ASYMPTOTIC BOUNDARY ELEMENT METHODS FOR THIN CONDUCTING SHEETS

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Abstract. Various asymptotic models for thin conducting sheets in computational electromagnetics describe them as closed hyper-surfaces equipped with linear local transmission conditions for the traces of electric and magnetic fields. The transmission conditions turn out to be singularly perturbed with respect to limit values of parameters depending on sheet thickness and conductivity.

We consider the reformulation of the resulting transmission problems into boundary integral equations (BIE) and their Galerkin discretization by means of low-order boundary elements. We establish stability of the BIE and provide a priori h-convergence estimates, with the dependence on model parameters made explicit throughout. This is achieved by a novel technique harnessing truncated asymptotic expansions of Galerkin discretization errors.

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1. Introduction

1.1. Thin Conducting Sheets

We consider a thin conducting sheet of constant thickness $d > 0$ and constant relative conductivity ξ, which is, for simplicity, the only conductor in otherwise non-conductive space. Its locus Ω_{int} can be expressed as all points with minimal distance $d/2$ to its mid-line Γ, which is supposed to be a closed cylinder. Figure 1 displays a cross-section.

Assume translation invariance in one direction. Then, for the transverse magnetic (TM) mode, the complex amplitude E of the out-of-plane component of the electric field solves the partial differential equation (PDE)

$$-\Delta E(x) + \xi^2(x)E(x) = F(x) \quad \text{in } \mathbb{R}^2 , \tag{1}$$

where

$$\xi^2(x) = \begin{cases} -\mathrm{i}\omega\mu\sigma , & \text{for } x \in \Omega_{\text{int}} , \\ 0 , & \text{for } x \notin \Omega_{\text{int}} , \end{cases} \tag{2}$$

with angular frequency $\omega > 0$, permeability μ and conductivity σ (of the sheet material). The PDE (1) describes the relation of an injected electric current J_0 (out-of-plane) via the source term $F(x) = -\mathrm{i}\omega\mu(x)J_0(x)$ and the induced electric field $E(x)$. The in-plane magnetic field $\underline{H}$ can be recovered as $\mathrm{i}\omega\mu^{-1}(x)(\nabla E(x))^{\perp}$, where we wrote $\underline{V}^{\perp} = (V_2, -V_1)$. The PDE (1) has to be supplemented with a decay condition at infinity.

Remark 1.1. *In the electromagnetic context,* (1) *is an intrinsically two-dimensional model and it will be the 2D setting for which the equations discussed in this paper have direct physical relevance. However,* (1) *still makes mathematical sense in* $\mathbb{R}^3$*, whence* Γ *is a closed 2-dimensional orientable manifold. Thus, in the remainder of this article we will treat* (1) *set in* $\mathbb{R}^n$*,* $n = 2, 3$*.*

Let $\gamma_0^{\pm} : H^1_{\text{loc}}(\Omega_{\text{ext}}) \to H^{\frac{1}{2}}(\Gamma)$ denote the Dirichlet trace operators from outside and inside of Γ, respectively. Similarly, write $\gamma_1^{\pm} : H^1_{\text{loc}}(\Delta, \Omega_{\text{ext}}) \to H^{-1/2}(\Gamma)$ for the standard Neumann traces. Their jumps and means are denoted by

$$[\gamma_\ell V] := (\gamma_\ell^+ V) - (\gamma_\ell^- V), \qquad \{\gamma_\ell V\} := \frac{1}{2}\left((\gamma_\ell^+ V) + (\gamma_\ell^- V)\right) . \tag{3}$$

1.2. Impedance Transmission Conditions

If $d \ll \operatorname{diam}(\Gamma)$, we may model the impact of the conducting sheet on the fields through so-called impedance transmission conditions (ITCs) connecting traces of electric and magnetic fields on both sides of Γ, see Section 2. The resulting transmission problem has the general form

$$\begin{aligned} -\Delta U &= F, && \text{in } \mathbb{R}^n \backslash \Gamma, \\ T_{11}[\gamma_0 U] + T_{12}\{\gamma_0 U\} + T_{13}\{\gamma_1 U\} &= 0, && \text{on } \Gamma, \\ T_{21}[\gamma_1 U] + T_{22}\{\gamma_0 U\} + T_{23}\{\gamma_1 U\} &= 0 && \text{on } \Gamma , \end{aligned} \tag{4}$$

where the T_{ij} may be mere (complex) coefficients but can also compromise (tangential) differential operators on Γ. We denote ∇_Γ the tangential gradient and Δ_Γ the Laplace-Beltrami operator on Γ, which are the first and second tangential derivative for $n = 2$, respectively. Also the equations (4) have to be supplemented with suitable (decay) conditions for U at infinity, see [11, Chap. 8, p. 259].

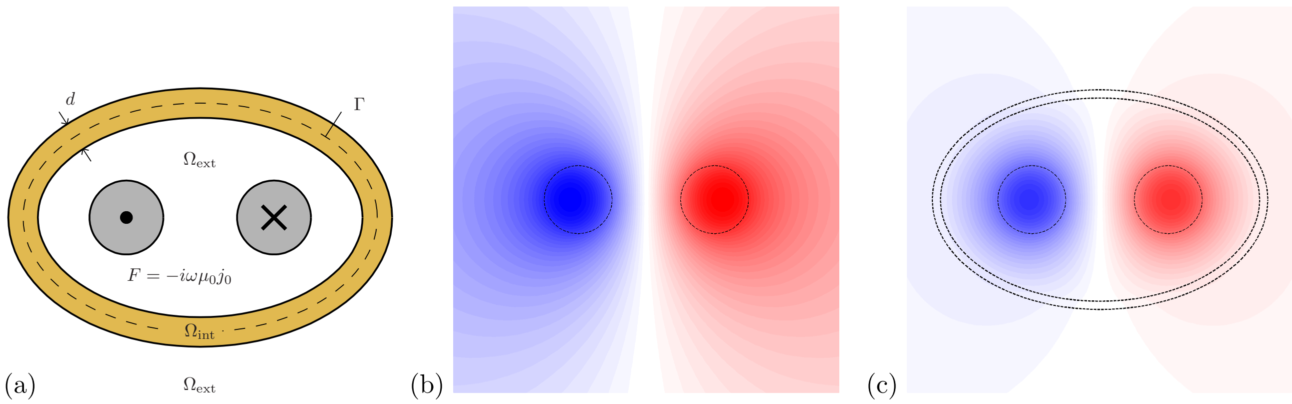

FIGURE 1. (a) Geometric setting of a thin conducting sheet in Ω_{int} of thickness d and with mid-line Γ. The exterior of the sheet $\Omega_{\text{ext}} = \mathbb{R}^2 \backslash \Omega_{\text{int}}$ where $\xi(x) = 0$ houses source currents. (b) Electric field (real part) of two current carrying circular wires without conducting shielding sheet. (c) Electric field (real part) of two current carrying circular wires in presence of a thin ellipsoidal conducting sheet.

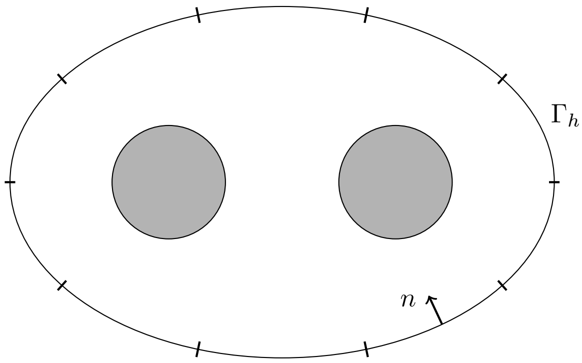

FIGURE 2. Mesh Γ_h of the mid-line Γ for boundary element methods for the different impedance transmission conditions.

1.3. **Overview**

In this article we are concerned with

- the derivation and analysis of *boundary integral equations* (BIE) equivalent to the transmission problem (4),
- a priori convergence estimates for low-order conforming *boundary element (BEM) Galerkin discretizations* of those.

This will be done for an array of concrete asymptotic shielding models presented in Section 2. Particular emphasis will be put on making explicit the dependence of stability and convergence estimates on model parameters, because these may become small or large (in modulus), which may cause singular perturbations.

In general, the asymptotic models are justified only for sufficiently smooth mid-lines Γ. On the other hand, both the boundary integral equations and the numerical methods remain meaningful, if Γ is merely Lipschitz continuous. Thus, Lipschitz continuity will be our minimal assumption on Γ. Several results will hinge on extra smoothness of Γ, which will be specified precisely in each case.

A brief survey of boundary integral operators and boundary element spaces is supplied in Section 3.1. Then transmission problems with particular structure and dependence on parameters are discussed in Sections 4 through 7. In each, we establish the stability of the derived BIE, then introduce suitable boundary element Galerkin discretizations, followed by investigations into their convergence. Our a priori error estimates for Galerkin BEM often rely on a novel technique for proving uniform stability of parameter dependent discrete variational problems, which is based on asymptotic expansions. Numerical experiments for a model problem are included to demonstrate, how the predictions of the theory manifest themselves in actual computations.

1.4. Model problem for numerical experiments

For illustration we will study the different ITCs by numerical experiments for a simple model problem. A thin conducting sheet of thickness $d = 3\,\mathrm{mm}$ has an ellipsoidal mid-line with the semi-axes $60\,\mathrm{mm}$ and $\sqrt{1500}\,\mathrm{mm} \approx 38.7\,\mathrm{mm}$, centered around the origin. The electromagnetic fields are excited by two cylindrical current carrying wires with radius $12.5\,\mathrm{mm}$ centered at positions $(\pm\,25\,\mathrm{mm}, 0)$ and with $F = \pm 1$ (arbitrary units), see the schematic sketch in Fig. 1(a). For one set of computations we choose $|\xi|^{-1} = 6.547\,\mathrm{mm}$ corresponding to a skin depth $d_{\mathrm{skin}} = \sqrt{2}/|\xi| \approx 9.26\,\mathrm{mm}$, about three times the sheet thickness. For copper, for which $\sigma = 5.91{\cdot}10^7\,\mathrm{A(Vm)}^{-1}$, $\mu = \mu_0 = 4\pi{\cdot}10^{-7}\,\mathrm{Vs(Am)}^{-1}$, this parameter ξ corresponds to a frequency of $50\,\mathrm{Hz}$ ($\omega = 314\,\mathrm{rad/s}$). We have computed reference solutions using a high order finite element discretisation with exactly curved cells with the numerical C++ library Concepts [5,6]. Our problem is stated in the unbounded space $\mathbb{R}^2$, which we model by exact Dirichlet-to-Neumann maps on the circular boundary of a bounded computational domain. This gives a reference solution on 327680 uniform intervals on Γ and error norms are computed with the trapezoidal rule. Note, that both the reference solution and the BEM solution are affected by a modelling error introduced by using the ITCs instead of modelling the thin conducting sheet, and we will compare the solution of the proposed BEM for the different ITCs with their respective reference solution.

2. Shielding models

We present a variety of ITCs of the form (4) that have been proposed by different authors. We forgo a discussion of their derivation, scope and performance and refer the reader to [15] or the original works cited below.

2.1. Impedance transmission conditions of type I [9, 15, 16]

The impedance transmission conditions of type I are

$$[\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma, \tag{5a}$$
$$[\gamma_0 U] = 0 \quad \text{on } \Gamma, \tag{5b}$$

where the complex coefficient β_1 is of the form $\beta_1 = \varphi_1(\xi d)/d$. It is given for the models ITC-1-0, ITC-1-1, ITC-2-0 (see [15]) by

$$\beta_1^{\text{ITC-1-0}} = \xi^2 d, \qquad \beta_1^{\text{ITC-1-1}} = \xi^2 d\left(1 + \tfrac{1}{6}\xi^2 d^2\right), \qquad \beta_1^{\text{ITC-2-0}} = \frac{2\xi \sinh\left(\xi \frac{d}{2}\right)}{\cosh\left(\xi\frac{d}{2}\right) - \xi\frac{d}{2}\sinh\left(\xi\frac{d}{2}\right)}. \tag{6}$$

The range of validity of the transmission conditions ITC-1-0, ITC-1-1 and ITC-2-0 is $|\xi d| \in (0, \infty)$, and β_1 may have small or large absolute values.

Lemma 2.1. *For any bounded $\xi \in (-1 + \mathrm{i})\,\mathbb{R}^+$ with $|\xi| > 0$ and any $d > 0$ it holds for the models ITC-1-0, ITC-1-1 and ITC-2-0 that $\mathrm{Im}(\beta_1) < 0$.*

Remark 2.2. *The imaginary part of $\beta_1^{ITC\text{-}2\text{-}0}$ tends to zero for $|\xi d| \to \infty$ and d fixed, whereas the real part tends to the negative value $-4/d$ while $\arg(\beta_1^{ITC\text{-}2\text{-}0}) \to -\pi$ [1] (see Fig. 3). Both imaginary and real part of $\beta_1^{ITC\text{-}1\text{-}1}$ tend to $-\infty$ for $|\xi d| \to \infty$ and d fixed while $\arg(\beta_1^{ITC\text{-}1\text{-}1}) \to -\pi$ as well.*

2.2. Impedance transmission conditions of type II [12]

The impedance transmission conditions of type II are

$$[\gamma_1 U] - (\beta_1 - \beta_2 \Delta_\Gamma)\{\gamma_0 U\} = 0 \quad \text{on } \Gamma, \tag{7a}$$
$$[\gamma_0 U] = 0 \quad \text{on } \Gamma, \tag{7b}$$

where the values of β_1, β_2 for the model NTFS (see [15]) are given by

$$\beta_1^{\text{NTFS}} = \xi^2 d, \qquad \beta_2^{\text{NTFS}} = d. \tag{8}$$

The parameters β_2 have the form $\beta_2 = \varphi_2(\xi d) d$. Note, that the ITCs of type I are of type II with $\beta_2 = 0$. The range of validity of NTFS is $|\xi d| \in (0, \infty)$.

[1] For a complex number $z \in \mathbb{C}$ we define $\arg(z) \in (-\pi, \pi]$ as the angle enclosed by its associated vector and the positive real axis in the complex plane.

2.3. Impedance transmission conditions of type III [19, Sec. 3.7], [10], [14]

The impedance transmission conditions of type III are

$$[\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma, \tag{9a}$$
$$[\gamma_0 U] - \beta_3 \{\gamma_1 U\} = 0 \quad \text{on } \Gamma, \tag{9b}$$

where the values of β_1 and β_3 for the models MB and ITC-2-1 for vanishing curvature (see [15]) are given by

$$\beta_1^{\text{MB}} = 2\xi \tanh(\xi \tfrac{d}{2}), \qquad \beta_3^{\text{MB}} = \tfrac{2}{\xi} \tanh(\xi \tfrac{d}{2}), \tag{10}$$

$$\beta_1^{\text{ITC-2-1}} = \frac{2\xi \sinh\left(\xi \frac{d}{2}\right)}{\cosh\left(\xi \frac{d}{2}\right) - \xi \frac{d}{2} \sinh\left(\xi \frac{d}{2}\right)}, \qquad \beta_3^{\text{ITC-2-1}} = -d\left(1 - \tfrac{2}{\xi d} \tanh(\xi \tfrac{d}{2})\right). \tag{11}$$

Note, that the parameters β_3 are of the form $\beta_3 = \varphi_3(\xi d) d$.

Lemma 2.3. *For any $\xi \in (-1+\mathrm{i})\mathbb{R}^+$ with $|\xi| > 0$ and any $d > 0$ it holds for the models MB and ITC-2-1 that* $\operatorname{Im}(\beta_1) < 0$, $\operatorname{Im}(\beta_3) > 0$.

Remark 2.4. *The imaginary parts of $\beta_1^{ITC\text{-}2\text{-}1}$ and $\beta_3^{ITC\text{-}2\text{-}1}$ tend to zero for $|\xi| \to \infty$ and d fixed, whereas the real part tends to the negative values $-4/d$ or $-d$, respectively, while $\arg(\beta_1^{ITC\text{-}2\text{-}1}) \to -\pi$ and $\arg(\beta_3^{ITC\text{-}2\text{-}1}) \to \pi$ (see Fig. 3).*

2.4. Impedance transmission conditions of type IV [14, 17]

The impedance transmission conditions of type IV are

$$[\gamma_1 U] - (\beta_1 - \beta_2 \Delta_\Gamma)\{\gamma_0 U\} + \beta_4 \kappa \{\gamma_1 U\} = 0 \quad \text{on } \Gamma, \tag{12a}$$
$$[\gamma_0 U] \qquad - \beta_4 \kappa \{\gamma_0 U\} \ - \beta_3 \{\gamma_1 U\} = 0 \quad \text{on } \Gamma, \tag{12b}$$

where κ is the signed (mean) curvature of Γ, which is assumed to be C^2. The signed curvature is a positive constant for a circular mid-line in two dimensions with $x_\Gamma(t) = (\cos(t), \sin(t))^\top$. The values of β_1, β_2, β_3 and β_4 for the model ITC-1-2 (see [15]) are given by

$$\beta_1^{\text{ITC-1-2}} = \xi^2 d\left(1 + \tfrac{1}{6}\xi^2 d^2 - \tfrac{7}{240}\xi^4 d^4\right), \quad \beta_2^{\text{ITC-1-2}} = -\tfrac{1}{12}\xi^2 d^3, \quad \beta_3^{\text{ITC-1-2}} = -\tfrac{1}{12}\xi^2 d^3, \quad \beta_4^{\text{ITC-1-2}} = \tfrac{1}{24}\xi^2 d^3,$$

and those for the model ITC-2-1 are given in (11) and

$$\beta_2^{\text{ITC-2-1}} = 0, \qquad \beta_4^{\text{ITC-2-1}} = \frac{d}{2}\left(1 - \tfrac{2}{\xi d}\tanh(\xi \tfrac{d}{2})\right).$$

Note, that the parameters β_4 are of the form $\beta_4 = \varphi_4(\xi d) d$.

3. Boundary integral equations

3.1. The representation formula

The solution $U(x)$ of (4) with any transmission condition (5), (7), (9), or (12) can be represented as [13, Thm. 3.1.8], [11, Thm. 6.10]

$$U = -S[\gamma_1 U] + D[\gamma_0 U] + N F \qquad \text{in } \mathbb{R}^n \setminus \Gamma, \tag{13}$$

where the single and double layer potential [13, Sec. 3.1] are defined by

$$(S\,\phi)(x) := \int_\Gamma G(x-y)\phi(y)\mathrm{d}y \qquad x \in \mathbb{R}^n \setminus \Gamma, \tag{14}$$

$$(D\,\psi)(x) := \int_\Gamma \gamma_{1,y} G(x-y)\psi(y)\mathrm{d}y \qquad x \in \mathbb{R}^n \setminus \Gamma, \tag{15}$$

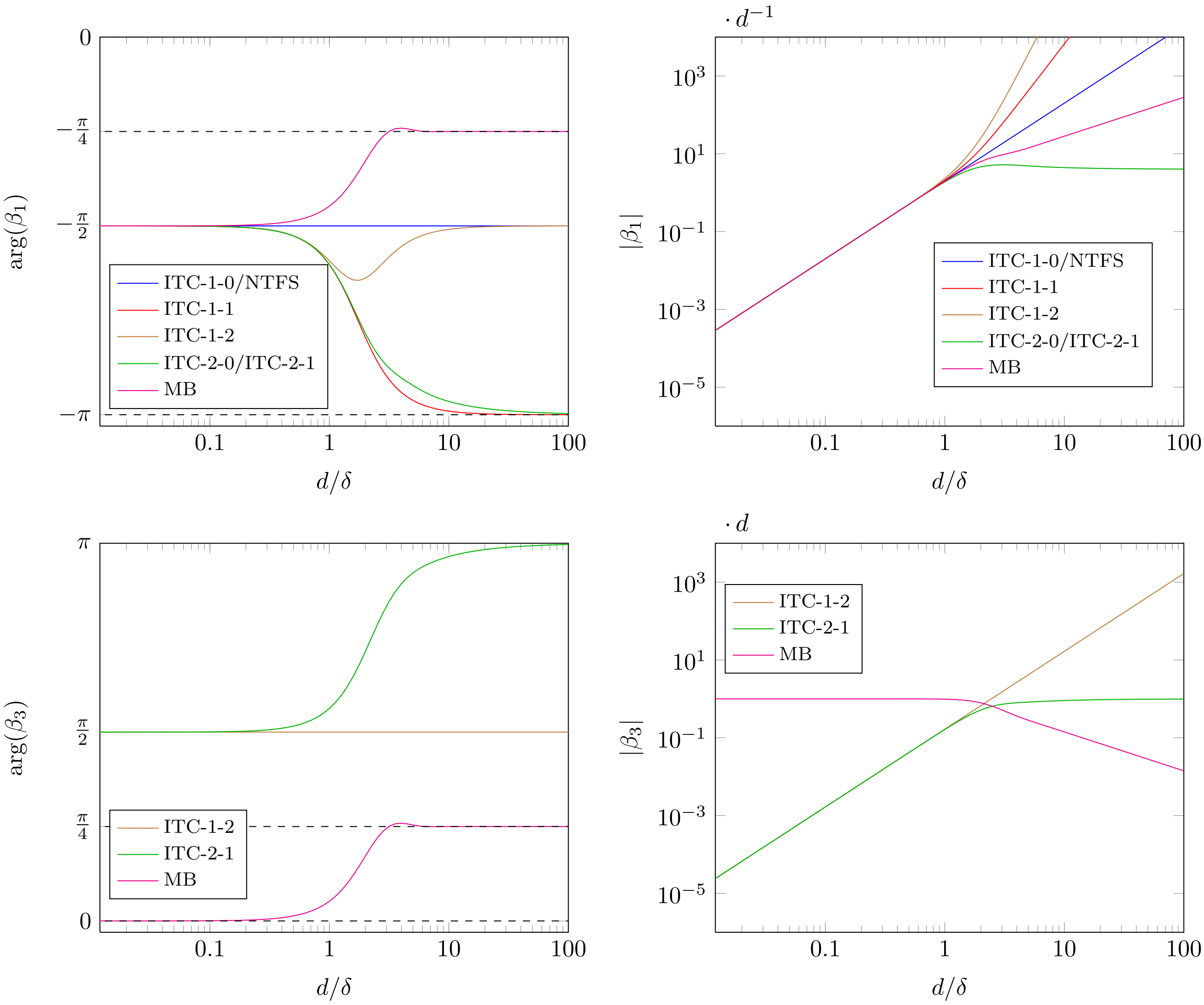

FIGURE 3. The phase angle and modulus of the parameters β_1 and β_3 for different impedance transmission conditions in dependence of the sheet thickness to skin depth ratio $d/\delta = |\xi d|/\sqrt{2}$.

and the Newton potential [13, Sec. 3.2] by

$$(N\,\rho)(x) := \int_{\mathbb{R}^n} G(x-y)\rho(y)\mathrm{d}y \qquad x \in \mathbb{R}^n. \tag{16}$$

The Green's kernel for the operator $-\Delta$ is

$$G(z) := \begin{cases} -\dfrac{1}{2\pi}\ln(|z|), & \text{if } n = 2, \\ \dfrac{1}{4\pi|z|}, & \text{if } n = 3. \end{cases}$$

See also the more general definition of S and D in [13, Def. 3.1.5].

The solution U may be discontinuous over Γ, which is reflected in the jump relations for the single and double layer potential [13, Thm. 3.3.1]

$$[\gamma_0 S\,\phi] = 0, \qquad [\gamma_0 D\,\psi] = \psi, \tag{17}$$

$$[\gamma_1 S\,\phi] = -\phi, \qquad [\gamma_1 D\,\psi] = 0. \tag{18}$$

The averages of single and double layer potential define the boundary integral operators with the following continuity properties (see [11, Thm. 7.1], [13, Thm. 3.1.16])

$$\begin{aligned} V &:= \{\gamma_0 S\,\cdot\} : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), & K &:= \{\gamma_0 D\,\cdot\} : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \\ K' &:= \{\gamma_1 S\,\cdot\} : H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma), & W &:= -\{\gamma_1 D\,\cdot\} : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma), \end{aligned} \tag{19}$$

for $0 \le s \le r+1$, if $\Gamma \in C^{r+1,1}$, where the latter notation means that Γ coincides with the graph of a function of smoothness $C^{r+1,1}$ locally, see [11, Chap. 2]. For more details on boundary integral operators consult [13, Sec. 3.1], [11, Chap. 7].

Lemma 3.1. *If $n = 3$, or $n = 2$ and the diameter of Γ is smaller than 1, then the boundary integral operator V is $H^{-1/2}(\Gamma)$-elliptic, i. e., there exists a constant $\gamma > 0$ such that*

$$\langle V\phi, \overline{\phi}\rangle \ge \gamma \, \|\phi\|^2_{H^{-1/2}(\Gamma)}\,, \quad \forall\, \phi \in H^{-1/2}(\Gamma). \tag{20}$$

Furthermore, the boundary integral operator W is $H^{1/2}(\Gamma)/\mathbb{C}$-elliptic, i. e., there exists a constant $\gamma > 0$ such that

$$\langle W j, \overline{j}\rangle \ge \gamma \, \|j\|^2_{H^{1/2}(\Gamma)}\,, \quad \forall\, j \in H^{1/2}(\Gamma)/\mathbb{C}. \tag{21}$$

Proof. See [13, Theorem 3.5.3] or [18, Satz 6.6]. ☐

Remark 3.2. *The assumption on the diameter of the interface Γ is needed only in two dimensions. Note, that this assumption can always be satisfied by appropiate coordinate scaling.*

For the remainder of this article we assume that the support of the source F is well-separated from the mid-line or mid-surface Γ of the sheet, *i. e.*, there is a $\varepsilon > 0$ such that $\mathrm{dist}(\mathrm{supp}(F), \Gamma) \ge \varepsilon$. Since the Newton potential is smooth away from $\mathrm{supp}(F)$ the traces of the Newton potential on Γ possess higher regularity.

Lemma 3.3 (Regularity of traces of the Newton potential)**.** *Let $\Gamma \in C^{r+1,1}$. Then, for any $s \in \mathbb{Z}$, $s \le r+1$ there exists a constant $C = C(s, \varepsilon, \Gamma)$ such that*

$$\|\gamma_0 NF\|_{H^{s+1/2}(\Gamma)} + \|\gamma_1 NF\|_{H^{s-1/2}(\Gamma)} \le C \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}.$$

Remark 3.4 (Computation of the Newton potential)**.** *Of practical relevance for the two-dimensional setting are sources $F \in L^2_{\mathrm{comp}}(\Omega)$ corresponding to currents flowing out of the symmetry plane in a bounded cross-section $\Omega_F \subset (\mathbb{R}^2 \setminus \Omega_{\mathrm{int}})$, i. e., their support has no contact to the thin conducting sheet. In general, the integral* (16) *and its derivative can be approximated by numerical quadrature. An important case are cylindrical wires with a circular cross-section and a constant (current) amplitude $F(M)$ (see Fig. 1(a)) for which we can use the analytic formulas for $|x - M| > R$*

$$(NF)(x) = -\frac{R^2}{2} \ln(|x - M|)\, F(M), \qquad \nabla(NF)(x) = -\frac{R^2}{2} \frac{x - M}{|x - M|^2}\, F(M), \tag{22}$$

where R is the radius and M are the coordinates of the mid-point of the circle. The Newton potential enters the right hand side functional of the variational boundary integral equations. Its boundary element discretization will entail integrating the Newton potential multiplied with a piecewise polynomial on Γ.

3.2. **Boundary element spaces**

Our boundary element approach is based on the approximation of Γ by a closed polygon Γ_h in 2D or polyhedron Γ_h in 3D, whose vertices x_j, $j \in \mathcal{I}_h := \{1, \ldots, N_h\}$, lie on Γ and whose straight line segments $K_j = [x_j, x_{j+1}]$ $(x_{N_h+1} := x_1)$ or flat triangular cells K_j we call panels. The set of panels is denoted by $\mathcal{T}_h$. The length of the largest panel is the mesh width h, and the shape regularity measure ρ_h is the largest ratio of diameter and radius of an inscribed ball of any panel $K \in \mathcal{T}_h$, which is 1 if $n = 2$.

Throughout this article Γ_h will be assumed to be a member of a shape-regular and quasi-uniform infinite family of triangulations of Γ whose mesh widths accumulate at zero [13, Sect. 4.1.2]. None of the "generic constants" in our estimates, usually denoted by C, will depend on the concrete Γ_h used for discretization. Sometimes, we are going to express this by the casual phrase "independent of h"

Remark 3.5. *The approximation of the curved mid-line by straight line segments or curved mid-surface by flat triangles contributes to the discretisation error, but does not affect the order of convergence of our low-order boundary element methods (see [13, Chap. 8]). Thus, in the subsequent analysis, we will assume that we use an exact resolution of* Γ *(see Fig. 2).*

We introduce the space of piecewise constant functions as

$$S_0^{-1}(\Gamma_h) := \Big\{ v_h \in L^2(\Gamma) : v_h \in \mathbb{P}_0(K)\ \forall K \in \mathcal{T}_h \Big\}, \tag{23}$$

and the space of piecewise linear, continuous functions as

$$S_1^{0}(\Gamma_h) := \Big\{ v_h \in L^2(\Gamma) \cap C(\Gamma) : v_h \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h \Big\}, \tag{24}$$

where $\mathbb{P}_p$ is the space of polynomials of degree $p \in \mathbb{N}_0$. For the sake of simplicity we consider only lowest order boundary element methods. An extension to higher polynomial degrees is, however, straightforward. In parts our analysis also covers spectral Galerkin discretisation based on Fourier modes ($n = 2$) or spherical harmonics ($n = 3$).

Let us shortly review some best-approximation estimates for these spaces [13, Sec. 4.3.4–5].

Lemma 3.6 (Best-approximation error)**.** *Let* $s, m \in \mathbb{R}$*. Then, there exist constants* $C = C(\Gamma, s, m, \rho_h)$ *such that*

$$\inf_{v_h \in S_1^0(\Gamma_h)} \|v - v_h\|_{H^s(\Gamma)} \le C\, h^{m-s} \|v\|_{H^m(\Gamma)}, \qquad -1 \le s \le 1, \quad 0 \le m \le 2 \tag{25a}$$

$$\inf_{v_h \in S_0^{-1}(\Gamma_h)} \|v - v_h\|_{H^s(\Gamma)} \le C\, h^{m-s} \|v\|_{H^m(\Gamma)}, \qquad -1 \le s \le 0, \quad 0 \le m \le 1. \tag{25b}$$

4. Second kind boundary element formulation for problems of type I

4.1. Boundary integral formulation

For the problem of type I, see (5), the representation formula (13) simplifies as the jump of the electric field vanishes by (5b). Taking the mean trace of (13) we get

$$\{\gamma_0 U\} = -V\,[\gamma_1 U] + \gamma_0 N F \tag{26}$$

and using (5a) we get for the new unknown $\phi = [\gamma_1 U]$ on Γ the boundary integral formulation of the second kind

$$\boxed{(Id + \beta_1 V)\phi = \beta_1 \gamma_0 N F.} \tag{27}$$

Testing by $\phi' \in L^2(\Gamma)$ we get the variational formulation: Seek $\phi \in L^2(\Gamma)$ such that

$$\boxed{\mathsf{a}_I(\phi, \phi') := \langle \phi, \phi' \rangle + \beta_1 \langle V\phi, \phi' \rangle = \beta_1 \langle \gamma_0 N F, \phi' \rangle \quad \forall \phi' \in L^2(\Gamma).} \tag{28}$$

Here, we denote by $\langle \cdot, \cdot \rangle$ the bilinear duality pairing w.r.t. $L^2(\Gamma)$, *i. e.*, $\langle u, v \rangle = \langle v, u \rangle$ and $\|u\|^2_{L^2(\Gamma)} = \langle u, \overline{u} \rangle$.

Obviously, $\phi = 0$ for $\beta_1 = 0$. As the parameter β_1 may attain small or large absolute values we are going to derive stability estimates which are robust for asymptotically small moduli, *i. e.*, $|\beta_1| \to 0$, as well as for asymptotically large moduli, *i. e.*, $|\beta_1| \to \infty$.

Theorem 4.1. *Let* $|\beta_1| > 0$ *and assume there exists a constant* $\theta_1^\star \in (-\pi, 0)$ *such that* $0 \ge \theta_1 := \arg(\beta_1) \ge \theta_1^\star$*. Furthermore, let* Γ *be Lipschitz. Then, the system* (28) *has a unique solution* $\phi \in L^2(\Gamma)$*, and there exists a constant* $C = C(\theta_1^\star)$ *independent of* $|\beta_1|$ *such that*

$$\|\phi\|_{L^2(\Gamma)} \le C |\beta_1| \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \qquad \|\phi\|_{H^{-1/2}(\Gamma)} \le C \min(1, |\beta_1|) \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}. \tag{29}$$

Proof. By assumption we have $\theta_1/2 \in [\theta_1^\star/2, 0]$. The bilinear form a_I is $L^2(\Gamma)$-elliptic with ellipticity constant $\cos(\theta_1^\star/2)$ and $H^{-1/2}(\Gamma)$-elliptic with ellipticity constant $|\beta_1|\cos(\theta_1^\star/2)$ since

$$\operatorname{Re}\left(\mathrm{e}^{-\frac{\mathrm{i}\theta_1}{2}}\mathsf{a}_I(\phi,\overline{\phi})\right) = \cos\left(\tfrac{\theta_1^\star}{2}\right)\left(\|\phi\|^2_{L^2(\Gamma)} + |\beta_1|\left\langle V\phi,\overline{\phi}\right\rangle\right). \tag{30}$$

Using the Lax-Milgram lemma and Lemma 3.3 we obtain the $L^2(\Gamma)$-estimate and the $H^{-1/2}(\Gamma)$-estimate with a constant C. Using the fact that $\|\phi\|_{H^{-1/2}(\Gamma)} \leq \|\phi\|_{L^2(\Gamma)}$, we get the $H^{-1/2}(\Gamma)$-estimate with $C|\beta_1|$, and the estimates with both constants are valid when the minimum is attained. □

The ITCs of type I may be used for lower frequencies, where β_1 is small and so is the jump of the normal derivative ϕ, or for large frequencies and, thus, large β_1. Assumptions that the sheet features certain smoothness, we can show that the normal derivative ϕ remains bounded even if $|\beta_1| \to \infty$.

Lemma 4.2. *Let the assumptions of Theorem 4.1 be satisfied,* $\Gamma \in C^{r+1,1}$*,* $r \in \mathbb{N}_0$*,* ϕ *the solution of* (28) *and* $u := \{\gamma_0 U\} = -V\phi + \gamma_0 NF$*. Then, for any* $0 \leq s \leq \frac{r}{2}$*, there exist constants* C_s *independent of* $|\beta_1|$ *such that the higher regularity estimates*

$$\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_s \min(1,|\beta_1|)\|\gamma_0 NF\|_{H^{2s+3/2}(\Gamma)}, \tag{31a}$$

$$\|u\|_{H^{s+1/2}(\Gamma)} \leq C_s \min(|\beta_1|^{-1},1)\|\gamma_0 NF\|_{H^{2s+3/2}(\Gamma)}, \tag{31b}$$

hold, where $\|\gamma_0 NF\|_{H^{2s+3/2}(\Gamma)} \leq C_s\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}$ *if* $2s+1 \leq r+1$*. Furthermore, for any* s, j *such that* $0 \leq j \leq s$*, there exist constants* C_s *independent of* $|\beta_1|$ *such that*

$$\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_{s,j} \max(|\beta_1|^{s+1-j},|\beta_1|)\|\gamma_0 NF\|_{H^{s+j+1/2}(\Gamma)}, \tag{31c}$$

$$\|u\|_{H^{s+1/2}(\Gamma)} \leq C_{s,j} \max(|\beta_1|^{s-j},\quad 1)\quad\|\gamma_0 NF\|_{H^{s+j+1/2}(\Gamma)}, \tag{31d}$$

where $\|\gamma_0 NF\|_{H^{s+j+1/2}(\Gamma)} \leq C_{s,j}\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}$ *if* $s+j \leq r+1$*.*

Proof. As (31b) and (31d) are direct consequences of (31a) and (31c) using $\phi = \beta_1 u$, we have to prove the estimates for ϕ only. Throughout the proof we assume $s \in \mathbb{N}_0$, the estimates for general s follow by interpolation. To obtain the higher regularity estimates we rewrite (27) as

$$\phi = \beta_1\left(-V\phi + \gamma_0 NF\right), \tag{32}$$

where we can assert in view of (19) that

$$\|\phi\|_{H^{s+1/2}(\Gamma)} \leq |\beta_1|\left(C_s\|\phi\|_{H^{s-1/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{s+1/2}(\Gamma)}\right). \tag{33}$$

Now, using the $H^{-1/2}(\Gamma)$-estimate in (29) for the right hand side we obtain for $s = 0$

$$\|\phi\|_{H^{1/2}(\Gamma)} \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)},$$

and repeated application of (33) leads to

$$\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_s \max(|\beta_1|^{s+1},|\beta_1|)\|\gamma_0 NF\|_{H^{s+1/2}(\Gamma)}. \tag{34}$$

Thus Lemma 3.3 gives (31a) for small $|\beta_1|$ and (31c) in the case of $j = 0$ for large $|\beta_1|$ as well. For large $|\beta_1|$, for which (27) is singularly perturbed, this estimates reflects the emergence of internal layers close to points on Γ where the regularity is reduced.

Using the regularity assumption on Γ we can improve this estimate for large $|\beta_1|$ using an asymptotic expansion in β_1^{-1}. For this we assume in the remainder of the proof that $|\beta_1| > 1$. We use the ansatz

$$\phi \sim \phi_0 + \beta_1^{-1}\phi_1 + \beta_1^{-2}\phi_2 + \ldots, \tag{35}$$

where $\phi_{-1} := -\gamma_0 NF$ and ϕ_n for $n = 0, 1, 2, \ldots$ are defined by

$$V\phi_{n+1} = -\phi_n.$$

The terms ϕ_n of the expansion are defined independently of β_1, and for any $s \geq 0$ we have

$$\|\phi_n\|_{H^{s-1/2}(\Gamma)} \leq C_n \|\gamma_0 NF\|_{H^{s+1/2+n}(\Gamma)}, \tag{36}$$

i. e., for given regularity of $\gamma_0 NF$ their regularity decays with increasing index n.

Now, let for some $N \in \mathbb{N}$

$$\delta\phi_N := \phi - \sum_{n=0}^{N} \beta_1^{-n} \phi_n,$$

which solves

$$(Id + \beta_1 V)\delta\phi_N = -\beta_1^{-N} \phi_N.$$

This equation is similar to (27) where $\beta_1 \gamma_0 NF$ on the right hand side is replaced by $-\beta_1^{-N}\phi_N$. Therefore, using (34) and (36) we have for $s \geq 0$

$$\|\delta\phi_N\|_{H^{s+1/2}(\Gamma)} \leq C_{s,N} |\beta_1|^{s-N} \|\phi_N\|_{H^{s+1/2}(\Gamma)} \leq C_{s,N} |\beta_1|^{s-N} \|\gamma_0 NF\|_{H^{s+3/2+N}(\Gamma)},$$

and, hence, if $N \leq s$

$$\|\phi\|_{H^{s+1/2}(\Gamma)} \leq \|\delta\phi_N\|_{H^{s+1/2}(\Gamma)} + \sum_{n=0}^{N} |\beta_1|^{-n} \|\phi_n\|_{H^{s+1/2}(\Gamma)} \leq C_{s,N} |\beta_1|^{s-N} \|\gamma_0 NF\|_{H^{s+3/2+N}(\Gamma)}.$$

Assuming $s + N \leq r$ and $N \leq s - 1$, using Lemma 3.3, and replacing N by j we obtain (31c) for large $|\beta_1|$. For $N = s$ we obtain (31a) for large $|\beta_1|$. This finishes the proof. □

4.2. **Boundary element formulation**

Let V_h be a finite-dimensional subspace of $L^2(\Gamma)$, in particular $S_0^{-1}(\Gamma_h)$ or $S_1^0(\Gamma_h)$. Then, the boundary element formulation reads: Seek $\phi_h \in V_h$ such that

$$\langle \phi_h, \phi_h' \rangle + \beta_1 \langle V\phi_h, \phi_h' \rangle = \beta_1 \langle \gamma_0 NF, \phi_h' \rangle \quad \forall \phi_h' \in V_h. \tag{37}$$

Theorem 4.3. *Let the assumptions of Theorem 4.1 be fulfilled and $V_h \subset L^2(\Gamma)$. Then, the linear variational equation* (37) *has a unique solution $\phi_h \in V_h$ that satisfies*

$$\|\phi_h\|_{L^2(\Gamma)} \leq C_0(\beta_1)\ \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}. \tag{38}$$

If $V_h = S_0^{-1}(\Gamma_h)$ and $\Gamma \in C^{2,1}$, then

$$\|\phi_h - \phi\|_{L^2(\Gamma)} \leq C_1(\beta_1)\, h \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \tag{39}$$

and if $V_h = S_1^0(\Gamma_h)$ and $\Gamma \in C^{3,1}$, then

$$\|\phi_h - \phi\|_{L^2(\Gamma)} \leq C_2(\beta_1)\, h^2 \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \tag{40}$$

where for $\ell = 0$ and Lipschitz Γ, $\ell = 1$ and $\Gamma \in C^{2,1}$, or $\ell = 2$ and $\Gamma \in C^{3,1}$, with a constant $C = C(\theta_1^\star)$ independent of $|\beta_1|$,

$$C_\ell(\beta_1) = C\, |\beta_1|\ ,$$

and for $\ell = 0$ and $\Gamma \in C^{1,1}$, $\ell = 1$ and $\Gamma \in C^{3,1}$, or $\ell = 2$ and $\Gamma \in C^{5,1}$,

$$C_\ell(\beta_1) = C\ \min(1, |\beta_1|)\ .$$

Proof. We conclude well-posedness of the Galerkin discretization (37) by the same arguments as that of the continuous variational formulation in the proof of Theorem 4.1. Thus we obtain so (38) for Lipschitz Γ.

The remainder of the proof is devoted to estimating the discretization error. To that end we have to strengthen the stability estimate for the discrete solution under higher smoothness assumptions on Γ. We proceed in two steps. Firstly, we show all the bounds with $C_\ell(\beta) = C|\beta_1|$ for general β_1. Afterwards we prove the bounds with $C_\ell(\beta_1) = C$ for $|\beta_1| \geq 1$ under higher smoothness assumptions. Below we write C for generic constants that depend on β_1 only through θ_1.

Step (i): The $L^2(\Gamma)$-norm of the bilinear form a_I defined in (28) is bounded by $C\max(1, |\beta_1|)$. So, with Cea's lemma [2] we can bound the discretisation error by the best-approximation error. This involves (39) and (40), respectively, using the approximation error estimates of Lemma 3.6, the regularity results of Lemma 4.2 and the fact that $|\beta_1| = \max(1, |\beta_1|)\min(1, |\beta_1|)$, because for $\Gamma \in C^{2,1}$ by (31a) with $s = \frac{1}{2}$ and Lemma 3.3

$$\|\phi\|_{H^1(\Gamma)} \leq C\min(1, |\beta_1|)\,\|\gamma_0 NF\|_{H^{5/2}(\Gamma)} \leq C|\beta_1|\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'},$$

and if $\Gamma \in C^{3,1}$ by (31c) with $s = \frac{3}{2}$, $j = \frac{1}{2}$

$$\|\phi\|_{H^2(\Gamma)} \leq C\min(1, |\beta_1|)\,\|\gamma_0 NF\|_{H^{7/2}(\Gamma)} \leq C|\beta_1|\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ ,$$

and so we have for $V_h = S_0^{-1}(\Gamma_h)$ and $\ell = 1$ or $V_h = S_1^0(\Gamma_h)$ and $\ell = 1, 2$ if $\Gamma \in C^{\ell+1,1}$

$$\|\phi_h - \phi_h\|_{L^2(\Gamma)} \leq C|\beta_1| h^\ell \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{41}$$

Step (ii): We assume $|\beta_1| \geq 1$. Then the continuity constant of a_I defined in (28) is bounded by $C|\beta_1|$. To avoid applying Cea's lemma directly as in Step (i), which would introduce another power of $|\beta_1|$, we decompose $\phi = \phi_0 + \delta\phi_0$ and $\phi_h = \phi_{0,h} + \delta\phi_{0,h}$. Note that $\phi_0 = V^{-1}(\gamma_0 NF)$ is the first term of the asymptotic expansion in β_1^{-1} of ϕ (see (35) in the proof of Lemma 4.2) and $\phi_{0,h} \in V_h$, the unique solution of

$$\langle V\phi_{0,h}, \phi_h'\rangle = \langle \gamma_0 NF, \phi_h'\rangle\,, \quad \forall \phi_h' \in V_h, \tag{42}$$

is its discrete approximation. Due to the fact that V^{-1} is a continuous operator $H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ we find for $s \geq 0$ that

$$\|\phi_0\|_{H^{s-1/2}(\Gamma)} \leq C\,\|\gamma_0 NF\|_{H^{s+1/2}(\Gamma)}. \tag{43}$$

By Cea's lemma and Lemma 3.6 we have for $V_h = S_0^{-1}(\Gamma_h)$ with $\ell = 0, 1$ and for $V_h = S_1^0(\Gamma_h)$ with $\ell = 0, 1, 2$

$$\|\phi_{0,h} - \phi_0\|_{H^{-1/2}(\Gamma)} \leq C\,h^{\ell+1/2}\|\phi_0\|_{H^\ell(\Gamma)}\ , \tag{44}$$

independently of $|\beta_1|$.

In order to obtain an estimate in $L^2(\Gamma)$ we have to rely on suitable projections $Q_h : L^2(\Gamma) \to V_h$:

- for $V_h = S_0^{-1}(\Gamma_h)$ we choose Q_h as the discrete dual projection introduced in Appendix A,
- for $V_h = S_1^0(\Gamma_h)$ the simple $L^2(\Gamma)$-orthogonal projection provides Q_h.

These projectors are continuous in $L^2(\Gamma)$ and can be extended to continuous mappings $H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, see Lemma A.1. In addition, we employ the triangle inequality, inverse estimates for functions in V_h, see [8, eq. (5.13)], and (44):

$$\begin{aligned}\|\phi_{0,h} - \phi_0\|_{L^2(\Gamma)} &\leq \|(\phi_{0,h} - Q_h\phi_0) - (\phi_0 - Q_h\phi_0)\|_{L^2(\Gamma)}\\ &\leq \|Q_h(\phi_{0,h} - \phi_0)\|_{L^2(\Gamma)} + \|\phi_0 - Q_h\phi_0\|_{L^2(\Gamma)}\\ &\overset{\text{inv. est.}}{\leq} C\,h^{-1/2}\|Q_h(\phi_{0,h} - \phi_0)\|_{H^{-1/2}(\Gamma)} + Ch^\ell\|\phi_0\|_{H^\ell(\Gamma)} \overset{(44)}{\leq} C\,h^\ell\|\phi_0\|_{H^\ell(\Gamma)}.\end{aligned} \tag{45}$$

Now, we observe that $\delta\phi_0 := \phi - \phi_0$ is the unique solution of

$$(Id + \beta_1 V)\,\delta\phi_0 = -\phi_0, \tag{46}$$

which agrees with (27) with $\beta_1\gamma_0 NF$ replaced by $-\phi_0$. Hence, by (31a), (43) for $s \geq 0$,

$$\|\delta\phi_0\|_{H^{s+1/2}(\Gamma)} \leq C|\beta_1|^{-1}\,\|\phi_0\|_{H^{2s+3/2}(\Gamma)} \leq C\,|\beta_1|^{-1}\|\gamma_0 NF\|_{H^{2s+5/2}(\Gamma)}. \tag{47}$$

Comparing (37) and (42) we observe that $\delta\phi_{0,h} = \phi_h - \phi_{0,h}$, is the unique solution of

$$\langle \delta\phi_{0,h}, \phi'_h \rangle + \beta_1 \langle V \delta\phi_{0,h}, \phi'_h \rangle = - \langle \phi_{0,h}, \phi'_h \rangle , \quad \forall \phi'_h \in V_h. \tag{48}$$

We cannot apply Cea's lemma directly, as the right hand side of (46) and (48) are not identical. As they are close, we can apply Strang's first lemma [2, Sec. 3.1] and use the $|\beta_1|$-uniform $L^2(\Gamma)$-ellipticity (30) (here the assumptions of Theorem 4.1 matter). We conclude

$$\begin{aligned} \|\delta\phi_{0,h} - \delta\phi_0\|_{L^2(\Gamma)} &\le C\Big(|\beta_1| \inf_{\psi_h \in V_h} \|\psi_h - \delta\phi_0\|_{L^2(\Gamma)} + \|\phi_{0,h} - \phi_0\|_{L^2(\Gamma)} \Big) \\ &\overset{(45)}{\le} C\, h^\ell \left(|\beta_1| \|\delta\phi_0\|_{H^\ell(\Gamma)} + \|\phi_0\|_{H^\ell(\Gamma)} \right) . \end{aligned} \tag{49}$$

Hence, in view of (49), (45), (43) and (47), we have for $\ell \ge 1$

$$\begin{aligned} \|\phi_h - \phi\|_{L^2(\Gamma)} &\le \|\phi_{0,h} - \phi_0\|_{L^2(\Gamma)} + \|\delta\phi_{0,h} - \delta\phi_0\|_{L^2(\Gamma)} \le C\, h^\ell \left(|\beta_1| \|\delta\phi_0\|_{H^\ell(\Gamma)} + \|\phi_0\|_{H^\ell(\Gamma)} \right) \\ &\le C\, h^\ell \Big(\|\gamma_0 NF\|_{H^{2\ell+3/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{\ell+1}(\Gamma)} \Big) \le C\, h^\ell \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \end{aligned} \tag{50}$$

if $\Gamma \in C^{2\ell+1,1}$ (see Lemma 3.3). Obviously, (50) provides the estimates (39) for $V_h = S_0^{-1}(\Gamma_h)$ and $\ell = 1$, and (40) for $V_h = S_1^0(\Gamma_h)$ and $\ell = 2$.

To obtain an improved stability estimate for the discrete solution ϕ_h we set $\ell = 0$. Then, in the same way as above we obtain

$$\|\phi_h - \phi\|_{L^2(\Gamma)} \le C \, \|\gamma_0 NF\|_{H^{5/2}(\Gamma)},$$

and, using the triangle inequality and (31a) with $s = 0$, we arrive at

$$\|\phi_h\| \le \|\phi_h - \phi\|_{L^2(\Gamma)} + \|\phi\|_{L^2(\Gamma)} \le C \, \Big(\|\gamma_0 NF\|_{H^{5/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{3/2}(\Gamma)} \Big) \le C\, h^\ell \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'} ,$$

if $\Gamma \in C^{2,1}$ which is (38). This completes the proof. □

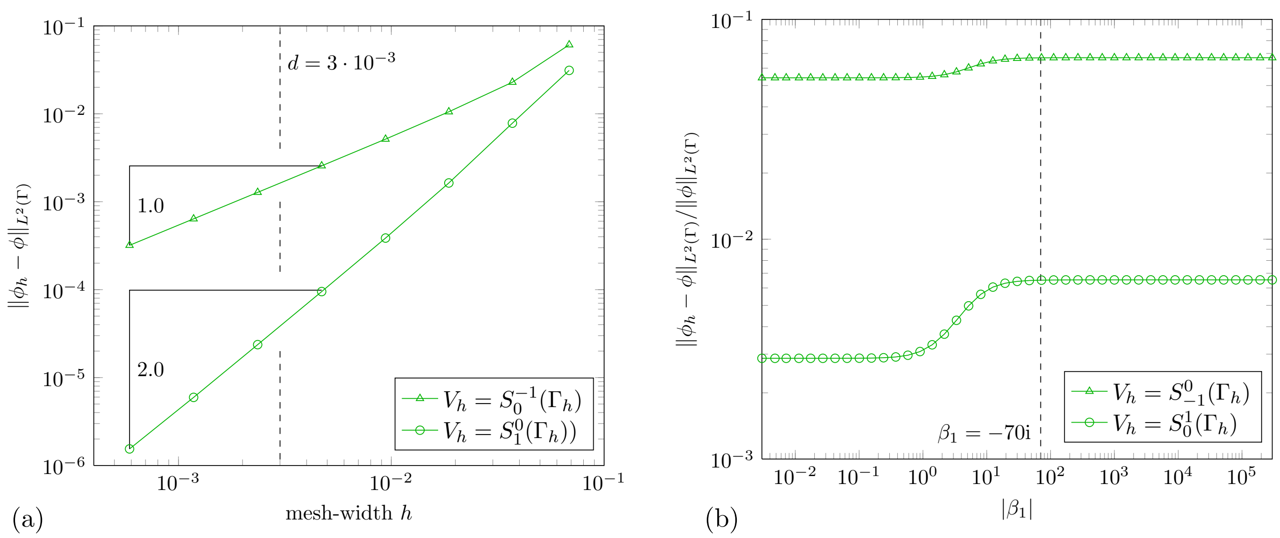

FIGURE 4. (a) Convergence of the discretisation error for the solution of the BEM for the ITC-1-0 (which is of type I) for the model problem from Section 1.4 ($\beta_1 = -70$i). (b) The accuracy of the discretisation is essentially independent of $|\beta_1|$ (shown for mesh width $h = 0.188$).

4.3. **Numerical experiments**

We have studied the proposed boundary element method for the numerical example described in Section 1.4 examplarily for ITC-1-0 for which β_1 is given in (6). The convergence of the discretisation error in the mesh width h is shown in Fig. 4(a) which confirms that the estimates of the discretisation error in h given in Theorem 4.3 are sharp. In Fig. 4(b) the relative discretisation error is plotted as a function of $|\beta_1|$, which gives evidence of the robustness of the discretisation error with $|\beta_1|$.

5. First kind boundary element formulation for problems of type II

5.1. **Boundary integral formulation**

Introducing the two unknowns $\phi = [\gamma_1 U]$ and $u = \{\gamma_0 U\}$ and using (26) we can convert (7a) into the boundary integral equation

$$\boxed{\begin{pmatrix} V & Id \\ -Id & \beta_1 Id - \beta_2 \Delta_\Gamma \end{pmatrix} \begin{pmatrix} \phi \\ u \end{pmatrix} = \begin{pmatrix} \gamma_0 NF \\ 0 \end{pmatrix}}\,. \tag{51}$$

Note that, in the limit $\beta_1, \beta_2 \to 0$ the solution of (51) is $\phi = 0$, $u = \gamma_0 NF$.

Now, testing the first line with ϕ' and the second line with u' we obtain the variational formulation: Seek $(\phi, u) \in H^{-1/2}(\Gamma) \times H^1(\Gamma)$ such that

$$\langle V\phi, \phi' \rangle + \quad \langle u, \phi' \rangle \qquad\qquad = \langle \gamma_0 NF, \phi' \rangle\,, \quad \forall \phi' \in H^{-1/2}(\Gamma), \tag{52a}$$

$$-\langle \phi, u' \rangle + \beta_1 \langle u, u' \rangle + \beta_2 \langle \nabla_\Gamma u, \nabla_\Gamma u' \rangle = 0, \qquad \forall u' \in H^1(\Gamma). \tag{52b}$$

Remark 5.1. *The mixed formulation* (52) *can be viewed as a saddle point problem with penalty term, see [2,* § *4, p. 138ff] for the case of purely real parameters.*

In the asymptotic models the parameter β_2 will attain only small absolute values, as it is scaled with the sheet thickness d. Thus, we are going to derive stability estimates which are robust for asymptotically small values, *i. e.*, $|\beta_2| \to 0$. In the case of the BIE for ITCs of type II we face singular perturbations not only for large $|\beta_1|$, as in the case of ITCs of type I, but also when $|\beta_2|$ is small. Hence, internal layers will emerge if Γ is not smooth, which leads to a blow-up of $|u|_{H^1(\Gamma)}$ for $\beta_2 \to 0$. Yet, as for the ITCs of type I, we obtain improved estimates for smoother interfaces Γ.

Theorem 5.2. *Let* $|\beta_1| > 0$ *and* assume *there exists a constant* $\theta_1^\star \in (-\pi, 0)$ *such that* $0 \geq \theta_1 := \arg(\beta_1) \geq \theta_1^\star$. *Furthermore, let* $|\beta_2| > 0$ *with* $\operatorname{Re}\beta_2 \geq 0$, $\operatorname{Im}\beta_2 \leq 0$, *and* Γ *be Lipschitz. Then, the system* (52) *has a unique solution* $\phi \in H^{-1/2}(\Gamma)$, $u \in H^1(\Gamma)$ *and there exists a constant* $C = C(\theta_1^\star)$ *independent of* $|\beta_1|$, $|\beta_2|$ *such that*

$$\|\phi\|_{H^{-1/2}(\Gamma)} \leq C \qquad\qquad \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \tag{53a}$$

$$\|u\|_{L^2(\Gamma)} \quad \leq C \min(|\beta_1|^{-1/2}, 1) \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \tag{53b}$$

$$|u|_{H^1(\Gamma)} \quad \leq C \qquad |\beta_2|^{-1/2} \qquad \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{53c}$$

Proof of Theorem 5.2. The proof is done in two steps. Firstly, we proof ellipticity with a constant which depends on β_1 and β_2, then we obtain the stability estimates (53).

Step (i): By assumption on β_1, β_2 we have with $\theta_2 := \arg(\beta_2) \in [-\pi/2, 0]$ and $\theta_m = \min(\theta_1, \theta_2) \in [-\theta_1^\star, 0)$ that $\theta_m/2 - \theta_1, \theta_m/2 - \theta_2 \in [\theta_1^\star/2, 0)$. Choosing $\phi' = \overline{\phi}$ and $u' = \overline{u}$ on the left side of (52) and summing (52a) and the complex conjugate of (52b), multiplying by $\mathrm{e}^{\mathrm{i}\theta_m/2}$, and taking the real part we get

$$\begin{aligned}
\operatorname{Re}\left(\mathrm{e}^{\frac{\mathrm{i}\theta_m}{2}} \left(\left\langle V\phi, \overline{\phi}\right\rangle + \overline{\beta_1}\|u\|^2_{L^2(\Gamma)} + \overline{\beta_2}|u|^2_{H^1(\Gamma)}\right)\right)& \\
= \cos\left(\tfrac{\theta_m}{2}\right)\left\langle V\phi, \overline{\phi}\right\rangle + |\beta_1| \cos\left(\tfrac{\theta_m}{2} - \theta_1\right) \|u\|^2_{L^2(\Gamma)} &+ |\beta_2| \cos\left(\tfrac{\theta_m}{2} - \theta_2\right) |u|^2_{H^1(\Gamma)} \\
&\geq C(\beta_1, \beta_2)\left(\|\phi\|^2_{H^{-1/2}(\Gamma)} + \|u\|^2_{H^1(\Gamma)}\right).
\end{aligned} \tag{54}$$

Hence, the bilinear form associated with (52) is $H^{-1/2}(\Gamma) \times H^1(\Gamma)$-elliptic and a unique solution $(\phi, u) \in H^{-1/2}(\Gamma) \times H^1(\Gamma)$ exists.

Step (ii): Now we write C for generic constants, which may depend on $\theta_1^\star$, but not on $|\beta_1|$ or $|\beta_2|$. As in Step (i), choosing $\phi' = \overline{\phi}$ and $u' = \overline{u}$ in (52) and summing (52a) and the complex conjugate of (52b), multiplying by $\mathrm{e}^{\mathrm{i}\theta_m/2}$ and taking the real part we obtain

$$\cos\left(\tfrac{\theta_m}{2}\right)\left\langle V\phi,\overline{\phi}\right\rangle + |\beta_1|\cos\left(\tfrac{\theta_m}{2}-\theta_1\right)\|u\|^2_{L^2(\Gamma)} + |\beta_2|\cos\left(\tfrac{\theta_m}{2}-\theta_2\right)|u|^2_{H^1(\Gamma)} \leq \left|\mathrm{Re}\left(\mathrm{e}^{\frac{\mathrm{i}\theta_m}{2}}\left\langle\gamma_0 NF,\overline{\phi}\right\rangle\right)\right|. \tag{55}$$

The $H^{-1/2}(\Gamma)$-ellipticity of V, see Lemma 3.1, the Cauchy-Schwarz inequality and Lemma 3.3 lead to (53a). Similarly, we obtain

$$|u|^2_{H^1(\Gamma)} \leq C|\beta_2|^{-1}\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}\|\phi\|_{H^{-1/2}(\Gamma)} \leq C|\beta_2|^{-1}\|\gamma_0 NF\|^2_{H^{1/2}(\Gamma)},$$

and so (53c), as well as

$$\|u\|^2_{L^2(\Gamma)} \leq C|\beta_1|^{-1}\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}\|\phi\|_{H^{-1/2}(\Gamma)} \leq C|\beta_1|^{-1}\|\gamma_0 NF\|^2_{H^{1/2}(\Gamma)}.$$

With the continuity of V by (19) and

$$\|u\|_{L^2(\Gamma)} \leq \|u\|_{H^{1/2}(\Gamma)} \leq \|V\phi\|_{H^{1/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{1/2}(\Gamma)} \leq C\|\phi\|_{H^{-1/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{1/2}(\Gamma)},$$

we get (53b). This finishes the proof. □

Lemma 5.3. *Let the assumptions of Theorem 5.2 be satisfied, $|\beta_2| \leq C$ independent of $|\beta_1|$, and (ϕ, u) the solution of* (52). *If $\Gamma \in C^{2,1}$, then there exists a constant C independent of $|\beta_1|$, $|\beta_2|$, and F such that*

$$\|\phi\|_{H^{-1/2}(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\left(\min(1,|\beta_1|) + |\beta_2|^{1/2}\right), \tag{56}$$

$$\|u\|_{H^{1/2}(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\min(|\beta_1|^{-1},1), \tag{57}$$

$$\|u\|_{H^2(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}|\beta_2|^{-1}. \tag{58}$$

If, furthermore, $\Gamma \in C^{3,1}$, then with a constant C independent of $|\beta_1|$, $|\beta_2|$, and F,

$$\|u\|_{H^1(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\min(|\beta_1|^{-1},1). \tag{59}$$

If, in addition, $\Gamma \in C^{4,1}$, then

$$\|u\|_{H^2(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\min(|\beta_1|^{-1},1)(1+|\beta_2|^{-1}). \tag{60}$$

If, moreover, $\Gamma \in C^{6,1}$, then

$$\|\phi\|_{H^{-1/2}(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\left(\min(1,|\beta_1|) + |\beta_2|\right), \tag{61}$$

and, if $\Gamma \in C^{10,1}$, then

$$\|u\|_{H^2(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\left(\min(|\beta_1|^{-1},1) + |\beta_2|\right), \tag{62}$$

where, throughout, C stands for a constant independent of $|\beta_1|$, $|\beta_2|$, and F.

Proof. (i) To cope with the singular perturbation for $|\beta_2| \to 0$ we use an asymptotic expansion for small β_2,

$$\begin{aligned}\phi &\sim \phi_0 + \beta_2\phi_1 + \beta_2^2\phi_2 + \dots,\\ u &\sim u_0 + \beta_2 u_1 + \beta_2^2 u_2 + \dots,\end{aligned} \tag{63}$$

with one or two terms to obtain improved estimates, which rely on higher regularity of Γ. The same idea has successfully been employed in the proof of Lemma 4.2 earlier.

We define (ϕ_0, u_0) by

$$(Id + \beta_1 V)\phi_0 = \beta_1\gamma_0 NF, \tag{64a}$$

$$u_0 = \beta_1^{-1}\phi_0 = -V\phi_0 + \gamma_0 NF, \tag{64b}$$

and (ϕ_j, u_j) for any $j > 0$ by

$$(Id + \beta_1 V)\phi_j = -\Delta_\Gamma u_{j-1}, \tag{65a}$$

$$u_j = \beta_1^{-1}\left(\phi_j + \beta_2 \Delta_\Gamma u_{j-1}\right) = -V\phi_j \ . \tag{65b}$$

We point out that the structure of (64a) and (65a) is that of a boundary integral equation arising from Type I transmission conditions, *cf.* (27). Therefore, we can apply the estimates in Theorem 4.1 and Lemma 4.2, to obtain

$$\begin{aligned}
&\|\phi_0\|_{H^{-1/2}(\Gamma)} \le C \min(1,|\beta_1|) \quad \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, && && (66a)\\
&\|\phi_0\|_{L^2(\Gamma)} \le C \quad |\beta_1| \quad \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, && && (66b)\\
&\|\phi_0\|_{H^{s+1/2}(\Gamma)} \le C \min(1,|\beta_1|) \quad \|\gamma_0 NF\|_{H^{2s+3/2}(\Gamma)}, && s \ge 0, && (66c)\\
&\|\phi_0\|_{H^{s+1/2}(\Gamma)} \le C \quad |\beta_1| \quad \|\gamma_0 NF\|_{H^{2s+1/2}(\Gamma)}, && s \ge 1\ , && (66d)\\
&\|u_0\|_{H^{s+1/2}(\Gamma)} \le C \min(|\beta_1|^{-1},1)\|\gamma_0 NF\|_{H^{2s+3/2}(\Gamma)}, && s \ge 0, && (66e)\\
&\|u_0\|_{H^{s+1/2}(\Gamma)} \le C \quad \|\gamma_0 NF\|_{H^{2s+1/2}(\Gamma)}, && s \ge 1\ , && (66f)
\end{aligned}$$

and for $j \ge 1$

$$\begin{aligned}
&\|\phi_j\|_{H^{-1/2}(\Gamma)} \le C \min(|\beta_1|^{-1},1) \quad \|u_{j-1}\|_{H^{5/2}(\Gamma)}, && && (67a)\\
&\|\phi_j\|_{L^2(\Gamma)} \le C \quad \|u_{j-1}\|_{H^{5/2}(\Gamma)}, && && (67b)\\
&\|\phi_j\|_{H^{s+1/2}(\Gamma)} \le C \min(|\beta_1|^{-1},1)\|u_{j-1}\|_{H^{2s+7/2}(\Gamma)}, && s \ge 0, && (67c)\\
&\|\phi_j\|_{H^{s+1/2}(\Gamma)} \le C \quad \|u_{j-1}\|_{H^{2s+5/2}(\Gamma)}, && s \ge 1\ . && (67d)
\end{aligned}$$

The terms u_j for $j \ge 1$ are not defined like u_0 and Lemma 4.2 does not apply. However, using the continuity properties of V given in (19) we obtain bounds for various norms of u_j in terms of other norms of u_{j-1}. Thus, we find for $j \ge 1$

$$\begin{aligned}
&\|u_j\|_{H^{1/2}(\Gamma)} \le C \min(|\beta_1|^{-1},1)\|\gamma_0 NF\|_{H^{2^{j+2}+1/2}(\Gamma)}\ , && && (68a)\\
&\|u_j\|_{H^{s+1/2}(\Gamma)} \le C \min(|\beta_1|^{-1},1)\|\gamma_0 NF\|_{H^{2^{j+1}\cdot s+3/2}(\Gamma)}\ , && s \ge \tfrac{1}{2} && (68b)\\
&\|u_j\|_{H^{s+1/2}(\Gamma)} \le C \quad \|\gamma_0 NF\|_{H^{2^{j+1}\cdot s+1/2}(\Gamma)}\ , && s \ge \tfrac{1}{2}\ . && (68c)
\end{aligned}$$

Furthermore, we define

$$\delta\phi_N := \phi - \sum_{j=0}^{N} \beta_2^j \phi_j, \qquad \delta u_N := u - \sum_{j=0}^{N} \beta_2^j u_j\ , \tag{69}$$

which solve

$$V\delta\phi_N + \quad \delta u_N \quad = 0, \tag{70a}$$

$$-\delta\phi_N + \beta_1 \delta u_N - \beta_2 \Delta_\Gamma \delta u_N = \beta_2^{N+1} \Delta_\Gamma u_N\ . \tag{70b}$$

Writing the variational formulation of (70) with test functions $\overline{\delta\phi}_N$ and $\overline{\delta u}_N$ and summing the first and the complex conjugate of the second equation, multiplying by $\mathrm{e}^{\mathrm{i}\theta_m/2}$ and taking the real part we obtain

$$\cos\left(\tfrac{\theta_m}{2}\right)\left\langle V\delta\phi_N, \overline{\delta\phi}_N\right\rangle + |\beta_1|\cos\left(\tfrac{\theta_m}{2} - \theta_1\right)\|\delta u_N\|^2_{L^2(\Gamma)} + |\beta_2|\cos\left(\tfrac{\theta_m}{2} - \theta_2\right)\|\nabla_\Gamma \delta u_N\|^2_{L^2(\Gamma)} \le |\beta_2|^{N+1}\left|\mathrm{Re}\left(\mathrm{e}^{\frac{\mathrm{i}\theta_m}{2}}\left\langle \nabla_\Gamma u_N, \nabla_\Gamma \overline{\delta u}_N\right\rangle\right)\right|.$$

The Cauchy-Schwarz inequality and the $H^{-1/2}(\Gamma)$-ellipticity of V yield

$$|\delta u_N|_{H^1(\Gamma)} \le C|\beta_2|^N |u_N|_{H^1(\Gamma)}, \tag{71a}$$
$$\|\delta\phi_N\|_{H^{-1/2}(\Gamma)} \le C|\beta_2|^{N+1/2} |u_N|_{H^1(\Gamma)}, \tag{71b}$$
$$|\beta_1|\|\delta u_N\|_{L^2(\Gamma)} \le C|\beta_2|^{N+1} |u_N|_{H^2(\Gamma)}, \tag{71c}$$

and with (70a) and (19) we get

$$\|\delta u_N\|_{H^{1/2}(\Gamma)} \le C\|\delta\phi_N\|_{H^{-1/2}(\Gamma)} \le C|\beta_2|^{N+1/2}|u_N|_{H^1(\Gamma)}. \tag{71d}$$

Furthermore, (70a) implies

$$\|\delta\phi_N\|_{L^2(\Gamma)} \le C\|\delta u_N\|_{H^1(\Gamma)} \le C|\beta_2|^N|u_N|_{H^1(\Gamma)}, \tag{71e}$$

and (70a) with (71e) and (71c) give us

$$\begin{aligned}\|\Delta_\Gamma \delta u_N\|_{L^2(\Gamma)} &\le |\beta_2|^N \|\Delta_\Gamma u_N\|_{L^2(\Gamma)} + |\beta_2|^{-1}\|\delta\phi_N\|_{L^2(\Gamma)} + |\beta_2|^{-1}|\beta_1|\|\delta u_N\|_{L^2(\Gamma)} \\ &\le C\left(|\beta_2|^{N-1}|u_N|_{H^1(\Gamma)} + |\beta_2|^N\|u_N\|_{H^2(\Gamma)}\right).\end{aligned} \tag{71f}$$

Applying (70a) once again we can assert that

$$\|\delta\phi_N\|_{H^1(\Gamma)} \le C\|\delta u_N\|_{H^2(\Gamma)} \le C\left(|\beta_2|^{N-1}|u_N|_{H^1(\Gamma)} + |\beta_2|^N\|u_N\|_{H^2(\Gamma)}\right). \tag{71g}$$

(ii) Next we aim for stronger estimates based on one term of the asymptotic expansion: Using the definition (65) of $\delta\phi_0$ and δu_0, the triangle inequality, (71b), (66a) and (66e) for $s = \frac{1}{2}$ we obtain, if $\Gamma \in C^{2,1}$,

$$\begin{aligned}\|\phi\|_{H^{-1/2}(\Gamma)} &\le \|\phi_0\|_{H^{-1/2}(\Gamma)} + \|\delta\phi_0\|_{H^{-1/2}(\Gamma)} \le \|\phi_0\|_{H^{-1/2}(\Gamma)} + C|\beta_2|^{1/2}|u_0|_{H^1(\Gamma)} \\ &\le \Big(\min(1,|\beta_1|) + |\beta_2|^{1/2}\min(|\beta_1|^{-1},1)\Big)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'},\end{aligned}$$

and **(56)** follows. Similarly, for u we obtain, using (71d) and (66e) for $s = \frac{1}{2}$,

$$\begin{aligned}\|u\|_{H^{1/2}(\Gamma)} &\le \|u_0\|_{H^{1/2}(\Gamma)} + \|\delta u_0\|_{H^{1/2}(\Gamma)} \le \|u_0\|_{H^{1/2}(\Gamma)} + C|\beta_2|^{1/2}|u_0|_{H^1(\Gamma)} \\ &\le C\min(|\beta_1|^{-1},1)(1+|\beta_2|^{1/2})\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'},\end{aligned}$$

which is **(57)**. Using (71a) and (66e) for $s = \frac{1}{2}$ we obtain a bound for the $H^1(\Gamma)$-seminorm of u

$$|u|_{H^1(\Gamma)} \le |u_0|_{H^1(\Gamma)} + |\delta u_0|_{H^1(\Gamma)} \le C|u_0|_{H^1(\Gamma)} \le C\min(|\beta_1|^{-1},1)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{72}$$

Now, using the first equation in (51), an elliptic shift theorem, (57) and (72), the second equation in (51) implies

$$\begin{aligned}|u|_{H^2(\Gamma)} &\le |\beta_2|^{-1}\left(\|\phi\|_{L^2(\Gamma)} + |\beta_1|\|u\|_{L^2(\Gamma)}\right) \le |\beta_2|^{-1}\left(C\|u\|_{H^1(\Gamma)} + |\beta_1|\|u\|_{L^2(\Gamma)}\right) \\ &\le C|\beta_2|^{-1}\left(\min(|\beta_1|^{-1},1) + \min(1,|\beta_1|)\right)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'},\end{aligned}$$

and so **(58)** follows. With a bound for the $L^2(\Gamma)$-norm of u,

$$\begin{aligned}\|u\|_{L^2(\Gamma)} &= \|u_0\|_{L^2(\Gamma)} + \|\delta u_0\|_{L^2(\Gamma)} \overset{(71c)}{\le} \|u_0\|_{L^2(\Gamma)} + C|\beta_1|^{-1}|\beta_2|\|u_0\|_{H^2(\Gamma)} \\ &\le C\Big(\min(|\beta_1|^{-1},1)\|\gamma_0 NF\|_{H^{3/2}(\Gamma)} + |\beta_1|^{-1}|\beta_2|\|\gamma_0 NF\|_{H^{7/2}(\Gamma)}\Big) \le C|\beta_1|^{-1}\|\gamma_0 NF\|_{H^{7/2}(\Gamma)},\end{aligned}$$

where we used the triangle inequality, (66e) for $s = 0$ and for $s = \frac{3}{2}$, and if $\Gamma \in C^{3,1}$ we conclude **(59)** using Lemma 3.3 and (72).

From the definition of δu_0, the triangle inequality, and (71f) we obtain

$$|u|_{H^2(\Gamma)} \le |u_0|_{H^2(\Gamma)} + |\delta u_0|_{H^2(\Gamma)} \le C(\|u_0\|_{H^2(\Gamma)} + |\beta_2|^{-1}|u_0|_{H^1(\Gamma)}),$$

and, if $\Gamma \in C^{4,1}$, **(60)** follows, using (66e) for $s = \frac{1}{2}$ and $s = \frac{3}{2}$ and (59).

(iv) Finally we establish stronger estimates based on two terms of the asymptotic expansion. Of course, the arguments will hinge on extra smoothness of Γ. Now, using the definition (65) of $\delta\phi_1$ and δu_1, the triangle inequality, (66a), (67a) for $j = 1$ and (68c) for $j = 1$ and $s = \frac{1}{2}$ we get, provided that $\Gamma \in C^{6,1}$,

$$
\begin{aligned}
\|\phi\|_{H^{-1/2}(\Gamma)} &\leq \|\phi_0\|_{H^{-1/2}(\Gamma)} + |\beta_2| \|\phi_1\|_{H^{-1/2}(\Gamma)} + \|\delta\phi_1\|_{H^{-1/2}(\Gamma)} \\
&\leq \|\phi_0\|_{H^{-1/2}(\Gamma)} + |\beta_2| \|\phi_1\|_{H^{-1/2}(\Gamma)} + \|\delta\phi_1\|_{H^{-1/2}(\Gamma)} + |\beta_2|^{3/2} |u_1|_{H^1(\Gamma)} \\
&\leq C \left(\min(1, |\beta_1|) + |\beta_2| \right) \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'},
\end{aligned}
$$

which is **(61)**. With definition of δu_1, the triangle inequality, and (71f) we obtain

$$
|u|_{H^2(\Gamma)} \leq |u_0|_{H^2(\Gamma)} + |\beta_2| |u_1|_{H^2(\Gamma)} + |\delta u_1|_{H^2(\Gamma)} \leq |u_0|_{H^2(\Gamma)} + C(|\beta_2| |u_1|_{H^2(\Gamma)} + |u_1|_{H^1(\Gamma)})
$$

and **(62)** follows, if $\Gamma \in C^{10,1}$, using (66e) for $s = \frac{3}{2}$, (68c) for $j = 1$ and $s = \frac{3}{2}$, (68b) for $j = 1$ and $s = \frac{1}{2}$, and (59). This completes the proof. □

In view of the first equation in (51) we have as a direct consequence of Lemma 5.3 the following corollary.

Corollary 5.4. *Let the assumption of Theorem 5.2 be satisfied, $|\beta_2| \leq C$ independent of $|\beta_1|$, and (ϕ, u) the solution of* (52). *Then, with a constant C independent of $|\beta_1|$, $|\beta_2|$, and F,*

$$
\|\phi\|_{H^1(\Gamma)} \leq C \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'} \cdot \begin{cases} |\beta_2|^{-1}, & \text{if } \Gamma \in C^{2,1}, \\ \min(|\beta_1|^{-1}, 1)(1 + |\beta_2|^{-1}), & \text{if } \Gamma \in C^{4,1}, \\ \min(|\beta_1|^{-1}, 1) + |\beta_2|, & \text{if } \Gamma \in C^{10,1}. \end{cases} \tag{73}
$$

5.2. Boundary element formulation

Let V_h be a finite-dimensional subspace of $H^{-1/2}(\Gamma)$, in particular $S_0^{-1}(\Gamma_h)$ or $S_1^0(\Gamma_h)$, and X_h a finite-dimensional subspace of $H^1(\Gamma)$, in particular $S_1^0(\Gamma_h)$. Then, the boundary element formulation reads: Seek $(\phi_h, j_h) \in V_h \times X_h$ such that

$$
\langle V\phi_h, \phi_h' \rangle + \quad \langle u_h, \phi_h' \rangle \qquad = \langle \gamma_0 N F, \phi_h' \rangle, \quad \forall \phi_h' \in V_h, \tag{74a}
$$

$$
-\langle \phi_h, u_h' \rangle + \beta_1 \langle u_h, u_h' \rangle + \beta_2 \langle \nabla_\Gamma u_h, \nabla_\Gamma u_h' \rangle = 0, \qquad \forall u_h' \in X_h. \tag{74b}
$$

The discretisation with $V_h = S_0^{-1}(\Gamma_h)$ is not stable for $\beta_1, \beta_2 \to 0$, see Figure 5, as the $L^2(\Gamma)$-pairing of $S_0^{-1}(\Gamma_h)$ and $S_1^0(\Gamma_h)$ is not uniformly stable on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. This choice for V_h may lead to large error in u_h if β_1 and β_2 are small. There is a simple remedy: As the solution and so ϕ are assumed to be smooth, an attractive choice is $V_h = S_1^0(\Gamma_h)$, which immediately ensures a stable pairing.

Remark 5.5. *Assuming standard choices of basis functions of the boundary element spaces, all blocks in the linear system of equations arising from the Galerkin discretization of* (74) *are sparse, except for one. Hence, the extra numerical effort involved in the BEM Galerkin discretization of models of type II compared to models of type I is essentially negligible.*

Theorem 5.6. *Let the assumptions of Theorem 5.2 be fulfilled, $|\beta_2| \leq C$ independent of $|\beta_1|$, and $V_h \subset H^{-1/2}(\Gamma)$, $X_h \subset H^1(\Gamma)$. Then, for $\Gamma \in C^{0,1}$, the linear system of equations* (74) *has a unique solution $(\phi_h, u_h) \in V_h \times X_h$ and there exists a constant $C = C(\arg(\beta_1), \arg(\beta_2))$ independent of $|\beta_1|$, F, and Γ_h, such that*

$$
\begin{aligned}
\|\phi_h\|_{H^{-1/2}(\Gamma)} &\leq C \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \\
\|u_h\|_{L^2(\Gamma)} &\leq C |\beta_1|^{-1/2} \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \\
|u_h|_{H^1(\Gamma)} &\leq C |\beta_2|^{-1/2} \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}.
\end{aligned} \tag{75}
$$

If, furthermore, $V_h = X_h = S_1^0(\Gamma_h)$ and Γ_h belongs to a quasi-uniform family of triangulations, then

$$
\|u_h\|_{H^{1/2}(\Gamma)} \leq C \qquad \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}, \tag{76a}
$$

$$
\|u_h\|_{H^1(\Gamma)} \leq C \big(\min(|\beta_1|^{-1/2}, 1) + |\beta_2|^{-1/2} \big) \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}. \tag{76b}
$$

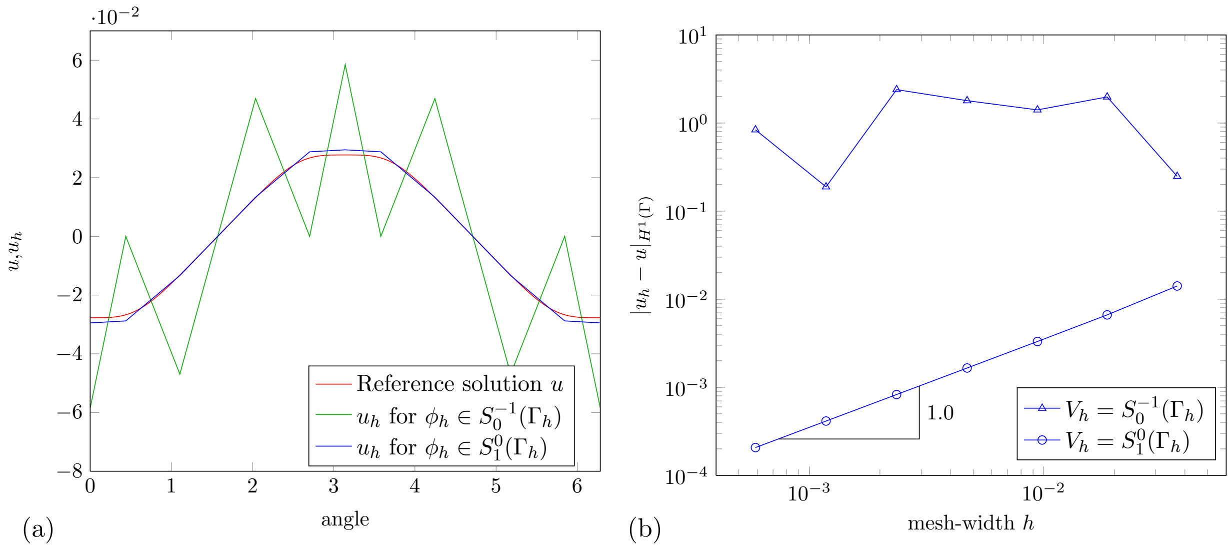

FIGURE 5. The discretisation with $V_h = S_0^{-1}(\Gamma_h)$ is not stable for $\beta_1, \beta_2 \to 0$: (a) The BEM solutions u_h for the same problem, but with $d = 0$, where $\beta_1 = \beta_2 = 0$, for illustration computed with 10 intervals, (b) the comparison of the discretisation error as a function of the mesh width.

If, in addition, $\Gamma \in C^{2,1}$ *then*

$$\|\phi_h\|_{H^{-1/2}(\Gamma)} \leq C\big(\min(1, |\beta_1|) + |\beta_2|^{1/2}\big)\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{77a}$$

$$\|u_h\|_{L^2(\Gamma)} \leq C\big(\min(|\beta_1|^{-1}, 1) + |\beta_2|^{1/2}\big)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\,, \tag{77b}$$

$$\|u_h\|_{H^1(\Gamma)} \leq C\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\,. \tag{77c}$$

If, moreover, $\Gamma \in C^{4,1}$ *then*

$$\|u_h\|_{H^1(\Gamma)} \leq C\big(\min(|\beta_1|^{-1}, 1) + |\beta_2|^{1/2}\big)\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{78a}$$

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} \leq C\big(|\beta_2|^{-1} \min(|\beta_1|^{-1}, 1) + |\beta_1|^{1/2}\big) h\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{78b}$$

$$\|u_h - u\|_{H^1(\Gamma)} \leq C\big(\min(|\beta_1|^{-1}, 1) + |\beta_1|^{1/2}|\beta_2|\big)\big(|\beta_1|^{-1/2} + |\beta_2|^{-1/2}\big)|\beta_2|^{-1}\, h\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{78c}$$

and, if $\Gamma \in C^{6,1}$, *then*

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} \leq C\big(1 + |\beta_1|^{1/2}|\beta_2|\big) h\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{79}$$

and, if $\Gamma \in C^{13,1}$, *then*

$$\|u_h - u\|_{H^1(\Gamma)} \leq C\big(\min(|\beta_1|^{-1}, 1) + |\beta_2| + |\beta_1|^{1/2}|\beta_2|^2\big) h\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{80}$$

with C *independent of* $|\beta_1|$, $|\beta_2|$, F, *and* h.

Proof. Due to the $H^{-1/2}(\Gamma)\times H^1(\Gamma)$-ellipticity of the bilinear form underlying (74), see the proof of Theorem 5.2, and the fact that $V_h \times X_h \subset H^{-1/2}(\Gamma) \times H^1(\Gamma)$, the well-posedness of (74) is immediate, and (75) follows.

For $V_h = X_h = S_1^0(\Gamma_h)$ the two spaces provide a uniformly stable $L^2(\Gamma)$-pairing. For ϕ_h given, (74a) can be regarded as an equation for u_h, and so we get

$$\|u_h\|_{H^{1/2}(\Gamma)} \leq C(\|\phi_h\|_{H^{-1/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}) \leq C\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\ , \tag{81}$$

which is **(76a)**. With (75) and for $\Gamma \in C^{0,1}$ **(76b)** follows.

For the remainder of the proof we restrict ourselves to $X_h = V_h = S_1^0(\Gamma_h)$. Following the proof of Lemma 5.3, in order to show stronger estimates for smoother Γ we use, for $N \geq 0$, the decompositions

$$\phi = \sum_{j=0}^{N} \beta_2^j \phi_j + \delta\phi_N, \quad u = \sum_{j=0}^{N} \beta_2^j u_j + \delta u_N, \quad \phi_h = \sum_{j=0}^{N} \beta_2^j \phi_{j,h} + \delta\phi_{N,h}, \quad u_h = \sum_{j=0}^{N} \beta_2^j u_{j,h} + \delta u_{N,h}. \tag{82}$$

The terms ϕ_j, u_j, $\delta\phi_N$, and δu_N are defined by (64), (65) and (70), respectively. The boundary element functions $\phi_{j,h}, u_{j,h} \in V_h$ will be specified below.
(i) **Estimates of $\phi_{0,h}$ and $u_{0,h}$.** The approximation $\phi_{0,h}$ to ϕ_0 is solution of (37) and we have

$$\Gamma \in C^{2,1} \quad \overset{(38)}{\Rightarrow} \quad \|\phi_{0,h}\|_{L^2(\Gamma)} \leq C \min(1, |\beta_1|) \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} , \tag{83}$$

$$\Gamma \in C^{3,1} \quad \overset{(41),(50)}{\Rightarrow} \quad \|\phi_{0,h} - \phi_0\|_{L^2(\Gamma)} \leq C \min(1, |\beta_1|)\, h \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} , \tag{84}$$

$$\Gamma \in C^{5,1} \quad \overset{(40)}{\Rightarrow} \quad \|\phi_{0,h} - \phi_0\|_{L^2(\Gamma)} \leq C \min(1, |\beta_1|)\, h^2 \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} . \tag{85}$$

The function $u_{0,h} \in X_h$ is the unique solution of

$$\langle u_{0,h}, u_h' \rangle = -\langle V\phi_{0,h}, u_h' \rangle + \langle \gamma_0 NF, u_h' \rangle = \beta_1^{-1} \langle \phi_{0,h}, u_h' \rangle , \quad \forall u_h' \in X_h. \tag{86}$$

Owing to the $H^1(\Gamma)$-stability of the $L^2(\Gamma)$-projections onto $S_1^0(\Gamma_h)$ for quasi-uniform families of meshes [3], plus the bound on $\phi_{0,h}$ from (83) and Lemma 3.3, we have, if $\Gamma \in C^{2,1}$,

$$\|u_{0,h}\|_{H^1(\Gamma)} \leq C\|\phi_{0,h}\|_{L^2(\Gamma)} + \|\gamma_0 NF\|_{H^1(\Gamma)} \leq C\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{87}$$

As $X_h = V_h$, we have $u_{0,h} = \beta_1^{-1}\phi_{0,h}$ and so (83) implies, if $\Gamma \in C^{2,1}$,

$$\|u_{0,h}\|_{L^2(\Gamma)} \leq C \min(|\beta_1|^{-1}, 1) \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} . \tag{88}$$

As $u_0 = \beta_1^{-1}\phi_0$ and with (85) if $\Gamma \in C^{5,1}$ we can assert that

$$\|u_{0,h} - u_0\|_{L^2(\Gamma)} \leq C \min(|\beta_1|^{-1}, 1)\, h^2 \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{89}$$

Similarly to the proof of Theorem 4.3 we have with the $L^2(\Gamma)$-projection $Q_h : L^2(\Gamma) \to V_h$, if $\Gamma \in C^{4,1}$,

$$\begin{aligned} \|u_{0,h} - u_0\|_{H^1(\Gamma)} &\leq \|(u_{0,h} - Q_h u_0) - (u_0 - Q_h u_0)\|_{H^1(\Gamma)} \leq \|Q_h(u_{0,h} - u_0)\|_{H^1(\Gamma)} + \|u_0 - Q_h u_0\|_{H^1(\Gamma)} \\ &\leq Ch^{-1} \|Q_h(u_{0,h} - u_0)\|_{L^2(\Gamma)} + Ch \|u_0\|_{H^2(\Gamma)} \leq C \min(|\beta_1|^{-1}, 1)\, h \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \end{aligned} \tag{90}$$

where we used an inverse inequality, the continuity and approximation properties of Q_h, and a bound on $\|u_0\|_{H^2(\Gamma)}$ by (66e) for $s = \frac{3}{2}$. Finally using the triangle inequality, (90), (66e) for $s = \frac{1}{2}$ and Lemma 3.3 we observe if $\Gamma \in C^{4,1}$

$$\|u_{0,h}\|_{H^1(\Gamma)} \leq \|u_0\|_{H^1(\Gamma)} + \|u_{0,h} - u_0\|_{H^1(\Gamma)} \leq C \min(|\beta_1|^{-1}, 1)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{91}$$

(ii) **Estimates for $\phi_{j,h}$ and $u_{j,h}$.** The approximations $\phi_{j,h} \in V_h$ to ϕ_j are solutions of

$$\langle \phi_{j,h}, \phi_h' \rangle + \beta_1 \langle V\phi_{j,h}, \phi_h' \rangle = \langle \nabla_\Gamma u_{j-1,h}, \nabla_\Gamma \phi_h' \rangle \quad \forall \phi_h' \in V_h . \tag{92}$$

We can rewrite

$$\langle f_{\phi,j}, \phi_h' \rangle := \langle \nabla_\Gamma u_{j-1,h}, \nabla_\Gamma \phi_h' \rangle = \langle \Delta_\Gamma u_{j-1}, \phi_h' \rangle + \langle \nabla_\Gamma (u_{j-1,h} - u_{j-1}), \nabla_\Gamma \phi_h' \rangle \quad \forall \phi_h' \in V_h . \tag{93}$$

Using an inverse estimate we conclude that

$$\langle \nabla_\Gamma (u_{j-1,h} - u_{j-1}), \nabla_\Gamma \phi_h' \rangle \leq |u_{j-1,h} - u_{j-1}|_{H^1(\Gamma)} |\phi_h'|_{H^1(\Gamma)} \leq Ch^{-1} |u_{j-1,h} - u_{j-1}|_{H^1(\Gamma)} \|\phi_h'\|_{L^2(\Gamma)}, \tag{94}$$

and so we can bound the right hand side of (92) by

$$\|f_{\phi,j}\|_{L^2(\Gamma)} \leq \|u_{j-1}\|_{H^2(\Gamma)} + Ch^{-1}|u_{j-1,h} - u_{j-1}|_{H^1(\Gamma)}, \tag{95}$$

and, hence, we can bound

$$\|\phi_{j,h}\|_{L^2(\Gamma)} \le C\left(\|u_{j-1}\|_{H^2(\Gamma)} + h^{-1}|u_{j-1,h} - u_{j-1}|_{H^1(\Gamma)}\right). \tag{96}$$

For $j = 1$ we have with (90) an estimate for the second term on the right hand side of (96) and with (66e) for $s = \frac{3}{2}$ we obtain if $\Gamma \in C^{5,1}$

$$\|\phi_{1,h}\|_{L^2(\Gamma)} \le C\min(|\beta_1|^{-1}, 1)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{97}$$

To obtain estimates for $\phi_{j,h}$, $j \ge 2$ we have to define $u_{j,h} \in X_h$ for $j \ge 1$, which we do in in analogy to the definition of $u_{0,h}$ by

$$\langle u_{j,h}, u'_h\rangle = -\langle V\phi_{j,h}, u'_h\rangle, \quad \forall u'_h \in X_h. \tag{98}$$

As u_j solves a similar equation, but with $V\phi_j$ instead of $V\phi_{j,h}$ on the right hand side, we find using Strang's lemma, the best-approximation properties in $X_h = S^0_1(\Gamma_h)$,

$$\begin{aligned}\|u_{j,h} - u_j\|_{H^1(\Gamma)} &\le C\big(\inf_{v_h\in X_h}\|v_h - u_j\|_{H^1(\Gamma)} + \|V(\phi_{j,h} - \phi_j)\|_{H^1(\Gamma)}\big)\\ &\le C\big(h\,\|u_j\|_{H^2(\Gamma)} + \|\phi_{j,h} - \phi_j\|_{L^2(\Gamma)}\big),\end{aligned} \tag{99}$$

and so we need to estimate $\|\phi_{j,h} - \phi_j\|_{L^2(\Gamma)}$ for $j \ge 1$. The equation defining ϕ_j is (27), where $\beta_1\gamma_0 NF$ is replaced by $-\Delta u_{j-1}$, and so by (50) for $\ell = 1$, (66f) we obtain

$$\|\phi_{j,h} - \phi_j\|_{L^2(\Gamma)} \le C\,h\|\Delta u_{j-1}\|_{H^{3/2}(\Gamma)} \le C\,h\|u_{j-1}\|_{H^{7/2}(\Gamma)}, \tag{100}$$

and inserted into (99),

$$\begin{aligned}\|u_{j,h} - u_j\|_{H^1(\Gamma)} &\le C\,h\big(\|u_j\|_{H^2(\Gamma)} + \|u_{j-1}\|_{H^{7/2}(\Gamma)}\big)\\ &\le C\min(|\beta_1|^{-1}, 1)\,h\big(\|\gamma_0 NF\|_{H^{2^j\cdot 3+3/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{2^j\cdot 3+3/2}(\Gamma)}\big) &(101)\\ &\le C\min(|\beta_1|^{-1}, 1)\,h\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} &(102)\end{aligned}$$

if $\Gamma \in C^{(2^j\cdot 3+1),1}$. Here, we have used (66e) or (68b) for $s = \frac{3}{2}$ and $s = 3$ and Lemma 3.3. Now, inserting (102) into (96) we find for $j \ge 2$ and if $\Gamma \in C^{(2^{j-1}\cdot 3+1),1}$

$$\|\phi_{j,h}\|_{L^2(\Gamma)} \le C\min(|\beta_1|^{-1}, 1)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{103}$$

Immediately from (98), we find, for $j \ge 1$,

$$\|u_{j,h}\|_{H^1(\Gamma)} \le C\,\|\phi_{j,h}\|_{L^2(\Gamma)}, \tag{104}$$

and so the bounds (97) and (103) apply to $\|u_{j,h}\|_{H^1(\Gamma)}$ as well. Finally, applying (**??**) for $\ell = 2$ and (66e), and Lemma 3.3 we find if $\Gamma \in C^{(2^{j+1}\cdot 7+1),1}$

$$\begin{aligned}\|\phi_{j,h} - \phi_j\|_{L^2(\Gamma)} &\le C\,h^2\|\Delta u_{j-1}\|_{H^{11/2}(\Gamma)} \le C\,h^2\|u_{j-1}\|_{H^{15/2}(\Gamma)}\\ &\le C\,h^2\min(|\beta_1|^{-1}, 1)\|\gamma_0 NF\|_{H^{2^{j+1}\cdot 7+3/2}(\Gamma)} \le C\,h^2\min(|\beta_1|^{-1}, 1)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}.\end{aligned} \tag{105}$$

(iii) **Estimates for $\delta\phi_{N,h}$ and $\delta u_{N,h}$.** The last terms in the decomposition of ϕ and u solve (see (70)): Seek $(\delta u_{N,h}, \delta\phi_{N,h}) \in X_h \times V_h$

$$\begin{aligned}\langle V\delta\phi_{N,h}, \phi'_h\rangle + \langle\delta u_{N,h}, \phi'_h\rangle &= 0, & \forall\phi'_h \in V_h, &\quad(106a)\\ -\langle\delta\phi_{N,h}, u'_h\rangle + \beta_1\langle\delta u_{N,h}, u'_h\rangle + \beta_2\langle\nabla_\Gamma\delta u_{N,h}, \nabla_\Gamma u'_h\rangle &= -\beta_2\langle\nabla_\Gamma u_{N,h}, \nabla_\Gamma u'_h\rangle, & \forall u'_h \in X_h. &\quad(106b)\end{aligned}$$

Choosing $\phi'_h = \overline{\delta\phi}_{N,h}$ and $u'_h = \overline{\delta u}_{N,h}$ in (106) and summing (106a) and the complex conjugate of (106b), multiplying by $\mathrm{e}^{\frac{\mathrm{i}\theta_m}{2}}$ and taking the real part we obtain

$$\cos\left(\tfrac{\theta_m}{2}\right)\left\langle V\delta\phi_{N,h}, \overline{\delta\phi}_{N,h}\right\rangle + |\beta_2|\cos\left(\tfrac{\theta_m}{2} - \theta_2\right)|\delta u_{N,h}|^2_{H^1(\Gamma)} \le \left|\mathrm{Re}(\beta_2\,\mathrm{e}^{\frac{\mathrm{i}\theta_m}{2}}\left\langle\nabla_\Gamma u_{N,h}, \nabla_\Gamma\overline{\delta u}_{N,h}\right\rangle)\right|.$$

Hence, by (87) for $N = 0$ and $\Gamma \in C^{2,1}$, (104) and (97) for $N = 1$ and $\Gamma \in C^{5,1}$, or (104) and (103) for $N > 1$ and $\Gamma \in C^{(2^{N-1}\cdot 3+1),1}$ that

$$|\delta u_{N,h}|_{H^1(\Gamma)} \le C|u_{N,h}|_{H^1(\Gamma)} \le C \,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,, \tag{107a}$$

$$\|\delta\phi_{N,h}\|^2_{H^{-1/2}(\Gamma)} \le C|\beta_2||u_{N,h}|_{H^1(\Gamma)}|\delta u_{N,h}|_{H^1(\Gamma)} \le C|\beta_2|\|F\|^2_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,. \tag{107b}$$

If $\delta\phi_{N,h}$ is known equation (106a) defines $\delta u_{N,h}$ and clearly

$$\|\delta u_{N,h}\|_{L^2(\Gamma)} \le C\|V\delta\phi_{N,h}\|_{L^2(\Gamma)} \le C\|\delta\phi_{N,h}\|_{H^{-1}(\Gamma)} \le C|\beta_2|^{\frac{1}{2}}\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,. \tag{107c}$$

The proof for the discretisation error estimate runs parallel to that for type I, see the proof of Theorem 4.3. The energy norm related to the bilinear form in (106) is defined by

$$|\!|\!|(\delta\phi_{N,h}, u_{N,h})|\!|\!|^2 := \|\delta\phi_{N,h}\|^2_{H^{-1/2}(\Gamma)} + |\beta_1|\|\delta u_{N,h}\|^2_{L^2(\Gamma)} + |\beta_2||\delta u_{N,h}|^2_{H^1(\Gamma)}.$$

In this norm the bilinear form is elliptic and continuous with constants depending on β_1 or β_2 only through $\theta_1^\star$. Applying Strang's lemma [2, Ch. III, Thm. 1.1] we therefore obtain,

$$\begin{aligned}
&\|\delta\phi_{N,h} - \delta\phi_N\|_{H^{-1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_{N,h} - \delta u_N\|_{L^2(\Gamma)} + |\beta_2|^{1/2}|\delta u_{N,h} - \delta u_N|_{H^1(\Gamma)} \\
&\quad\le C \inf_{(\psi_h,v_h)\in V_h\times X_h} \big(\|\psi_h - \delta\phi_N\|_{H^{-1/2}(\Gamma)} + |\beta_1|^{1/2}\|v_h - \delta u_N\|_{L^2(\Gamma)} + |\beta_2|^{1/2}\|v_h - \delta u_N\|_{H^1(\Gamma)}\big) \\
&\qquad + C|\beta_2|^{1/2}|u_{N,h} - u_N|_{H^1(\Gamma)} \\
&\quad\le C\,h\,\big(\|\delta\phi_N\|_{H^{1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_N\|_{H^1(\Gamma)} + |\beta_2|^{1/2}\|\delta u_N\|_{H^2(\Gamma)}\big) + C|\beta_2|^{1/2}|u_{N,h} - u_N|_{H^1(\Gamma)}.
\end{aligned}$$

Using (71g), (71a), (71d), (71f) we bound

$$\begin{aligned}
&\|\delta\phi_N\|_{H^{1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_N\|_{H^1(\Gamma)} + |\beta_2|^{1/2}\|\delta u_N\|_{H^2(\Gamma)} \\
&\quad\le C\big(|\beta_2|^{N-1}|u_N|_{H^1(\Gamma)} + |\beta_2|^N\|u_N\|_{H^2(\Gamma)} + |\beta_1|^{1/2}|\beta_2|^N|u_N|_{H^1(\Gamma)} + |\beta_1|^{-1/2}|\beta_2|^{N+1}|u_N|_{H^2(\Gamma)}\big)
\end{aligned}$$

and so for $N = 0$ if $\Gamma \in C^{4,1}$

$$\|\delta\phi_0\|_{H^{1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_0\|_{H^1(\Gamma)} + |\beta_2|^{1/2}\|\delta u_0\|_{H^2(\Gamma)} \le C\big(|\beta_2|^{-1}\min(|\beta_1|^{-1}, 1) + |\beta_1|^{1/2}\big)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,,$$

and for $N \ge 1$ if $\Gamma \in C^{(2^N\cdot 3+1),1}$

$$\|\delta\phi_N\|_{H^{1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_N\|_{H^1(\Gamma)} + |\beta_2|^{1/2}\|\delta u_N\|_{H^2(\Gamma)} \le C\big(|\beta_2|^{N-1} + |\beta_1|^{1/2}|\beta_2|^N\big)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,.$$

Hence, using (90) we obtain if $\Gamma \in C^{4,1}$

$$\begin{aligned}
&\|\delta\phi_{0,h} - \delta\phi_0\|_{H^{-1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_{0,h} - \delta u_0\|_{L^2(\Gamma)} + |\beta_2|^{1/2}|\delta u_{0,h} - \delta u_0|_{H^1(\Gamma)} \\
&\quad\le C\big(|\beta_2|^{-1}\min(|\beta_1|^{-1}, 1) + |\beta_1|^{1/2}\big)h\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,,
\end{aligned} \tag{108}$$

and using (102) we find for $N \ge 1$ if $\Gamma \in C^{(2^N\cdot 3+1),1}$

$$\begin{aligned}
&\|\delta\phi_{N,h} - \delta\phi_N\|_{H^{-1/2}(\Gamma)} + |\beta_1|^{1/2}\|\delta u_{N,h} - \delta u_N\|_{L^2(\Gamma)} + |\beta_2|^{1/2}|\delta u_{N,h} - \delta u_N|_{H^1(\Gamma)} \\
&\quad\le C\big(|\beta_2|^{1/2}\min(|\beta_1|^{-1}, 1)) + |\beta_2|^{N-1} + |\beta_1|^{1/2}|\beta_2|^N\big)\,h\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,.
\end{aligned} \tag{109}$$

(iv) **Estimates for ϕ_h and u_h.** Now, we can use the decomposition of ϕ_h and the estimates (83) and (107b) to bound the discrete solution ϕ_h if $\Gamma \in C^{2,1}$ as

$$\|\phi_h\|_{H^{-1/2}(\Gamma)} \le \|\phi_{0,h}\|_{H^{-1/2}(\Gamma)} + \|\delta\phi_{0,h}\|_{H^{-1/2}(\Gamma)} \le C\big(\min(1, |\beta_1|) + |\beta_2|^{\frac{1}{2}}\big)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,,$$

which is **(77a)**. Similarly with (88) and (107c) we get for $\Gamma \in C^{2,1}$

$$\|u_h\|_{L^2(\Gamma)} \le \|u_{0,h}\|_{L^2(\Gamma)} + \|\delta u_{0,h}\|_{L^2(\Gamma)} \le C\big(\min(|\beta_1|^{-1}, 1) + |\beta_2|^{1/2}\big)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \,,$$

which is **(77b)**. In the same way using (87), (107a), (107c) we get for $\Gamma \in C^{2,1}$

$$\|u_h\|_{H^1(\Gamma)} \le \|u_{0,h}\|_{H^1(\Gamma)} + \|\delta u_{0,h}\|_{H^1(\Gamma)} \le C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}$$

which is **(77c)**. We obtain **(78a)** as **(77c)**, while only using (91) instead of (87).

Furthermore, we can bound the discretisation error as

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} \le \|\phi_{0,h} - \phi_0\|_{H^{-1/2}(\Gamma)} + \|\delta\phi_{0,h} - \delta\phi_0\|_{H^{-1/2}(\Gamma)},$$

and if $\Gamma \in C^{4,1}$ the estimates (84) and (108) and the fact that $\min(1, |\beta_1|) \le |\beta_1|^{1/2}$ result in **(78b)**. Moreover, the estimate **(78c)** is obtained in the same way using (90) and (108).

To obtain estimates for the discretisation error for ϕ_h, which are robust even for $\beta_2 \to 0$, we use one more term of the expansion, which gives

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} \le \|\phi_{0,h} - \phi_0\|_{H^{-1/2}(\Gamma)} + |\beta_2|\|\phi_{1,h} - \phi_1\|_{H^{-1/2}(\Gamma)} + \|\delta\phi_{1,h} - \delta\phi_1\|_{H^{-1/2}(\Gamma)}\ ,$$

and so **(79)** when we use (84), (100) for $j = 1$, (66f) for $s = 3$ and (109) for $N = 1$. To obtain an estimate for the discretisation error of u_h, which does not blow up for $\beta_2 \to 0$, we have to take another term in the expansion,

$$\|u_h - u\|_{H^1(\Gamma)} \le \|u_{0,h} - u_0\|_{H^1(\Gamma)} + |\beta_2|\|u_{1,h} - u_1\|_{H^1(\Gamma)} + |\beta_2|^2\|u_{2,h} - u_2\|_{H^1(\Gamma)} + \|\delta u_{2,h} - \delta u_2\|_{H^1(\Gamma)}\ ,$$

and using (90), (102) for $j = 1, 2$ and (109) for $N = 2$ we obtain **(80)**. This completes the proof. □

5.3. Numerical experiments

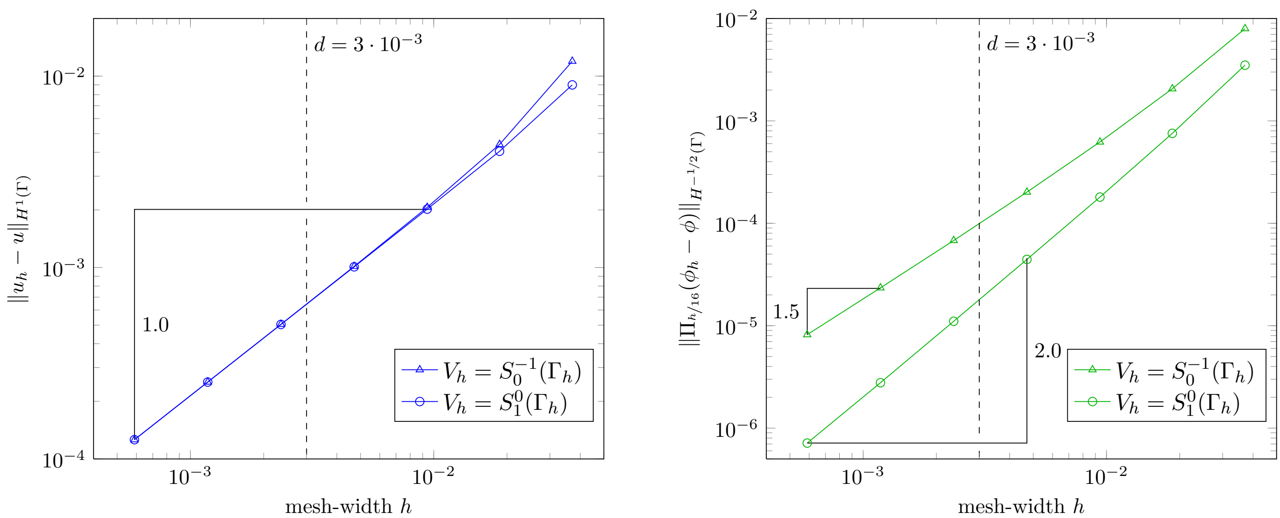

FIGURE 6. Convergence of the discretisation error for the solution of the BEM for the NTFS model (which is of type II) for the model problem from Section 1.4 ($\beta_1 = -70\mathrm{i}$, $\beta_2 = 3 \cdot 10^{-3}$). The $H^{-1/2}(\Gamma)$-norm is computed via the single layer potential operator for the L^2-projection onto $V_{h/16}$ on the mesh $\Gamma_{h/16}$, for which each interval of Γ_h has been refined four times.

We have studied the proposed boundary element method for the numerical example described in Section 1.4 and the NTFS condition by Nakata *et al.* [12] for which β_1 and β_2 are given in (8). In this example, in which β_1 and β_2 are not too small, we observe for $V_h = S_0^{-1}(\Gamma_h)$ the convergence orders given by the best-approximation, which are 1 for u_h in $H^1(\Gamma)$ and $\frac{3}{2}$ for ϕ_h in $H^{-1/2}(\Gamma)$, see Figure 6. For $V_h = S_1^0(\Gamma_h)$ the behavior of the discretisation error as a function of the mesh width h is also shown in Figure 6. The results confirm that the estimates of the discretisation error of u_h in h given in Theorem 5.6 are sharp.

Note that the evident quadratic convergence of ϕ_h is better than the predictions of our theory. We believe that, for all $h > h^\star(\beta_2)$ for some $h^\star(\beta_2)$, which decreases with $|\beta_2|$, the discretization error behaves like $O(h^{5/2})$,

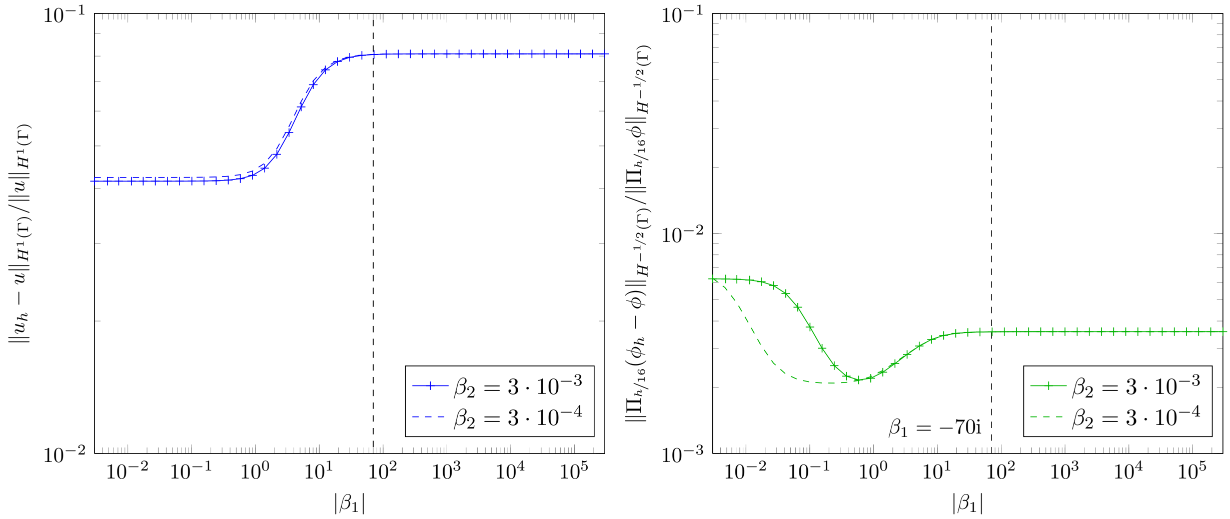

FIGURE 7. The relative discretisation errors for the solution of the BEM ($V_h = S_0^1(\Gamma_h)$ with $h = 0.188$) for the NTFS model (which is of type II) for the model problem from Section 1.4 as a function of the model parameter $|\beta_1|$, and for two different β_2. The $H^{-1/2}(\Gamma)$-norm is computed via the single layer potential operator for the L^2-projection onto $V_{h/16}$ on the mesh $\Gamma_{h/16}$ where each interval of Γ_h is four times refined.

that is, like the corresponding best-approximation error for $V_h = S_1^0(\Gamma_h)$. Yet, the low order polygonal boundary approximation of Γ may limit the order of convergence.

In Fig. 7 the relative discretisation error for $V_h = S_1^0(\Gamma_h)$ is shown as a function of $|\beta_1|$ for two different values of β_2. The data confirm that the relative discretization error enjoys the same moderate dependence on $|\beta_1|$ and $|\beta_2|$ as the exact solution. Thus we have strong evidence of the robustness of the discretization with respect $|\beta_1|$ and $|\beta_2|$.

6. Second kind boundary element formulation for problems of type III

6.1. **Boundary integral formulation**

In comparison to the type I problems the jump of the Dirichlet trace does not vanish for the problems of type III. Taking the mean Dirichlet and Neumann traces of (13) we get with the relations (3) that

$$\{\gamma_0 U\} = -V[\gamma_1 U] + K[\gamma_0 U] + \gamma_0 NF, \tag{110}$$
$$\{\gamma_1 U\} = -K'[\gamma_1 U] - W[\gamma_0 U] + \gamma_1 NF \tag{111}$$

and inserting into (9a) and (9b) we obtain $\phi = [\gamma_1 U]$, $j = [\gamma_0 U]$ from the boundary integral equations

$$\begin{pmatrix} Id + \beta_1 V & -\beta_1 K \\ \overline{\beta_1} K' & \overline{\beta_1}\beta_3^{-1} Id + \overline{\beta_1} W \end{pmatrix} \begin{pmatrix} \phi \\ j \end{pmatrix} = \begin{pmatrix} \beta_1 \gamma_0 NF \\ \overline{\beta_1} \gamma_1 NF \end{pmatrix}, \tag{112}$$

where we multiplied the second equation with $\overline{\beta_1}\beta_3^{-1}$. Note, that in the limit $\beta_1 \to 0$, where $\overline{\beta_1}\beta_3^{-1}$ does not converge to zero, the solution of (112) is $\phi = j = 0$.

Now, testing the first line by ϕ' and the second line by j' we obtain the variational formulation: Seek $(\phi, j) \in L^2(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\langle \phi, \phi' \rangle + \beta_1 \langle V\phi, \phi' \rangle - \beta_1 \langle Kj, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle, \quad \forall \phi' \in L^2(\Gamma), \tag{113a}$$
$$\overline{\beta_1} \langle K'\phi, j' \rangle + \overline{\beta_1}\beta_3^{-1} \langle j, j' \rangle + \overline{\beta_1} \langle Wj, j' \rangle = \overline{\beta_1} \langle \gamma_1 NF, j' \rangle, \quad \forall j' \in H^{1/2}(\Gamma). \tag{113b}$$

We may assume that the parameter β_3 can attain only small absolute values as it is scaled with the sheet thickness d. The BIE for ITCs of type III are not only singularly pertubed for large $|\beta_1|$ as those for ITCs of type I, but they are also singularly pertubed if $|\beta_3|$ is small. Hence, there will be internal layers if the sheet mid-line or mid-surface Γ is not smooth enough, which leads to a blow-up of $|\beta_3|^{-1}\|j\|_{H^{1/2}(\Gamma)}$ for $\beta_3 \to 0$. For sake of simplicity and unlike in the case of the ITCs of type I and II we are not going to derive sharper estimates in $|\beta_1|$ and $|\beta_3|$ for smoother interfaces Γ.

Theorem 6.1. *Let $0 < |\beta_1|$, $0 < |\beta_3| < C$ with a constant C independent of $|\beta_1|$ and assume there exist constants $\theta_1^\star, \theta_3^\star \in (0,\pi)$ such that $0 \le \theta_1 := \arg(\beta_1^{-1}) \le \theta_1^\star$ and $0 \le \theta_3 := \arg(\overline{\beta_3^{-1}}) \le \theta_3^\star$. Furthermore, let Γ be Lipschitz. Then, the system (113) has a unique solution $\phi \in L^2(\Gamma)$, $j \in H^{1/2}(\Gamma)$ and there exists a constant $C = C(\theta_1^\star, \theta_3^\star)$ of $|\beta_1|, |\beta_3|$ such that*

$$\|\phi\|_{L^2(\Gamma)} \le C|\beta_1|\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \qquad \|j\|_{H^{1/2}(\Gamma)} \le C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{114}$$

Furthermore,

$$\|\phi\|_{H^{-1/2}(\Gamma)} \le C\min(1,|\beta_1|)\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \qquad \|j\|_{L^2(\Gamma)} \le C|\beta_3|^{1/2}\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{115}$$

Remark 6.2. *For any $0 < |\xi| < \infty$ and $0 < d < \infty$ the assumptions of Theorem 6.1 are satisfied for the impedance boundary conditions MB and ITC-2-1 (neglecting curvature).*

Proof of Theorem 6.1. Similarly to the proof of Theorem 5.2 we first show ellipticity with a constant depending on the parameters, here β_1 and β_3, and then the stability estimates.

Step (i): By assumption on β_1 and β_3 we can define $\theta := -\frac{1}{2}\max(\theta_1,\theta_3) \in (-\frac{\pi}{2}, 0)$, and $\theta_1+\theta, \theta_3+\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Choosing $\phi' = \overline{\phi}$ and $j' = \overline{j}$ in the left sides of (113), summing (113a) and the complex conjugate of (113b), multiplying by $\beta_1^{-1}\,\mathrm{e}^{\mathrm{i}\theta}$ and we obtain

$$\begin{aligned}\mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta}\Big(\beta_1^{-1}\|\phi\|^2_{L^2(\Gamma)} + \langle V\phi,\overline{\phi}\rangle + \overline{\beta_3^{-1}}\|j\|^2_{L^2(\Gamma)} + \langle Wj,\overline{j}\rangle\Big)\Big)&\\ = |\beta_1|^{-1}\cos(\theta_1+\theta)\|\phi\|^2_{L^2(\Gamma)} + \cos(\theta)\langle V\phi,\overline{\phi}\rangle + |\beta_3|^{-1}\cos(\theta_3+\theta)\|j\|^2_{L^2(\Gamma)} + \cos(\theta)\langle Wj,\overline{j}\rangle&\\ \ge C(\beta_1,\beta_3)\Big(\|\phi\|^2_{L^2(\Gamma)} + \|j\|^2_{H^{1/2}(\Gamma)}\Big)&\end{aligned}$$

where the terms with K and K' cancel each other since $\langle Kj',\overline{\phi'}\rangle = \overline{\langle K'\phi',\overline{j'}\rangle}$ for all $\phi' \in H^{-1/2}(\Gamma)$, $j' \in H^{1/2}(\Gamma)$. We also used the symmetry of W which is $\langle Wj',\overline{j''}\rangle = \overline{\langle Wj'',\overline{j'}\rangle}$ for all $j', j'' \in H^{1/2}(\Gamma)$. Hence, the product bilinear form associated to (113) is $L^2(\Gamma)\times H^{1/2}(\Gamma)$-elliptic by Lemma 3.1 and a unique solution $(\phi, j) \in L^2(\Gamma)\times H^{1/2}(\Gamma)$ exists.

Step (ii): In analogy to the proof of Theorem 5.2 we write C for generic constants, which may depend on $\theta_1^\star$, $\theta_3^\star$, but not on $|\beta_1|$ or $|\beta_3|$. Applying the same steps as in Step (i) to both sides of (113) we obtain

$$\begin{aligned}&\mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta}\big(\beta_1^{-1}\|\phi\|^2_{L^2(\Gamma)} + \langle V\phi,\overline{\phi}\rangle + \overline{\beta_3^{-1}}\|j\|^2_{L^2(\Gamma)} + \langle Wj,\overline{j}\rangle\big)\Big)\\ &\quad= |\beta_1|^{-1}\cos(\theta_1+\theta)\|\phi\|^2_{L^2(\Gamma)} + \cos(\theta)\langle V\phi,\overline{\phi}\rangle + |\beta_3|^{-1}\cos(\theta_3+\theta)\|j\|^2_{L^2(\Gamma)} + \cos(\theta)\langle Wj,\overline{j}\rangle\\ &\quad\le \big|\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}\langle\gamma_0 NF,\overline{\phi}\rangle\big)\big| + \Big|\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}\overline{\langle\gamma_1 NF,\overline{j}\rangle}\big)\Big| \le C\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}\Big(\|\phi\|_{H^{-1/2}(\Gamma)} + \|j\|_{H^{1/2}(\Gamma)}\Big),\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Applying Young's inequality and Lemma 3.3 we obtain

$$\|\phi\|_{H^{-1/2}(\Gamma)} \le C\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}$$

and the stability estimate for $\|j\|_{H^{1/2}(\Gamma)}$ in (114). As consequence we find the estimate for $\|j\|_{L^2(\Gamma)}$ in (115). Rewriting (113a) as equation for ϕ

$$\langle\phi,\phi'\rangle + \beta_1\langle V\phi,\phi'\rangle = \beta_1\langle\gamma_0 NF + Kj,\phi'\rangle, \quad \forall\phi' \in L^2(\Gamma),$$

both estimates for ϕ in (29) follow. This completes the proof. ☐

In the case of higher smoothness of Γ the solution (ϕ, j) of (112) possesses higher regularity, which we are going to state in the following lemma, where we do not study the dependence of constants on the parameters β_1 and β_3.

Lemma 6.3. *Let the assumption of Theorem 6.1 be satisfied and let* $\Gamma \in C^{r+1,1}$, $r \geq -1$. *Then for any* $0 \leq s \leq r+1$ *there exist constants* $C_s = C_s(\beta_1, \beta_3)$ *such that*

$$\|\phi\|_{H^{s+1/2}(\Gamma)} + \|j\|_{H^{s+1/2}(\Gamma)} \leq C_s \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}.$$

Proof. In this proof we denote C constants which may depend on $|\beta_1|$ or $|\beta_3|$. We may write (112) as

$$\phi = \beta_1(-V\phi + Kj + \gamma_0 NF), \tag{116a}$$
$$Wj = -K'\phi - \beta_3^{-1} j + \gamma_1 NF. \tag{116b}$$

As Γ is Lipschitz (116a), (19) and Theorem 6.1 imply

$$\|\phi\|_{H^{1/2}(\Gamma)} \leq C\|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}. \tag{117}$$

If $\Gamma \in C^{1,1}$ (116b), (117), (19) and Theorem 6.1 imply

$$\|j\|_{H^{3/2}(\Gamma)} \leq C\|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'},$$

and in view of (116a) we have

$$\|\phi\|_{H^{3/2}(\Gamma)} \leq C\|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}.$$

By a bootstrapping argument we find the statement of the lemma. □

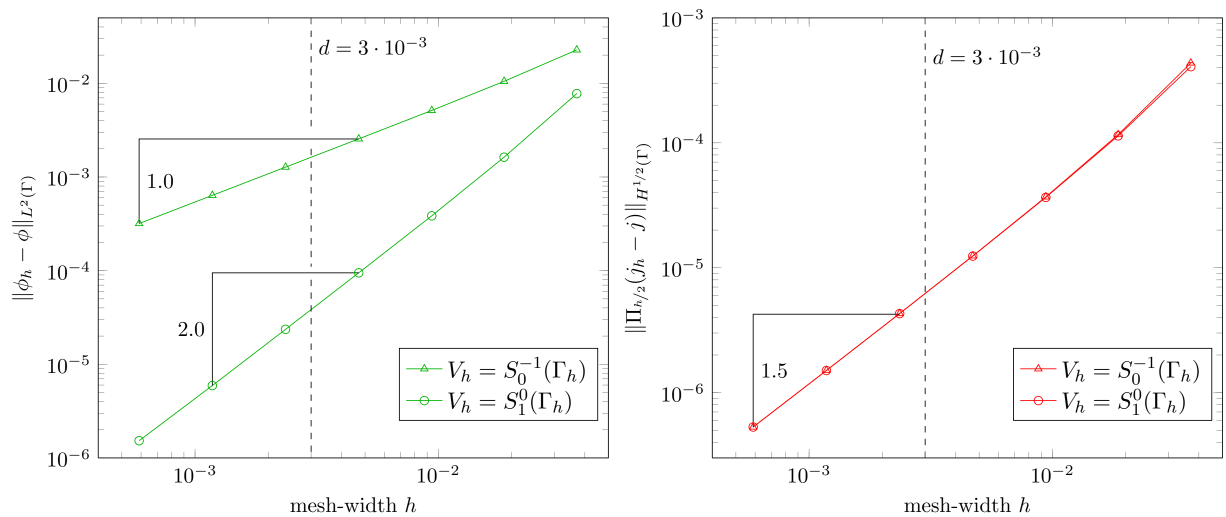

FIGURE 8. Convergence of the discretisation error for the solution of the BEM for the MB (which is of type III) for the model problem from Section 1.4 ($\beta_1 = 1.2 - 70\mathrm{i}$, $\beta_3 = 3 \cdot 10^{-3} + 5.2 \cdot 10^{-5}\mathrm{i}$). The $H^{1/2}(\Gamma)$-norm is computed for the L^2-projection onto $W_{h/2}$ on the mesh $\Gamma_{h/2}$ where each interval of Γ_h is refined twice.

6.2. Boundary element formulation

Let V_h be a finite-dimensional subspace of $L^2(\Gamma)$, in particular $S_0^{-1}(\Gamma_h)$ or $S_1^0(\Gamma_h)$, and W_h a finite-dimensional subspace of $H^{1/2}(\Gamma)$, in particular $S_1^0(\Gamma_h)$. Then, the boundary element formulation reads: Seek $(\phi_h, j_h) \in V_h \times W_h$ such that

$$\langle \phi_h, \phi_h' \rangle + \beta_1 \langle V\phi_h, \phi_h' \rangle - \beta_1 \langle Kj_h, \phi_h' \rangle = \beta_1 \langle \gamma_0 NF, \phi_h' \rangle, \quad \forall \phi_h' \in V_h, \tag{118a}$$
$$\overline{\beta_1} \langle K'\phi_h, j_h' \rangle + \overline{\beta_1}\beta_3^{-1} \langle j_h, j_h' \rangle + \overline{\beta_1} \langle Wj_h, j_h' \rangle = \overline{\beta_1} \langle \gamma_1 NF, j_h' \rangle, \quad \forall j_h' \in W_h. \tag{118b}$$

Theorem 6.4. *Let the assumptions of Theorem 6.1 be fulfilled and* $V_h \subset L^2(\Gamma)$, $W_h \subset H^{1/2}(\Gamma)$. *Then, the linear system of equations* (118) *has a unique solution* $(\phi_h, j_h) \in V_h \times W_h$, *and there exists a constant* $C = C(\beta_1, \beta_3)$ *such that*

$$\|\phi_h\|_{L^2(\Gamma)} + \|j_h\|_{H^{1/2}(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \tag{119a}$$

If $\Gamma \in C^{1,1}$ *and* $V_h \in \{S_0^{-1}(\Gamma_h), S_1^0(\Gamma_h)\}$, $W_h = S_1^0(\Gamma_h)$, *then*

$$\|\phi_h - \phi\|_{L^2(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} \leq C\,h\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{119b}$$

If $\Gamma \in C^{2,1}$ *and* $V_h = W_h = S_1^0(\Gamma_h)$, *then*

$$\|\phi_h - \phi\|_{L^2(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} \leq C\,h^{3/2}\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{119c}$$

Proof. The well-posedness follows similar to the proof of Theorem 6.1. The remainder of the proof is for the proof of the discretisation error, which can be bounded by the best-approximation error by Cea's lemma

$$\|\phi_h - \phi\|_{L^2(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} \leq C \inf_{(\psi_h,k_h)\in V_h\times W_h} \left(\|\psi_h - \phi\|_{L^2(\Gamma)} + \|k_h - j\|_{H^{1/2}(\Gamma)}\right).$$

We distinguish two cases for different smoothness assumption on Γ.

(i) Assume that $\Gamma \in C^{1,1}$. Then, by Lemma 6.3 we can assert that $\phi \in H^1(\Gamma)$ and $j \in H^{3/2}(\Gamma)$ and so Lemma 3.6 implies for both combinations of discrete spaces the estimate (119b).

(ii) Assume that $\Gamma \in C^{2,1}$. Then, by Lemma 6.3 we can assert that $\phi \in H^{3/2}(\Gamma)$ and $j \in H^2(\Gamma)$ and so Lemma 3.6 implies for $V_h = W_h = S_1^0(\Gamma_h)$ the estimate (119c).

This finishes the proof. □

6.3. Numerical experiments

We have studied the proposed boundary element method for the numerical example described in Section 1.4 examplarily for the MB condition by Mayergoyz *et al.* [10, 19] for which β_1 and β_3 are given in (10). The convergence of the discretisation error in the mesh width h is shown in Fig. 8 which confirms that the estimates of the discretisation error of ϕ_h for $V_h \in S_0^{-1}(\Gamma_h)$ and of j_h for $V_h \in S_1^0(\Gamma_h)$ in h given in Theorem 6.4 are sharp. Note that the evident quadratic convergence order 2.0 for $\phi_h \in S_1^0(\Gamma_h)$ in $L^2(\Gamma)$ and 1.5 for j_h in $H^{1/2}(\Gamma)$ in the case $V_h = S_0^{-1}(\Gamma_h)$ are better than the predictions of our theory. We believe similarly to Sec. 5 that we observe the best-approximation error due to the smallness of $|\beta_3|$ and that the asymptotic convergence rates will be reached for much smaller mesh widths.

7. First kind boundary element formulation for problems of type IV

7.1. Boundary integral formulation

As for the type III problems the jump of the Dirichlet trace is not zero, but the extra terms do not allow for a second kind formulation and we have to keep the unknown $u = \{\gamma_0 U\}$ as in the first kind formulation for the type II problems.

We start by expressing $\{\gamma_1 U\}$ using (12b)

$$\{\gamma_1 U\} = \beta_3^{-1} j - \beta_3^{-1}\beta_4\kappa u. \tag{120}$$

Taking (110), inserting (120) into (12a) and into (111) we get the mixed system

$$\begin{pmatrix} V & -K & Id \\ K' & W + \beta_3^{-1} Id & -\beta_3^{-1}\beta_4\kappa \\ -Id & -\beta_3^{-1}\beta_4\kappa & \beta_1 Id + \beta_3^{-1}\beta_4^2\kappa^2 - \beta_2\Delta_\Gamma \end{pmatrix} \begin{pmatrix} \phi \\ j \\ u \end{pmatrix} = \begin{pmatrix} \gamma_0 NF \\ \gamma_1 NF \\ 0 \end{pmatrix} \tag{121}$$

To give a meaning to the (mean) curvature κ we need to assume Γ to be C^2, such that $\|\kappa\|_{L^\infty(\Gamma)} < C$.

We proceed and analyse the two (distinct) cases $\beta_2 = 0$ and $|\beta_2| > 0$ together in this section, where we highlight only the differences. The natural space for u in case of $\beta_2 = 0$ is $H^{1/2}(\Gamma)$ where for $|\beta_2| > 0$ we need to search for $u \in H^1(\Gamma)$. Let us therefore define the Sobolev spaces

$$H^{1/2}_{\beta_2}(\Gamma) := \{v \in H^{1/2}(\Gamma) : \sqrt{\beta_2}\nabla_\Gamma v \in L^2(\Gamma)\} \subset L^2_{\beta_2}(\Gamma) := \{v \in L^2(\Gamma) : \sqrt{\beta_2}\nabla_\Gamma v \in L^2(\Gamma)\}$$

with the β_2-dependent norms defined by

$$\|v\|^2_{H^{1/2}_{\beta_2}(\Gamma)} := \|v\|^2_{H^{1/2}(\Gamma)} + |\beta_2||v|^2_{H^1(\Gamma)}, \qquad \|v\|^2_{L^2_{\beta_2}(\Gamma)} := \|v\|^2_{L^2(\Gamma)} + |\beta_2||v|^2_{H^1(\Gamma)}.$$

Note, that the spaces $H^{1/2}_{\beta_2}(\Gamma)$, $L^2_{\beta_2}(\Gamma)$ and $H^1(\Gamma)$ are equivalent for $|\beta_2| > 0$ fixed.

Now, testing the first line by ϕ', the second line by u', and the third by j' we obtain the variational formulation: Seek $(\phi, j, u) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{1/2}_{\beta_2}(\Gamma)$ such that for all $(\phi', j', u') \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{1/2}_{\beta_2}(\Gamma)$

$$\langle V\phi, \phi'\rangle - \langle Kj, \phi'\rangle + \langle u, \phi'\rangle = \langle \gamma_0 NF, \phi'\rangle, \tag{122a}$$

$$\langle K'\phi, j'\rangle - \langle Wj, j'\rangle + \beta_3^{-1}\langle j, j'\rangle + \beta_3^{-1}\beta_4\langle \kappa u, j'\rangle = \langle \gamma_1 NF, j'\rangle, \tag{122b}$$

$$-\langle \phi, u'\rangle - \beta_3^{-1}\beta_4\langle \kappa j, u'\rangle + \beta_1\langle u, u'\rangle + \beta_3^{-1}\beta_4^2\langle \kappa^2 u, u'\rangle + \beta_2\langle \nabla_\Gamma u, \nabla_\Gamma u'\rangle = 0. \tag{122c}$$

Remark 7.1. *In absence of tangential derivatives, i. e., $\beta_2 = 0$, which, for instance, occurs for ITC-2-1, we may derive a system with the unknowns ϕ and j only in a similar way as* (112). *This would lead to*

$$\begin{pmatrix} Id + \beta_1 V - \beta_4\kappa K' & -\beta_1 K - \beta_4\kappa W \\ \beta_3 K' + \beta_4\kappa V & Id + \beta_3 W - \beta_4\kappa K\end{pmatrix}\begin{pmatrix}\phi \\ j\end{pmatrix} = \begin{pmatrix}\beta_1\gamma_0 NF - \beta_4\kappa\gamma_1 NF \\ \beta_3\gamma_1 NF + \beta_4\kappa\gamma_0 NF\end{pmatrix}.$$

Besides the issue of a proper evaluation or approximation of the operators κV, κK, $\kappa K'$ and κW, the system cannot be used with the space $L^2(\Gamma) \times H^{1/2}(\Gamma)$ for which the duality product $\langle \kappa W j, \phi'\rangle$ is not well-defined. For the space $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ the associate bilinear form is not elliptic and an answer to the question of well-posedness is not obvious. Alternatively, one may reduce (121) *by solving for ϕ using its second equation, which gives*

$$\begin{pmatrix} Id + \beta_1 V + \beta_4^2\beta_3^{-1}V\kappa^2 & -K - \beta_4\beta_3^{-1}V\kappa \\ \beta_1 K' + \beta_4^2\beta_3^{-1}K'\kappa^2 - \beta_4\beta_3^{-1}\kappa & W + \beta_3^{-1}Id - \beta_3\beta_3^{-1}K'\kappa\end{pmatrix}\begin{pmatrix}u \\ j\end{pmatrix} = \begin{pmatrix}\gamma_0 NF \\ \gamma_1 NF\end{pmatrix}.$$

Here, we would have to discretise the operators $V\kappa$, $V\kappa^2$, $K'\kappa$ and $K'\kappa^2$. It is not straightforward to see if the system is well-posed in $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

We are going to derive stability estimates for Γ being Lipschitz and piecewise smooth and will as for the ITC of type III for sake of simplicity not derive sharper estimates in $|\beta_1|$, $|\beta_2|$, $|\beta_3|$, and $|\beta_4|$ for smoother interfaces Γ.

Theorem 7.2. *Let Γ be Lipschitz and $\|\kappa\|_{L^\infty(\Gamma)} < C$ for some constant C. Furthermore, let $0 < |\beta_1|, 0 < |\beta_3| < C$, where C does not depend on $|\beta_1|$, $\operatorname{Re}\beta_2 \geq 0$, $\operatorname{Im}\beta_2 \leq 0$, assume there exist constants $\theta_1^\star, \theta_3^\star \in (0, \pi)$ such that $0 \leq \theta_1 := \arg(\overline{\beta_1}) \leq \theta_1^\star$ and $0 \leq \theta_3 := \arg(\overline{\beta_3^{-1}}) \leq \theta_3^\star$ and for $\theta := -\frac{1}{2}\max(\theta_1, \theta_3) \in (-\frac{\pi}{2}, 0)$ that*

$$|\beta_4|^2(\cos(\theta_3^\star + \theta) + 1)\|\kappa\|^2_{L^\infty(\Gamma)} \leq \tfrac{1}{2}|\beta_1||\beta_3|\cos(\theta_1^\star + \theta)\cos(\theta_3^\star + \theta). \tag{123}$$

Then, the system (122) *has a unique solution $(\phi, u, j) \in H^{-1/2}(\Gamma) \times H^{1/2}_{\beta_2}(\Gamma) \times H^{1/2}(\Gamma)$ and there exists a constant $C = C(\theta_1^\star, \theta_3^\star)$ independent of $|\beta_i|$, $i = 1, \ldots, 4$ such that*

$$\begin{aligned}\|\phi\|_{H^{-1/2}(\Gamma)} + \|j\|_{H^{1/2}(\Gamma)} + \|u\|_{H^{1/2}(\Gamma)} + |\beta_2|^{1/2}\|u\|_{H^1(\Gamma)} &\leq C \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \\ \|j\|_{L^2(\Gamma)} &\leq C|\beta_3|^{1/2} \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \\ \|u\|_{L^2(\Gamma)} &\leq C\min(|\beta_1|^{-1/2}, 1)\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}.\end{aligned} \tag{124}$$

Remark 7.3. *For any $0 < |\xi| < \infty$ and $0 < d < \infty$ the assumptions of Theorem 7.2 are satisfied for the impedance boundary conditions ITC-1-2 and ITC-2-1 if $\|\kappa\|_{L^\infty(\Gamma)}$ or d/δ is small enough. The larger d/δ the closer the angles θ_1, θ_3 approach π and so $\cos(\theta_1+\theta), \cos(\theta_3+\theta)$ approach zero, which restrict the well-posedness to small curvatures κ.*

Proof of Theorem 7.2. Different to the previous models the bilinear form is not elliptic in the associated space, which is here $H^{-1/2}(\Gamma) \times H^{1/2}_{\beta_2}(\Gamma) \times H^{1/2}(\Gamma)$, but only in $H^{-1/2}(\Gamma) \times L^2_{\beta_2}(\Gamma) \times H^{1/2}(\Gamma)$. Therefore, we start by proofing stability and, hence, uniqueness, and finish by observing that the associated operator is Fredholm of index 0.
Step (i): By assumption on β_2 we can define $\theta_2 = 0$ if $\beta_2 = 0$ and $\theta_2 := \arg(\overline{\beta_2}) \in [0, \frac{\pi}{2}]$ otherwise, and we have $\theta_1 + \theta, \theta_2 + \theta, \theta_3 + \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Choosing $\phi' = \overline{\phi}$, $j' = \overline{j}$ and $u' = \overline{u}$ in (122), summing (122a) and the complex conjugate of (122b) and (122c), respectively, multiplying by $\mathrm{e}^{\mathrm{i}\theta}$ and taking the real part we obtain

$$\begin{aligned}
\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'} \big(\|\phi\|_{H^{-1/2}(\Gamma)} + \|j\|_{H^{1/2}(\Gamma)} + |\beta_3|^{-1/2}\|j\|_{L^2(\Gamma)} + |\beta_1|^{1/2}\|u\|_{L^2(\Gamma)} + |\beta_2|^{1/2}|u|_{H^1(\Gamma)}\big)&\\
\geq \mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta}\big(\langle \gamma_0 NF, \overline{\phi}\rangle + \overline{\langle \gamma_1 NF, \overline{j}\rangle}\big)\Big) = & \qquad (125)\\
\mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta}\big(\langle V\phi, \overline{\phi}\rangle + \overline{\beta}_1\|\phi\|^2_{L^2(\Gamma)} + \overline{\beta}_2|\phi|^2_{H^1(\Gamma)} + \overline{\beta_3^{-1}}\|j\|^2_{L^2(\Gamma)} + \langle Wj, \overline{j}\rangle + \overline{\beta_3^{-1}}\overline{\beta}_4^2\|\kappa u\|^2_{L^2(\Gamma)} - 2\overline{\beta_3^{-1}}\overline{\beta}_4 \mathrm{Re}\,\langle j, \kappa\overline{u}\rangle\big)\Big)&\\
\geq \cos(\theta)\,\langle V\phi, \overline{\phi}\rangle + |\beta_1|\,\cos(\theta_1+\theta)\|u\|^2_{L^2(\Gamma)} + |\beta_2|\,\cos(\theta_2+\theta)|u|^2_{H^1(\Gamma)} & \qquad (126)\\
+ |\beta_3|^{-1}\,\cos(\theta_3+\theta)\|j\|^2_{L^2(\Gamma)} + \cos(\theta)\,\langle Wj, \overline{j}\rangle &\\
- |\beta_3|^{-1}|\beta_4|^2\left(1 + \frac{2}{\cos(\theta_3+\theta)}\right)\|\kappa u\|^2_{L^2(\Gamma)} - \tfrac{1}{2}|\beta_3|^{-1}\,\cos(\theta_3+\theta)\|j\|^2_{L^2(\Gamma)} &\\
\geq 5\,C\big(\|\phi\|^2_{H^{-1/2}(\Gamma)} + \|j\|^2_{H^{1/2}(\Gamma)} + |\beta_3|^{-1}\|j\|^2_{L^2(\Gamma)} + |\beta_1|\|u\|^2_{L^2(\Gamma)} + |\beta_2||u|^2_{H^1(\Gamma)}\big)&\\
\geq \quad C\big(\|\phi\|_{H^{-1/2}(\Gamma)} + \|j\|_{H^{1/2}(\Gamma)} + |\beta_3|^{-1/2}\|j\|_{L^2(\Gamma)} + |\beta_1|^{1/2}\|u\|_{L^2(\Gamma)} + |\beta_2|^{1/2}|u|_{H^1(\Gamma)}\big)^2,&
\end{aligned}$$

with a constant C independent of $|\beta_j|$, $j = 1, \ldots, 4$. The terms with K and K' cancelled each other since $\langle Kj', \overline{\phi'}\rangle = \overline{\langle K'\phi', \overline{j'}\rangle}$ for all $\phi' \in H^{-1/2}(\Gamma)$, $j' \in H^{1/2}(\Gamma)$. We also used the symmetry of W which is $\langle Wj', \overline{j''}\rangle = \overline{\langle Wj'', \overline{j'}\rangle}$ for all $j', j'' \in H^{1/2}(\Gamma)$, Young's inequality, the $H^{-1/2}(\Gamma)$-ellipticity of V, the $H^{1/2}(\Gamma)/\mathbb{C}$-ellipticity of W, and assumption (123), Hence,

$$\|\phi\|_{H^{-1/2}(\Gamma)} + \|j\|_{H^{1/2}(\Gamma)} + |\beta_3|^{-1/2}\|j\|_{L^2(\Gamma)} + |\beta_1|^{1/2}\|u\|_{L^2(\Gamma)} + |\beta_2|^{1/2}|u|_{H^1(\Gamma)} \leq C\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \qquad (127)$$

It remains to show the bound on $\|u\|_{H^{1/2}(\Gamma)}$, which we get in view of (122a), (127) and (19)

$$\|u\|_{H^{1/2}(\Gamma)} \leq \|\gamma_0 NF\|_{H^{1/2}(\Gamma)} + \|V\phi\|_{H^{1/2}(\Gamma)} + \|Kj\|_{H^{1/2}(\Gamma)} \leq C\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}.$$

Step (ii): Let us call a the bilinear form associated to the variational formulation (122), where the complex conjugate of the second and of third equation are added to the first. Let us furthermore denote the negative $H^{1/2}(\Gamma)$-inner product as

$$\mathsf{k}\big((\phi, j, u)^\top, (\phi', j', u')^\top\big) := -(u, u')_{H^{1/2}(\Gamma)}.$$

Then, $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times L^2_{\beta_2}(\Gamma)$-ellipticity of a implies the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{1/2}_{\beta_2}(\Gamma)$-ellipticity of $\mathsf{a}_0 := \mathsf{a} - \mathsf{k}$. Hence, the associated operator A_0 is an isomorpishm. As $H^{1/2}(\Gamma)$ is by the Rellich-Kondrachov theorem compactly embedded in $L^2(\Gamma)$ [1, Chap. 6] the operator K associated to k is compact. So, the operator $A = A_0 + K$ associated to $\mathsf{a} = \mathsf{a}_0 + \mathsf{k}$ is Fredholm with index 0 and by the Fredholm alternative [13, Sec. 2.1.4] the uniqueness of a solution implies its existence. As we have shown stability, and so uniqueness, in Step (i) we can assert the statement of the lemma. □

In case of higher smoothness of Γ the solution (ϕ, j, u) of (121) possesses higher regularity, which we are going to state in the following lemma, where we do not study the dependence of constants on the parameters β_j, $j = 1, \ldots, 4$.

Lemma 7.4. *Let the assumption of Theorem 7.2 be satisfied and let $\Gamma \in C^{r+1,1}$, $r \geq -1$. If $\beta_2 = 0$, then for any $0 \leq s \leq r+1$ there exist constants $C_s = C_s(\beta_1, \beta_3, \beta_4)$ such that*

$$\|\phi\|_{H^{s-3/2}(\Gamma)} + \|j\|_{H^{s-1/2}(\Gamma)} + \|u\|_{H^{s-1/2}(\Gamma)} \leq C_s \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}.$$

If $|\beta_2| > 0$, then for any $0 \leq s \leq r+1$ there exist constants $C_s = C_s(\beta_1, \beta_2, \beta_3, \beta_4)$ such that

$$\|\phi\|_{H^{s-5/2}(\Gamma)} + \|j\|_{H^{s-1/2}(\Gamma)} + \|u\|_{H^{s-1/2}(\Gamma)} \leq C_s \|F\|_{(H^1(\mathbb{R}^n \setminus \Gamma))'}.$$

Proof. In this proof we denote C generic constants which may depend on $|\beta_j|$, $j = 1, \ldots, 4$. Note, that $\nabla_\Gamma^s \kappa, \nabla_\Gamma^s(\kappa^2) \in L^\infty(\Gamma)$ for $0 \leq s \leq r-1$.

The proof divides into two cases, for $\beta_2 = 0$ and $|\beta_2| > 0$, where we first show one step of an iteration in s.
Case $\beta_2 = 0$: We may rewrite (121) as

$$\phi = -\beta_3^{-1}\beta_4 K j + \beta_1 u + \beta_3^{-1}\beta_4^2 \kappa^2 u, \tag{128a}$$
$$W j = -K'\phi - \beta_3^{-1} j + \beta_3^{-1}\beta_4 \kappa u + \gamma_1 N F, \tag{128b}$$
$$u = -V\phi + K j + \gamma_0 N F. \tag{128c}$$

If $\Gamma \in C^{3,1}$ then $\kappa^2, \nabla_\Gamma(\kappa^2) \in L^\infty(\Gamma)$ and so $\kappa^2 u \in H^{1/2}(\Gamma)$. Using (19) and Theorem 7.2 the right hand side of (128a) is in $H^{1/2}(\Gamma)$, which implies $\phi \in H^{1/2}(\Gamma)$. Using $\phi \in H^{1/2}(\Gamma)$ we can assert that the right hand side of (128b) is in $H^{1/2}(\Gamma)$ as well, and we find using (19) that $j \in H^{3/2}(\Gamma)$. Now, the right hand side of (128c) is in $H^{3/2}(\Gamma)$ and so $u \in H^{3/2}(\Gamma)$.
Case $|\beta_2| > 0$: Then, we may rewrite (121) as

$$\beta_2 \Delta_\Gamma u = -\phi - \beta_3^{-1}\beta_4 K j + \beta_1 u + \beta_3^{-1}\beta_4^2 \kappa^2 u, \tag{129a}$$
$$(W + K'V^{-1}K) j = -\beta_3^{-1} + K'V^{-1} u + \beta_3^{-1}\beta_4 \kappa u - K'V^{-1}\gamma_0 N F + \gamma_1 N F, \tag{129b}$$
$$\phi = -V^{-1} u + V^{-1} K j + V^{-1}\gamma_0 N F. \tag{129c}$$

If $\Gamma \in C^{2,1}$ then $\kappa^2 \in L^\infty(\Gamma)$. Using (19) and Theorem 7.2 the right hand side of (129a) is in $H^{-1/2}(\Gamma)$, which implies $u \in H^{3/2}(\Gamma)$. If $\Gamma \in C^{4,1}$ then $\Delta_\Gamma \kappa \in L^\infty(\Gamma)$ and so $\kappa u \in H^{3/2}(\Gamma)$. Hence, we can assert that the right hand side of (129b) is in $H^{1/2}(\Gamma)$ as well, and we find using (19) that $j \in H^{3/2}(\Gamma)$. Knowing that $j \in H^{3/2}(\Gamma)$ we find that the right hand side of (129b) is even in $H^{3/2}(\Gamma)$, and so $j \in H^{5/2}(\Gamma)$. Now, the right hand side of (129c) is in $H^{1/2}(\Gamma)$ and so $\phi \in H^{1/2}(\Gamma)$. Now, the right hand side of (129a) is in $H^{1/2}(\Gamma)$ which implies $u \in H^{5/2}(\Gamma)$. For both cases we iterate in the regularity assumption, where we find the the statement of the lemma. □

7.2. Boundary element formulation

Let V_h be a finite-dimensional subspace of $H^{-1/2}(\Gamma)$, in particular $S_0^{-1}(\Gamma_h)$ or $S_1^0(\Gamma_h)$, W_h a finite-dimensional subspace of $H^{1/2}(\Gamma)$, in particular $S_1^0(\Gamma_h)$, and X_h a finite-dimensional subspace of $H_{\beta_2}^{1/2}(\Gamma)$, in particular $S_1^0(\Gamma_h)$. Then, the boundary element formulation reads: Seek $(\phi_h, j_h, u_h) \in V_h \times W_h \times X_h$ such that for all $(\phi_h', j_h', u_h') \in V_h \times W_h \times X_h$

$$\begin{aligned}
\langle V\phi_h, \phi_h'\rangle \quad - \langle K j_h, \phi_h'\rangle \quad + \langle u_h, \phi_h'\rangle &= \langle \gamma_0 N F, \phi_h'\rangle, \\
\langle K'\phi_h, j_h'\rangle \quad - \langle W j_h, j_h'\rangle + \beta_3^{-1}\langle j_h, j_h'\rangle + \beta_3^{-1}\beta_4\langle \kappa u_h, j_h'\rangle &= \langle \gamma_1 N F, j_h'\rangle, \\
-\langle \phi_h, u_h'\rangle - \beta_3^{-1}\beta_4\langle \kappa j_h, u_h'\rangle \quad + \beta_1\langle u_h, u_h'\rangle + \beta_3^{-1}\beta_4^2\langle \kappa^2 u_h, u_h'\rangle + \beta_2\langle \nabla_\Gamma u_h, \nabla_\Gamma u_h'\rangle &= 0.
\end{aligned} \tag{130}$$

The variational formulation (52) for type II degenerates for the model parameters $\beta_1, \beta_2 \to 0$ to a saddle point problem and the boundary element formulation is only stable with spaces V_h and X_h that provide a uniformly stable $L^2(\Gamma)$-pairing. The instability manifests itself in a blow-up of the stability constant even for $\|u_h\|_{L^2(\Gamma)}$ for $\beta_1, \beta_2 \to 0$. A similar behaviour can be observed for the variational formulation (122) for type IV, where in difference we allow $\beta_2 = 0$. As we are not analysing the constants in the stability and error estimates in terms of the model parameters β_i, $i = 1, \ldots, 4$ we will include the instable pairing $V_h = S_0^{-1}(\Gamma_h)$, $X_h = S_1^0(\Gamma_h)$ if $|\beta_2| > 0$ in the following theorem.

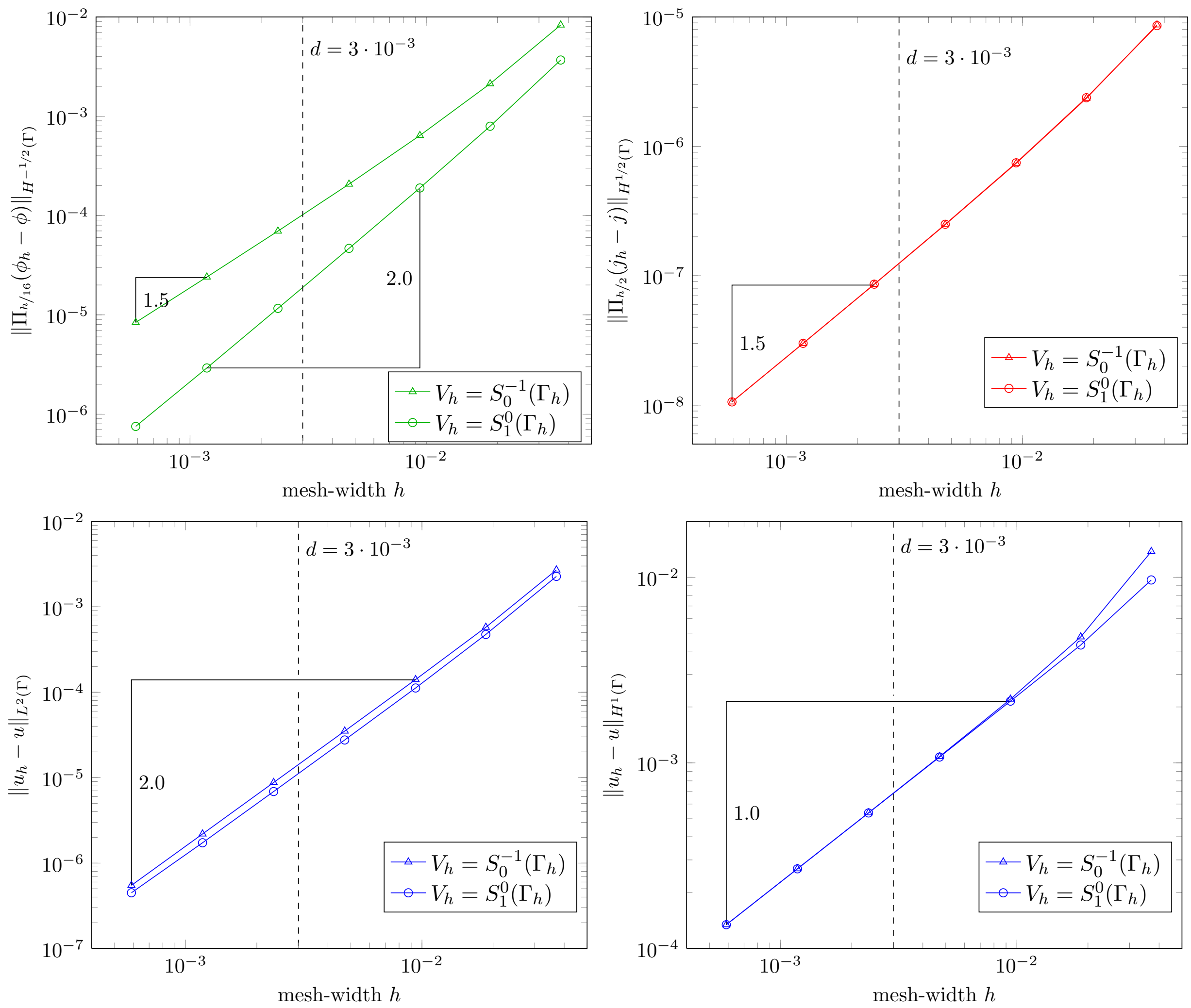

FIGURE 9. Convergence of the discretisation error for the solution of the BEM for the ITC-2-1 (which is of type IV and with $\beta_2 = 0$) for the model problem in Section 1.4 ($\beta_1 = -2.4 - 70\mathrm{i}$, $\beta_3 = -1.1 \cdot 10^{-6} + 5.2 \cdot 10^{-5}\mathrm{i}$, $\beta_4 = 5.5 \cdot 10^{-7} - 2.6 \cdot 10^{-5}\mathrm{i}$), where $X_h = W_h = S_1^0(\Gamma_h)$. The $H^{-1/2}(\Gamma)$-norm is computed via the single layer potential operator for the L^2-projection onto $V_{h/16}$ on the mesh $\Gamma_{h/16}$ where each interval of Γ_h is refined four times, and the $H^{1/2}(\Gamma)$-norm via the L^2-projection onto $W_{h/2}$ on the mesh $\Gamma_{h/2}$ where each interval of Γ_h is refined twice.

Theorem 7.5. *Let the assumptions of Theorem 7.2 be fulfilled and $V_h \subset H^{-1/2}(\Gamma)$, $W_h \subset H^{1/2}(\Gamma)$, $X_h \subset H^{1/2}_{\beta_2}(\Gamma)$. Then, the linear system of equations* (130) *has a unique solution $(\phi_h, j_h, u_h) \in V_h \times W_h \times X_h$, and there exists a constant $C = C(\beta_1, \beta_2, \beta_3, \beta_4)$ such that*

$$\|\phi_h\|_{H^{-1/2}(\Gamma)} + \|j_h\|_{H^{1/2}(\Gamma)} + \|u_h\|_{L^2_{\beta_2}(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}, \tag{131a}$$

If $V_h = X_h = S_1^0(\Gamma_h)$, then

$$\|u_h\|_{H^{1/2}(\Gamma)} \leq C\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{131b}$$

If $\beta_2 = 0$, $V_h = W_h = X_h = S_1^0(\Gamma_h)$, and $\Gamma \in C^{4,1}$, then

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^{1/2}(\Gamma)} \leq C\,h^{3/2}\,\|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{131c}$$

If $|\beta_2| > 0$, $V_h \in \{S_0^{-1}(\Gamma_h), S_1^0(\Gamma_h)\}$, $W_h = X_h = S_1^0(\Gamma_h)$, *and* $\Gamma \in C^{4,1}$, *then*

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^1(\Gamma)} \leq C\, h\, \|F\|_{(H^1(\mathbb{R}^n\setminus\Gamma))'}. \tag{131d}$$

Proof. The stability and uniqueness of solutions follows similar to the proof of Theorem 7.2. The duality pairing of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ to prove the $H^{1/2}(\Gamma)$-stability of u is replaced by the uniformly stable $L^2(\Gamma)$-pairing of the spaces V_h and X_h, which implies the $H^{1/2}(\Gamma)$-stability (131b) of u_h. As the matrix related to (130) is quadratic and of finite size, the uniqueness implies the existence of a solution.

The remainder of the proof is for the estimates of the discretisation error, which we can bound using Cea's lemma [2] by the best-approximation error

$$\|\phi_h - \phi\|_{L^2(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^{1/2}_{\beta_2}(\Gamma)} \leq C \inf_{(\psi_h,k_h,v_h)\in V_h\times W_h\times X_h} \left(\|\psi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|k_h - j\|_{H^{1/2}(\Gamma)} + \|v_h - u\|_{H^{1/2}_{\beta_2}(\Gamma)} \right).$$

We distinguish the two cases of $\beta_2 = 0$ and $|\beta_2| > 0$.

(i) Assume that $\beta_2 = 0$. If $\Gamma \in C^{4,1}$ then by Lemma 7.4 we have $\phi \in H^{3/2}(\Gamma)$, $j, u \in H^{5/2}(\Gamma)$ and so by Lemma 3.6 for both combinations of discrete spaces we obtain (131c).

(ii) Assume $|\beta_2| > 0$. $\Gamma \in C^{4,1}$ then by Lemma 7.4 $\phi \in H^{1/2}(\Gamma)$, $j, u \in H^{5/2}(\Gamma)$ and so by Lemma 3.6 for both combinations of discrete spaces we obtain (131d).

This finishes the proof. □

7.3. Numerical experiments

We have studied the proposed boundary element method for the numerical example described in Section 1.4 examplarily for the ITC-2-1 condition by Schmidt and Chernov [15], for which β_1 and β_3 are given in Sec. 2.3, $\beta_2 = 0$, and β_4 is given in Sec. 2.4. The convergence of the discretisation error in the mesh width h is shown in Fig. 9 which confirms that the estimates of the discretisation error of j_h and u_h in h given in Theorem 7.5 for the case $\beta_2 = 0$ are sharp. Note that the evident quadratic convergence order 2.0 for $\phi_h \in S_1^0(\Gamma_h)$ in $H^{-1/2}(\Gamma)$ is better than the prediction of our theory. We believe similarly to Sec. 5 that we observe the best-approximation error due to the smallness of $|\beta_3|$ and that the asymptotic convergence rates will be reached for much smaller mesh widths.

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Appendix A. Discrete dual projections

Given a triangulation Γ_h of Γ let $S_1^0(\widehat{\Gamma}_h)$ be a space of continuous piecewise linear functions on its barycentric refinement $\widehat{\Gamma}_h$ for which nodal values at the barycentres of cells of Γ_h provide valid degrees of freedom. The construction of suitable $S_1^0(\widehat{\Gamma}_h)$ on a triangulated curve is described in [7, Sect. 4.4.1], and in [4, Sect. 2] for a triangulated surface. In each case simple local computations establish

$$\sup_{v_h\in S_1^0(\widehat{\Gamma}_h)\setminus\{0\}} \frac{|\langle \psi_h, v_h\rangle|}{\|v_h\|_{L^2(\Gamma)}} \geq C_{\mathrm{ST}}\,\|\psi_h\|_{L^2(\Gamma)} \quad \forall \psi_h \in S_0^{-1}(\Gamma_h)\;, \tag{132}$$

with a constant depending only on the shape-regularity of Γ_h. As an immediate consequence of (132) we have the dual inf-sup condition

$$\sup_{\psi_h\in S_0^{-1}(\Gamma_h)\setminus\{0\}} \frac{|\langle \psi_h, v_h\rangle|}{\|\psi_h\|_{L^2(\Gamma)}} \geq C_{\mathrm{ST}}\,\|v_h\|_{L^2(\Gamma)} \quad \forall v_h \in S_1^0(\widehat{\Gamma}_h)\;. \tag{133}$$

Thus we can define two projectors $Q_h : L^2(\Gamma) \to S_0^{-1}(\Gamma_h)$ and $Q_h^* : L^2(\Gamma) \to S_1^0(\widehat{\Gamma}_h)$ through

$$\langle Q_h\phi, v_h\rangle = \langle \phi, v_h\rangle \quad \forall v_h \in S_1^0(\widehat{\Gamma}_h) \quad , \quad \langle \psi_h, Q_h^* v\rangle = \langle \psi_h, v\rangle \quad \forall \psi_h \in S_0^{-1}(\Gamma_h)\;. \tag{134}$$

Both, Q_h and Q_h^* will be $L^2(\Gamma)$-continuous with norms bounded by C_{ST}^{-1}. Moreover,

$$\|Q_h^* v\|_{H^1(\Gamma)} \leq C\,\|v\|_{H^1(\Gamma)} \quad \forall v \in H^1(\Gamma)\;, \tag{135}$$

where $C > 0$ may also depend on the quasi-uniformity of Γ_h. This estimate is a consequence of the continuity of the $L^2(\Gamma)$-orthogonal projection onto $S_1^0(\widehat{\Gamma}_h)$ in $H^1(\Gamma)$. Then, interpolation between $H^1(\Gamma)$ and $L^2(\Gamma)$ immediately yields

$$\|Q_h^* v\|_{H^{1/2}(\Gamma)} \leq C\,\|v\|_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma)\;. \tag{136}$$

Next, we appeal to the definition of the norm of $H^{1/2}(\Gamma)$ and get

$$\begin{aligned}
\|Q_h\phi\|_{H^{-1/2}(\Gamma)} &= \sup_{v\in H^{1/2}(\Gamma)} \frac{|\langle Q_h\phi, v\rangle|}{\|v\|_{H^{1/2}(\Gamma)}} = \sup_{v\in H^{1/2}(\Gamma)} \frac{|\langle Q_h\phi, Q_h^* v\rangle|}{\|v\|_{H^{1/2}(\Gamma)}} \\
&\overset{(136)}{\leq} C \sup_{v\in H^{1/2}(\Gamma)} \frac{|\langle Q_h\phi, Q_h^* v\rangle|}{\|Q_h^* v\|_{H^{1/2}(\Gamma)}} = C \sup_{v_h\in S_1^0(\widehat{\Gamma}_h)} \frac{|\langle Q_h\phi, v_h\rangle|}{\|v_h\|_{H^{1/2}(\Gamma)}} = C \sup_{v_h\in S_1^0(\widehat{\Gamma}_h)} \frac{|\langle \phi, v_h\rangle|}{\|v_h\|_{H^{1/2}(\Gamma)}} \\
&\leq C\,\|\phi\|_{H^{-1/2}(\Gamma)}\;.
\end{aligned} \tag{137}$$

Thus we have established the following result.

Lemma A.1. *The projection Q_h defined in* (134) *can be extended to a bounded operator $H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$.*

We remark that with the same arguments, this result can also be established for the standard $L^2(\Gamma)$-orthogonal projection onto $S_1^0(\Gamma_h)$.