplicities disappear while the others remain, and that the eigenvalues of $\widehat{A} + \widehat{U}\widehat{V}^T$ which are not already eigenvalues of $\widehat{A}$ are all simple and non-real.

THEOREM 4.1. *Let $\widehat{A} \in \mathcal{Q}_{n,n}$, and let the Jordan canonical form of $\widehat{A}$ be given by $A_1 \oplus \overline{A_1}$, where*

$$\begin{aligned} A_1 = &\left(\bigoplus_{i=1}^{r_1} \mathcal{J}_{n_{i,1}}(\lambda_1)\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{r_p} \mathcal{J}_{n_{i,p}}(\lambda_p)\right) \\ &\oplus \left(\bigoplus_{i=1}^{r_{p+1}} \mathcal{J}_{n_{i,p+1}}(\lambda_{p+1})\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{r_m} \mathcal{J}_{n_{i,m}}(\lambda_m)\right) \end{aligned}$$

where the eigenvalues $\lambda_1, \dots, \lambda_m$ are pairwise distinct with $\lambda_1, \dots, \lambda_p$ being real and $\lambda_{p+1}, \dots, \lambda_m$ having positive imaginary part, and where the partial multiplicities are ordered in decreasing order: $n_{1,j} \geq \cdots \geq n_{r_j,j}$ for all $j = 1, \dots, m$.

Then, there exists a generic set $\Omega \subseteq \mathcal{Q}_{n,k} \times \mathcal{Q}_{n,k}$ such that for all $(\widehat{U}, \widehat{V}) \in \Omega$ the Jordan form of $\widehat{A} + \widehat{U}\widehat{V}^T$ is given by $C_1 \oplus \overline{C_1}$, where

$$\begin{aligned} C_1 = &\left(\bigoplus_{i=k+1}^{r_1} \mathcal{J}_{n_{i,1}}(\lambda_1)\right) \oplus \cdots \oplus \left(\bigoplus_{i=k+1}^{r_p} \mathcal{J}_{n_{i,p}}(\lambda_p)\right) \\ &\oplus \left(\bigoplus_{i=2k+1}^{r_{p+1}} \mathcal{J}_{n_{i,p+1}}(\lambda_{p+1})\right) \oplus \cdots \oplus \left(\bigoplus_{i=2k+1}^{r_m} \mathcal{J}_{n_{i,m}}(\lambda_m)\right) \oplus \widetilde{\mathcal{J}}, \end{aligned}$$

where $\widetilde{\mathcal{J}}$ has simple nonreal eigenvalues with positive imaginary part that are different from any of the eigenvalues of A.

Proof. Without loss of generality we can assume that A is already equal to its Jordan canonical form $A_1 \oplus \overline{A_1}$. We then aim to apply Theorem 3.2 for the case $\kappa = 2k$ and $\ell = 1$, and for the function $\widehat{P} = \widehat{U}\widehat{V}^T$ which is interpreted as a function of the $r = 8nk$ real and imaginary parts of the entries of U_1, U_2, V_1 and V_2, where

$$\widehat{U} = \begin{bmatrix} U_1 & U_2 \\ -\overline{U_2} & \overline{U_1} \end{bmatrix} \quad \text{and} \quad \widehat{V} = \begin{bmatrix} V_1 & V_2 \\ -\overline{V_2} & \overline{V_1} \end{bmatrix}.$$

Hence, it remains to find for each eigenvalue λ_j and any $\widetilde{\varepsilon} > 0$ a particular choice of matrices $\widehat{U}_0, \widehat{V}_0 \in \mathcal{Q}_{n,k}$ with $\|\widehat{U}_0\|, \|\widehat{V}_0\| < \widetilde{\varepsilon}$ such that $\widehat{A} + \widehat{U}_0\widehat{V}_0^T$ satisfies parts 3a) and 3b) of Theorem 3.2. Then Theorem 3.2 yields the existence of a generic set $\Omega_j \subseteq \mathcal{Q}_{n,k} \times \mathcal{Q}_{n,k}$ (canonically identified with a subset of $\mathbb{R}^{8nk}$) such that for all $(\widehat{U}, \widehat{V}) \in \Omega_j$ the parts 3a) and 3b) of Theorem 3.2 are satisfied. Taking then the intersection $\Omega = \Omega_1 \cap \cdots \cap \Omega_m$ yields the desired generic set. Concerning the matrix $\widetilde{\mathcal{J}}$, note that since $\widehat{A} + \widehat{U}\widehat{V}^T$ is in $\mathcal{Q}_{n,n}$, the set of the new simple eigenvalues that are not eigenvalues of $\widehat{A}$ does not contain real eigenvalues (as those would have even multiplicity) and is necessarily symmetric with respect to the real line. We can thus order the new eigenvalues in the Jordan canonical form in such a way that all eigenvalues with positive imaginary part are collected in $\widetilde{\mathcal{J}}$. In the following we will consider two cases.

Case 1: $k = 1$. We first consider the subcase that λ_j is real, that is $j \in \{1, \dots, p\}$. Let $B_1 \in \mathbb{C}^{n,n}$ be the matrix that has zero entries everywhere, except for the position $(a_j + n_{1,j}, a_j + 1)$ where the entry $\varepsilon \cdot e^{\mathrm{i}\varphi}$. Here, ε is a sufficiently small positive number, φ satisfies $0 < \varphi < \frac{\pi}{n_{1,j}}$, and we have

$$a_j = \sum_{s=1}^{j-1} \sum_{i=1}^{r_s} n_{i,s}.$$

Thus, in $A_1 + B_1$ only a single Jordan block of partial multiplicity $n_{1,j}$ associated with the eigenvalue λ_j of A_1 is perturbed by the rank-one perturbation B_1 as

$$\begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ \varepsilon \cdot e^{\mathrm{i}\varphi} & & & \lambda_j \end{bmatrix}.$$

The characteristic polynomial χ of this block in the independent variable t is given by $(t-\lambda_j)^{n_{1,j}} - \varepsilon \cdot e^{\mathrm{i}\varphi}$ and thus its roots are the vertices of a regular polygon on a circle of radius $\varepsilon^{1/n_{1,j}}$ with center λ_j. Since $0 < \varphi < \frac{\pi}{n_{1,j}}$ the set of roots of χ is conjugate-free. In particular, all roots of χ are nonreal. Furthermore, choosing ε small enough guarantees that all roots of χ are distinct from each of the eigenvalues of A.

Now let $\widehat{B} = B_1 \oplus \overline{B_1}$. Then $\widehat{B} \in \mathcal{Q}_{n,n}$ has rank two, so by Proposition 2.2 there exist rank two matrices $\widehat{U}_0, \widehat{V}_0 \in \mathcal{Q}_{n,1}$ such that $\widehat{B} = \widehat{U}_0\widehat{V}_0^T$. Then it is easy to check that the Jordan canonical form of $\widehat{A}+\widehat{U}_0\widehat{V}_0^T = (A_1+B_1)\oplus(\overline{A_1}+\overline{B_1})$ corresponding to λ_j is as desired. In particular, by the choice of ε and φ above and since the eigenvalues of $\overline{A_1}+\overline{B_1}$ are the conjugates of those of A_1+B_1, all eigenvalues of $\widehat{A}+\widehat{B}$ that are not eigenvalues of $\widehat{A}$ are simple and nonreal.

Next, consider an eigenvalue λ_j with $j \geq p$. If there is just one Jordan block associated with λ_j, then we can proceed as for real eigenvalues, where here we can choose $\varphi = 0$. Then λ_j will not be an eigenvalue of the perturbed matrix. If the geometric multiplicity of λ_j is at least 2, then consider the submatrix

$$S := \begin{bmatrix} \mathcal{J}_{n_{1,j}}(\lambda_j) & 0 & 0 & 0 \\ 0 & \mathcal{J}_{n_{2,j}}(\lambda_j) & 0 & 0 \\ 0 & 0 & \mathcal{J}_{n_{1,j}}(\overline{\lambda_j}) & 0 \\ 0 & 0 & 0 & \mathcal{J}_{n_{2,j}}(\overline{\lambda_j}) \end{bmatrix}.$$

We aim to find a rank two perturbation of S such that all eigenvalues of the perturbed matrix are simple and nonreal. To achieve this, we use an idea from pole-placement in control theory. Consider the submatrix

$$S_1 = \begin{bmatrix} \mathcal{J}_{n_{1,j}}(\lambda_j) & 0 \\ 0 & \mathcal{J}_{n_{2,j}}(\overline{\lambda_j}) \end{bmatrix}$$

of S. Since λ_j is nonreal, this matrix is nonderogatory and thus similar to the companion form of its characteristic polynomial, i.e., there exists a nonsingular T such that

$$S_1 = T \begin{bmatrix} 0 & & & -\beta_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -\beta_{\nu-2} \\ & & 1 & -\beta_{\nu-1} \end{bmatrix} T^{-1},$$

where $\beta_0, \ldots, \beta_{\nu-1}$ are the coefficients of the polynomial

$$\chi = (t-\lambda_j)^{n_{1,j}}(t-\overline{\lambda_j})^{n_{2,j}} = t^\nu + \beta_{\nu-1}t^{\nu-1} + \cdots + \beta_1 t + \beta_0,$$

and where we used the abbreviation $\nu = n_{1,j} + n_{2,j}$. Now choose ν values $\mu_1, \dots, \mu_\nu$ such that $n_{1,j}$ of them are close to λ_j and the remaining $n_{2,j} = \nu - n_{1,j}$ are close to $\overline{\lambda_j}$, and such that the set $\{\mu_1, \dots, \mu_\nu\}$ is conjugate-free and does not intersect the spectrum of A. Let $\gamma_0, \dots, \gamma_{\nu-1}$ be the coefficients of the polynomial

$$\prod_{i=1}^{\nu}(t - \mu_i) = t^\nu + \gamma_{\nu-1}t^{\nu-1} + \cdots + \gamma_1 t + \gamma_0.$$

Then setting

$$\widetilde{B} := \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} := T \begin{bmatrix} 0 & & & \beta_0 - \gamma_0 \\ 0 & \ddots & & \vdots \\ & \ddots & 0 & \beta_{\nu-2} - \gamma_{\nu-2} \\ & & 0 & \beta_{\nu-1} - \gamma_{\nu-1} \end{bmatrix} T^{-1}$$

with $B_{ik} \in \mathbb{C}^{n_{i,j}, n_{k,j}}$, $i, k \in \{1, 2\}$, we obtain that $S_1 + \widetilde{B}$ has exactly the eigenvalues $\mu_1, \dots, \mu_\nu \in \mathbb{C}^+$. In particular, the eigenvalues of $S_1 + \widetilde{B}$ are conjugate-free (and nonreal). Thus, setting

$$\widehat{B} = \begin{bmatrix} B_{11} & 0 & 0 & B_{12} \\ 0 & \overline{B_{22}} & \overline{B_{21}} & 0 \\ 0 & \overline{B_{12}} & \overline{B_{11}} & 0 \\ B_{21} & 0 & 0 & B_{22} \end{bmatrix},$$

we find that $\widehat{B} \in \mathcal{Q}_{\nu,\nu}$ has rank two and the eigenvalues of $S + \widehat{B}$ are given by $\mu_1, \dots, \mu_\nu, \overline{\mu_1}, \dots, \overline{\mu_\nu}$. In particular, the eigenvalues of $S + \widehat{B}$ are all simple. By choosing the values $\mu_1, \dots, \mu_\nu$ sufficiently close to the values λ_j and $\overline{\lambda_j}$, respectively, we can guarantee that the coefficients γ_i can be chosen to be arbitrarily close to the coefficients β_i for $i = 1, \dots, \nu$, and thus, $\widehat{B}$ can be chosen to be of arbitrarily small norm.

Case 2: $k > 1$. By using the result for the already proved case 1, we can find a sequence of k matrices $\widehat{U}_i\widehat{V}_i^\top$ with $\widehat{U}_i, \widehat{V}_i \in \mathcal{Q}^{n,1}$ being of arbitrarily small norm such that in

$$\widehat{A} + \widehat{U}_1\widehat{V}_1^\top + \cdots + \widehat{U}_k\widehat{V}_k^\top$$

the change in the Jordan canonical form with respect to the eigenvalue λ from the matrix $\widehat{A} + \widehat{U}_1\widehat{V}_1^\top + \cdots + \widehat{U}_{i-1}\widehat{V}_{i-1}^\top$ to $\widehat{A} + \widehat{U}_1\widehat{V}_1^\top + \cdots + \widehat{U}_i\widehat{V}_i^\top$ is that the largest two Jordan block associated with λ disappear from the Jordan canonical form while all smaller ones remain (or λ is no longer an eigenvalue if there were at most two Jordan blocks left in the previous step), and all newly generated eigenvalues are simple. In particular, $\widehat{A} + \widehat{U}_1\widehat{V}_1^\top + \cdots + \widehat{U}_k\widehat{V}_k^\top$ then has the Jordan canonical form as claimed in the theorem. If

$$\widehat{U}_i = \begin{bmatrix} u_{i1} & u_{i2} \\ -\overline{u_{i2}} & \overline{u_{i1}} \end{bmatrix} \quad \text{and} \quad \widehat{V}_i = \begin{bmatrix} v_{i1} & v_{i2} \\ -\overline{v_{i2}} & \overline{v_{i1}} \end{bmatrix}$$

then choosing

$$\widehat{U} = \begin{bmatrix} u_{11} & \cdots & u_{k1} & u_{12} & \cdots & u_{k2} \\ -\overline{u_{12}} & \cdots & -\overline{u_{k2}} & \overline{u_{11}} & \cdots & \overline{u_{k1}} \end{bmatrix} \in \mathcal{Q}_{n,k}$$

and

$$\widehat{V} = \left[\begin{array}{cccccc} v_{11} & \cdots & v_{k1} & v_{12} & \cdots & v_{k2} \\ -\overline{v_{12}} & \cdots & -\overline{v_{k2}} & \overline{v_{11}} & \cdots & \overline{v_{k1}} \end{array}\right] \in \mathcal{Q}_{n,k}$$

gives the desired example, because $\widehat{U}\widehat{V}^{\top} = \widehat{U}_1\widehat{V}_1^{\top} + \cdots + \widehat{U}_k\widehat{V}_k^{\top}$ as one can easily check. ☐

5. Rank one perturbations of quaternion matrices. As a direct application of the main theorem in the previous section, we immediately obtain the following theorem that describes the generic change in the Jordan structure of a given quaternion matrix under a generic rank-k perturbation.

THEOREM 5.1. *Let A be an $n \times n$ quaternion matrix, and let its Jordan canonical form be given by*

$$\left(\bigoplus_{i=1}^{r_1} \mathcal{J}_{n_{i,1}}(\lambda_1)\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{r_p} \mathcal{J}_{n_{i,p}}(\lambda_p)\right)$$
$$\oplus \left(\bigoplus_{i=1}^{r_{p+1}} \mathcal{J}_{n_{i,p+1}}(\lambda_{p+1})\right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{r_m} \mathcal{J}_{n_{i,m}}(\lambda_m)\right)$$

where $\lambda_1, \ldots, \lambda_p$ are real, and $\lambda_{p+1}, \ldots, \lambda_m$ are non-real and in the open upper half plane, and where for each $j = 1, \ldots, m$ the partial multiplicities are ordered in decreasing order: $n_{1,j} \geq \cdots \geq n_{r_j,j}$.

Then there exists a generic set $\Omega \subseteq \mathbb{H}^{n,k} \times \mathbb{H}^{n,k}$ such that for each $(U, V) \in \Omega$ the Jordan canonical form of $A + UV^T$ is given by

$$\left(\bigoplus_{i=k+1}^{r_1} \mathcal{J}_{n_{i,1}}(\lambda_1)\right) \oplus \cdots \oplus \left(\bigoplus_{i=k+1}^{r_p} \mathcal{J}_{n_{i,p}}(\lambda_p)\right)$$
$$\oplus \left(\bigoplus_{i=2k+1}^{r_{p+1}} \mathcal{J}_{n_{i,p+1}}(\lambda_{p+1})\right) \oplus \cdots \oplus \left(\bigoplus_{i=2k+1}^{r_m} \mathcal{J}_{n_{i,m}}(\lambda_m)\right) \oplus \widetilde{\mathcal{J}},$$

where $\widetilde{\mathcal{J}}$ has simple non-real eigenvalues not equal to any of the eigenvalues of A.

Proof. The proof is based on reduction to the complex structured case treated in the previous section. The matrix $\omega(A)$ is in the class $\mathcal{Q}_{n,n}$ and for $U, V \in \mathbb{H}^{n,k}$ we have $\omega(U), \omega(V) \in \mathcal{Q}_{n,k}$. Moreover, genericity of subsets $\mathbb{H}^{n,k} \times \mathbb{H}^{n,k}$ with respect to the four real components of each matrix pair means exactly the same as genericity of the corresponding subset $\mathcal{Q}_{n,k} \times \mathcal{Q}_{n,k}$ with respect to the real and imaginary parts of each matrix pair. Also, we have $\omega(A + UV^T) = \omega(A) + \omega(U)\omega(V)^T$ by (2.2).

Observe that the Jordan canonical form given in this theorem leads to the Jordan canonical form of the matrix $\omega(A)$ as given in Theorem 4.1. Applying the results of that theorem, and translating back via ω^{-1} we see that for a non-real eigenvalue λ of A the partial multiplicities of $A+UV^T$ corresponding to λ are given by the $(2k+1)$st and following partial multiplicities of A corresponding to λ (if any), while for a real eigenvalue λ of A the partial multiplicities of $A + UV^T$ corresponding to λ are given by the $(k + 1)$st and following partial multiplicities of A corresponding to λ (if any). So non-real eigenvalues lose the largest $2k$ partial multiplicities, but real eigenvalues only the largest k ones. In addition, eigenvalues of $A+UV^T$ which are not eigenvalues of A are simple and non-real. ☐

6. Rank-k perturbations of Hermitian quaternion matrices. In this section, we will focus on Hermitian quaternion matrices, i.e., matrices $A \in \mathbb{H}^{n,n}$ satisfying $A^* = A$. In that case, the corresponding matrix $\omega(A) \in \mathcal{Q}_{n,n}$ is a complex Hermitian matrix, and consequently all its eigenvalues are real, and all its partial multiplicities are equal to one. Since it is a matrix in $\mathcal{Q}_{n,n}$, the geometric multiplicity of each eigenvalue of $\omega(A)$ is even.

While the result on the generic behavior of Hermitian matrices in $\mathcal{H}_{n,n}$ under arbitrary perturbations still follows from Theorem 5.1, it is a natural question to ask whether this remains true under *structure-preserving* transformations. Observe that a rank-k Hermitian quaternion perturbation of A takes the form $A+B$ with $B \in \mathbb{H}^{n,n}$ being Hermitian and of rank k. Thus, in $\mathcal{Q}_{n,n}$ we should be considering $\omega(A)+\omega(B)$ with $\omega(B)$ being Hermitian of rank $2k$.

Note that a-priori this is a restriction on the type of rank-$2k$ Hermitian perturbations we are allowed to make to $\omega(A)$, because our perturbation matrix does not only have to be Hermitian, but also to be in the range of ω. Indeed, for a $2n \times 2n$ Hermitian matrix of rank $2k$ it is possible that the number of positive (or the number of negative) eigenvalues is odd, but a matrix in $\mathcal{Q}_{n,n}$ always has eigenvalues with even geometric multiplicities. We therefore start our investigation by characterizing the set of Hermitian matrices of rank $2k$ that are in the range of ω. We do this by using the following results that generalize the well-known results on the spectral decomposition and Sylvester's Law of Inertia to the case of Hermitian quaternion matrices.

PROPOSITION 6.1 ([20, Theorem 5.3.6.(c) and Theorem 4.1.6.(a)]). *Let $A \in \mathbb{H}^{n,n}$ be Hermitian. Then the following statements hold.*

1. *There exists a unitary matrix $Q \in \mathbb{H}^{n,n}$ such that*
$$Q^*AQ = \mathrm{diag}(\alpha_1, \dots, \alpha_n),$$
where $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.
2. *There exists an invertible matrix $S \in \mathbb{H}^{n,n}$ and uniquely defined integers π, ν such that*
$$S^*AS = \begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.1}$$

Note that 1) confirms our observation at the beginning of this section that the eigenvalues of a Hermitian quaternion matrix are all real and semisimple. Part 2) immediately yields a characterization of Hermitian quaternion matrices of rank k.

COROLLARY 6.2. *Let $A \in \mathbb{H}^{n,n}$ be a Hermitian quaternion matrix of rank k. Then there exists an integer π and a matrix $U \in \mathbb{H}^{n,k}$ of full rank such that*
$$A = U\Sigma U^*, \quad \text{where } \Sigma = \begin{bmatrix} I_\pi & 0 \\ 0 & -I_{k-\pi} \end{bmatrix}. \tag{6.2}$$

Proof. This follows immediately from part 2) of Proposition 6.1 by noting that A is of rank k if and only if $\pi + \nu = k$ in (6.1). The result then follows from letting U be the part of S^* that consists of its first k columns. □

COROLLARY 6.3. *Let $\widehat{A} \in \mathcal{Q}_{n,n}$ be Hermitian and of rank $2k$. Then there exists a matrix $\widehat{U} \in \mathcal{Q}_{n,k}$ of full rank $2k$ and a diagonal matrix $\widehat{\Sigma} \in \mathcal{Q}_{n,n}$ satisfying $\widehat{\Sigma}^2 = I_{2n}$ such that $\widehat{A} = \widehat{U}\widehat{\Sigma}\widehat{U}^*$.*

Proof. The result follows immediately by using Corollary 6.2 on $A := \omega^{-1}(\widehat{A})$ to obtain a decomposition $A = U\Sigma U^*$ as in (6.2). Then applying ω yields the desired decomposition with $\widehat{U} = \omega(U)$ and $\widehat{\Sigma} = \omega(\Sigma) = \Sigma \oplus \Sigma$. $\square$

We now obtain the following result on generic rank-$2k$ perturbations of Hermitian matrices in $\mathcal{Q}_{n,n}$.

THEOREM 6.4. *Let $\widehat{A} \in \mathcal{Q}_{n,n}$ be Hermitian, and let $\lambda_1, \ldots, \lambda_p$ be the pairwise distinct (necessarily real) eigenvalues of $\widehat{A}$, with multiplicities $2r_1, \ldots, 2r_p$ (where necessarily the algebraic multiplicities coincide with the geometric multiplicities). Furthermore, let $\widehat{\Sigma} \in \mathcal{Q}_{k,k}$ be diagonal such that $\widehat{\Sigma}^2 = I_{2k}$. Then there exists a generic set $\Omega \subseteq \mathcal{Q}_{n,k}$ such that for all $\widehat{U} \in \Omega$ the following statements hold:*

1. *For all $j \in \{1, \ldots, p\}$ the eigenvalue λ_j of $\widehat{A} + \widehat{U}\widehat{\Sigma}\widehat{U}^*$ has multiplicity $2r_j - 2k$ if $r_j > k$, and if $r_j \leq k$, then λ_j is not an eigenvalue of $\widehat{A} + \widehat{U}\widehat{\Sigma}\widehat{U}^*$.*
2. *All eigenvalues of $\widehat{A} + \widehat{U}\widehat{\Sigma}\widehat{U}^*$ which are not eigenvalues of $\widehat{A}$ have multiplicity precisely two.*

Proof. We will apply Theorem 3.2 for the case $\kappa = 2k$ and $\ell = 2$, and for the function $P = \widehat{U}\widehat{\Sigma}\widehat{U}^*$ which is a polynomial in the $4nk$ real and imaginary part of the entries of $U_1, U_2 \in \mathbb{C}^{n,k}$ when we write

$$\widehat{U} = \begin{bmatrix} U_1 & U_2 \\ -\overline{U_2} & \overline{U_1} \end{bmatrix}.$$

Note that we are indeed in the case $\ell = 2$, because each eigenvalue of a Hermitian matrix in $\mathcal{Q}_{n,k}$ has even multiplicity. Thus, it remains to find for each eigenvalue λ_j a particular matrix $\widehat{U}_0 \in \mathcal{Q}_{n,k}$ such that $\widehat{A} + \widehat{U}_0\widehat{\Sigma}\widehat{U}_0^*$ satisfies parts 3a) and 3b) of Theorem 3.2 and such that the norm of $\widehat{U}_0$ can be chosen to be arbitrarily small.

To this end we may assume that $k = 1$, because as in the proof of Theorem 4.1 an example in the case $k > 1$ can be constructed via k consecutive rank-2 perturbations from $\mathcal{Q}_{n,k}$. Furthermore, we may assume without loss of generality that $\widehat{A}$ has the form

$$\widehat{A} = \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_n),$$

where $\alpha_1 = \lambda_j$. Then choosing the value $c \in \mathbb{R} \setminus \{0\}$ such that $\alpha_1 + c$ and $\alpha_1 - c$ are different from all the other eigenvalues of $\widehat{A}$, setting $u_0 := ce_1 + ce_{n+1} \in \mathbb{Q}_{n,1}$, and considering $\widehat{A} + u_0\widehat{\Sigma}u_0^* = \widehat{A} \pm u_0u_0^*$ yields the desired example as c can clearly be chosen such that $\|u_0\|$ has arbitrarily small norm. $\square$

An immediate consequence is the following result on rank-k perturbations of Hermitian quaternion matrices which will be proved completely analogously to Theorem 5.1.

THEOREM 6.5. *Let $A \in \mathbb{H}_{n,n}$ be a Hermitian and let $\lambda_1, \ldots, \lambda_p$ its pairwise distinct (necessarily real) eigenvalues with multiplicities $r_1, \ldots, r_p$ (where algebraic and geometric multiplicities coincide). Furthermore, let $\Sigma = I_\pi \oplus (-I_{k-\pi})$. Then there exists a generic set $\Omega \subseteq \mathbb{H}^{n,k}$ such that for all $U \in \Omega$ the following statements hold:*

1. *For all $j \in \{1, \ldots, p\}$ the eigenvalue λ_j of $A + U\Sigma U^*$ has multiplicity $r_j - k$ if $r_j > k$, and if $r_j \leq k$, then λ_j is not an eigenvalue of $A + U\Sigma U^*$.*
2. *All eigenvalues of $A + U\Sigma U^*$ which are not eigenvalues of $\widehat{A}$ are simple.*

7. Rank-k perturbations of skew-Hermitian quaternion matrices. In this section, we will focus on skew-Hermitian quaternion matrices, i.e., quaternion matrices $A \in \mathbb{H}^{n,n}$ satisfying $A^* = -A$. Again, the corresponding matrix $\omega(A) \in \mathcal{Q}_{n,n}$ will have the corresponding structure, i.e., it will be skew-Hermitian. It is important to note that a common trick that is used in complex matrix algebra is no longer available when dealing with the quaternions: while a complex Hermitian matrix becomes skew-Hermitian when it is multiplied by the imaginary unit i and vice versa, this need not be the case for a Hermitian matrix $A \in \mathbb{H}^{n,n}$, because we obtain $(\mathsf{i}A)^* = A^*\mathsf{i}^* = -A\mathsf{i}$ and the matrix $A\mathsf{i}$ may be different from $\mathsf{i}A$ as the following example shows.

EXAMPLE 7.1. Consider the matrix

$$A = \begin{bmatrix} 1 & \mathsf{j} \\ -\mathsf{j} & 1 \end{bmatrix}.$$

Then $A^* = A$ is Hermitian, but $\mathsf{i}A$ is not skew-Hermitian as

$$(\mathsf{i}A)^* = \begin{bmatrix} \mathsf{i} & \mathsf{k} \\ -\mathsf{k} & \mathsf{i} \end{bmatrix}^* = \begin{bmatrix} -\mathsf{i} & \mathsf{k} \\ -\mathsf{k} & -\mathsf{i} \end{bmatrix} \neq \begin{bmatrix} -\mathsf{i} & -\mathsf{k} \\ \mathsf{k} & -\mathsf{i} \end{bmatrix} = -\mathsf{i}A.$$

When Theorem 6.1 is adapted to the skew-Hermitian case one should have in mind that by the previous example the transition from Hermitian matrices to skew-Hermitian matrices is not a trivial task. Nevertheless, observe that part (a) in the following theorem looks exactly like the corresponding results on complex matrices that can be obtained from the corresponding result on Hermitian matrices via the "multiplying with i"-trick. On the other hand, comparing the parts (b) we see that in the skew-Hermitian case "*Sylvester's Law of Inertia*" turns out to be substantially different from the corresponding result in the case of Hermitian quaternion matrices.

PROPOSITION 7.2 ([20, Theorem 5.3.6.(d) and Theorem 4.1.6.(b)]). *Let $A \in \mathbb{H}^{n,n}$ be skew-Hermitian. Then the following statements hold.*

1. *There exists a unitary matrix $Q \in \mathbb{H}^{n,n}$ such that*

$$Q^*AQ = \operatorname{diag}(\mathsf{i}\alpha_1, \ldots, \mathsf{i}\alpha_n),$$

   where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

2. *There exists an invertible matrix $S \in \mathbb{H}^{n,n}$ and a uniquely defined integer r such that*

$$S^*AS = \begin{bmatrix} \mathsf{i}I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.1}$$

As a corollary of Theorem 7.2, we immediately obtain the following characterizations of skew-Hermitian matrices of rank k in $\mathbb{H}^{n,n}$ or rank $2k$ in $\mathcal{Q}_{n,n}$.

COROLLARY 7.3. *Let $A \in \mathbb{H}^{n,n}$ be a skew-Hermitian quaternion matrix of rank k. Then there exists a matrix $U \in \mathbb{H}^{n,k}$ of full rank such that $A = U(\mathsf{i}I_k)U^*$.*

COROLLARY 7.4. *Let $\widehat{A} \in \mathcal{Q}^{n,n}$ be a skew-Hermitian quaternion matrix of rank $2k$. Then there exists a matrix $\widehat{U} \in \mathcal{Q}^{n,k}$ of full rank $2k$ such that*

$$\widehat{A} = \widehat{U}\omega(\mathsf{i}I_k)\widehat{U}^* = \widehat{U} \begin{bmatrix} \mathsf{i}I_k & 0 \\ 0 & -\mathsf{i}I_k \end{bmatrix} \widehat{U}^*.$$

It is easily seen that the "multiplying with i"-trick will also not work in the set $\mathcal{Q}_{n,n}$ as this set is only closed under scalar multiplication with real numbers. We therefore need an analogue of Theorem 6.4 for the case of skew-Hermitian matrices.

THEOREM 7.5. *Let $\widehat{A} \in \mathcal{Q}_{n,n}$ be skew-Hermitian, and let $\lambda_1, \ldots, \lambda_p$ be the pairwise distinct (necessarily purely imaginary) eigenvalues of $\widehat{A}$, with multiplicities $r_1, \ldots, r_p$ (where necessarily the algebraic multiplicities coincide with the geometric multiplicities, and r_j is even if $\lambda_j = 0$). Then there exists a generic set $\Omega \subseteq \mathcal{Q}_{n,k}$ such that for all $\widehat{U} \in \Omega$ the following statements hold for all $j \in \{1, \ldots, p\}$:*

1. *If $\lambda_j \neq 0$, then λ_j is an eigenvalue of $\widehat{A} + \widehat{U}(\mathrm{i}I_k)\widehat{U}^*$ with multiplicity $r_j - 2k$ if $r_j > k$, and if $r_j \leq k$, then λ_j is not an eigenvalue of $\widehat{A} + \widehat{U}(\mathrm{i}I_k)\widehat{U}^*$.*
2. *If $\lambda_j = 0$, then λ_j is an eigenvalue of $\widehat{A} + \widehat{U}(\mathrm{i}I_k)\widehat{U}^*$ with (necessarily even) multiplicity $r_j - 2k$ if $r_j > 2k$, and if $r_j \leq 2k$, then λ_j is not an eigenvalue of $\widehat{A} + \widehat{U}(\mathrm{i}I_k)\widehat{U}^*$.*
3. *All eigenvalues of $\widehat{A} + \widehat{U}(\mathrm{i}I_k)\widehat{U}^*$ which are not eigenvalues of $\widehat{A}$ are nonzero and simple.*

Proof. We will apply Theorem 3.2 for the case $\kappa = 2k$ and $\ell = 1$, and for the function $P = \widehat{U}(\mathrm{i}I_k)\widehat{U}^*$ which is a polynomial in the $4nk$ real and imaginary part of the entries of $U_1, U_2 \in \mathbb{C}^{n,k}$ when we write

$$\widehat{U} = \begin{bmatrix} U_1 & U_2 \\ -\overline{U_2} & \overline{U_1} \end{bmatrix}.$$

Indeed, note that in contrast to the Hermitian case we have $\ell = 1$ instead of $\ell = 2$. The remainder of the proof uses the same strategy as the proof of Theorem 6.4. Thus, we may again assume that $k = 1$ and that A is diagonal, i.e.,

$$A = \operatorname{diag}(\mathrm{i}\alpha_1, \ldots, \mathrm{i}\alpha_n, -\mathrm{i}\alpha_1, \ldots, -\mathrm{i}\alpha_n),$$

where $\mathrm{i}\alpha_1 = \cdots = \mathrm{i}\alpha_{r_j} = \lambda_j$ if $\lambda_j \neq 0$, or $\mathrm{i}\alpha_1 = \cdots = \mathrm{i}\alpha_{r_j/2} = 0$ if $\lambda_j = 0$. It remains to find one particular matrix $\widehat{U} \in \mathcal{Q}_{n,1}$ such that its norm can be chosen to be arbitrarily small and such that 3a) and 3b) of Theorem 3.2 are satisfied. Constructing such an example is the part where the proof of this theorem will differ substantially from the corresponding part of the proof of Theorem 6.4. We will distinguish two cases.

Case 1: $\lambda_j = 0$. In this case, let

$$\widehat{U} = \begin{bmatrix} ce_1 & \mathrm{i}ce_1 \\ \mathrm{i}ce_1 & ce_1 \end{bmatrix} \in \mathcal{Q}_{n,1},$$

where $c \in \mathbb{R}$. Then we obtain

$$\widehat{P} := \widehat{U} \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} \widehat{U}^* = \begin{bmatrix} 0 & 2c^2 e_1 e_1^T \\ -2c^2 e_1 e_1^T & 0 \end{bmatrix},$$

and the eigenvalues of $\widehat{A} + \widehat{P}$ are given by the values $\pm\mathrm{i}\alpha_2, \ldots, \pm\mathrm{i}\alpha_n$ and in addition by the eigenvalues of

$$\begin{bmatrix} 0 & 2c^2 \\ -2c^2 & 0 \end{bmatrix}$$

which are $\pm\mathrm{i}2c^2$. Thus, if the sufficiently small c is chosen such that $2c^2$ is different from $\pm\alpha_2, \ldots, \pm\alpha_n$ then we have found our example such that 3a) and 3b) of Theorem 3.2 are satisfied.

Case 2: $\lambda_j \neq 0$. If $r_j = 1$, then choosing the same perturbation as above will produce a perturbed matrix with the eigenvalues $\pm\mathrm{i}\alpha_2, \ldots, \mathrm{i}\alpha_n$ and $\pm\mathrm{i}\sqrt{\alpha_1^2 + 4c^4}$ which shows that λ_j is not an eigenvalue of the perturbed matrix. Furthermore, choosing c appropriately guarantees that the newly generated eigenvalues are all simple. Thus, let $r_j > 1$ which implies $\mathrm{i}\alpha_2 = \lambda_j$. Now choose $c > 0$ sufficiently small such that in particular we have $\alpha_1^2 - c^2 > 0$ and set

$$\widehat{P} = \left[\begin{array}{cc|cc} \mathrm{i}ce_1 & -\mathrm{i}ce_2 & ce_2 & ce_1 \\ \hline -ce_2 & -ce_1 & -\mathrm{i}ce_1 & \mathrm{i}ce_2 \end{array}\right].$$

Then we have $\widehat{P} \in \mathcal{Q}_{n,n}$ and in addition $\widehat{P}$ is skew-Hermitian and of rank 2. Note that in $\widehat{A} + \widehat{P}$ only the 4×4 submatrix is perturbed that consists of the rows and columns with indices $1, 2, n+1, n+2$ and which is given by

$$\begin{bmatrix} \mathrm{i}(\alpha_1 + c) & 0 & 0 & c \\ 0 & \mathrm{i}(\alpha_1 - c) & c & 0 \\ 0 & -c & -\mathrm{i}(\alpha_1 + c) & 0 \\ -c & 0 & 0 & -\mathrm{i}(\alpha_1 - c) \end{bmatrix}.$$

Since the eigenvalues of this submatrix are the four pairwise distinct complex numbers $\mathrm{i}c \pm \sqrt{\alpha_1^2 - c^2}$ and $-\mathrm{i}c \pm \sqrt{\alpha_1^2 - c^2}$ which are clearly also mutually distinct from the values $\pm\mathrm{i}\alpha_3, \ldots, \pm\mathrm{i}\alpha_n$ if c had been chosen sufficiently small, we have constructed our desired example that can also be constructed to be of arbitrarily small norm.

Note that the additional statement that the newly generated eigenvalues are nonzero is implied by their simplicity, since the eigenvalue zero must have even multiplicity as a real eigenvalue. $\square$

As a direct consequence of Theorem 7.5, we obtain the following analogue of Theorem 6.5 and its proof is again analogous to the one of Theorem 5.1.

THEOREM 7.6. *Let $A \in \mathbb{H}_{n,n}$ be a skew-Hermitian quaternion matrix and let $\lambda_1, \ldots, \lambda_p$ its pairwise distinct (necessarily purely imaginary) eigenvalues with multiplicities $r_1, \ldots, r_p$. Then there exists a generic set $\Omega \subseteq \mathbb{H}^{n,k}$ such that for all $U \in \Omega$ the following statements hold for all $j \in \{1, \ldots, p\}$:*

1. *If $\lambda_j \neq 0$, then λ_j is an eigenvalue of $A + U(\mathrm{i}I_k)U^*$ with multiplicity $r_j - 2k$ if $r_j > 2k$, and if $r_j \leq 2k$, then λ_j is not an eigenvalue of $A + U(\mathrm{i}I_k)U^*$.*
2. *If $\lambda_j = 0$, then λ_j is an eigenvalue of $A + U(\mathrm{i}I_k)U^*$ with multiplicity $r_j - k$ if $r_j > k$, and if $r_j \leq k$, then λ_j is not an eigenvalue of $A + U(\mathrm{i}I_k)U^*$.*
3. *All eigenvalues of $A + U(\mathrm{i}I_k)U^*$ which are not eigenvalues of $\widehat{A}$ are nonzero and simple.*

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