Consistency Decision I
Self-Inconsistency

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Abstract

The consistency formula for gödelian Arithmetics **T** can be stated as free-variable predicate in terms of the categorical theory **PR** of primitive recursive functions/maps/predicates. Free-variable p. r. predicates are decidable by gödelian theory **T**, key result, built on recursive evaluation of p. r. map codes and *soundness* of that evaluation into theories **T** : internal, *arithmetised* p. r. map code equality is evaluated into map equality of **T**. In particular the free-variable p. r. consistency predicate of **T** is decided by **T**. Therefore, by Gödel's second incompleteness theorem, gödelian quantified Arithmetics **T** turn out to be self-inconsistent.

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Contents

Introduction

The formula which expresses in a gödelian (quantified) arithmetical theory **T** the consistency of **T** can be stated as free-variable primitive recursive predicate in terms of the categorical theory **PR** of primitive recursive ("p. r.") maps/predicates.

Free-variable p. r. predicates are decided by a quantified arithmetical theory **T** satisfying an axiom of *finite descent* – of chains in linearly ordered semiring $\mathbb{N}[\omega]$ of polynomials in one indeterminate ω (thought "big"), coefficients in $\mathbb{N}$.[1]

This *Decision Theorem* is our key result. It builds on recursive, *Complexity Controlled Iterative* evaluation of primitive recursive map codes and *Stimmigkeit*/soundness of that evaluation into **T** : Internal, *arithmetised* primitive map code equality is evaluated into map equality of theory **T**.

In particular the free-variable primitive recursive *consistency predicate* $\mathrm{Con}_{\mathbf{T}}$ of theory **T** is decided by **T**. This decision gives, by Gödel's second incompleteness theorem, self-inconsistency of theory **T** as final result.

The Appendix is to give a detailed construction and proof for resolution of double recursive PR map code evaluation into a CCI: Complexity Controlled Iteration with complexity values in $\mathbb{N}[\omega]$ which has only finite descending chains: is an *ordinal* in terms of first order **set** theory, subsystem of Principia Mathematica **PM** "und verwandte Systeme" as in particular Zermelo-Fraenkel **set** theory **ZF.**

Theory basis both for present *negative* approach to classical foundations as well as for *self-consistency* of (recursive) *itera-*

[1] **set** theories satisfy *finite descent* since ω^ω there is an *ordinal*

tive descent theory $\pi\mathbf{R}$ strengthening **PR** (within the monography *Arithmetical Foundations*[2]) is exposition of fully formalised free-variables cartesian categorical theory **PR** of primitive recursion.[3]

It comes in two levels: First Categorical cartesian language **CA** generated over a (proto) natural numbers object $\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ – zero and successor functions – and second, built on this, axioms and fundamental theorems making $\mathbb{N}$ into an NNO (*Natural Numbers Object*) in the sense of availability of endo map iteration and the schemata of primitive recursion proper: Theory **PR** of primitive recursion.

These explicit axioms, lemmata and theorem are stated for reference in later sections of present paper as well as for the corresponding sections of *positive* second paper.

1 Cartesian language CA

1.1 Fundamental object language symbols

$$\{\mathbb{1}, \mathbb{N}, \times, 0, \mathrm{s}, \mathrm{id}, \circ, \Pi, \ell, \mathrm{r}\}$$

[2] Pfender 2016c

[3] Manin 2010 "treats, among other things, a categorical approach to the theory of computation, quantum computation, and the P/NP problem." In Foundations let us use here categorical language with its absence of formal variables for individuals: categories have only objects and maps as fundamental notions. This circumstance makes coding – *gödelisation* – of categorical theories comparably simple. In cartesian categories, *free variables* (re)enter as names for identic maps and projections out of (cartesian) products. So Free-Variables primitive recursive Arithmetic comes back in a conveniently codable way.

$\mathbb{1}$ is the *one-element object,* $\mathbb{N}$ the *natural numbers object* of theories **CA** and **PR** to come, $\times$ the cartesian product of objects and of maps.

0 is the *zero constant* $0 : \mathbb{1} \to \mathbb{N}$, s is the "fundamental" *successor function* $s : \mathbb{N} \to \mathbb{N}$

Identity id is the family of *identity maps* to all objects obained out of $\mathbb{1}$ and $\mathbb{N}$ by *cartesian product* $\times$, $\circ$ is map *composition* occasionally replaced by concatenation, Π symbolises the family of *terminal maps* into object $\mathbb{1}$, ℓ and r are left resp. right *projections* out of cartesian product(s) $A \times B$

Theory **PR** of primitive recursion below will come with an additional symbol § for endomap *iteration.*

1.2 Cartesian category axioms

• $\underline{\text{Ax}}\,[\,\mathbb{N}\,]$ ———

$\{\underline{\text{Obj}}\ \mathbb{1},\ \mathbb{N}\}$

one-element object and *natural numbers object*

$\underline{\text{map}}\ 0 : \mathbb{1} \to \mathbb{N}$ *zero constant*

$\underline{\text{map}}\ \mathrm{s} : \mathbb{N} \to \mathbb{N}$ *successor function*

$$\underline{\text{Ax}}\,[\,\text{id}\,]\quad \frac{\underline{\text{Obj}}\ A}{\underline{\text{map}}\ \text{id}_A = \text{id} : A \to A\ \textit{identity map}}$$

$$\underline{\text{Ax}}\ [\,\textit{reflexivity}\,]\quad \frac{\underline{\text{map}}\ f}{f = f}$$

$$\underline{\text{Ax}}\ [\,\textit{symmetry}\,]\quad \frac{\begin{array}{l}\underline{\text{map}}\ f, g;\\ f = g\end{array}}{g = f}$$

$$\underline{\text{Ax}}\ [\,\textit{transitivity}\,]\quad \frac{\begin{array}{l}\underline{\text{map}}\ f, g, h;\\ f = g;\ \ g = h\end{array}}{f = h}$$

$$\underline{\text{Ax}}\ [\,\circ\,]\quad \frac{f : A \to B;\ \ g : B \to C}{\begin{array}{l}\underline{\text{map}}\ (g\, f) = (g \circ f) = g(f) : A \to C\\ (g \circ f) : A \to B \to C\ \ \textit{composition}\\ \text{outmost brackets may be omitted}\end{array}}$$

$$\underline{\text{Ax}}\ [\,\circ\,\text{sub}\,]\quad \frac{f, \tilde{f} : A \to B;\ \ g : B \to C;\ f = \tilde{f}}{g \circ f = g \circ \tilde{f}\ \ \text{Leibniz' } \textit{substitutivity}}$$

$$\underline{\text{Ax}}\ [\,\text{sub}\,\circ\,]\quad \frac{f : A \to B;\ \ g, \tilde{g} : B \to C;\ g = \tilde{g}}{g \circ f = \tilde{g} \circ f\ \ \textit{second}\ \text{Leibniz}\ \textit{substitutivity}}$$

$$\underline{\text{Ax}}\,[\circ\ \text{id}]\quad \frac{f : A \to B}{\begin{array}{l} f \circ \text{id} = f \circ \text{id}_A = f \\ \text{id} \circ f = \text{id}_B \circ f = f \end{array}}$$

neutrality of identities to composition

$$\underline{\text{Lem}}\,[\circ\ \underline{\text{var}}]\quad \frac{\begin{array}{l} f : A \to B; \\ \underline{\text{var}}\ a \in A,\ \ a := \text{id}_A \end{array}}{f(a) = f(\text{id}_A) = f \circ \text{id}_A = f}$$

free variable as identity

$f(a) \in B$ *"dependent variable"*

$$\bullet\ \underline{\text{Ax}}\,[\text{ass}\ \circ]\quad \frac{\begin{array}{l} f : A \to B;\ \ g : B \to C;\ \ h : C \to D \\ \underline{\text{var}}\ a \in A \end{array}}{\begin{array}{l} (h \circ g) \circ f = h \circ (g \circ f) : A \to D \\ = h \circ g \circ f = h\,g\,f = h(g(f(a))) \end{array}}$$

associativity of composition

Numerals:

$0 : \mathbb{1} \to \mathbb{N}$ numeral.

$$\frac{\boldsymbol{n} : \mathbb{1} \to \mathbb{N}\ \text{numeral}}{\boldsymbol{n} + 1 = \text{s}\,\boldsymbol{n} = (\text{s} \circ \boldsymbol{n}) : \mathbb{1} \to \mathbb{N}\ \text{numeral}}$$

Cartesian structure:

Ax [Π]
$$\frac{\underline{\text{Obj}}\ A}{\underline{\text{map}}\ \Pi = \Pi_A : A \to \mathbb{1}\ \textit{terminal map}}$$

Ax [!Π]
$$\frac{f : A \to \mathbb{1}}{f = \Pi_A}$$

uniqueness, naturality of family Π

DIAGRAM:
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \Pi} & = & \downarrow{\scriptstyle \Pi} \\ \mathbb{1} & \xrightarrow{\text{id}} & \mathbb{1} \end{array}$$

Remark: This naturality axiom for family Π will not hold in *half*-terminal *monoidal* categories introduced in Budach & Hoehncke 1975, and to be considered "marginally": Theory $\mathbf{P\widehat{R}a}$ of *partially defined* recursive maps.

- Ax [Obj ×]
$$\frac{\underline{\text{Obj}}\ A, B}{\underline{\text{Obj}}\ (A \times B)}$$

cartesian product of objects

[Outmost brackets may be omitted]

- $\underline{\text{Ax}}\ [\,\ell, \mathrm{r}\,]$ $\dfrac{\underline{\text{Obj}}\ A, B \qquad \underline{\text{var}}\ a \in A,\ \underline{\text{var}}\ b \in B}{\underline{\text{map}}\ \ a = \ell = \ell_{A,B} : A \times B \to A \qquad \underline{\text{map}}\ \ b = \mathrm{r} = \mathrm{r}_{A,B} : A \times B \to B}$

  left resp. right projection,

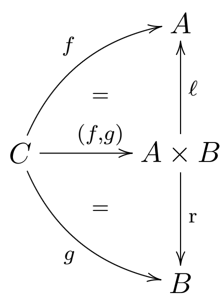
  variables as projections

- $\underline{\text{Ax}}\ [\,\text{indu}\,]$ $\dfrac{\underline{\text{map}}\ f : C \to A,\ \ g : C \to B}{\underline{\text{map}}\ (f, g) : C \to A \times B}$

  induced map into product

  $\ell \circ (f, g) = f,\ \ \mathrm{r} \circ (f, g) = g$

DIAGRAM



$$\underline{\text{Ax}}\ [\,\text{sub}(\ ,\ )\,]\quad \frac{f, \tilde{f} : C \to A;\ g, \tilde{g} : C \to B;\quad f = \tilde{f};\ g = \tilde{g}}{(f, g) = (\tilde{f}, \tilde{g})}$$

compatibility of inducing with '='

$$\underline{\text{Ax}}\ [\,\text{distr}\,]\quad \frac{h : D \to C,\ f : C \to A,\ g : C \to B}{(f, g) \circ h = (f \circ h, g \circ h) : D \to (A \times B)}$$

distributivity of $\circ$ *over forming*

induced map into product

$$[\,\underline{\text{Lem}}\,]\quad \frac{\underline{\text{var}}\ c \in C,\ c := \mathrm{id}_C}{\begin{gathered}\ell \circ (f, g)(c) = \ell \circ (f(c), g(c)) = f(c)\\ \mathrm{r} \circ (f, g)(c) = \mathrm{r} \circ (f(c), g(c)) = g(c)\end{gathered}}$$

<u>q. e. d.</u>

$$\underline{\text{Ax}}\ [\,\text{retr. pairing}\,]\quad \frac{h : C \to (A \times B)}{(\ell_{A,B} \circ h, \mathrm{r}_{A,B}\, h) = h}$$

pairing *is* ***retractive***

even ***isomorphic***

$$\text{\underline{Lem} } [\,!(\ ,\ )\,] \quad \frac{f : C \to A;\ g : C \to B;\ h : C \to (A \times B);\quad \ell_{A,B} \circ h = f;\ \mathrm{r}_{A,B} \circ h = g}{h = (f, g)}$$

uniqueness of induced map

Proof:

$$\begin{aligned} h &= \mathrm{id}_{A\times B} \circ h \\ &= (\ell_{A,B} \circ \mathrm{id}_{A,B}, \mathrm{r}_{A,B} \circ \mathrm{id}_{A,B}) \circ h \quad \text{[retr. pairing]} \\ &= (\ell_{A,B}, \mathrm{r}_{A,B}) \circ h \\ &= (\ell_{A,B} \circ h, \mathrm{r}_{A,B} \circ h) \quad \text{[distr]} \\ &= (f, g) : C \to A \times B \quad \text{[sub(,)]} \end{aligned}$$

$$\text{\underline{Lem} } [\,(\ell, \mathrm{r})\,] \quad \frac{\underline{\mathrm{Obj}}\ A, B}{(\ell_{A,B}, \mathrm{r}_{A,B}) = \mathrm{id}_{A\times B}}$$

Proof: uniqueness of induced into product $A \times B$

$$\text{\underline{Def} } [\,\times \text{ maps}\,] \quad \frac{f : A \to A',\ g : B \to B' \quad \underline{\mathrm{var}}\ a := \ell_{A,B},\ b := \mathrm{r}_{A,B}}{(f \times g) = (f \circ \ell, g \circ \mathrm{r}) : (A \times B) \to (A' \times B')}$$

$$f \times g = (f \times g)(a, b) = (f(a), g(b))$$

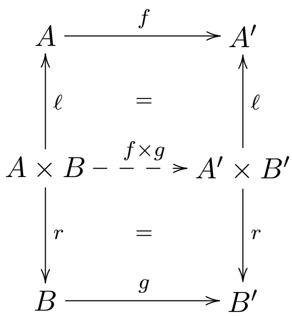
cartesian map product

$$[\,\underline{\text{unary}}\ \times\,]\quad \frac{f : A \to A',\ g : B \to B'}{\begin{array}{l}(A \times g)\ =_{\text{def}}\ (\text{id}_A \times g) : A \times B \to A \times B' \\ (f \times B)\ =_{\text{def}}\ (f \times \text{id}_B) : A \times B \to A' \times B\end{array}}$$

$$\underline{\text{Thm}}\ [\,\text{nat}_{\ell,r}\,]\quad \frac{\underline{\text{map}}\ f : A \to A',\ g : B \to B'}{\begin{array}{l}\ell \circ (f \times g) = f \circ \ell;\ \text{r} \circ (f \times g) = g \circ \text{r} \\ \textit{naturality of projection families } \ell \textit{ and } \text{r}\end{array}}$$

<u>Proof</u>: uniqueness of induced map into product $A' \times B'$

DIAGRAM:



$f : A \to A',\ f' : A' \to A'';$

$g : B \to B',\ g' : B' \to B''$

<u>Thm</u> [× ∘] ———————————————

$\mathrm{id}_A \times \mathrm{id}_B = \mathrm{id}_{A\times B} : A \times B \to A \times B$

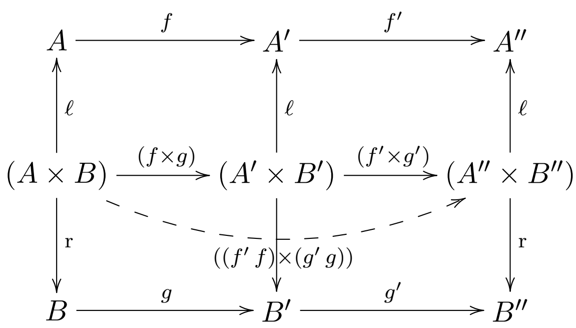
$(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g) :$

$(A \times B) \to (A'' \times B'')$

bifunctoriality of cartesian product

<u>Proof</u>: uniqueness of induced maps into products

COMMUTING DIAGRAM:



2 Primitive Recursion PR

2.1 Iteration axioms added

$$f : A \to A,\ \underline{\text{var}}\ a \in A,\ \underline{\text{var}}\ n \in \mathbb{N}$$

$\underline{\text{Ax}}\ [\,\S\,]$ ————————————————

$$f^{\S} = f^{\S}(a, n) : A \times \mathbb{N} \to A$$

$$f^{\S}(a, 0) := f^{\S}(\text{id}_A, 0\,\Pi_A) = a = \text{id}_A$$

$$f^{\S} \circ (A \times \text{s}) = f^{\S}(a, \text{s}\,n)$$

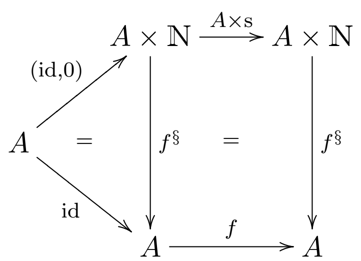
$$= f \circ f^{\S} = f(f^{\S}(a, n)) : A \times \mathbb{N} \to A \to A$$

$$f^n(a) := f^{\S}(a, n)$$

apply iteratively n *times endomap* f

to initial argument a

DIAGRAM:



(§)

$f : A \to B;\ g : B \to B;\ h : A \times \mathbb{N} \to B;$

$\underline{\text{var}}\ a \in A,\ \underline{\text{var}}\ n \in \mathbb{N};$

$h(a, 0) = f(a);$

$h(a, \mathrm{s}\, n) = g\, h(a, n)$

<u>Ax</u> [FR!] ───────────────────────

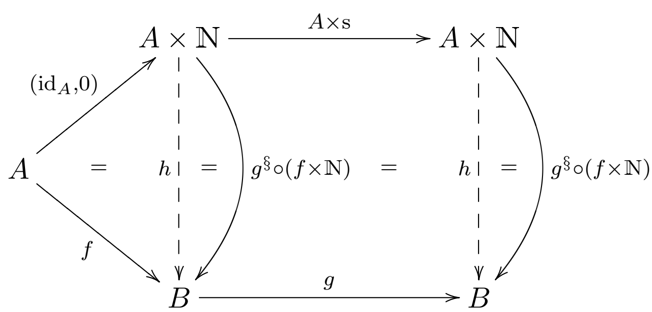
$h = g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}})$ i. e.

$h(a, n) = g^n(f(a))$:

Freyd*'s uniqueness of iterated endomap g initialised by a map f*

[$g^{\S}\,(f \times \mathrm{id}_{\mathbb{N}})$ does the job]

DIAGRAM[4]



$f, \tilde{f} : A \to A;\ f = \tilde{f}$

<u>Lem</u> [§] ───────────────────────

$f^{\S} = \tilde{f}^{\S} : A \times \mathbb{N} \to A$

uniqueness of "simple" iterated endo f

[4] FREYD 1972

Proof:

$\tilde{f} = f$

entails $\tilde{f}^{\S}(a, 0) = \mathrm{id}_A$

and $\tilde{f}^{\S}(a, s\,n) = \tilde{f} \circ \tilde{f}^{\S}(a, n) = f \circ \tilde{f}^{\S}(a, n)$; [sub ∘]

entails

$\tilde{f}^{\S} = f^{\S} \circ (\mathrm{id}_A \times \mathbb{N})$ by [FR!]

$= f^{\S} \circ (\mathrm{id}_{A \times \mathbb{N}}) = f^{\S}$

2.2 Full schema of primitive recursion

Def Thm [pr]

$g = g(a) : A \to B$

$h = h((a, n), b) : (A \times \mathbb{N}) \times B \to B$

$f = f(a, n) : A \times \mathbb{N} \to B$ s. t.

(anchor) $f(a, 0) = g(a)$ and

(step) $f(a, \mathrm{s}\,n) = h((a, n), f(a, n))$

$f =: \mathrm{pr}[g, h]$

+

(pr!) *uniqueness of* f to satisfy

these (anchor) and (step) equations.

Interpretation:

general primitive recursive map $f = f(a, b)$ *initialised by a map* $g = g(a)$ *and iteratively extended using a* ***step*** *map* $h =$

$h((a,n),b)$ which depends on previous value b but (possibly) also from initial argument a as well as from running recursion parameter n

Schema (pr) without use of free variables:

$$
\text{(pr)} \quad \frac{g : A \to B, \quad h : (A \times \mathbb{N}) \times B \to B}{\begin{array}{l} \text{pr}[g,h] := f : A \times \mathbb{N} \to B \\ f(\text{id}_A, 0) = g : A \to B \\ f\,(\text{id}_A \times \text{s}) = h\,(\text{id}_{A \times \mathbb{N}}, f) : \\ \quad (A \times \mathbb{N}) \to (A \times \mathbb{N}) \times B \to B \\ (\text{pr!}) : f \ \textit{unique} \end{array}}
$$

Schema (pr) is a consequence of iteration schema (§) and *uniqueness of the initialised iterated h*, this taken above as **axiom** (FR!)

Remark: Full schema (pr) of primitive recursion is an **axiom** in the classical theory of primitive recursion, subsystem of any arithmetical theory **T**.

Free-Variables Arithmetics of the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$ can be based on the axioms of the cartesian theory **PR** of primitive recursion as defined by the axioms introduced in the above. Goodstein's[5] uniqueness axioms U_1 to U_4 basic for his *Free-Variables Arithmetics* are theorems of **PR**.

[5] Goodstein 1971

2.3 Predicate abstraction PRa

We discuss a p. r. **abstraction scheme** as a definitional enrichment of **PR** into theory **PRa** of *PR decidable objects and PR maps in between,* decidable subobjects of the objects of **PR**.

The extension **PRa** is given by adding schemes $(\mathrm{Ext}_{\mathbf{Obj}})$, $(\mathrm{Ext}_{\mathbf{Map}})$, and $(\mathrm{Ext}_{=})$ below. Together they correspond to the *scheme of abstraction* in **set** theory, and they are referred as schemes of *PR* ***a****bstraction.*

Our first predicate-into-object *abstraction* scheme is

$$(\mathrm{Ext}_{\mathbf{Obj}})\quad \frac{\begin{array}{l}\chi : A \to \mathbb{N} \text{ a } \mathbf{PR}\text{-predicate:} \\ \mathrm{sign} \circ \chi = \neg\neg\chi = \chi : A \to \mathbb{N} \to \mathbb{N}\end{array}}{\{A : \chi\} \text{ object (of emerging theory } \mathbf{PRa})}$$

Subobject $\{A : \chi\} \subseteq A$ may be written alternatively with *bound* variable a as $\{A : \chi\} = \{a \in A : \chi(a)\}$

Example: The subdiagonal grid

$$\{\mathbb{N} \times \mathbb{N} : \leq\} = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \leq b\} \subset \mathbb{N} \times \mathbb{N}$$

The *maps* of $\mathbf{PRa} = \mathbf{PR} + (\mathrm{abstr})$ come in by

$$\frac{\begin{array}{l}\{A:\chi\},\ \{B:\varphi\}\ \mathbf{PRa}\text{-objects}\\ f:A\to B \text{ a } \mathbf{PR}\text{-map}\\ \mathbf{PR}\vdash\ \chi(a)\implies\varphi\, f\,(a)\ \text{i.e.}\\ [\,\chi\implies\varphi\circ f\,]=^{\mathbf{PR}}\mathrm{true}_A:A\xrightarrow{\Pi}\mathbb{1}\xrightarrow{1}\mathbb{N}\end{array}}{f\text{ is a }\mathbf{PRa}\text{-map } f:\{A:\chi\}\to\{B:\varphi\}}\quad(\mathrm{Ext}_{\mathbf{Map}})$$

A posteriori we introduce, following Reiter 1982, the formal *truth Algebra* $\mathbb{2}$ as

$$\mathbb{2}\ =_{\mathrm{def}}\ \{n\in\mathbb{N}:n\le \mathrm{s}\,0\}$$

with proto Boolean operations on $\mathbb{N}$ restricting – in codomain and domain – to *boolean* operations on $\mathbb{2}$ resp.

$$\mathbb{2}\times\mathbb{2}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:m,n\le s\,0\}$$

Definition of cartesian product of objects within **PRa**.

PRa-maps with common **PRa** domain and codomain are considered equal, if their values are equal on their defining *domain predicate.* This is expressed by the scheme

$$\frac{\begin{array}{l}f,\,g:\{A:\chi\}\to\{B:\varphi\}\ \mathbf{PRa}\text{-maps}\\ \mathbf{PR}\vdash\ \chi(a)\implies f(a)\doteq_B g(a)\end{array}}{f=g:\{A:\chi\}\to\{B:\varphi\}}\quad(\mathrm{Ext}_=)$$

explicitly:

$f=^{\mathbf{PRa}}g:\{A:\chi\}\to\{B:\varphi\},$ also noted

$\mathbf{PRa}\vdash\ f=g:\{A:\chi\}\to\{B:\varphi\}$

Structure Theorem for the theory **PRa** of *primitive recursion with predicate abstraction:*[6]

- **PRa** is a cartesian p. r. theory
- Theory **PR** is cartesian p. r. embedded
- Theory **PRa** has (universal) extensions of all of its predicates and boolean truth object noted $\mathbb{2}$ as codomain of these predicates, with truth values false $\equiv 0$, true $\equiv 1 \equiv s\,0 : \mathbb{1} \to \mathbb{2}$ as well as predicative equalities $m \doteq n : \mathbb{N} \times \mathbb{N} \to \mathbb{2}$, $a \doteq \tilde{a} : A \times A \to \mathbb{2}$ and map definition by case distinction
- **PRa** has all finite (projective) limits, in particular *equalisers, pullbacks* and *kernel pairs*
- **PRa** has (binary) *sums* (coproducts) and *coequalisers* of kernel pairs, of *equivalence predicates.*

3 Numerals and universal set

Objective numerals revisited

$$\begin{aligned}
&\mathrm{num}(0) \equiv 0 : \mathbb{1} \to \mathbb{N} \\
&\mathrm{num}(1) = s(0) = s \circ 0 : \mathbb{1} \to \mathbb{N} \to \mathbb{N} \\
&\mathrm{num}(\mathbf{n}+1) \equiv \mathbf{n}+1 = s \circ \mathbf{n} = s(\mathbf{n}) : \mathbb{1} \to \mathbb{N} \\
&\mathbf{n} \in \mathcal{N} \ \text{ meta-variable over “naive” natural numbers}
\end{aligned}$$

[6] see Pfender, Kröplin, Pape 1994

Internal numerals

Numeralisation $\nu = \nu_{\mathbb{N}}(n) : \mathbb{N} \to \mathbb{N}$ is p. r. defined by

$\nu(0) = \ulcorner 0 \urcorner : \mathbb{1} \to \mathbb{N}$

gödel number, `utf8 code of` 0

$\nu(1) = \ulcorner s \urcorner * \ulcorner \circ \urcorner * \ulcorner 0 \urcorner : \mathbb{1} \to \mathbb{N}$

string concatenation of symbol codes,

$\nu(s\,n) = \ulcorner s \urcorner \odot \nu(n) \in \mathbb{N},\ \odot \equiv \ulcorner \circ \urcorner$

This internal numeralisation distributes the "elements" (numbers) of the NNO $\mathbb{N}$ over $\mathbb{N}$, with suitable gaps to receive in particular any other symbols of object language **PR**.

Predicate Lemma: Enumeration $\nu : \mathbb{N} \to \mathbb{N}$ defines a characteristic p. r. *image* predicate $\mathrm{im}[\nu] : \mathbb{N} \to \mathbb{2}$ and by this **PRa** object

$\dot{\mathbb{N}} = \nu\mathbb{N} = \{\mathbb{N} : \mathrm{im}[\nu]\} \cong \mathbb{N}$

of (enumerated) internal numerals

Proof of Lemma: Use iterative 'or' for definition of $\mathrm{im}[\nu]$:

$$\mathrm{im}[\nu](c) = \vee_{n \le c}[c \doteq \nu(n)]$$

$\nu : \mathbb{N} \to \mathbb{N}$ has codomain restriction $\nu : \mathbb{N} \to \dot{\mathbb{N}} = \{\mathbb{N} : \mathrm{im}[\nu]\}$ and is then an iso with p. r. inverse

$$\nu^{-1} = \nu^{-1}(c) = \min\{n \le c : \nu(n) \doteq c\} : \dot{\mathbb{N}} \xrightarrow{\cong} \mathbb{N}$$ **q. e. d.**

Extend numeralisation to object $\mathbb{1}$ by

$$\nu_{\mathbb{1}} = \nu_{\mathbb{1}}(0) = \ulcorner 0 \urcorner \in \dot{\mathbb{1}} = \nu\mathbb{1} = \{\ulcorner 0 \urcorner\} \cong \mathbb{1}$$

and definition of (nested) numpairs and predicative numpair sets by

$$\frac{\begin{array}{l} A, B \ \mathbf{PR} \text{ objects}, \ \nu_A : \mathbb{N} \to A, \ \nu_B : \mathbb{N} \to B \\ \dot{A} = \{A : \mathrm{im}[\nu_A]\}, \ \dot{B} = \{B : \mathrm{im}[\nu_B]\} \subset \mathbb{N} \text{ given} \end{array}}{\begin{array}{l} \langle A \dot{\times} B \rangle = \{\mathbb{N} : \mathrm{im}[\nu_{A\times B}]\} \subset \mathbb{N} \text{ constructed by} \\ \nu_{A\times B}(a, b) = \langle \nu(a); \nu_B(b) \rangle : A \times B \to \mathbb{N} \\ \mathrm{im}[\nu_{A\times B}](c) = \vee_{n \le c}[c = \nu_{A\times B}(n)] : \mathbb{N} \to \mathbb{2} \end{array}}$$

Abbreviations:

- $\langle \ldots \rangle = \ulcorner (\urcorner \ldots \ulcorner) \urcorner$
- $\odot = \ulcorner \circ \urcorner$
- $\dot{\times} = \ulcorner \times \urcorner$
- $; = \ulcorner , \urcorner$

Universal set $\mathbb{X}$

Define *universal set* $\mathbb{X} = \{\mathbb{N} : \mathbb{X}\} \subset \mathbb{N}$ of all *numerals* $\nu(n)$ and (possibly nested) *numpairs* first by p. r. enumeration

$$\begin{array}{l} \ulcorner 0 \urcorner \in \mathbb{X} \\ n \in \mathbb{N} \implies \ulcorner \mathrm{s} \urcorner \odot \nu(n) \in \mathbb{X} \\ x \in \mathbb{X} \wedge y \in \mathbb{X} \implies \langle x; y \rangle \in \mathbb{X} \end{array}$$

This enumeration has characteristic p. r. image predicate

$\mathbb{X} = \mathbb{X}(c) : \mathbb{N} \to \mathbb{2}$ defined as follows:

$$\mathbb{X}(c) = \begin{cases} \text{true if } \vee_{n \leq c} \,\mathrm{ct}_{\mathbb{X}}(n) = c \\ \text{false otherwise, i.e. if } \wedge_{n \leq c} \,\mathrm{ct}_{\mathbb{X}}(n) \neq c \end{cases}$$

$\mathrm{ct}_{\mathbb{X}} : \mathbb{N} \to \mathbb{N}$ is the p. r. enumeration/counting process given by cyclic application of the rules above generating $\mathbb{X}$ as a "set", analogon to Cantor count

$$\mathrm{ct}_{\mathbb{N}\times\mathbb{N}} : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N} \xrightarrow{\cong} \langle \mathbb{N} \dot{\times} \mathbb{N} \rangle \subset \mathbb{N}$$

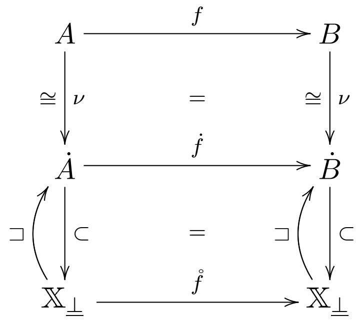
Variable $c \in \mathbb{N}$ works in fact as an upper bound, since obviously $\mathrm{ct}_{\mathbb{X}}(n) > n, \ n \in \mathbb{N}$ free.

Numeral Theorem

- $\nu : \mathbb{N} \to \mathbb{N}$ has a retraction $\nu^{-} : \mathbb{N} \twoheadrightarrow \dot{\mathbb{N}} \xrightarrow{\cong} \mathbb{N}$
- analogously for all objects A of theory **PR** :
  $\nu_A : A \to \mathbb{N}$ has a retraction $\nu^{-} : \mathbb{N} \twoheadrightarrow \dot{A} \xrightarrow{\cong} A$
- these make up a *natural equivalence* $\nu = [\nu_A]_A$ (out of **PRa** iso maps), see commutative diagram below, where $\dot{f} = \nu_B \circ f \circ \nu_A^{-1} : \dot{A} \to \dot{B}$, and

$$\mathring{f} = \mathring{f}(x) = \begin{cases} \dot{f}(x) \in \dot{B} \subset \mathbb{X} \subset \mathbb{X}_{\underline{\perp}} \\ \quad = \mathbb{X} \cup \{ \ulcorner \perp \urcorner \} \text{ for } x \in \dot{A} \\ x \text{ for } x \in \mathbb{X}_{\underline{\perp}} \smallsetminus \dot{A} \\ \quad \text{(undefined argument)} \end{cases}$$

DIAGRAM:



4 Evaluation

From now on we place ourselves in a "*gödelian*" quantified arithmetical frame theory **T** strengthened by the following **axiom** schema (fin desc) of *finite descent* of chains in linear order of semiring $\mathbb{N}[\omega]$ of polynomials in indeterminate ω (thought "big"), coefficients in $\mathbb{N}$:

$$\text{(fin desc)} \quad \frac{\begin{array}{l} q_n = q_n(\omega) : \mathbb{N} \to \mathbb{N}[\omega] \\ \text{descending chain above } 0 \text{ in } \mathbb{N}[\omega] \end{array}}{\exists m \in \mathbb{N}\ q_m(\omega) \equiv 0}$$

Such *frame* theory **T** is a subsystem of Principia Mathematica **PM**, of Zermelo-Fraenkel **set** theory **ZF**, and of von Neumann-Gödel-Bernays **set** theory **NGB**. First-order subsystems of these theories suffice. Cf. the classical arithmetical theories **T** as considered in Smorynski 1977, part D.1 in the *Handbook of Mathematical Logic,* here strengthened by schema (fin desc).

These "**set**" theories **T** are considered as extensions (in language and theorems) of cartesian p. r. theory **PRa** with predicate abstracction. Such classical extension **T** is to have in particular p. r. enumerated subsets of NNO $\mathbb{N}$ and p. r. countable unions of such subsets as subsets of $\mathbb{N}$. Schema (fin desc) is to guarantee *termination* of evaluation of PR map codes.

4.1 Evaluation of PR into "set" theory T

We **define** **T**-recursively an evaluation

$$\boldsymbol{eva} : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \longrightarrow \mathbb{X}_{\underline{\perp}}$$

of **PR** *map code set*

$\mathrm{PR} = \cup_{A,B}[A, B]_{\mathrm{PR}} \subset \mathbb{N}$

on *numerals* and (nested) *numpairs* out of

$\mathbb{X} = \cup_A \dot{A} \subset \mathbb{N}$ of forgoing section,

universal set for theories **CA** as well as **PR** (and **PRa**) within theory **T**, augmented by symbol $\underline{\perp} = \ulcorner \perp \urcorner \in \mathbb{N}$ for *trash* element – into $\mathbb{X}_{\underline{\perp}} = \mathbb{X} \cup \{\underline{\perp}\} \subset \mathbb{N}$

With objects $\mathbb{1}, \mathbb{N}, A, B, C, A', B'$ considered as **PR** objects as well as **T** sets, with coding – *gödelisation* – $\ulcorner f \urcorner$ of (**CA** and) **PR**, and with

'$;$' $= \ulcorner , \urcorner$ $\langle \ldots \rangle = \ulcorner (\urcorner \ldots \ulcorner) \urcorner$, $\odot = \ulcorner \circ \urcorner$, $\dot{\times} = \ulcorner \times \urcorner$, $\dot{\S} = \ulcorner \S \urcorner$

we **define**

Basic map/function code evaluation $\boldsymbol{eva}$:

$$\boldsymbol{eva}(\ulcorner 0 \urcorner, \ulcorner 0 \urcorner) = \ulcorner 0 \urcorner \in \dot{\mathbb{N}} \subset \mathbb{X}$$

$$\Downarrow \quad \frac{\underline{\mathrm{Obj}}(A),\ a \in \dot{A} \subset \mathbb{X} \subset \mathbb{N}}{\boldsymbol{eva}(\ulcorner \mathrm{id} \urcorner, a) = a}$$

$$\Downarrow \quad \frac{n \in \mathbb{N}}{\boldsymbol{eva}(\ulcorner \mathrm{s} \urcorner, \nu(n)) = \nu(\mathrm{s}\, n) \in \dot{\mathbb{N}} \subset \mathbb{X}}$$

$$\Downarrow \quad \frac{\underline{\mathrm{Obj}}(A),\ a \in \dot{A}}{\boldsymbol{eva}(\ulcorner \Pi \urcorner, a) = \ulcorner 0 \urcorner \in \dot{\mathbb{1}} \subset \mathbb{X}}$$

$$\Downarrow \quad \frac{\underline{\mathrm{Obj}}(A),\ \underline{\mathrm{Obj}}(B),\ a \in \dot{A},\ b \in \dot{B}}{\begin{array}{l}\boldsymbol{eva}(\ulcorner \ell \urcorner, \langle a; b \rangle) = a \\ \boldsymbol{eva}(\ulcorner \mathrm{r} \urcorner, \langle a; b \rangle) = b\end{array}}$$

Put together:

$$\Downarrow \quad \frac{\begin{array}{l}\mathrm{ba} \in \mathrm{bas} = \{\mathrm{id}, 0, s, \Pi, \ell, \mathrm{r}\} \\ = \{\mathrm{id}_A, 0, s, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : \underline{\mathrm{Obj}}\ A, B\} \subset \mathrm{PR} \\ A = \mathrm{Dom}[\mathrm{ba}],\ B = \mathrm{Codom}[\mathrm{ba}],\ a \in \dot{A}\end{array}}{\begin{array}{l}\boldsymbol{eva}(\ulcorner \mathrm{ba} \urcorner, a) = \nu_B(\mathrm{ba}(\nu_A^{-1}(a))) \in \dot{B} \subset \mathbb{X} \\ \boldsymbol{eva}(\mathrm{ba}, x) = \underline{\perp} \text{ for } x \in (\mathbb{X}_{\underline{\perp}} \smallsetminus \dot{A})\end{array}}$$

Evaluation of composed map codes:

$$(\text{compos}) \Downarrow \quad \frac{f \in [A,B],\ g \in [B,C],\ a \in \dot{A}}{\boldsymbol{eva}(g \odot f, a) = \boldsymbol{eva}(g, \boldsymbol{eva}(f,a)) \in \dot{C} \subset \mathbb{X}}$$

formally (and for **PR** instead of **CA** in fact)

double recursive

$$(\text{indu}) \Downarrow \quad \frac{f \in [C,A] \subset \text{PR} \subset \mathbb{N},\ g \in [C,B] \subset \text{PR} \quad c \in \dot{C} \subset \mathbb{X}}{\boldsymbol{eva}(\langle f; g\rangle, c) = \langle \boldsymbol{eva}(f,c); \boldsymbol{eva}(g,c)\rangle \in \langle A \dot{\times} B\rangle = \nu(A \times B) \subset \mathbb{X}}$$

primitive recursive

$$(\times) \Downarrow \quad \frac{f \in [A,A'] \subset \text{PR},\ g \in [B,B'] \subset \text{PR} \quad a \in \dot{A},\ b \in \dot{B}}{\boldsymbol{eva}(\langle f \dot{\times} g\rangle, \langle a; b\rangle) = \langle \boldsymbol{eva}(f,a); \boldsymbol{eva}(g,b)\rangle \in \langle A' \dot{\times} B'\rangle \subset \mathbb{X}}$$

(*redundant*)

$$(\text{anchor it}) \Downarrow \quad \frac{f \in [A,A],\ a \in \dot{A}}{\boldsymbol{eva}(f^{\dot{\S}}, (a, \ulcorner 0 \urcorner)) = a}$$

$$(\text{step it}) \Downarrow \quad \frac{f \in [A, A],\ a \in \dot{A},\ n \in \mathbb{N}}{\begin{array}{l} \boldsymbol{eva}(f^{\dot{\S}}, (a, \ulcorner \mathrm{s} \urcorner \odot \nu n) \\ = \boldsymbol{eva}(f, \boldsymbol{eva}(f^{\dot{\S}}, (a, \nu n)) \in \dot{A} \\ \textit{\textbf{double recursive}} \\ \text{inner recursion on } n \end{array}}$$

$$(\text{trash}) \Downarrow \quad \frac{f \in [A, B],\ x \in \mathbb{X}_{\perp} \setminus A}{\boldsymbol{eva}(f, x) = \perp \in \mathbb{X}_{\perp}}$$

in particular:

$$(\text{trash it}) \Downarrow \quad \frac{f \in [A, A],\ x \in \mathbb{X}_{\perp} \setminus \langle A \dot{\times} \mathbb{N} \rangle}{\boldsymbol{eva}(f^{\dot{\S}}, x) = \perp}$$

4.2 Evaluation Theorem

(i) Double recursion above defines a *total* **T**-map

$$\boldsymbol{eva} = \boldsymbol{eva}(f, x) : \mathrm{PR} \times \mathbb{X}_{\perp} \to \mathbb{X}_{\perp}$$

(ii) $\boldsymbol{eva}$ is **characterised** within gödelian Arithmetics **T** by

$$\boldsymbol{eva}(\ulcorner \mathrm{ba} \urcorner, x) = \mathrm{ba}(x)$$

for ba $\in$ bas (basic map constants)

$$\boldsymbol{eva}(g \odot f, a) = \boldsymbol{eva}(g, \boldsymbol{eva}(f, a))$$

$$\boldsymbol{eva}(\langle f; g \rangle, c) = \langle \boldsymbol{eva}(f, c); \boldsymbol{eva}(g, c) \rangle$$

$$\boldsymbol{eva}(\langle f \dot{\times} g \rangle, \langle a; b \rangle) = \langle \boldsymbol{eva}(f, a); \boldsymbol{eva}(g, b) \rangle$$

as well as

$$\boldsymbol{eva}(f^{\dot{\S}}, \langle a; \ulcorner 0 \urcorner \rangle) = a \text{ and}$$
$$\boldsymbol{eva}(f^{\dot{\S}}, \langle a; \nu(\mathrm{s}\, n\rangle) = \boldsymbol{eva}(f, \boldsymbol{eva}(f^{\dot{\S}}, \langle a, \nu n\rangle)$$
$$a \in \dot{A},\ b \in \dot{B},\ c \in \dot{C},\ n \in \mathbb{N} \text{ all free}$$
$$\boldsymbol{eva}(f, x) = \underline{\perp} \text{ for } x \in (\mathbb{X}_{\underline{\perp}} \setminus \mathrm{Dom}[f])$$

(iii) $\boldsymbol{eva}$ defines within theory **T** a (natural) **family**

$$\boldsymbol{ev} = \boldsymbol{ev}_{A,B} = \boldsymbol{ev}_{A,B}(f, a) : [A, B] \times A \to B$$
A, B **PR** objects by
$$\boldsymbol{ev}_{A,B}(f, a) = \nu_B^{-1} \circ \boldsymbol{eva}(f, \nu_A(a)) :$$
$$[A, B] \times A \xrightarrow{\cong} [A, B] \times \dot{A} \xrightarrow{\boldsymbol{eva}} \dot{B} \xrightarrow{\cong} B$$

(iv) This family $\boldsymbol{ev} = \boldsymbol{ev}_{A,B}$ is (jointly) *objective:*

$$\frac{f : A \to B \ \textbf{PR} \text{ map}, \ a \in A \text{ free}}{\boldsymbol{ev}(\ulcorner f \urcorner, a) = \boldsymbol{ev}_{A,B}(\ulcorner f \urcorner, a) = f(a) \in B}$$

Totality of this map – this map family – defined by an Ackermann type double recursion is certainly believed in **set** theory, but bears a **problem** constructively. In the **Appendix** we resolve this map into a CCI, a *Complexity Controlled Iteration* which always *terminates,* at least within quantified arithmetical theories **T** – with finite descent (fin desc) – as frame. The corresponding *Evaluation Resolution Theorem* of the Appendix then infers present Evaluation Theorem.

5 PR Stimmigkeit with T

5.1 Internal notion of equality

For cartesian p. r. theory **PR** we have the objective notion of map equality

$$
\begin{aligned}
&=_{\boldsymbol{k}} : \mathcal{N} \to \mathbf{PR} \times \mathbf{PR} \\
&(\boldsymbol{k} : \mathbb{1} \to \mathrm{N}) \mapsto \langle f =_{\boldsymbol{k}} g \rangle
\end{aligned}
$$

externally p. r. enumerated. Numeral

$\boldsymbol{k} : \mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{s} \ldots \xrightarrow{s} \mathbb{N}$ "in $\mathcal{N}$" is a meta free external counting index.

PR map equality pairs $\langle f =_{\boldsymbol{k}} g \rangle = \langle f, g \rangle_{\boldsymbol{k}}$ come in sequentially by (external) p. r. enumeration of (binary) deduction trees.

This enumeration has an internal p. r. equality enumeration analogon

$$
\begin{aligned}
&\check{\doteq}_k : \mathbb{N} \to \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N} \times \mathbb{N} \\
&k \mapsto (f \check{\doteq}_k g), \ \ k \in \mathbb{N} \text{ free} \\
&\quad \text{where we write } f \check{\doteq}_k g \text{ for} \\
&\check{\doteq}_k \ \ \doteq \ \ (f, g) \in \mathrm{PR} \times \mathrm{PR}
\end{aligned}
$$

given by p. r. count of internal deduction trees, **example:** For $k \in \mathbb{N}$ suitable a transitivity-of-equality deduction tree has form

$$\mathrm{dtree}_k \quad = \quad \Uparrow\frac{f \,\check{=}_k\, h}{\Uparrow\dfrac{f \,\check{=}_i\, g}{\mathrm{dtree}_{ii} \quad \mathrm{dtree}_{ji}} \qquad \Uparrow\dfrac{g \,\check{=}_j\, h}{\mathrm{dtree}_{ij} \quad \mathrm{dtree}_{jj}}}$$

$$i, j < k, \;\; ii, ji < i, \;\; ij, jj < j$$

5.2 PR evaluation *Stimmigkeits* Theorem

framed by quantified arithmetical theory **T** with finite descent in $\mathbb{N}[\omega]$:

For p. r. theory **PR** with its internal notion of equality '$\check{=}$' we have for evaluation family

$\boldsymbol{ev} = [\boldsymbol{ev}_{A,B} : [A, B] \times A \to B]_{A,B}$:

(i) **PR** to **T** evaluation *Stimmigkeit:*

$$\mathbf{T} \vdash f \,\check{=}_k\, g \implies \boldsymbol{ev}(f, x) = \boldsymbol{ev}(g, x) \qquad (\bullet)$$

$$k \in \mathbb{N}, \;\; f, g \in \mathrm{PR}, \;\; x \in \mathbb{X} \text{ all free}$$

Substituting "concrete" **PR** codes into f resp. g we get by *objectivity* of evaluation $\boldsymbol{ev}$:

(ii) **T**-framed objective *soundness* of **PR** to **T** :

For p. r. maps $f, g : A \to B$

$$\mathbf{T} \vdash \ulcorner f \urcorner \,\check{=}_k\, \ulcorner g \urcorner \implies f(a) = g(a)$$

$$k \in \mathbb{N}, \;\; a \in A \text{ both free}$$

(iii) Specialising to case $f := \chi : A \to \mathbb{2} = \{0, 1\}$ a p. r. *predicate* and to $g :=$ true we get

$\mathbf{T}$-framed *logical Stimmigkeit of* $\mathbf{PR}$:

$$\mathbf{T} \vdash \ \exists k \, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall\, a\, \chi(a)$$

***If** a p. r. predicate is – within* $\mathbf{T}$ – $\mathbf{PR}$*-internally* provable *by say* k*th* proof, *then it holds in* $\mathbf{T}$ *for all of its arguments.*

(iv) what we will need for decidability and consistency considerations is equivalent subcase (equivalent via Cantor count $\mathrm{ct}_A : \mathbb{N} \xrightarrow{\cong} A$)

$$\mathbf{T} \vdash \ \exists k \, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall n \, \chi(n)$$

$$\chi = \chi(n) : \mathbb{N} \to \mathbb{2} \text{ a numerical } \mathbf{PR} \text{ predicate}$$

Proof of assertion ($\bullet$) by primitive recursion on k, dtree_k the k th deduction tree of the theory *proving* its root equation $f \check{=}_k g$ These (argument-free) deduction trees are counted in lexicographical order.

Super Case of *equational* internal axioms, in particular

- associativity of (internal) composition:
  $\langle h \odot g \rangle \odot f \check{=} h \odot \langle g \odot f \rangle \implies$

$$\begin{aligned} &\boldsymbol{ev}(\langle h \odot g \rangle \odot f, a) = \boldsymbol{ev}(\langle h \odot g \rangle, \boldsymbol{ev}(f, a)) \\ &= \boldsymbol{ev}(h, \boldsymbol{ev}(g, \boldsymbol{ev}(f, a))) \\ &= \boldsymbol{ev}(h, \boldsymbol{ev}(\langle g \odot f \rangle, a)) = \boldsymbol{ev}(h \odot \langle g \odot f \rangle, a) \end{aligned}$$

This **proves** assertion ($\bullet$) in present *associativity-of-composition* case.

- Analogous proof for the other flat – equational – cases, namely *reflexivity of equality, left and right neutrality* of identities, Godement's equations for the induced map:

  $$\ulcorner\ell\urcorner \odot \langle f; g\rangle \doteq f, \ \ \ulcorner \mathrm{r}\urcorner \odot \langle f; g\rangle \doteq g$$

  and definition of cartesian product of maps via induced map, as well as *retractive pairing*

  $$\langle \ulcorner\ell\urcorner \odot h; \ulcorner \mathrm{r}\urcorner \odot h\rangle \doteq h$$

  and distributivity equation

  $$\langle f; g\rangle \odot h \doteq \langle f \odot h; g \odot h\rangle$$

  for composition with an induced.

- proof of (•) for the last equational cases, iteration equations:

  - *iteration anchoring, equation*

  $$\begin{aligned} & f^{\dot{\S}} \odot \langle \ulcorner \mathrm{id}\urcorner; \ulcorner 0\urcorner \odot \ulcorner \Pi\urcorner\rangle \doteq \ulcorner \mathrm{id}\urcorner : \\ \mathbf{T} \vdash\ & \boldsymbol{ev}(f^{\dot{\S}} \odot \langle \ulcorner \mathrm{id}\urcorner; \ulcorner 0\urcorner \odot \ulcorner \Pi\urcorner\rangle, a) \\ & = \boldsymbol{ev}(f^{\dot{\S}}, (\boldsymbol{ev}(\ulcorner \mathrm{id}\urcorner, a), \boldsymbol{ev}(\ulcorner 0\urcorner, \boldsymbol{ev}(\ulcorner \Pi\urcorner, a))) \\ & = \boldsymbol{ev}(f^{\dot{\S}}, (a, \boldsymbol{ev}(\ulcorner 0\urcorner, 0)) \\ & = \boldsymbol{ev}(f^{\dot{\S}}, (a, 0)) = a = \boldsymbol{ev}(\ulcorner \mathrm{id}\urcorner, a) \end{aligned}$$

  - *iteration step,* case of *genuine iteration equation*

  $$f^{\dot{\S}} \odot \langle \ulcorner \mathrm{id}\urcorner \dot{\times} \ulcorner \mathrm{s}\urcorner\rangle \doteq (f \odot f^{\dot{\S}})$$

where $\dot{\times}$ is the internal cartesian product of map codes:

$$\begin{aligned}
\mathbf{T} \vdash \boldsymbol{ev}(f^{\dot{\S}} \odot \langle \ulcorner \mathrm{id} \urcorner \dot{\times} \ulcorner \mathrm{s} \urcorner \rangle, (a, n))
&= \boldsymbol{ev}(f^{\dot{\S}}, \boldsymbol{ev}(\ulcorner \mathrm{id} \urcorner \dot{\times} \ulcorner \mathrm{s} \urcorner, (a, n))) \\
&= \boldsymbol{ev}(f^{\dot{\S}}, (a, \mathrm{s}\, n)) \\
&= \boldsymbol{ev}(f, \boldsymbol{ev}(f^{\dot{\S}}, (a, n))) \\
&= \boldsymbol{ev}(f \odot f^{\dot{\S}}, (a, n))
\end{aligned}$$

[Internal cartesian map product is defined as an internal induced]

Proof of **PR** to **T** evaluation *Stimmigkeit* for the genuine HORN case axioms, of form

$$f \check{\doteq}_i g \wedge \tilde{f} \check{\doteq}_j \tilde{g} \implies h \check{\doteq}_k \tilde{h}, \ i, j < k$$

Transitivity-of-equality case

$$f \check{\doteq}_i g \wedge g \check{\doteq}_j h \implies f \check{\doteq}_k h$$

Evaluate here at argument $a \in A$ and get in fact

$$\begin{aligned}
&\mathbf{T} \vdash f \check{\doteq}_k h \\
&\qquad \implies \boldsymbol{ev}(f, a) = \boldsymbol{ev}(g, a) \wedge \boldsymbol{ev}(g, a) = \boldsymbol{ev}(h, a) \\
&\qquad\qquad \text{by hypothesis } f \check{\doteq}_i g, \ g \check{\doteq}_j h \\
&\qquad \implies \boldsymbol{ev}(f, a) = \boldsymbol{ev}(h, a) :
\end{aligned}$$

transitivity export q. e. d. in this case

Compatibility case of composition with equality

$$g \check{=}_i g,\ f \check{=}_j \tilde{f} \implies g \odot f \check{=}_k g \odot \tilde{f} :$$
$$\boldsymbol{ev}(g \odot f, a) = \boldsymbol{ev}(g, \boldsymbol{ev}(f, a)) = \boldsymbol{ev}(g, \boldsymbol{ev}(\tilde{f}, a))$$
$$= \boldsymbol{ev}(g \odot \tilde{f}, a)$$

by hypothesis on $f \check{=} \tilde{f}$ and by Leibniz' substitutivity in **T** q. e. d. in this first compatibility case.

Case of compatibility of composition with equality in second factor:

$$g \check{=}_i \tilde{g} \implies g \odot f \check{=}_k \tilde{g} \odot f :$$
$$\boldsymbol{ev}(g \odot f, a) = \boldsymbol{ev}(g, \boldsymbol{ev}(f, a)) = \boldsymbol{ev}(\tilde{g}, \boldsymbol{ev}(f, a)) \quad (*)$$
$$= \boldsymbol{ev}(\tilde{g} \odot f, a)$$

$(*)$ holds by $g \check{=}_i \tilde{g}$ and induction hypothesis on i : arbitrary argument, here $\boldsymbol{ev}(f, a)$

This proves *Stimmigkeits* assertion $(\bullet)$ in this 2nd compatibility case.

Compatibility case of internal formation of the induced map with internal equality:

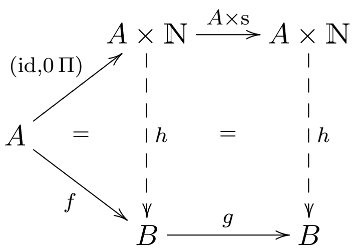
$$f \check{=}_i \tilde{f},\ g \check{=}_j \tilde{g} \implies \langle f; g \rangle \check{=}_k \langle \tilde{g}; \tilde{f} \rangle :$$
$$\boldsymbol{ev}(\langle f; g \rangle, c) = (\boldsymbol{ev}(f, c), \boldsymbol{ev}(g, c)) = (\boldsymbol{ev}(\tilde{f}, c), \boldsymbol{ev}(\tilde{g}, c))$$
$$\text{by hypothesis } f \check{=}_i \tilde{f},\ g \check{=}_j \tilde{g}$$
$$= \boldsymbol{ev}(\langle \tilde{f}; \tilde{g} \rangle, c)$$

Same for compatibility of internal cartesian map product with equality (redundant).

(Final) case of Freyd's (internal) uniqueness of the *initialised iterated* is **case**

$$
\begin{aligned}
& h \odot \langle \ulcorner \mathrm{id} \urcorner ; \ulcorner 0 \urcorner \odot \ulcorner \Pi \urcorner \rangle \check{\doteq}_i f \\
& \wedge\ h \odot \langle \ulcorner \mathrm{id} \urcorner \dot{\times} \ulcorner \mathrm{s} \urcorner \rangle \check{\doteq}_j g \odot h \\
& \implies h \check{\doteq}_k g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle \qquad (**)
\end{aligned}
$$

internal version of h unique, $h =^{\mathbf{PR}} g^{\S} \circ (f \times \mathrm{id})$ in



Comment: h is an internal *comparison candidate* fullfilling the same internal p. r. equations as the *initialised iterated* $g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle$ It should be – **is**: *Stimmigkeit* – evaluated equal to the latter on $A \times \mathbb{N}$; h corresponds to h, f to f, g to g, and $g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle$ to $g^{\S} \circ (f \times \mathrm{id}_{\mathbb{N}})$

Stimmigkeits **proof** in this case

$$
\begin{aligned}
& h \odot \langle \ulcorner \mathrm{id} \urcorner , 0 \rangle \check{\doteq}_i f \wedge h \odot \langle \ulcorner \mathrm{id} \urcorner \dot{\times} \mathrm{s} \rangle \check{\doteq}_j g \odot h \\
& \implies h \check{\doteq}_k g^{\dot{\S}} \odot \langle \ulcorner \mathrm{id} \urcorner \dot{\times} f \rangle
\end{aligned}
$$

is the following, by (structural) recursion on k :

$$\boldsymbol{ev}(h,(a,0)) = \boldsymbol{ev}(f,a) \qquad \text{(hypothesis on } i < k)$$
$$= \boldsymbol{ev}(g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle, (a,0))$$

as well as – *induction on n* –

$$\boldsymbol{ev}(h,(a,\mathrm{s}\,n))$$
$$= \boldsymbol{ev}(h \odot \langle \ulcorner \mathrm{id} \urcorner \dot{\times} \ulcorner \mathrm{s} \urcorner \rangle, (a,n))$$
$$= \boldsymbol{ev}(g \odot h, (a,n)) \qquad \text{(hypothesis on } j < k)$$
$$= \boldsymbol{ev}(g, \boldsymbol{ev}(h,(a,n)))$$
$$= \boldsymbol{ev}(g, \boldsymbol{ev}(g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle, (a,n)))$$

by induction hypothesis on n

$$= \boldsymbol{ev}(g \odot \langle g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle \rangle, (a,n))$$
$$= \boldsymbol{ev}(g^{\dot{\S}} \odot \langle f \dot{\times} \ulcorner \mathrm{id} \urcorner \rangle, (a,\mathrm{s}\,n))$$

q. e. d.

6 Decision of PR predicates

We consider **PR** predicates χ for decision by quantified arithmetical theorie(s) **T** (with finite descent in $\mathbb{N}[\omega]$), without restriction of generality just predicates $\chi = \chi(n) : \mathbb{N} \to \mathbb{N}$

Basic tool for decision is **T**-framed evaluation-*Stimmigkeit* of **PR** above, namely

$$\frac{\chi = \chi(n) : \mathbb{N} \to \mathbb{N} \ \mathbf{PR} \text{ predicate}}{\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall n\, \boldsymbol{ev}(\ulcorner \chi \urcorner, n) = \mathrm{true}}$$
$$\mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner \chi \urcorner) \implies \forall n\, \chi(n)$$

Within **T define** for $\chi : \mathbb{N} \to \mathbb{2}$ a predicate out of **PR** a partially defined **predicate decision**

$$\nabla\chi = \begin{cases} \text{false if } \exists k \, \neg\chi(k) \\ \text{true if } \exists k \, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \\ \bot \ \textit{otherwise} \text{ i. e.} \\ \text{if } \forall k \, \chi(k) \ \wedge \ \forall k \, \neg\mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \\ \text{(derivable but not } \textit{provable}\text{)} \end{cases}$$

$$: \mathbb{1} \to \mathbb{2} \cup \{\bot\} = \{0, 1, \bot\}$$

well defined by **Stimmigkeit**/**soundness**

– – – – – – – – – – – – – – – – – – – –

$\mathbf{T} \vdash \ \exists k \, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \implies \forall n \, \chi(n)$

– – – – – – – – – – – – – – – – – – – –

and same as

$$\nabla\chi = \begin{cases} \text{false if } \exists k \, \neg\chi(k) \\ \text{true if } \forall k \, \chi(k) \ \wedge \ \exists k \, \mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \\ \bot \text{ if } \forall k \, \chi(k) \ \wedge \ \forall k \, \neg\mathrm{Prov}_{\mathbf{PR}}(k, \ulcorner\chi\urcorner) \end{cases}$$

Union of latter two cases gives

Decidability Theorem:

Complete **T**-alternative for PR predicates $\chi = \chi(n) : \mathbb{N} \to \mathbb{2}$:

- *counterexample* $\mathbf{T} \vdash \exists n \, \neg\chi(n)$

  or else

- *theorem* $\mathbf{T} \vdash \forall n \, \chi(n)$

Decision Remark: this does not mean a priori that *decision algorithm* $\nabla\chi$ *terminates* for all such predicates χ. The theorem says only that χ is ***decidable*** "by", *within* theory $\mathbf{T}$, that it is ***not independent*** of $\mathbf{T}$.

7 Gödel's incompleteness theorems

We visit §2. Gödel's theorems, in Smorynski 1977, Handbook of Mathematical Logic.

FIRST INCOMPLETENESS THEOREM. *Let* $\mathbf{T}$ *be a formal theory containing arithmetic. Then there is a sentence* φ *which asserts its own unprovability and such that:*

(i) *If* $\mathbf{T}$ *is consistent,* $\mathbf{T} \nvdash \varphi$

(ii) *If* $\mathbf{T}$ *is* ω*-consistent,* $\mathbf{T} \nvdash \neg\varphi$

In §3.2.6 Smorynski discusses possible choices of *arithmetic* theory, namely

(a) **PRA** = classical primitive recursive arithmetic.

(b) **PA** = Peano Arithmetic.
Conjecture: $\mathbf{PA} \cong \mathbf{PR} + \exists$

(c) **ZF** = Zermelo-Fraenkel set theory. "This is both a good and a bad example. It is bad because the whole encoding problem is more easily solved in a set theory than in an arithmetical theory. By the same token, it is a good example."

We take for formal extension $\mathbf{T}$ of $\mathbf{PR}$ one of the categorical pendants to gödelian quantified arithmetical theories – with finite descent – (subsystems of **ZF**, see OSIUS 1974).

A minimal choice for our purposes – **conjecture** – is quantified arithmetical theory $\mathbf{T} = \mathbf{PA} + \omega^\omega$:

$\mathbf{PA}$ + the lexicographic order on $\omega^\omega \supset \mathbb{N}[\omega]$ a well-order.

Smorynski's **proof** gives the *First Gödel's incompleteness theorem* for $\mathbf{T}$, and from that the following

Second incompleteness theorem: Let $\mathbf{T}$ be one of the quantified arithmetical theory extensions above of $\mathbf{PR}$ and $\mathbf{T}$ consistent. Then

$$\mathbf{T} \nvdash \mathrm{Con}_{\mathbf{T}}$$

where here $\mathrm{Con}_{\mathbf{T}} = \neg\exists\, k\ \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner)$ is the sentence asserting the consistency of $\mathbf{T}$.

The consistency formula $\mathrm{Con}_{\mathbf{T}}$ *of* $\mathbf{T}$ *is not derivable in Metamathematics, even if theory* $\mathbf{T}$ *itself is taken as metamathematical frame,* provided that $\mathbf{T}$ is consistent.

8 Inconsistency provability

Predicate $\mathrm{Prov}_{\mathbf{T}}(x, y)$ corresponds to Gödel formula

45. xBy, x ist *Beweis* von y.

Gödel proves that this formula is *rekursiv,* primitive recursive in contemporary terms.

[Later Ackermann found "Ackermann recursive" functions growing faster than any "primitive recursive" function, evaluation ***eva*** above is of this type]

Formula 46. $\exists x\, xBy$ 'y ist *beweisbar'* is a priori, formally not primitive recursive, same as for "undecidable" formula 17 Gen r

But $\mathrm{Con}_{\mathbf{T}} = \neg\exists k\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) = \forall k\, \neg\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner)$ corresponds to the free-variable PR predicate $\neg\mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner) : \mathbb{N} \to \mathbb{2}$, decidable by Decision Theorem above.

From the above Gödel's **2nd theorem** and **PR decision theorem** for quantified arithmetical theories we conclude

Inconsistency provability theorem for quantified arithmetical theories $\mathbf{T}$ (with finite descent): Such theory $\mathbf{T}$ derives its own inconsistency formula:

$$\mathbf{T} \vdash \neg\,\mathrm{Con}_{\mathbf{T}} \ \text{ i.e. } \ \mathbf{T} \vdash \exists k\, \mathrm{Prov}_{\mathbf{T}}(k, \ulcorner \mathrm{false} \urcorner)$$

8.1 Proof Résumé:

- The consistency formula for "any" theory, in particular for an arithmetical theory, can be stated in terms of a free variable PR predicate: *For any number k ($k \in \mathbb{N}$ free), k is not the enumeration index of a proof code for (code of)* false.
- Quantified arithmetics $\mathbf{T}$ with finite descent admit (correctly terminating) *evaluations* of their PR map code sets.
- Such theory $\mathbf{T}$ is able to decide any PR predicate on *counterexample vs.* **PR** *provability: Decision Theorem.*
- In particular the consistency formula of such theory $\mathbf{T}$ is **decided** by theory $\mathbf{T}$ taken as "metamathematical" frame.
- This result leads to self-inconsistency of quantified arithmetical theory (with finite descent) by the second Gödel's *Incompleteness Theorem on non-derivability of theorie's* $\mathbf{T}$ *consistency formula, theory* $\mathbf{T}$ *assumed consistent.*

[If **T** is inconsistent, it derives everything, in particular its own inconsistency formula]

Note: Observe that Gödel's undecidable formula 17 Gen r is not primitive recursive.

Remark: A way out will be given in part II on *Iterative Self-Consistency.* Choose as "metamathematical" frame *iterative descent theory* $\pi\mathbf{R}$: p. r. theory **PR** with extension objects of predicates – among these *universal object* $\mathbb{X}$ of all (internal) numerals and nested numpairs –, and additional **axiom schema** of *non-infinite descent* of all complexity controlled iterations.

8.2 Discussion

Background for the discussion are the books of Yu. I. Manin 2010 and K. Sigmund 2015.

"Vorbilder des Wiener Kreises sind der Physiker Albert Einstein, der Mathematiker David Hilbert und der Philosoph Bertrand Russell." [7]

Russell had discovered a first contradiction in Frege-Cantor's **set** theory, namely availability of "set" $R = \{x : x \notin x\}$ with paradoxical property $R \in R \iff R \notin R$, and authored with Whitehead 1900 the (typified) Principia Mathematica **PM** in order to exclude this paradoxon from **set** theory.

The first two of Hilbert's famous 10 (later 23) **problems**[8] ask for a provably consistent foundation of Mathematics (and decision of the Continuum Hypothesis **CH**). Hilbert: "Wir

[7] Sigmund 2015

[8] talk at ICM conference Paris 1900, *Gesammelte Abhandlungen.* Springer 1970

wollen wissen, wir werden wissen. ... Niemand wird uns aus dem Paradies vertreiben, in das Cantor uns geführt hat." Hilbert devoted himself to a solution of these first (and second) problems.

In the opinion of the majority of Mathematicians, Gödel has "erledigt" Hilbert's formalistisches **Programm** with the publication of his two **incompleteness** theorems for *Principia Mathematica* **PM** *und verwandte Systeme,* such as in particular Zermelo-Fraenkel **set** theory **ZF.**

The *anti-idealistic anti-metaphysical Wiener Kreis*, Gödel's intellectual home, celebrated Gödel for his [vermeintliche] Relativierung of the GREEK identity of truth with *provability* in axiomatic Mathematics.[9]

Gödel himself was said to have doubts on his ***assumption*** of ω-consistency, of non-ω-inconsistency. Did he even have doubts on *consistency* of **PM**? As K. Sigmund reports, Gödel became deeply depressive, after his death answers to letters (not given to mail) were found in his desk revealing his platonic convictions.

A. Grothendieck told us after his "green" talk in the 1980ties in Berlin, that S. Eilenberg had proposed to N. BOURBAKI a categorical approach to Foundations "but A. Weyl n'en voulait pas."

[9] cf. MANIN 2010, II 11.7. **Gödel's Incompleteness Theorem for Arithmetic.** ... $\{$*true formulas*$\} \neq \{$*deducible formulas*$\}$

Appendix: Iterative evaluation

We resolve *uniform* evaluation $\boldsymbol{eva}$ into a **CCI** (*Complexity Controlled Iteration*) and **show** within framework of quantified arithmetical theory **T** with finite descent of chains of polynomials in $\mathbb{N}[\omega]$ the following

Evaluation Resolution:

Evaluation $\boldsymbol{eva} = \boldsymbol{eva}(f, x) : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}}$ can be resolved into a *Complexity Controlled Iteration* (CCI):

$$\underline{\text{while}}\ \boldsymbol{cx}\, f > 0\ \underline{\text{do}}\ (f, x) := \boldsymbol{e}(f, x)\ \underline{\text{od}}$$

where $\boldsymbol{cx} = \boldsymbol{cx}\, f : \mathrm{PR} \to \mathbb{N}[\omega]$ is a suitable map code *complexity* within the linearily ordered semiring of polynomials with coefficients in $\mathbb{N}$. This complexity descends, eventually down to 0, with each application of *evaluation step* $\boldsymbol{e} = \boldsymbol{e}(f, x) : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR} \times \mathbb{X}_{\underline{\perp}}$ and is to give evaluation result as value in right component $\mathbb{X}$ upon reaching complexity $0 = \boldsymbol{cx}\, \ulcorner \mathrm{id} \urcorner$ in left component PR.

Iterative evaluation of cartesian theory CA

evaluation step

$$\boldsymbol{e} = \boldsymbol{e}\,(f, a) = (\boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a)) :$$

$$\mathrm{CA} \times \mathbb{X}_{\underline{\perp}} \longrightarrow \mathrm{CA} \times \mathbb{X}_{\underline{\perp}}$$

$$\mathbb{X}_{\underline{\perp}} = \mathbb{X} \cup \{\underline{\perp}\} \subset \mathbb{N},\ \underline{\perp} = \ulcorner \perp \urcorner\ (\textit{trash})$$

$$\mathbb{X} = \cup_A \dot{A} = \{\mathbb{N} : \mathbb{N} \xrightarrow{\mathbb{X}} 2\}\ \text{universal set}$$

of internal *numerals* and (nested) *numpairs*

$\boldsymbol{e}_{\mathrm{arg}}(f, a)$ *is the intermediate argument obtained by one evaluation step applied to the pair* (f, a), *and* $\boldsymbol{e}_{\mathrm{map}}(f, a)$ *is the remaining map code still to be evaluated on intermediate argument* $\boldsymbol{e}_{\mathrm{arg}}(f, a)$, *same then iteratively applied to pair* $(\boldsymbol{e}_{\mathrm{map}}, \boldsymbol{e}_{\mathrm{arg}})$

This evaluation step $\boldsymbol{e}$ is **defined** by recursive case distinction below, *controlled* by $\mathbb{N}$-valued descending **complexity**

$$\boldsymbol{cx} = \boldsymbol{cx}\, f \ \in \mathbb{N}$$

in turn p. r. defined by

$$\begin{aligned}
&\boldsymbol{cx}\, \ulcorner \mathrm{id} \urcorner := 0 \\
&\boldsymbol{cx}\, \ulcorner \mathrm{ba} \urcorner := 1,\ \mathrm{ba} \in \mathrm{bas} \smallsetminus \{\mathrm{id}\} = \{0, \mathrm{s}, \Pi, \ell, \mathrm{r}\} \\
&\boldsymbol{cx}\, \langle g \odot f \rangle := \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1 \\
&\boldsymbol{cx}\, \langle f ; g \rangle := \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1 \\
&\boldsymbol{cx}\, \langle f \dot{\times} g \rangle := \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1
\end{aligned}$$

evaluation step $\boldsymbol{e} = \boldsymbol{e}(f, a)$ is p. r. defined (and is iteration complexity-controlled) as follows:

- **basic map cases:**

$$\begin{aligned}
&\boldsymbol{e}\,(\ulcorner \mathrm{id} \urcorner, a) := (\ulcorner \mathrm{id} \urcorner, a),\ \boldsymbol{cx}\, \ulcorner \mathrm{id} \urcorner = 0,\ \textit{stationary}; \\
&\boldsymbol{e}\,(\ulcorner \mathrm{ba} \urcorner, a) := (\ulcorner \mathrm{id} \urcorner, \boldsymbol{eva}(\ulcorner \mathrm{ba} \urcorner, a)) \\
&\quad \text{with } \boldsymbol{eva}(\ulcorner \mathrm{ba} \urcorner, a) = \nu_B\, \mathrm{ba}\, \nu_A^{-1}\, a \\
&\quad A = \mathrm{Dom}\, \mathrm{ba},\ \ B = \mathrm{Codom}\, \mathrm{ba} \\
&\quad \mathrm{ba} \in \mathrm{bas}' = \{0, \mathrm{s}, \Pi_A, \ell_{A,B}, \mathrm{r}_{A,B} : A, B\ \textbf{PR}\ \text{objects}\} \\
&\nu = \nu_A : A \xrightarrow{\cong} \dot{A}\ \text{(internal)}\ \textit{numeralisation;} \\
&\boldsymbol{cx}(\ulcorner \mathrm{id} \urcorner) = 0 < \boldsymbol{cx}(\ulcorner \mathrm{ba} \urcorner) = 1,\ \mathrm{ba} \in \mathrm{bas}'
\end{aligned}$$

- **composition cases:**
  - identity subcase:

$$\boldsymbol{e}(g \odot \ulcorner \mathrm{id}_A \urcorner, a) := (g, a)$$
$$\boldsymbol{cx}\, g < \boldsymbol{cx}\, g + 0 + 1 = \boldsymbol{cx}\, \langle g \odot \ulcorner \mathrm{id}_A \urcorner \rangle$$

  - For $f \in [A, B],\ g \in [B, C],\ a \in A,\ \boldsymbol{cx}\, f > 0:$

$$\boldsymbol{e}\,(g \odot f, a) = (\boldsymbol{e}_{\mathrm{map}}(g \odot f, a), \boldsymbol{e}_{\mathrm{arg}}(g \odot f, a))$$
$$:= (g \odot \boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a))$$

Complexity descent:

$$\boldsymbol{cx}\ \boldsymbol{e}_{\mathrm{map}}(g \odot f, a)$$
$$= \boldsymbol{cx}\,(g \odot \boldsymbol{e}_{\mathrm{map}}(f, a), a)$$
$$= \boldsymbol{cx}\, \boldsymbol{e}_{\mathrm{map}}(f, a) + \boldsymbol{cx}\, g + 1$$
$$< \boldsymbol{cx}\, f + \boldsymbol{cx}\, g + 1$$
$$= \boldsymbol{cx}\, \langle g \odot f \rangle$$

- **cases of an induced:**
  - identities case:

$$\boldsymbol{e}(\langle \ulcorner \mathrm{id}_C \urcorner; \ulcorner \mathrm{id}_C \urcorner \rangle, c) := (\ulcorner \mathrm{id}_{C \times C} \urcorner, \langle c; c \rangle)$$
$$\boldsymbol{cx}\, \ulcorner \mathrm{id}_{C \times C} \urcorner = \boldsymbol{c}(\ulcorner \mathrm{id} \urcorner) = 0$$
$$< 1 = \boldsymbol{cx}(\langle \ulcorner \mathrm{id}_C \urcorner; \ulcorner \mathrm{id}_C \urcorner \rangle)$$

– case $f \in [C, A]$, $g \in [C, B]$, not both equal to $\ulcorner \mathrm{id}_C \urcorner$:

$$
\begin{aligned}
&\boldsymbol{e}\,(\langle f; g\rangle, c)\\
&:= (\langle \boldsymbol{e}_{\mathrm{map}}(f, c); \boldsymbol{e}_{\mathrm{map}}(g, c)\rangle, \langle \boldsymbol{e}_{\mathrm{arg}}(f, c); \boldsymbol{e}_{\mathrm{arg}}(g), c\rangle)\\
&\boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(\langle f; g\rangle, c)\\
&= \boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(f, c) + \boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(g, c) + 1\\
&< \boldsymbol{cx}\ f + \boldsymbol{cx}\ g + 1 = \boldsymbol{cx}\,\langle f; g\rangle
\end{aligned}
$$

since in this case $\boldsymbol{cx}\ f > 0$ *and/or* $\boldsymbol{cx}\ g > 0$,
and therefore $\boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}\,(f, c) < \boldsymbol{cx}\ f$
and/or $\boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(g, c) < \boldsymbol{cx}\ g$

cartesian cases:

$$
\begin{aligned}
&\boldsymbol{e}\,(\ulcorner \mathrm{id}_A \urcorner \dot{\times} \ulcorner \mathrm{id}_B \urcorner, \langle a; b\rangle) := (\ulcorner \mathrm{id}_{A\times B} \urcorner, \langle a; b\rangle)\\
&\boldsymbol{cx}\ \ulcorner \mathrm{id}_{A\times B} \urcorner = 0\\
&< 1 = \boldsymbol{cx}\ \ulcorner \mathrm{id}_A \urcorner + \ulcorner \mathrm{id}_B \urcorner + 1 = \boldsymbol{cx}\,\langle \ulcorner \mathrm{id}_A \urcorner \dot{\times} \ulcorner \mathrm{id}_B \urcorner \rangle
\end{aligned}
$$

For $f \in [A, A']$, $g \in [B, B]$ not both identity codes:

$$
\begin{aligned}
&\boldsymbol{e}\,(f \dot{\times} g, \langle a; b\rangle)\\
&:= (\boldsymbol{e}_{\mathrm{map}}(f, a) \dot{\times} \boldsymbol{e}_{\mathrm{map}}(g, a), \langle \boldsymbol{e}_{\mathrm{arg}}(f, a); \boldsymbol{e}_{\mathrm{arg}}(g, b)\rangle)
\end{aligned}
$$

one-step-evaluate both components f *and* g *in parallel.*

Complexity descent:

$$
\begin{aligned}
&\boldsymbol{cx}\,\langle \boldsymbol{e}_{\mathrm{map}}(f, a) \dot{\times} \boldsymbol{e}_{\mathrm{map}}(g, b)\rangle\\
&= \boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(f, a) + \boldsymbol{cx}\,\boldsymbol{e}_{\mathrm{map}}(g, b) + 1\\
&< \boldsymbol{cx}\ f + \boldsymbol{cx}\ g + 1 = \boldsymbol{cx}\,\langle f \dot{\times} g\rangle.
\end{aligned}
$$

Evaluation of theory PR

Let $\dot{\S} = ^{\ulcorner \S \urcorner}$ code the iteration symbol of **PR**

CA evaluation **step** $\boldsymbol{e}$ is **extended** by clause:

For endomap code $f \in [A, A] = [A, A]_{\mathbf{PR}}$ and $a \in \dot{A}$

$$\begin{aligned}
&\boldsymbol{e}\,(f^{\dot{\S}}, \langle a; \ulcorner 0 \urcorner \rangle) := (f^0, a) \\
&\boldsymbol{e}\,(f^{\dot{\S}}, \langle a, \nu(\mathrm{s}\,n) \rangle) := (\langle f \odot f^n \rangle, a)
\end{aligned}$$

where $f^0 := \ulcorner \mathrm{id} \urcorner$

$f^{\mathrm{s}\,n} := \langle f \odot f^n \rangle$ recursively:

code expansion

Complexity extension:

$$\boldsymbol{cx}\, f^{\dot{\S}} := (\boldsymbol{cx}\, f + 1) \cdot \omega \; \in \; \mathbb{N}[\omega]$$

$\mathbb{N}[\omega]$ the well-ordered semiring of polynomials in one indeterminate over $\mathbb{N}$, pendant to set theoretic ordinal ω^ω : Within **set** theory **T**, $\mathbb{N}[\omega]$ has only *finite descending chains.*

In this "acute" iteration case we have
complexity descent

$$\boldsymbol{cx}\, f^0 = \boldsymbol{cx}\, \ulcorner \mathrm{id} \urcorner = 0 < (\boldsymbol{cx}\, f + 1) \cdot \omega = \boldsymbol{cx}\, f^{\dot{\S}}$$

and further inductively

$$\begin{aligned}
&\boldsymbol{cx}\, f^{\mathrm{s}\,n} = \boldsymbol{cx}\, \langle f \odot \langle f \dots f \rangle \dots \rangle \\
&= \boldsymbol{cx}\, f \cdot \mathrm{s}\,n + n \\
&< (\boldsymbol{cx}\, f + 1) \cdot (n + 1) \\
&< (\boldsymbol{cx}\, f + 1) \cdot \omega = \boldsymbol{cx}\, f^{\dot{\S}}
\end{aligned}$$

Explication: $\boldsymbol{cx}$ now takes values within the linearly ordered semiring $\mathbb{N}[\omega] \supset \mathbb{N}$ of polynomials in one indeterminate ω, ω thought to represent (arbitrarily) big natural numbers. So in fact $\boldsymbol{cx}(f^{\mathrm{s}\,n}) < \boldsymbol{cx}(f^{\dot{\S}})$ since the former polynomial has lower degree than the latter.

Linear order of polynomials $p, q \in \mathbb{N}[\omega]$ is defined hierarchically by first comparison of the *degrees* of p and q, second in case of equal degrees by comparison of the pivot coefficients, and third, if the pivot monomials are equal, recursively by comparison of the polynomials p and q with the two pivot monomials deleted.

Note: A first approach to evaluate arbitrary constants $c : \mathbb{1} \to A$ of **PR** into numerals/nested numpairs has been given in Lassmann 1981.

Evaluation Resolution Theorem:

- Evaluation $\boldsymbol{eva}$ of **PR** map code variable $f \in [A, B] = [A, B]_{\mathbf{PR}} \subset \mathrm{PR}$ on (fitting) arguments $a \in \dot{A} \subset \mathbb{X}$ is **totally defined** by the ***complexity controlled iteration*** (CCI)

$$\boldsymbol{eva} = \boldsymbol{eva}(f, a) := \begin{cases} \underline{\text{init}} \left\{ (h, x) := (f, a) \right. \\ * \\ \underline{\text{while}}\, [\boldsymbol{cx}\, h > 0] \\ \underline{\text{do}}\ (h, x) := \boldsymbol{e}\,(h, x)\ \underline{\text{od}} \\ * \\ \underline{\text{result}} := x \end{cases}$$

which always **terminates,** (at least) within quantified arithmetical theories $\mathbf{T}$ with finite descent since there complexity (co)domain $\mathbb{N}[\omega]$ has only *finite descending chains* **whence**

$$\text{(term)}\quad \frac{f \in [A, A] \text{ (endo)map code variable}}{\begin{array}{l}(\exists m \in \mathbb{N})\, \boldsymbol{e}^m(f, a) = (\ulcorner \mathrm{id} \urcorner, \boldsymbol{eva}\,(f, a)) \\ [m = m(f, a) = \mu\{\tilde{m} : \boldsymbol{cx}\, \boldsymbol{e}^{\tilde{m}}(f, a) = 0\}] \\ \textbf{so}\quad \boldsymbol{eva}\,(f, a) = \mathrm{r}\, \boldsymbol{e}^m(f, a)\end{array}}$$

- $\boldsymbol{eva}$ is **characterised** by the double recursion ("Ackermann")

$$\boldsymbol{eva}(\mathrm{ba}, a) = \nu_B(\mathrm{ba}(\nu_A^{-1}(a)))$$
$$\text{for ba} \in \text{bas},\ A = \mathrm{Dom}[\mathrm{ba}],\ B = \mathrm{Codom}[\mathrm{ba}]$$
$$\boldsymbol{eva}(g \odot f, a) = \boldsymbol{eva}(g, \boldsymbol{ev}(f, a))$$
$$\boldsymbol{eva}\langle f; g\rangle, c) = \langle \boldsymbol{eva}(f, c); \boldsymbol{eva}(g, c)\rangle$$
$$\boldsymbol{eva}(f \dot{\times} g, \langle a; b\rangle) = \langle \boldsymbol{eva}(f, a); \boldsymbol{eva}(g, b)\rangle$$

as well as

$$\boldsymbol{eva}(f^{\dot{\S}}, \langle a; \ulcorner 0 \urcorner\rangle) = a \text{ and}$$
$$\boldsymbol{eva}(f^{\dot{\S}}, \langle a; \nu(\mathrm{s}\, n)\rangle) = \boldsymbol{eva}(f, \boldsymbol{ev}(f^{\dot{\S}}, \langle a; \nu n\rangle)$$

- **define** (natural) *evaluation family*

$$\boldsymbol{ev} = \boldsymbol{ev}_{A,B} = \boldsymbol{ev}_{A,B}(f, a) : [A, B] \times A \to B \text{ by}$$
$$\boldsymbol{ev}_{A,B}(f, a) = \nu_B^{-1}(\boldsymbol{eva}(f, \nu_A(a))$$

This family $\boldsymbol{ev}$ is *objective:*

$$\frac{f : A \to B \ \ \mathbf{PR} \text{ map}}{\boldsymbol{ev}\,(\ulcorner f \urcorner, a) = f(a) : A \to B}$$

"evaluation is application."

Proof of evaluation resolution theorem

by (external) Peano induction on *iteration-index-until-termination* $m = m(h, x) \in \mathbb{N}$, via *case distinction* on **PR** map h and (fitting) $x \in \mathbb{X}$ appearing in the different cases of the asserted conjunction.

- anchor $m = 0, 1 : h = \ulcorner \mathrm{ba} \urcorner,\ \mathrm{ba} \in \mathrm{bas} = \{\mathrm{id}, 0, s, \Pi, \ell, \mathrm{r}\}$ see evaluation definition above.

cases $\mu = \mu\{\tilde{m} : \boldsymbol{e}^{\tilde{m}}(h, x) = (\ulcorner \mathrm{id} \urcorner, \boldsymbol{ev}\,(h, x))\} = m + 1 :$

- case $(h, x) = (g \odot f, a)$ of an (internally) *composed,* subcase $f = \ulcorner \mathrm{id} \urcorner$: obvious.
- non-trivial subcase $(h, x) = (g \odot f, a),\ f \neq \ulcorner \mathrm{id} \urcorner$:

$$\boldsymbol{eva}(g \odot f, a) = \mathrm{r}\,\boldsymbol{e}^m(g \odot \boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a))$$

by iterative definition of $\boldsymbol{eva}$ *in this case,*

m *fold iteration*

$$= \boldsymbol{eva}(g, \boldsymbol{eva}(\boldsymbol{e}_{\mathrm{map}}(f, a), \boldsymbol{e}_{\mathrm{arg}}(f, a)))$$

$$= \boldsymbol{eva}(g, \mathrm{r}\,\boldsymbol{e}^m\,(f, a))$$

$$= \boldsymbol{eva}(g, \boldsymbol{eva}(f, a))$$

The latter three equations hold (backwards) by induction hypothesis on m

Objectivity in this case, substitute $\ulcorner f : A \to B \urcorner$ into $f \in [A, B]$, $\ulcorner g : B \to C \urcorner$ into $g \in [B, C]$:

$$\boldsymbol{ev}(\ulcorner g \circ f \urcorner, a) = \boldsymbol{ev}(\ulcorner g \urcorner \odot \ulcorner f \urcorner, a)$$
$$= \boldsymbol{ev}(\ulcorner g \urcorner, \boldsymbol{ev}(\ulcorner f \urcorner, a)) \text{ see } \boldsymbol{eva} \text{ just above}$$
$$= \boldsymbol{ev}(\ulcorner g \urcorner, f(a)) = g(f(a))$$
both by hypothesis on m
$$= (g \circ f)(a) \text{ q. e. d. in this case}$$

- case $(h, x) = (\langle f; g \rangle, c)$ of an (internal) *induced:* Obvious by definition of $\boldsymbol{eva}$ and then of $\boldsymbol{ev}$ on an induced into a product.
- case $(h, x) = (f \dot{\times} g, \langle a; b \rangle)$ of an (internal) *cartesian product:* Obvious by definition of $\boldsymbol{eva}$ and then of $\boldsymbol{ev}$ on a cartesian product of maps.
- anchor case $(h, x) = (f^{\dot{\S}}, \langle a; \ulcorner 0 \urcorner \rangle)$ of an iterated:

  $$\boldsymbol{eva}(f^{\dot{\S}}, (a, \ulcorner 0 \urcorner)) = a = \boldsymbol{eva}(\ulcorner \mathrm{id} \urcorner, a)$$

- step case $(h, x) = (f^{\dot{\S}}, \langle a; \nu(\mathrm{s}\, n) \rangle)$ of a genuine (internally) iterated:

$$\begin{aligned}
&\boldsymbol{eva}(f^{\dot{\S}}, \langle a; \nu(\mathrm{s}\, n)\rangle) \\
&= \boldsymbol{eva}(\boldsymbol{e}(f^{\dot{\S}}, \langle a; \nu(\mathrm{s}\, n)\rangle)) \\
&= \boldsymbol{eva}(f^{s\, n}, a) \ (\textit{definition of evaluation step } \boldsymbol{e}) \\
&= \boldsymbol{eva}(f \odot f^{n}, a) \ (\textit{recursive definition of } f^{s\, n}) \\
&= \boldsymbol{eva}(f, \boldsymbol{eva}(f^{n}, a)) \ \textit{by induction hypothesis on } m \\
&\quad\quad \textit{case of a composed map} \\
&= \boldsymbol{eva}(f, \boldsymbol{eva}(f^{\dot{\S}}, \langle a; \nu n\rangle))
\end{aligned}$$

Proof of **objectivity** in this last case: substitute $\ulcorner f \urcorner$ into $f \in [A, A]$ and get from the above

$$\begin{aligned}
&\boldsymbol{ev}(\ulcorner f^{\S} \urcorner, (a, \mathrm{s}\, n)) \\
&\boldsymbol{ev}(\ulcorner f \urcorner^{\dot{\S}}, (a, \mathrm{s}\, n)) \\
&= \nu_A^{-1}(\boldsymbol{eva}(\ulcorner f \urcorner^{\dot{\S}}, \langle \nu_A(a); \nu(\mathrm{s}\, n)\rangle)) \\
&= \nu_A^{-1}(\boldsymbol{eva}(\ulcorner f \urcorner \odot \ulcorner f \urcorner^{\dot{\S}}, \langle \nu_A(a); \nu n\rangle)) \text{ by the above} \\
&= \nu_A^{-1}(\boldsymbol{eva}(\ulcorner f \circ f^{\S} \urcorner, \langle \nu_A(a); \nu n\rangle)) \\
&= (f \circ f^{\S})(a, n) = f^{\S}(a, \mathrm{s}\, n) \text{ by } \textit{naturality of } \nu
\end{aligned}$$

This shows the theorem in the remaining iteration case **q. e. d.**

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