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# A projection-based formulation of the Implicit Function Theorem and its application to time-varying manifolds

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# A projection-based formulation of the Implicit Function Theorem and its application to time-varying manifolds

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**Abstract**

In this paper, we derive a projection-based formulation of the Implicit Function Theorem. We give conditions, when an algebraic, time-parameterized equation $G(t, x) = 0$ is solvable for components $P^c x$ that are selected by a projection $P^c$ and we derive an implicit function $g_P$ that specializes $P^c x$ in terms of the complementary components $Px$, where $P = I_n - P^c$. We apply this result to construct a projection-based parameterization of time-varying submanifolds and to generalize the concept of projections to these sets. We illustrate our results by several examples.

The results are motivated by the positivity analysis of differential-algebraic equations (DAEs). These are implicit systems $F(t, x, \dot{x}) = 0$ whose solutions $x$ are supposed to remain componentwise nonnegative whenever the initial value is nonnegative. To entangle the differential and algebraic components in $F(t, x, \dot{x}) = 0$ without changing the coordinate system, we pursue the presented projection-based solution of implicit algebraic equations.

**Keywords:** Algebraic Equations, Implicit Function Theorem, Projections, Embeddings, Embedded Submanifolds

**AMS(MOS) subject classification:** 26B10, 58C15, 57R40, 47A67

## 1 Introduction

In this paper, we study the solvability and the solution representation of a time-parameterized algebraic equation

$$G(t, x) = 0 \tag{1}$$

in terms of a projection $P$. The function $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^d)$ is defined on an open set $\mathcal{I} \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$. Hence, if (1) is solvable, then the solution set $G^{-1}(0) = \{(t, x^*) \in \mathcal{I} \times \Omega | G(t, x^*) = 0\}$ is specified only up to $n - d$ free components.

Typically, these components are filtered out by a variable transformation, i.e., choosing a pointwise nonsingular matrix $[T_1, T_2]$ such that $G_x T_2$ is nonsingular, the Implicit Function Theorem [26, p. 128], allows to solve (1) locally for the components $x_2$, where $[T_1, T_2]^{-1} x = [x_1^T, x_2^T]^T$. Hence, the solution set $G^{-1}(0)$ can be locally parameterized by the components $(t, x_1)$.

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If, however, we want to characterize intersection $G_+^{-1}(0) := G^{-1}(0) \cap \mathbb{R} \times \mathbb{R}_+^n$ to specify the *nonnegative* solutions of (1), then this transformation approach is problematic as it changes the coordinates but positivity is a property strongly related with the coordinate system. To avoid a change of variables, we pursue a projection approach that parameterizes $G^{-1}(0)$ in terms of components lying in the original space $\mathbb{R}^n$. For a pointwise projection $P \in C^k(\mathcal{I}, \mathbb{R}^d)$ and its complement $P^c = I_n - P$, we give conditions when (1) is solvable for the components $x_{P^c} := P^c x$ and we explain how to choose the parameterizing components $x_P := Px$ to obtain a nonnegative solution $(t, x) \in G_+^{-1}(0)$. We construct a solution representation $\mathcal{P}$ for (1), that, $G^{-1}(0)$, acts similarly to a locally defined projection.
For time-varying submanifolds $\mathbb{S} \subset \mathbb{R} \times \mathbb{R}^n$ and their intersections with the nonnegative orthant in particular, we obtain a description by projections that is similar to that of linear subspaces and projections.
We use the given results to construct a closed, explicit solution formula for differential-algebraic equations (DAEs), i.e., differential equations whose dynamics are restricted by algebraic constraints, cp. eg. [1, 9, 23, 24], in [4]. Based on projections, i.e., avoiding a change of variables, this solution formula allows to characterize positivity for DAEs, cp. [2].

The outline is as follows. In Section 2, we introduce and prove some auxiliary results for pointwise projections, in particular of projections induced by the Moore-Penrose inverse and introduce the concept of time-varying submanifolds. In Section 3.1, we give conditions, when (1) is solvable using projections and we derive a solution representation that acts like a local projection on the solution set $G^{-1}(0)$. Pointing out the issues in characterizing the set $G_+^{-1}(0)$ via the transformation approach, in Section 3.2 we specify the free components to obtain a nonnegative solution of (1). In Section 4, we apply these results to parametrize time-varying submanifolds in terms. We illustrate our results with several examples in Section 5.

# 2 Preliminaries

In this section, we introduce and prove some auxiliary results for pointwise projections, in particular of projections induced by the Moore-Penrose inverse and introduce the concept of time-varying submanifolds as they occur in our analysis.

## 2.1 Projections and the Moore-Penrose inverse

On an open set $\mathcal{I} \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$, we consider *time* or *time-state dependent projections*, i.e., matrix functions $P \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n \times n})$, $k \geq 0$, that satisfy $P^2(t, x) = P(t, x)$ for every $(t, x) \in \mathcal{I} \times \Omega$. Then, the classical properties of constant projections pointwise extend to the function $P$, cp. [3]. In particular, $P \in \mathbb{R}^{n \times n}$ is called *orthogonal* if $P$ is pointwise symmetric, i.e., $P^T(t, x) = P(t, x)$ on $\mathcal{I} \times \Omega$. The *complement* $P^c := I_n - P$ of a projection $P$ is again a projection and satisfies $\operatorname{range}(P^c(t, x)) = \ker(P(t, x))$ and $\ker(P^c(t, x)) = \operatorname{range}(P(t, x))$. We note the following identities for the total time derivative $\dot{P}(t, x) = \frac{d}{dt} P(t, x(t))$.

**Lemma 2.1.** *Let $P \in C^1(\mathcal{I} \times \Omega, \mathbb{R}^{n \times n})$ be a projection and $P^c$ its complement. On $\mathcal{I} \times \Omega$, then $\dot{P}(t, x) = -\dot{P}^c(t, x)$, $(P\dot{P})(t, x) = (\dot{P}P^c)(t, x)$ and $(\dot{P}P)(t, x) = (P^c\dot{P})(t, x)$ as well as $(P\dot{P}P)(t, x) = 0$ and $(P^c\dot{P}P^c)(t, x) = 0$.*

*Proof.* The first assertion follows from the definition $P = I_n - P^c$. For the second identity, we note that $(PP^c)(t, x) = 0$, $(t, x) \in \mathcal{I} \times \Omega$, implies that $(P\dot{P}^c)(t, x) = -(\dot{P}P^c)(t, x)$. With $\dot{P}(t, x) =$

$-\dot{P}^c(t,x)$, we have proved the assertion. For the third identity, we differentiate $(P^cP)(t,x)=0$ and obtain that $(\dot{P}^cP)(t,x) = -(P^c\dot{P})(t,x)$. In combination with $\dot{P}(t,x) = -\dot{P}^c$, we have proved the assertion. $\square$

If $P \in \mathbb{R}^{n\times n}$ is a projection, then $P$ is diagonalizable with respect to a basis of range$(P)$ and ker$(P)$ [14, p. 22]. A time-state dependent projection $P \in C^k(\mathcal{I}\times\Omega,\mathbb{R}^{n\times n})$ is locally diagonalizable if rank$(P)$ is constant on a (not necessarily open) subset $S \subset \Omega$.

**Lemma 2.2.** *Let $P \in C^k(\mathcal{I}\times\Omega,\mathbb{R}^{n\times n})$ be a projection with* $\operatorname{rank}(P(t,x)) = d$ *on a subset $\mathcal{J}\times S \subset \mathcal{I}\times\Omega$. For every $(t_0,x_0^*) \in \mathcal{J}\times S$, there exist a neighborhood $\mathcal{I}_0\times\mathcal{U}(x_0^*) \subset \mathcal{J}\times\Omega$ and a pointwise nonsingular function $T=[T_1,T_2] \in C^k(\mathcal{I}_x 0\times\mathcal{U}(x_0),\mathbb{R}^{n\times n})$ with* $\operatorname{range}(T_1(t,x)) = \operatorname{range}(P(t,x))$ *and* $\operatorname{range}(T_2(t,x)) = \ker(P(t,x))$*, such that*

$$P(t,x) = T(t,x)\begin{bmatrix} I_d & 0 \\ 0 & 0\end{bmatrix}T^{-1}(t,x), \quad P^c(t,x) = T(t,x)\begin{bmatrix} 0 & 0 \\ 0 & I_{n-d}\end{bmatrix}T^{-1}(t,x). \tag{2}$$

*If $P$ is orthogonal, then $T$ is pointwise orthogonal. If $P \in C^k(\mathcal{I},\mathbb{R}^{n\times n})$ and* $\operatorname{rank}(P(t)) = d$ *on $\mathcal{J}$, then $T=[T_1,T_2] \in C^k(\mathcal{J},\mathbb{R}^{n\times n})$ and the decomposition* (2) *is globally defined on $\mathcal{J}$.*

*Proof.* If $P \in C^k(\mathcal{I}\times\Omega,\mathbb{R}^{n\times n})$ and $\operatorname{rank}(P(t,x)) = d$ on $\mathcal{J}\times S$, then there exists a factorization of $P$ resembling the smooth SVD [8] except that the middle factor is not diagonal [23, Thm. 4.3, p. 155]. More exactly, for every $(t_0,x_0) \in \mathcal{J}\times S$, there exist a neighborhood $\mathcal{I}_0\times\mathcal{U}(x_0) \subset \mathcal{J}\times\Omega$ and a pointwise orthogonal function $\tilde{T} = [\tilde{T}_1,\tilde{T}_2] \in C^k(\mathcal{I}_0\times\mathcal{U}(x_0),\mathbb{R}^{n\times n})$, such that

$$P(t,x) = \tilde{T}(t,x)\begin{bmatrix} \Sigma_{11}(t,x) & 0 \\ 0 & 0\end{bmatrix}T^T(t,x),$$

is satisfied pointwise on $\mathcal{I}_0\times\mathcal{U}(x_0)$. The matrix $\Sigma_{11} \in C^k(\mathcal{I}_0\times\mathcal{U}(x_0),\mathbb{R}^{d\times d})$ is pointwise nonsingular. Setting $T := T\operatorname{diag}(\Sigma_{11}, I_{n-d})$, we obtain the proposed decomposition. $\square$

Partitioning the inverse $T^{-1}$ conformably to $T=[T_1,T_2]$ by setting

$$T_1^- = [I_d, 0]T^{-1}, \quad T_2^- = [0, I_{n-d}]T^{-1}, \tag{3}$$

the identity (2) implies that $P = T_1T_1^-$ and $P^c = T_2T_2^-$. If $P$ is orthogonal, then $P = T_1T_1^T$ and $P^c = T_2T_2^T$.

In particular, we consider projections that are induced by the Moore-Penrose inverse. For $E \in \mathbb{R}^{m\times n}$, a matrix $E^+ \in \mathbb{R}^{n\times m}$ is called the *Moore-Penrose inverse* of $E$, if the following conditions are satisfied [5, 10, 16]

$$(i)\ EE^+E = E, \quad (ii) E^+EE^+ = E^+, \quad (iii)\ (E^+E)^T = E^+E, \quad (iv)\ (EE^+)^T = EE^+. \tag{4}$$

For every matrix $E \in \mathbb{R}^{n\times n}$, there exists a unique Moore-Penrose inverse [15]. If $E$ is nonsingular, then $E^+ = E^{-1}$ [27]. The Moore-Penrose inverse induces orthogonal projections onto range$(E)$ and corange$(E)$, respectively, i.e., $EE^+$ projects along corange$(E)$ onto range$(E)$ and $E^+E$ projects along ker$(E)$ onto coker$(E)$ [10, p. 9].
For a matrix function $E \in C^k(\mathcal{I}\times\Omega,\mathbb{R}^{n\times n})$, the Moore-Penrose inverse is pointwise defined by $E^+(x) := (E(x))^+$ for $x \in \Omega$. If $E$ has constant rank on a subset $\mathcal{J}\times S$, then $E^+$ is as smooth as $E$.

**Lemma 2.3.** *Consider $E \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{m\times n})$ and let $\operatorname{rank}(E(t,x)) = d$ on a subset $\mathcal{J} \times S \subset \mathcal{I} \times \Omega$. For every $(t_0, x_0) \in \mathcal{J} \times S$, there exists a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_0) \subset \mathcal{J} \times \Omega$, such that $E^+ \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{n\times m})$.*
*If $E \in C^k(\mathcal{I}, \mathbb{R}^{m\times n})$ and $\operatorname{rank}(E(t)) = d$ on $\mathcal{J}$, then $E^+ \in C^k(\mathcal{J}, \mathbb{R}^{n\times m})$.*

*Proof.* For a constant matrix $E \in \mathbb{R}^{m\times n}$, the Moore-Penrose inverse $E^+$ can equivalently be defined by the Singular Value Decomposition (SVD) [15, p. 70]. By [23, Thm. 4.3, p. 155],
If $\operatorname{rank}(E(t,x)) = d$ on $\mathcal{I} \times S$, for every $(t_0, x_0) \in \mathcal{I} \times S$, there exists a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_0) \subset \mathcal{J} \times \Omega$ and pointwise orthogonal functions $U \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{m\times m})$, $V \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{n\times n})$, such that

$$E(t,x) = U(t,x) \begin{bmatrix} E_{11}(t,x) & 0 \\ 0 & 0 \end{bmatrix} V^T(t,x), \tag{5}$$

is satisfied pointwise on $\mathcal{I}_0 \times \mathcal{U}(x_0)$, where $E_{11} \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{d\times d})$ is pointwise nonsingular. Then, on $\mathcal{I}_0 \times \mathcal{U}(x_0)$, the Moore-Penrose inverse $(E(t,x))^+$ is given by

$$(E(t,x))^+ = V(t,x) \begin{bmatrix} E_{11}^{-1}(t,x) & 0 \\ 0 & 0 \end{bmatrix} U^T(t,x), \tag{6}$$

which can be verified by checking the characteristic properties (4). Using Cramer's rule [18, p. 21], and noting that the determinant of a matrix is multilinear in the entries, it follows that $E_{11}^{-1} \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{d\times d})$, i.e., $E^+ \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{m\times n})$.
For time-varying functions $E \in C^k(\mathcal{I}, \mathbb{R}^{m\times n})$ with $\operatorname{rank}(E) = d$ on $\mathcal{J}$, the decomposition (5) is defined on $\mathcal{J}$ with $U \in C^k(\mathcal{J}, \mathbb{R}^{m\times m})$, $V \in C^k(\mathcal{J}, \mathbb{R}^{n\times n})$ and $E_{11} \in C^k(\mathcal{J}, \mathbb{R}^{d\times d})$ [23, p. 62]. Then, also $E^+ \in C^k(\mathcal{I}, \mathbb{R}^{m\times n})$ is globally defined. □

In particular, Lemma 2.3 implies that the Moore-Penrose projections $EE^+$ and $E^+E$ are locally as smooth if $E$ has constant rank on a subset $\mathcal{J} \times S$.

For matrix products, we observe the following [2, Lem. 2.3.7].

**Lemma 2.4.** *Consider $E \in \mathbb{R}^{m\times n}$. If $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal, then $(U^T EV)^+ = V^T E^+ U$. If $U_1 \in \mathbb{R}^{m\times k}$, $V_1 \in \mathbb{R}^{n\times l}$, $k \le m$, $l \le n$ have orthogonal columns and $E_{11} \in \mathbb{R}^{k\times l}$, then $(UE_{11}V^T)^+ = VE_{11}^+U^T$.*

*Proof.* If $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal, then the assertion follows by checking the properties (4) by direct computation, cp. [2]. If $U_1 \in \mathbb{R}^{m\times k}$ and $V_1 \in \mathbb{R}^{n\times l}$, $k \le m$, $l \le n$ have orthogonal columns, then $U^T U = I_k$, $V^T V = I_l$ and the assertion follows by checking the properties (4) by direct computation [2, Lem. 2.3.7]. □

## 2.2 Time-varying subsets and manifolds

We consider time-varying subsets $S \subset \mathbb{R} \times \mathbb{R}^n$ as they arise as time-parameterized level set $G^{-1}(0)$ of a time-state dependent function $G \in C(\mathcal{I} \times \mathbb{R}^n, \mathbb{R}^{n-d})$. For an interval $\mathcal{I} \subset \mathbb{R}$ and a family of subsets $\{\mathbb{S}(t)\}_{t\in\mathcal{I}}$ we call

$$S := \bigcup_{t\in\mathcal{I}} \{\{t\} \times S(t)\} \tag{7}$$

a *time-varying subset* on $\mathcal{I}$. The subsets $S(t)$ are called the *t-sections* of $S$ [20]. As a subset of the product space $\mathbb{R} \times \mathbb{R}^n$, a time-varying subset $S$ is equipped with the subspace topology induced by the product metric [22, p. 9-10]

$$d\big((t_1, x_1), (t_2, x_2)\big) = \max\big\{|t_2 - t_1|, \, \|x_2 - x_1\|\big\}. \tag{8}$$

Then, the notion of open, closed and compact subsets as well as of continuous functions directly extend to time-varying sets.

In particular, we are interested in time-varying subsets $\mathbb{S}$ displaying a similar structure as embedded submanifolds, as subsets of $\mathbb{R} \times \mathbb{R}^n$ as well as of $\mathbb{R}^n$ via its $t$-sections $S(t)$. For a function $\phi \colon \mathcal{I} \times \mathcal{U} \to \mathbb{R}^n$, we define the *autonomization* $\phi_{\text{aut}} \colon \mathcal{I} \times \mathcal{U} \to \mathbb{R} \times \mathbb{R}^n$ by $\phi_{\text{aut}}(t, x) := (t, \phi(t, x))$. We call the pair $(\mathcal{I} \times \mathcal{U}, \phi)$ a *time-varying $C^k$-chart* if, for $t \in \mathcal{I}$, $\phi_{aut}, \phi(t; \cdot)$, are $C^k$-diffeomorphisms on their domains of definitions, i.e., $\phi_{aut}, \phi(t; \cdot)$ are bijective and the functions as well as their inverses are $C^k$-continuous differentiable on their domain of definitions, e.g. [22, p. 110]. A time-varying $C^k$-chart $(\mathcal{I} \times \mathcal{U}, \phi)$ is called a *time-varying slice chart* if, for $t \in \mathcal{I}$,

$$\phi(t; \mathcal{U} \cap \mathbb{S}(t)) = \phi(t; \mathcal{U}) \cap \mathbb{R}^n_{d,0},$$
$$\phi\big((\mathcal{I} \times \mathcal{U}) \cap \mathbb{S}\big) = \phi\big(\mathcal{I} \times \mathcal{U}\big) \cap \mathbb{R}^n_{d,0},$$

where $\mathbb{R}^n_{d,0} = \{x \in \mathbb{R}^n | x_i = 0, \, i = d+1, ..., n\}$ is a linear subspace with $\dim(\mathbb{R}^n_{d,0}) = d$ equipped with the subspace topology induced by $\mathbb{R}^n$, i.e., $\tilde{\mathcal{U}}$ is open in $\mathbb{R}^n_{d,0}$ if and only if there exists an open set $\mathcal{U} \subset \mathbb{R}^n$ such that $\tilde{\mathcal{U}} = \mathcal{U} \cap \mathbb{R}^n_{d,0}$ [23, 25]. Two time-varying $C^k$-charts $(\mathcal{I}_1 \times \mathcal{U}_1, \phi_1)$, $(\mathcal{I}_2 \times \mathcal{U}_2, \phi_2)$ are called $C^k$-compatible if the transition maps $\phi_{2,\text{aut}} \circ \phi^{-1}_{1,\text{aut}}$ and $\phi_2(t; \cdot) \circ \phi_1^{-1}(t; \cdot)$, $t \in \mathcal{I}_1 \cap \mathcal{I}_2$, are $C^k$-diffeomorphisms, respectively. Dropping the time-dependancy, these definitions agree with the standard definition of local (slice) charts and compatibility, cp. e.g., [13, pp. 5], [25, pp. 97], [23, pp. 198], leading to the standard definition of an embedded $C^k$-submanifold $\mathbb{S} \subset \mathbb{R}^n$ as a subset locally diffeomorphic to $\mathbb{R}^n_{d,0}$, cp. [13, pp. 3],[25, pp. 97],[21, p. 10]. Accordingly, we call a time-varying subset $\mathbb{S}$ that can be covered by a set of time-varying $C^k$-compatible charts $\{(\mathcal{I}_i \times \mathcal{U}_i, \phi_i)\}_{i \in I}$, such that $\mathbb{S} = \bigcup_{i \in I} \{\mathcal{I}_i \times \mathcal{U}_i\}$ a *time-varying embedded $C^k$-submanifold*. Then, every $t$-section $\mathbb{S}(t)$ is an embedded submanifold in $\mathbb{R}^n$ via the parameterized slice charts $\{(\mathcal{U}_i, \phi_i(t; \cdot))\}_{i \in I}$ and the *autonomization* $\mathbb{S}_{\text{aut}} = \{[t, x^T]^T \,|\, (t, x) \in \mathbb{S}\}$ is an embedded submanifold in $\mathbb{R}^{n+1}$ via the slice charts $\bar{\phi}_{i,\text{aut}} := \phi^T_{i,\text{aut}}$. If every slice chart $(\mathcal{U}_i, \phi_i(t; \cdot))$ maps onto $\mathbb{R}^n_{d,0}$ for $t \in \mathcal{I}$, we say that $\mathbb{S}$ has the *dimension* $\dim(\mathbb{S}) = d$.

For an embedded $C^1$-submanifold $\mathbb{S} \subset \mathbb{R}^n$, the *tangent space* $\mathrm{T}_x \mathbb{S}$ in $x \in \mathbb{S}$ is the linear subspace

$$\mathrm{T}_x \mathbb{S} := D\phi^{-1}(\phi(x)) \cdot \mathbb{R}^n_{d,0},$$

where $(\mathcal{U}, \phi)$ is a slice chart containing $x \in \mathbb{S}$ [13, pp. 12], [25, p. 102], [21, p. 11]. Accordingly, for a time-varying submanifold $\mathbb{S} \subset \mathbb{R} \times \mathbb{R}^n$, we define the tangent space $\mathrm{T}_x \mathbb{S}$ in $(t, x) \in \mathbb{S}$ as the linear subspace

$$\mathrm{T}_{(t,x)} \mathbb{S} := \left( \big[[e_1^T, \, 0] \, D\phi^{-1}(\phi(t,x)) \cdot \begin{bmatrix} \mathbb{R} \\ \mathbb{R}^n_{d,0} \end{bmatrix}, \quad \big[0, \, I_n\big] \, D\phi^{-1}(\phi(t,x)) \cdot \begin{bmatrix} \mathbb{R} \\ \mathbb{R}^n_{d,0} \end{bmatrix} \right),$$

where $(\mathcal{U}, \phi)$ is a slice chart containing $(t, x) \in \mathbb{S}$. Like for the standard case [22, p. 122], the orthogonal complement

$$\mathrm{N}_{(t,x)} \mathbb{S} := \mathrm{T}_{(t,x)} \mathbb{S}^{\perp}$$

is called the *normal space* of $\mathrm{T}_{(t,x)} \mathbb{S}$.

**Lemma 2.5.** *Let $\mathbb{S} \subset \mathbb{R} \times \mathbb{R}^n$ be a time-varying, embedded $C^1$-submanifold. For $(t,x) \in \mathbb{S}$, the tangent space $\mathrm{T}_{(t,x)}\,\mathbb{S}$ is the product*

$$\mathrm{T}_{(t,x)}\,\mathbb{S} = \mathbb{R} \times \big(\, \mathrm{T}_x\,\mathbb{S}(t) - \dot{\mathbb{S}}_{(t,x)}\big), \tag{9}$$

*where $\mathbb{R} = \mathrm{T}_t\,\mathbb{R}$, $\mathrm{T}_x\,\mathbb{S}(t)$ is the tangent space of the $t$-section $\mathbb{S}(t)$ and $\dot{\mathbb{S}}_{(t,x)} \subset \mathrm{N}_x\,\mathbb{S}(t)$ is the change in time of $\mathbb{S}(t)$ into the direction of the normal space. For a time-varying slice chart $(\mathcal{I} \times \mathcal{U}, \phi)$ containing $(t,x)$, then*

$$\mathrm{T}_x\,\mathbb{S}(t) = \ker\big([0, I_{n-d}]\phi_x(t,x)\big), \tag{10}$$

$$\dot{\mathbb{S}}_{(t,x)} = \mathrm{span}(\phi_x^{-1}\phi_t)(t,x). \tag{11}$$

*Proof.* For $(t,x) \in \mathbb{S}$, let $(I \times \mathcal{U}, \phi)$ be a time-varying slice chart containing $(t,x)$. Considering the autonomized chart $\bar{\phi}_{i,\mathrm{aut}} = \phi_{i,\mathrm{aut}}^T$, we find that

$$D\phi_{\mathrm{aut}} = \begin{bmatrix} 1 & 0 \\ \phi_t & \phi_x \end{bmatrix}, \quad D\phi_{\mathrm{aut}}^{-1} = \begin{bmatrix} 1 & 0 \\ -\phi_x^{-1}\phi_t & \phi_x^{-1} \end{bmatrix}, \tag{12}$$

implying that $\mathrm{T}_{(t,x)}\,\mathbb{S}$ can be written as

$$\mathrm{T}_{(t,x)}\,\mathbb{S} = \big(\mathbb{R}, \quad \phi_x^{-1} \cdot \mathbb{R}^n_{d,0} - \phi_x^{-1}\phi_t \cdot \mathbb{R}\big)\,.$$

Noting that the $t$-section $\mathbb{S}(t)$ is an embedded submanifold in $\mathbb{R}^n$ whose tangent space in $x$ is given by $\mathrm{T}_x\,\mathbb{S}(t) = \phi_x^{-1}(t,x) \cdot \mathbb{R}^n_{d,0}$ and setting $\dot{\mathbb{S}}_{(t,x)} := \mathrm{span}(\phi_x^{-1}\phi_t)$, we have proven (9), (11). In particular, the slice chart $\phi(t,\cdot)$ may be chosen as [25, p. 107]

$$\phi(t,x) = \begin{bmatrix} x_1 \\ \theta(t,x_1,x_2) \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

for $x_1 \in \mathbb{R}^d,\ x_2 \in \mathbb{R}^{n-d}$. This implies that

$$\phi_t(t,x) = \begin{bmatrix} 0 \\ \theta_t(t,x_1,x_2) \end{bmatrix} \in {\mathbb{R}^n_{d,0}}^{\perp}$$

and noting that $\mathrm{N}_x\,\mathbb{S}(t) = \phi_x^{-1}(t,x) \cdot {\mathbb{R}^n_{d,0}}^{\perp}$, this proves that $\dot{\mathbb{S}}_{(t,x)} \subset \mathrm{N}_x\,\mathbb{S}(t)$. For (10), we note that for every $v \in \mathrm{T}_x\,\mathbb{S}(t)$ there exists $[v_1^T, 0]^T \in \mathbb{R}^n_{d,0}$ such that $v = \phi_x^{-1}(t,x)[v_1^T, 0]^T$, implying that $v \in \ker([0, I_{n-d}]\phi_x(t,x))$. Conversely, if $v \in \ker([0, I_{n-d}]\phi_x(t,x))$, there exists $v_1 \in \mathbb{R}^d$ such that $\phi_x(t,x)v = [v_1^T, 0]^T$. Hence, $v = \phi_x^{-1}(t,x)[v_1^T, 0]^T$, implying that $v \in \mathrm{T}_x\,\mathbb{S}(t)$. ☐

Like standard embedded submanifolds [25, p. 116 and p. 118], [21, p. 10,], [22, p. 122], time-varying submanifolds and their tangent sets can be characterized as level sets of locally defining functions.

**Lemma 2.6.** *A time-varying subset $\mathbb{S} \subset \mathbb{R} \times \mathbb{R}^n$ is a time-varying, embedded $C^k$-submanifold with $\dim(\mathbb{S}) = d$ if and only if for every $(t_0, x_0) \in \mathbb{S}$, there exist neighborhoods $\mathcal{I}_0 \subset \mathbb{R}$, $\mathcal{U}(x_0) \subset \mathbb{R}^n$ and a function $G \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{n-d})$ that satisfies $G^{-1}(0) = (\mathcal{I}_0 \times \mathcal{U}(x_0)) \cap \mathbb{S}$ with $\mathrm{rank}(DG(t,x)) = \mathrm{rank}(G_x(t,x)) = n-d$ on $G^{-1}(0)$. The tangent spaces associated with $\mathbb{S}$ are given by*

$$\mathrm{T}_{(t,x)}\,\mathbb{S} = \ker(DG(t,x)), \tag{13a}$$

$$\mathrm{T}_x\,\mathbb{S}(t) = \ker(G_x(t,x)), \tag{13b}$$

$$\dot{\mathbb{S}}_{(t,x)} = -(G_x^+ G_t)(t,x). \tag{13c}$$

*Proof.* $\Rightarrow$ Let $\mathbb{S}$ be a time-varying, embedded $C^k$-submanifold. For every $(t_0, x_0) \in \mathbb{S}$, there exists a time-varying $C^k$-slice chart $(\mathcal{I}_0 \times \mathcal{U}(x_0), \phi)$ satisfying $[0, I_{n-d}]\phi(t, x) = 0$ on $\mathcal{I}_0 \times \mathcal{U}(x_0)$. Setting $G := [0, I_{n-d}]\phi$ and noting that $\operatorname{rank}([0, I_{n-d}]D\phi(t, x)) = \operatorname{rank}([0, I_{n-d}]\phi_x(t, x))$ on $\mathcal{I}_0 \times \mathcal{U}(x_0)$, $G$ has the desired properties.
$\Leftarrow$ Consider $G \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0), \mathbb{R}^{n-d})$ with $\operatorname{rank}(DG(t, x)) = \operatorname{rank}(G_x(t, x)) = n - d$ on $G^{-1}(0)$. For every $(t_0, x_0) \in G^{-1}(0)$, there exists a permutation $[\Pi_1, \Pi_2] \in \mathbb{R}^{n\times n}$, such that $(G_x\Pi_2)(t_0, x_0)$ is nonsingular and by the Inverse Function Theorem [22, p. 108], $(G_x\Pi_2)(t_0, x_0)$ is nonsingular on a neighborhood $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0) \subset \mathcal{I}_0 \times \mathcal{U}(x_0)$. Setting

$$\phi(t, x) = \begin{bmatrix} \Pi_1^T x \\ G(t, x) \end{bmatrix}, \tag{14}$$

we get that $\phi \in C^k(\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0), \mathbb{R}^n)$ with $\operatorname{rank}(D\phi_{aut}(t, x)) = n + 1$, $\operatorname{rank}(\phi_x(t, x)) = n$ on $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0)$, i.e., $\phi_{\text{aut}}$, $\phi(t; \cdot)$ are $C^k$-diffeomorphisms on their domain of definition, respectively. As

$$\phi\big((\tilde{\mathcal{I}}_0 \tilde{\times} \mathcal{U}(x_0)) \cap \mathbb{S}\big) = \phi\big(\tilde{\mathcal{I}}_0 \tilde{\times} \mathcal{U}(x_0)\big) \cap \mathbb{R}^n_{d,0},$$

we find that $(\mathcal{I}_0 \times \mathcal{U}(x_0), \phi)$ is a time-varying slice chart for $G^{-1}(0)$. Repeating these arguments on $\mathbb{S}$, it follows that $\mathbb{S}$ is a time-varying, embedded submanifold.
To prove (13), we consider the chart $(\mathcal{I}_0 \times \mathcal{U}(x_0), \phi)$ defined by $G$ according to (14). Noting that (om. arg.)

$$\phi_x = \begin{bmatrix} \Pi_1^T \\ G_x \end{bmatrix}, \quad \phi_t = \begin{bmatrix} 0 \\ G_t \end{bmatrix}, \qquad \phi_x^{-1} = \Pi \begin{bmatrix} I_d & 0 \\ -(G_x\Pi_2)^{-1}G_x\Pi_1 & (G_x\Pi_2)^{-1} \end{bmatrix},$$

then (13a), (13b) follow from Lemma 2.5, while for $\dot{\mathbb{S}}_{t,x)}$, we get that

$$\dot{\mathbb{S}}_{(t,x)} = (\phi_x^{-1}\phi_t)(t, x) = (\Pi_2(G_x\Pi_2)^{-1}G_t)(t, x).$$

Noting that $(G_x\Pi_2)^{-1} = \Pi_2^T G_x^+$ by Lemma 2.4, we get that $\Pi_2(G_x\Pi_2)^{-1}G_t = \Pi_2\Pi_2^T G_x^+ G_t$, where $\Pi_2\Pi_2^T$ is a projection onto $\operatorname{coker}(G_x)$ by the choice of $\Pi_2$. For the Moore-Penrose projection $G_x^+G_x$, we also have that $\operatorname{range}(G_x^+G_x) = \operatorname{coker}(G_x)$, implying that $G_x^+G_x = \Pi_2\Pi_2^T G_x^+ G_x$ [17]. Using that $G_x^+G_xG_x^+ = G_x^+$, cp. (4), this implies that $\Pi_2\Pi_2^T G_x^+ G_t = G_x^+ G_t$ and we have proved the assertion. □

A function $G$ satisfying the assertions of Lemma 2.6 is called a *time-varying locally defining function.*

# 3 A projection-based solution representation and its application to nonnegative solutions

In Section (3.1), we give conditions, when an algebraic equation is solvable in terms of projections and we derive a solution representation acting like a local projection on its solution set. In Section 3.2, we use these results to characterize the nonnegative solutions of the algebraic equation.

### 3.1 A projection-based formulation of the Implicit Function Theorem

For a function $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$, $k \geq 0$, defined on an open set $\mathcal{I} \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$, we consider the algebraic equation

$$G(t, x) = 0 \tag{15}$$

and consider the intersection of the solution set $G^{-1}(0) = \{(t, x^*) \in \mathcal{I} \times \Omega | G(t, x^*) = 0\}$ with the nonnegative orthant

$$G_+^{-1}(0) := G^{-1}(0) \cap \mathbb{R} \times \mathbb{R}_+^n. \tag{16}$$

If $\operatorname{rank}(G_x(t_0, x_0^*)) = n - d$ for $(t_0, x_0^*) \in G^{-1}(0)$, then by the Implicit Function Theorem [26, p. 128], equation (33) is locally solvable for $n - d$ variables and we can parameterize $G^{-1}(0)$ locally by the remaining $d$ free variables. To filter out the dependent and free components, we can either use a *variable transformation* or a *projection approach*.

For a pointwise nonsingular matrix $T = [T_1, T_2] \in C(\mathcal{I}, \mathbb{R}^{n \times n})$ and given $t \in \mathcal{I}$, every $x \in \mathbb{R}^n$ has a unique *basis representation with respect to* $T$ given by

$$x = T_1(t)\, x_1(t) + T_2(t)\, x_2(t), \tag{17a}$$

$$x_1(t) := T_1^-(t)x, \quad x_2(t) := T_2^-(t)x, \tag{17b}$$

where the coefficients $x_1, x_2$ are specified by the inverse $T^{-1} = [T_1^{-T}, T_2^{-T}]^T$ that is partitioned according to $T$, cp. (3). For suitable transformations, we can solve equation (33) locally for the coefficients $x_1, x_2$, cp. [26, p. 128].

**Theorem 3.1.** *Consider $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$, $k \geq 0$, defined on an open set $\mathcal{I} \times \Omega \subset \mathbb{R} \times \mathbb{R}^n$. For $(t_0, x_0^*) \in G^{-1}(0)$ and a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$, let $T = [T_1, T_2] \in C^k(\mathcal{I}_0, \mathbb{R}^{n \times n})$ be pointwise nonsingular, such that $G_x T_2$ exists on $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ and is continuously in $(t_0, x_0^*)$. If $(G_x T_2)(t_0, x_0^*)$ is nonsingular, then there exist neighborhoods $\mathcal{U}(x_{1,0}^*) \subset \mathbb{R}^d$, $\mathcal{U}(x_{2,0}^*) \subset \mathbb{R}^{n-d}$ of $x_{i,0}^* = T_i^-(t_0)x_0$, $i = 1, 2$, and a function $g \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_{1,0}^*), \mathcal{U}(x_{2,0}^*))$, such that $(t, x) \in G^{-1}(0)$ if and only if its coefficients $x_1, x_2$ satisfy*

$$x_2 = g(t, x_1) \text{ and } (t, x_1) \in \mathcal{I}_0 \times \mathcal{U}(x_{1,0}^*). \tag{18}$$

*If $(G_x T_1)(t_0, x_0^*)$ exists, then $Dg(t_0, x_{1,0}^*)$ exists and the partial derivatives are given by*

$$g_{x_1}(t_0, x_{1,0}^*) = -\big((G_x T_2)^{-1}(G_x T_1)\big)(t_0, x_0^*), \tag{19a}$$

$$g_t(t_0, x_{1,0}^*) = -\big((G_x T_2)^{-1} G_t\big)(t_0, x_0^*). \tag{19b}$$

*Proof.* If $T = [T_1, T_2] \in C^k(\mathcal{I}_0, \mathbb{R}^{n \times n})$ pointwise nonsingular, then it induces a variable transformation (17b). Setting $\tilde{G}(t, x_1, x_2) := G\big(t, T_1^-(t)x_1 + T_2^-(t)x_2\big)$, we find that $G_{x_2}(t, x_1, x_2)$ exists on $\mathcal{I}_0 \times (T_1^-(t_0) \cdot \mathcal{U}(x_0^*)) \times (T_2^-(t_0) \cdot \mathcal{U}(x_0^*))$ and $G_{x_2}(t_0, x_{1,0}, x_{2,0})$ is nonsingular. Then, the assertions follow from the Implicit Function Theorem [26, p. 128]. □

Combining the relation (25) and the basis representation (17), we obtain a locally defined mapping from the coefficient space $\mathbb{R}^d$ onto the solution set $G^{-1}(0)$.

**Corollary 3.1.** *Under the assertions of Theorem 3.1, there exists a function* $\mathcal{T} \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_{1,0}^*), G^{-1}(0))$ *that, for* $(t, x_1) \in \mathcal{I}_0 \times \mathcal{U}(x_{1,0}^*)$*, is pointwise defined by*

$$\mathcal{T}(t, x_1) := T_1(t)\, x_1(t) + T_2(t)\, g\big(t, x_1(t)\big). \tag{20}$$

*On* $\mathcal{I}_0 \times \mathcal{U}(x_{1,0}^*)$*,* $\mathcal{T}$ *satisfies* $G(t, \mathcal{T}(t, x_1)) = 0$*, implying that*

$$x^* = \mathcal{T}(t, x_1^*) \text{ if and only if } (t, x^*) \in G^{-1}(0). \tag{21}$$

The function $\mathcal{T}$ parameterizes solutions $(t, x^*) \in G^{-1}(0)$ in terms of the coefficients $x_1^*$ of the subspace basis $T_1$. As $(T_i^- T_j)(t) = \delta_{ij}$ on $\mathcal{I}$, for every $t \in \mathcal{I}$, we can recover the argument $x_1$ by multiplication with $T_1^-(t)$, i.e., $(T_1^- \mathcal{T})(t, x_1) = x_1$.
Studying solutions properties of $G^{-1}(0)$, the function $\mathcal{T}$ allows to characterize these properties in terms of the coefficients $x_1^*$. On the other hand, choosing $(t, x_1) \in \mathcal{I}_0 \times \mathcal{U}(x_{1,0}^*)$, we can explicitly construct a solution $(t, x^*) \in G^{-1}(0)$ by setting $x^* := \mathcal{T}(t, x_1)$. We call $\mathcal{T}$ the *solution representation* of (33) *with respect to* $T$.

Instead of representing solutions of (33) by a variable transformation and the associated coefficients, we can use components that are filtered out by projection. More exactly, for a projection $P \in C(\mathcal{I}, \mathbb{R}^{n\times n})$ with complement $P^c = I_n - P$, a point $(t, x) \in \mathcal{I} \times \mathbb{R}^n$ has a unique *projection representation* with respect to $P$ given by

$$x = x_P + x_{P^c}, \tag{22a}$$

$$x_P := P(t)x, \quad x_{P^c} := P^c(t)x. \tag{22b}$$

If $T = [T_1, T_2] \in C(\mathcal{I}_0, \mathbb{R}^{n\times n})$ is a pointwise nonsingular transformation diagonalizing $P$, then $P = T_1 T_1^-$, $P^c = T_2 T_2^-$, cp. Lemma 2.2, and the projected components $x_P, x_{P^c}$ are the images of the coefficients $x_1, x_2$ under $T_1, T_2$, i.e.,

$$x_P = T_1(t, x)x_1, \quad x_{P^c} = T_2(t, x)x_2. \tag{23}$$

To derive a solution representation of (33) in terms of the components $x_P, x_{P^c}$, we formulate Theorem 3.1 in terms of projections.

**Theorem 3.2.** *Consider* $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$*. For* $(t_0, x_0^*) \in G^{-1}(0)$ *and a neighborhood* $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$*, let* $P \in C^k(\mathcal{I}_0, \mathbb{R}^{n\times n})$ *be an orthogonal projection with complement* $P^c$ *and* $\operatorname{rank}(P) = d$ *on* $\mathcal{I}_0$*, such that* $G_x P^c$ *exists on* $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ *and is continuously in* $(t_0, x_0^*)$*. If*

$$(G_x P^c)^+ (G_x P^c)(t_0, x_0^*) = P^c(t_0), \tag{24a}$$

$$(G_x P^c)(G_x P^c)^+ (t_0, x_0^*) = I_{n-d}, \tag{24b}$$

*then there exist neighborhoods* $\mathcal{U}(x_{P,0}^*), \mathcal{U}(x_{P^c,0}^*) \subset \mathbb{R}^n$ *of* $x_{P,0}^* := P(t_0)x_0$*,* $x_{P^c,0}^* := P^c(t_0)x_0$ *and a function* $g_P \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*), \mathcal{U}(x_{P^c,0}^*))$*, such that* $(t, x)$ *solves* (33) *if and only if its components* $x_P, x_{P^c}$ *satisfy*

$$x_{P^c}(t) = g_P(t, x_P) \text{ and } (t, x_P) \in \mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*). \tag{25}$$

*On* $\mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*)$*, the function* $g_P$ *satisfies* $g_P(t, x_P) \in \ker(P(t))$*.In* $(t_0, x_{P,0}^*)$*, its partial derivatives are given by*

$$g_{x_P}(t_0, x_{P,0}^*) = -((G_x P^c)^+ G_x P^c)(t, x_P), \tag{26a}$$

$$g_t(t_0, x_{P,0}^*) = -((G_x P^c)^+ G_t)(t, x_P). \tag{26b}$$

*Proof.* We prove the assertion using Theorem 3.1 and relation (23). If $P \in C^k(\mathcal{I}_0, \mathbb{R}^{n\times n})$ is an orthogonal projection with $\text{rank}(P) = d$ on $\mathcal{I}_0$, then there exist a pointwise orthogonal function $T = [T_1, T_2] \in C^k(\mathcal{I}_0, \mathbb{R}^{n\times n})$, such that $P = T_1 T_1^T$, $P^c = T_2 T_2^T$, cp. Lemma 2.2. Using that $T_2^T T_2 = I_d$, it follows from the assumption that $G_x T_2$ exists on $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ and is continuous in $(t_0, x_0^*)$. Condition (24) implies that

$$\big(T_2^T (G_x P^c)^+ (G_x P^c) T_2\big)(t_0, x_0^*) = I_d, \tag{27a}$$

$$\big((G_x P^c)(G_x P^c)^+\big)(t_0, x_0^*) = I_d. \tag{27b}$$

Noting that $(G_x T_2 T_2^T)^+ = T_2 (G_x T_2)^+$, cp. Lemma 2.4, from (27) it follows that

$$\big((G_x T_2)^+ (G_x T_2)\big)(t_0, x_0^*) = \big((G_x T_2)(G_x T_2)^+\big)(t_0, x_0^*) = I_d,$$

implying that $(G_x T_2^T)^+(t_0, x_0^*) = (G_x T_2^T)^{-1}(t_0, x_0^*)$, i.e., the Jacobian $G_x(t_0, x_0^*)$ is nonsingular if restricted to $\ker(P(t_0))$. Considering the variable transformation induced by $T$, we consider $G$ as a function of the coefficients $x_1 := T_1^T(t)x$, $x_2 := T_2^T(t)x$ and set

$$\tilde{G}(t, x_1, x_2) := G\big(t, T_1^T(t)x_1 + T_2^T(t)x_2\big).$$

On $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$, then

$$\tilde{G}_{x_2}(t, x_1, x_2) = (G_x T_2)(t, x_1, x_2),$$

and we find that $\tilde{G}_{2,x_2}(t_0, x_{1,0}, x_{2,0})$ is nonsingular. Applying Theorem 3.1, there exist neighborhoods $\tilde{\mathcal{I}}_0 \subset \mathcal{I}_0$, $\mathcal{U}(x_{1,0}^*) \subset \mathbb{R}^d$, $\mathcal{U}(x_{2,0}^*) \subset \mathbb{R}^{n-d}$ and a function $g \in C(\tilde{\mathcal{I}}_0 \times \mathcal{U}(x_{1,0}^*), \mathcal{U}(x_{2,0}^*))$, such that $(t, x_1, x_2)$ solves $\tilde{G}(t, x_1, x_2) = 0$ if and only if $(t, x_1) \in \tilde{\mathcal{I}}_0 \times \mathcal{U}(x_{1,0}^*)$ and

$$x_2(t) = g(t, x_1). \tag{28}$$

To formulate (28) in terms of the components $x_P = P(t)x$, $x_{P^c} = P^c(t)x$, we note that $x_1 = T_1^T(t)x_P$, $x_2 = T_2^T(t)x_{P^c}$, cp. (23), and get

$$g_P(t, x_P) := T_2(t)\, g(t, T_1^T(t)x_P). \tag{29}$$

For the neighborhoods, we set $\mathcal{U}(x_{P,0}^*) = \bigcup_{t\in\tilde{\mathcal{I}}_0} T_1(t)\mathcal{U}(x_{1,0}^*)$ and $\mathcal{U}(x_{P^c,0}^*) = \bigcup_{t\in\tilde{\mathcal{I}}_0} T_2(t_0)\mathcal{U}(x_{2,0}^*)$. Since $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$, $T \in C^k(\mathcal{I}_0, \mathbb{R}^{n\times n})$, then $g_P \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*), \mathbb{R}^n)$. Hence, $(t, x)$ solves $G(t, x) = 0$ if and only if the components $x_P, x_{P^c}$ satisfy condition (25).
Noting that $(P^c T_1)(t) = 0$ on $\mathcal{I}_0$ as $\text{span}(T_1(t)) = \ker(P^c(t))$, it follows that

$$P(t)\, g_P(t, x_P) = (PT_2)(t)\, g(t, T_1^T(t)x_P) = 0$$

for every $(t, x_P) \in \hat{\mathcal{I}}_0 \times \mathcal{U}(x_{P,0}^*)$. This proves that $g_P(t, x_P) \in \ker(P(t, x_P + g_P(t, x_P)))$.
By construction, $G(t, x_P + g_P(t, x_P)) = 0$ on $(t, x_P) \in \hat{\mathcal{I}}_0 \times \mathcal{U}(x_{P,0}^*)$ and the partial derivatives satisfy

$$(G_x P^c + G_x P^c g_{P_{x_P}})(t, x_P) = 0,$$
$$(G_t + G_x P^c g_{P_t})(t, x_P) = 0.$$

In $(t_0, x_0^*)$, condition (25) holds and we can solve these equations towards

$$(P^c g_{P_{x_P}})(t, x_P) = -((G_x P^c)^+ G_x P^c)(t, x_P),$$
$$(P^c g_{P_t})(t, x_P) = -((G_x P^c)^+ G_t)(t, x_P).$$

As $g_P(t, x_P) \in \ker(P(t, x_P + g_P(t, x_P)))$, we have proved (26). □

**Remark 3.1.** *Under the assertions of Theorem 3.2, there exists a neighborhood* $\hat{\mathcal{I}}_0 \times \hat{\mathcal{U}}(x_0^*) \subset \mathcal{I}_0 \times \mathcal{U}(x_0^*)$ *on which condition* (24) *is satisfied for every* $(t,x) \in \hat{\mathcal{I}}_0 \times \hat{\mathcal{U}}(x_0^*)$. *This follows from the Inverse Function Theorem, cp. [22, p. 108], implying* $(G_xT_2)(t,x)$ *is nonsingular on* $\hat{\mathcal{I}}_0 \times \hat{\mathcal{U}}(x_0^*)$ *if* $(G_xT_2)(t_0,x_0^*)$ *is. Then,* $(G_xT_2^T)^+(t,x) = (G_xT_2^T)^{-1}(t,x)$ *on* $\hat{I}_0 \times \hat{\mathcal{U}}(x_0^*)$ *and* $GP^c$ *satisfies condition* (24) *on* $\hat{I}_0 \times \hat{\mathcal{U}}(x_0^*)$.

If $P^c = I_n$, then condition (24) implies that $G_x(t_0, x_0^*)$ is nonsingular and Theorem 3.2 coincides with the Inverse Function Theorem [22, p. 108]. In contrast to Theorem 3.1, Theorem 3.2 allows to solve (33) for components selected by a projection, i.e., for the components $x_{P^c}$ lying in the linear subspace range$(P^c)$. These components are specified by the implicit function $g_P$ in terms of the free parameters $(t, x_P)$, where $x_P$ lies in the complement range$(P)$.
Combining relation (29) with the projection representation (22), we obtain a solution representation in terms of projections. Setting

$$\mathcal{U}_P(x_0^*) = \{x \in \mathbb{R}^n | (t, P(t)x) \in \mathcal{I} \times \mathcal{U}(x_{P,0}^*)\},$$

for a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*) \in$ range$(P)$, this operator can be extended to on open set in $\mathbb{R}^n$.

**Corollary 3.2.** *Under the assertions of Theorem 3.2, there exists a function* $\mathcal{P} \in C^k(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*), G^{-1}(0))$ *that, for* $(t,x) \in \mathcal{I}_0 \times \mathcal{U}_P(x_0^*))$, *is pointwise defined by*

$$\mathcal{P}(t,x) := P(t)x + g_P(t, P(t)x). \tag{30}$$

*The function* $\mathcal{P}$ *satisfies*

1. $\mathcal{I}_0 \times \mathcal{P}(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) = (\mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*)) \cap G^{-1}(0)$ *and* $\mathcal{P}|_{(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) \cap G^{-1}(0)} = I_n$.
2. *On* $\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)$, $(P\mathcal{P})(t,x) = P(t)x$ *and* $(P^c\mathcal{P})(t,x) = g_P(t, P(t)x)$.

*On* $\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)$, $\mathcal{P}$ *satisfies* $G(t, \mathcal{P}(t, P(t)x^*) = 0$, *implying that*

$$x^* = \mathcal{P}(t, P(t)x^*) \text{ if and only if } (t, x^*) \in G^{-1}(0). \tag{31}$$

*Proof.* 2. Noting that $g_P(t, P(t)x) \in \ker(P(t))$, cp. Theorem 3.2, and range$(P(t)) = \ker(P^c(t))$, we verify the projection properties $(P\mathcal{P})(t,x) = P(t)x$ and $(P^c\mathcal{P})(t,x) = g_P(t, P(t)x)$.
1. Under the assertions of Theorem 3.2, $(t,x) \in G^{-1}(0)$ if and only if its components $x_P, x_{P^c}$ satisfy relation (25). Inserting (25) into the the projection representation (22) we find that every $(t,x^*) \in G^{-1}(0)$ satisfies $x^* = \mathcal{P}(t,x^*)$. Hence, $\mathcal{P}|_{(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) \cap G^{-1}(0)} = I_n$ and (31) in particular. Now, let $(t,x) \in (\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \setminus G^{-1}(0)$. Noting that $P^c(t)g_P(t, P(t)x) = g_P(t, P(t)x)$, from $\mathcal{P}(t,x) = P(t)\mathcal{P}(t,x) + P^c(t)\mathcal{P}(t,x)$ we find that $P^c(t)\mathcal{P}(t,v) = g_P(t, P(t)\mathcal{P}(t,x))$, i.e., $\mathcal{P}(t,x) \in G^{-1}(0)$. Hence, $\mathcal{I}_0 \times \mathcal{P}(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) = (\mathcal{I}_0 \times \mathcal{U}(x_{P,0}^*)) \cap G^{-1}(0)$. □

For the solution set $G^{-1}(0)$, the function $\mathcal{P}$ locally parameterizes $(t,x^*) \in G^{-1}(0)$ in terms of the components $P(t)x^*$ lying in range$(P(t))$. The projection properties in Corollary 3.2, 2. allow to access these components $P(t)x$ by projecting with $P(t)$, i.e., $(P\mathcal{P})(t,x) = P(t)x$. Studying solutions properties of $G^{-1}(0)$, the function $\mathcal{P}$ allows to characterize these properties in terms of the components $P(t)x^*$. On the other hand, choosing $(t,x) \in \mathcal{I}_0 \times \mathcal{U}_P(x_0^*)$, we can explicitly construct a solution $(t,x^*) \in G^{-1}(0)$ by setting $x^* := \mathcal{P}(t,x)$. We call $\mathcal{P}$ the *solution representation* of (33) *with respect to* $P$.

Furthermore, the function $\mathcal{P}$ acts like a locally defined projection on $(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) \cap G^{-1}(0)$. The function $\mathcal{P}$ is idempotent, i.e., $\mathcal{P}(t, \mathcal{P}(t, v)) = \mathcal{P}(t, v)$ on $\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)$, and its image satisfies $\mathcal{I}_0 \times \mathcal{P}(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) = (\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) \cap G^{-1}(0)$. Similarly to an affine projection, the homogeneous and affine components $P(t)x$ and $g_P(t, P(t)x)$ are obtained by projection with $P(t)$ and $P^c(t)$, respectively. Indeed, if $\mathbb{L}$ is a time-varying linear subspace, then $\mathcal{P}$ is the affine projection onto $\mathbb{L}$, cp. Corollary 3.3. Setting $\mathcal{P}^c := I_n - \mathcal{P}$, we also recover the projection property $\mathcal{P}\big((\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) \cap G^{-1}(0)\big) = (\mathcal{P}^c)^{-1}\{0\}$, thinking of $\mathcal{P}^c$ as complement of $\mathcal{P}$. Note, however, that $\mathcal{P}(\mathcal{P}^c(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)) \cap G^{-1}(0)) = 0$ if and only if $g(t, 0) = 0$ as $\mathcal{P}(\mathcal{I}_0 \times \mathcal{P}^c(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*))) = g(\mathcal{I}_0 \times \mathcal{P}^c(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*)))$. Hence, we call $\mathcal{P}$ a *local projection* onto the solution setsolution set $G^{-1}(0)$.

If the Jacobian $G_x$ has locally constant rank, we can always construct a solution projection using the Moore-Penrose projection or projections induced by permutations.

**Lemma 3.1.** *Consider $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$. For $(t_0, x_0^*) \in G^{-1}(0)$, let $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ be a neighborhood on which $G_x$ exists with* $\operatorname{rank}(G_x(t, x)) = n - d$ *on $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap G^{-1}(0)$.*

1. *For the Moore-Penrose projection $P^c := G_x^+ G_x$, there exists a neighborhood $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*) \subset \mathcal{I}_0 \times \mathcal{U}(x_0^*)$ on which condition* (24) *is pointwise satisfied by $G_x P^c$.*

2. *For the projection $P^c = \Pi_2 \Pi_2^T$ induced by a permutation $[\Pi_1, \Pi_2]$ for which $(G_x \Pi_2)(t_0, x_0^*)$ is nonsingular, there exists a neighborhood $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*) \subset \mathcal{I}_0 \times \mathcal{U}(x_0^*)$ on which condition* (24) *is pointwise satisfied by $G_x P^c$.*

*Proof.* 1. If $G_x$ exists on a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ with $\operatorname{rank}(G_x(t, x)) = n - d$ on $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap G^{-1}(0)$, then there exists a neighborhood $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*) \subset \mathcal{I}_0 \times \mathcal{U}(x_0^*)$, such that $G^+ \in C^k(\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*), \mathbb{R}^{n \times m})$, cp. Lemma 2.3. Then, $P := I_n - G_x^+ G_x \in C^k(\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*), \mathbb{R}^{n \times n})$ and $\operatorname{rank}(P) = d$ on $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*)$, implying that the product $G_x P^c$ exists on $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*)$ and is continuously in $(t_0, x_0^*)$. Since $G_x P^c = G_x$ due to the properties of the Moore-Penrose inverse (4), implying that $G, P$ satisfy the solvability condition (24) of Theorem 3.2 pointwise on $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*)$.
2. If $G_x$ exists on a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ with $\operatorname{rank}(G_x(t, x)) = n - d$ on $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap G^{-1}(0)$, then there exists a permutation $\Pi = [\Pi_1, \Pi_2]$ such that $(G_x \Pi_2)(t_0, x_0^*)$ is nonsingular. By the Inverse Function Theorem [p. 108]Koe00, there exists a neighborhood $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*) \subset \mathcal{I}_0 \times \mathcal{U}(x_0^*)$ on which $\bar{\Pi}_2^T G_x(t, x) \Pi_2$ is nonsingular. Setting $P_\Pi^c := \Pi_2^T \Pi_2$, then $G_x P^c$ satisfies the solvability condition (24) on $\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*)$. □

For the linear system

$$G(t)x = b(t), \tag{32}$$

with $G \in C^k(\mathcal{I}, \mathbb{R}^{n \times n})$ and $b \in C^k(\mathcal{I}, \mathbb{R}^n)$, the solution set is the time-varying linear subspace $G^{-1}(0) = \ker(G) = \bigcup_{t \in \mathcal{I}} \{t\} \times \ker(G(t))$ and the intersection with the nonnegative orthant is the time-varying cone

$$G_+^{-1}(0) = \ker^+(G) = \bigcup_{t \in \mathcal{I}} \{t\} \times \ker^+(G(t)). \tag{33}$$

For (32), we can compute the implicit function $g_P$ as well as the solution representation $\mathcal{P}$ explicitly.

**Corollary 3.3.** *Consider $G \in C^k(\mathcal{I}, \mathbb{R}^{n\times n})$ and $b \in C^k(\mathcal{I}, \mathbb{R}^n)$. Let $P \in C^k(\mathcal{I}, \mathbb{R}^{n\times n})$ be an orthogonal projection with complement $P^c$ and $\operatorname{rank}(P) = d$ on $\mathcal{I}$. If $GP^c$ satisfies condition (24) in $t_0 \in \mathcal{I}$, then there exists an interval $\mathcal{I}_0 \subset \mathcal{I}$, such that $(t, x) \in \mathcal{I}_0 \times \mathbb{R}^n$ solves (32) if and only if*

$$x_{P^c}(t) = -\big((GP^c)^+G\big)(t)x_P + \big((GP^c)^+b\big)(t). \tag{34}$$

*Then, the solution representation $\mathcal{P}$ is given by*

$$\mathcal{P}(t, x) = \mathcal{P}_{hom}(t)x + \big((GP^c)^+b\big)(t), \tag{35}$$

*for $(t, x) \in \mathcal{I}_0 \times \mathbb{R}^n$, where $\mathcal{P}_{hom} = P - (GP^c)^+GP$ is a projection onto $\ker(G)$.*

*Proof.* If $(GP^c)(t_0)$ satisfies condition (24) for $t_0 \in \mathcal{I}$, then there exists an interval $\tilde{\mathcal{I}}_0 \subset \mathcal{I}$, such that $(GT_2)(t)$ is nonsingular for $t \in \tilde{\mathcal{I}}_0$, where $[T_1, T_2]$ is a nonsingular transformations diagonalizing $P$, cp. Lemma 2.2. Following the arguments in Theorem 3.2, it follows that $(GP^c)(t)$ satisfies condition (24) for $t \in \mathcal{I}_0$.
Now, let $(t, x)$ solve (32). Multiplying (32) by $(GP^c)^+(t)$, we get that

$$\big((GP^c)^+G\big)(t)x = \big((GP^c)^+b\big)(t).$$

Using that $x = x_P + x_P$ and the idempotency of $P, P^c$, it follows that

$$\big((GP^c)^+GP^c\big)(t)x_{P^c} = -\big((GP^c)^+GP\big)(t)x_P + \big((GP^c)^+b\big)(t)$$

and as $(GP^c)^+GP^c = P^c$ by (24a), we have verified the relation (34).
Conversely, let $(t, x)$ be such that $x_P, x_{P^c}$ satisfy (34). Multiplying (34) by $GP^c$ and using that $GP^c(GP^c)^+ = P^c$ by (24a), we get that

$$(GP^c)(t)x_{P^c} = -(GP)(t)x_P + \big((GP^c)^+b\big)(t).$$

Noting that $x = x_P + x_P$, we find that $(t, x)$ solves (32).
Formula (35) follows from the relation (34). Noting that $(GP^c)^+(GP^c) = P^c$ on $\mathcal{I}_0$ by (24a) and exploiting the Moore-Penrose property (ii), we get that $P(GP^c)^+ = 0$, implying that $\mathcal{P}_{hom}$ is idempotent. Using (24), we observe that

$$G(GP^c)^+ = G(GP^c)^+(GP^c)(GP^c)^+ = GP^c(GP^c)^+ = I_{n_d}$$

such that $G\mathcal{P}_{hom} = 0$, i.e., $\operatorname{range}(\mathcal{P}_{hom}) \subset \ker(G)$. On the other hand, if $x \in \ker(G)$, then $x = \mathcal{P}(t, x)$, i.e., $x \in \operatorname{range}(\mathcal{P}_{hom})$. In conclusion, $\mathcal{P}$ is a projection onto $\ker(G)$. □

Hence, the solution representation $\mathcal{P}$ associated with a linear equation is an affine linear projection onto the solution subspace. Choosing $Q^c = GG^+$, $P^c = G^+G$, then

$$\mathcal{P}(t, x) = P(t)x + (G^+b)(t), \tag{36}$$

i.e., $\mathcal{P}$ maps $v \in \mathbb{R}^n$ onto the least squares solution of $G(t)x = b(t)$ [19, p. 93].

### 3.2 Nonnegative solutions of algebraic equations

The solution representation (20) obtained by variable transformation describes the set $G^{-1}(0)$ in terms of the coefficients $x_1$, the implicit function $g$ and the variable transformation $T$. Characterizing the set of nonnegative solutions $G_+^{-1}(0)$, however, we thus find that $\mathcal{T}(t,x_1) \geq 0$ if and only if

$$T_1(t)x_1 \geq -T_2(t)g(t,x_1). \tag{37}$$

To solve (37) for the parameterizing components $x_1$ while preserving the componentwise inequality, the matrix $T_1^-$ must be componentwise nonnegative. Conversely, to recover (37) from a given componentwise relation of $x_1, g(t,x_1)$, we need $T_1 \geq 0$. Similarly, to solve (37) for $g(t,x_1)$, we need that $T_2T_2^- \geq 0$. But then, $T$ can only be a general permutation [6].
Hence, to characterize $G_+^{-1}(0)$ in terms of the coefficients of a variable transformation $T = [T_1, T_2]$, we have to include and impose possibly severe conditions on the transformation $T$. This is because a variable transformation changes the coordinates, while positivity is a property that is intrinsically related with the coordinate system.
Describing $G^{-1}(0)$ in terms of the components $x_P = P(t)x$ that are filtered out by a projection $P$, we can avoid such a change of variables as the parameterizing components $x_P$ are taken from the original space. Characterizing $G_+^{-1}(0)$, we find that the associated solution representation $\mathcal{P}$ satisfies $\mathcal{P}(t,x) \geq 0$ if and only if

$$x_P \geq -g_P(t,x_P), \tag{38}$$

i.e., we can directly impose condition on the parameterizing components. We summarize the pairs $(t,x) \in \mathcal{I} \times \mathcal{U}_P(x_0^*)$ satisfying (38) in the set

$$\mathcal{I} \times \mathcal{U}_P^+(x_0^*) := \big\{(t,x) \in \mathcal{I} \times \mathcal{U}_P(x_0^*) \,|\, \mathcal{P}(t,x) \geq 0\big\}. \tag{39}$$

Depending on the sign of the inducing projections $P, P^c$, we can specify the set $\mathcal{I} \times \mathcal{U}_P^+(x_0^*)$.

**Lemma 3.2.** *Consider $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$. For $(t_0, x_0^*) \in G^{-1}(0)$, let $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ be a neighborhood on which $G_x$ exists with $\operatorname{rank}(G_x(t,x)) = n-d$ on $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap G^{-1}(0)$. Let $\mathcal{P} \in C^k(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*), G^{-1}(0))$ be a local projection onto $G^{-1}(0))$ and consider $\mathcal{I} \times \mathcal{U}_P^+(x_0^*)$ as defined in* (39)*.*

1. *If $P(t) \geq 0$ on $\mathcal{I}$, then $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) \subset \ker^+(P)$.*
2. *If $P^\perp(t) \geq 0$ on $\mathcal{I}$, then $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) \subset {g_P}_+^{-1}$.*
3. *If $P(t), P^\perp(t) \geq 0$ on $\mathcal{I}$, then $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) = \ker^+(P) \cap {g_P}_+^{-1}$.*

*Proof.* If $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$ satisfies the proposed assertions, then there exists a local projection $\mathcal{P} \in C^k(\mathcal{I}_0 \times \mathcal{U}_P(x_0^*), G^{-1}(0))$ on $G^{-1}(0)$ that is induced, e.g., by a projection obtained from the Moore-Penrose projections or a permutation, cp. Lemma 3.1. We prove the assertion using the projection properties $(P\mathcal{P})(t,x) = P(t)x$ and $(P^\perp\mathcal{P})(t,x) = g_P(t,P(t)x)$, cp. Theorem 4.1.
1. If $P(t) \geq 0$ on $\mathcal{I}$, then $(P\mathcal{P})(t,x) = P(t)x \geq 0$ on $\mathcal{I} \times \mathcal{U}_P^+(x_0^*)$. Hence, $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) \subset \ker^+(P)$.
2. If $P^\perp(t) \geq 0$ on $\mathcal{I}$, then $(P^\perp\mathcal{P})(t,x) = g(t,P(t)x) \geq 0$ on $\mathcal{I} \times \mathcal{U}_P^+(x_0^*)$. Hence, $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) \subset {g_P}_+^{-1}$.
3. If $P(t), P^\perp(t) \geq 0$ on $\mathcal{I}$, then 1., 2. imply that $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) \subset \mathcal{I} \times (\ker^+(P) \cap {g_P}_+^{-1})$. Conversely, $\mathcal{P}(t,x) = P(t)x + g(t,P(t)x) \geq 0$ on $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) \subset \ker^+(P) \cap {g_P}_+^{-1}$. □

The set $\ker^+(P)$ is a time-varying polyhedral cone, cp., e.g., [11, p. 149], [7, p. 31].. By Minkowski's Theorem, cp. [12], if $P \in C^1(\mathcal{I}, \mathbb{R}^{n\times m})$, then there exists a generating matrix function $B \in C^1(\mathcal{I}, \mathbb{R}^{n\times m})$ that satisfies $\ker^+(P) = \text{cone}(B)$, i.e., every $x \in \text{range}(P(t)) \cap \mathbb{R}^n_+$ can be represented as $x = B(t)v^+$ for suitable coefficients $v^+ \in \mathbb{R}^m_+$. Considering projections $P, P^c \geq 0$ obtained from a permutation, cp. Lemma 3.1, we can explicitly compute a such generating set $B$.

**Lemma 3.3.** *Consider $G \in C^k(\mathcal{I} \times \Omega, \mathbb{R}^{n-d})$. For $(t_0, x_0^*) \in G^{-1}(0)$, let $\mathcal{I}_0 \times \mathcal{U}(x_0^*)$ be a neighborhood on which $G_x$ exists with $\text{rank}(G_x(t,x)) = n - d$ on $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap G^{-1}(0)$. Let $\mathcal{P}$ be the local projection induced by $P = \Pi_1\Pi_1^T$, where $[\Pi_1, \Pi_2]$ is a permutation such that $(G_x\Pi_2)(t_0, x_0^*)$ is nonsingular. Then,*

$$\mathcal{I} \times \mathcal{U}_P^+(x_0^*) = \text{cone}(P) \cap {g_P}_+^{-1}.$$

*Proof.* Considering a solution representation $\mathcal{P}$ induced by a projection $P = \Pi_1^T\Pi_1$, $P^c = \Pi_2^T\Pi_2$, where $[\Pi_1, \Pi_2]$ is a permutation such that $G_x(t_0, x_0^*)\Pi_2$ is nonsingular, cp. Lemma 3.1, we have that $P, P^c \geq 0$ and $\mathcal{I} \times \mathcal{U}_P^+(x_0^*) = \ker^+(P) \cap {g_P}_+^{-1}$, cp. Lemma 3.2. We show that

$$\ker^+(P) = \text{cone}(P) + \text{range}(P^c).$$

If $(t, x) \in \text{cone}(P)$, then there exists $v^+ \in \mathbb{R}^m_+$ such that $x = P(t)v^+$. As $P(t) \geq 0$, it follows that $x \geq 0$, and thus $P(t)x \geq 0$. Hence, $(t, x) \in \ker^+(P)$. If $(t, x) \in \text{range}(P^c)$, then $P(t)x = 0$, i.e., $(t, x) \in \ker^+(P)$ in particular. In conclusion, we find that $\text{cone}(P) + \text{range}(P^c) \subset \ker^+(P)$. Conversely, if $(t, x) \in \ker^+(P)$, we have that $x = P(t)x + P^c(t)x$ with $P(t)x \geq 0$. As $P$ is idempotent, this implies that $P(t)x \in \text{cone}(P(t))$. With $P^c(t)x \in \text{range}(P^c(t))$, it thus follows that $\ker^+(P) \subset \text{cone}(P) + \text{range}(P^c)$. Hence,

$$\mathcal{I} \times \mathcal{U}_P^+(x_0^*) = (\text{cone}(P) + \text{range}(P^c)) \cap {g_P}_+^{-1},$$

Noting that $g_P(t, x_P) = g_P(t, P(t)x_P)$, it follows that $g_P|_{\text{cone}(P)+\text{range}(P^c)} = g_P|_{\text{cone}(P)}$, which proves the assertion. $\square$

# 4 Projection-based parameterizations of time-varying submanifolds

Time-varying embedded manifolds can be locally described as level sets of locally defining functions, cp. Lemma 2.6. The locally defining functions satisfy the rank assertions of Theorem 3.2. Hence, every embedded submanifold can be locally described by a local projection. In particular, we can describe the tangent space and the change in time of $\mathbb{S}$.

**Lemma 4.1.** *Let $\mathbb{S}$ be a time-varying, embedded $C^k$-submanifold, $k \geq 1$, with $\dim(\mathbb{S}) = d$. For every $(t_0, x_0^*) \in \mathbb{S}$, there exists a neighborhood $\mathcal{I}_0 \times \mathcal{U}(x_0^*))$ and a local projection $\mathcal{P} \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0^*), \mathbb{S}$, such that $\mathcal{P}(\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*)) = (\tilde{\mathcal{I}}_0 \times \tilde{\mathcal{U}}(x_0^*)) \cap \mathbb{S}$. On $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap \mathbb{S}$, $\text{span}(\mathcal{P}_x(t,x)) \subset \mathrm{T}_x\,\mathbb{S}(t)$. If $\text{span}(\mathcal{P}_x(t,x)) = \mathrm{T}_x\,\mathbb{S}(t)$ if $\text{rank}(\mathcal{P}_x) = d$. The change in time of $\mathbb{S}$ is given by $\dot{\mathbb{S}}_{(t,x)} = (G_x^+ G_x \dot{\mathcal{P}})(t,x)$.*

*Proof.* If $\mathbb{S}$ is a time-varying, embedded $C^k$-submanifold, for every $(t_0, x_0^*) \in \mathbb{S}$, then there exists a locally defining function $G \in C^k(\mathcal{I}_0 \times \mathcal{U}(x_0^*), \mathbb{R}^{n-d})$ with $G^{-1}(0) = (\mathcal{I}_0 \times \mathcal{U}(x_0^*) \cap \mathbb{S}$,

cp. Lemma 2.6. Choosing an orthogonal projection $P$, e.g. $P = I_n - G_x^+ G_x$ or $P = \Pi_1 \Pi_1^T$ for a suitable permutaion, cp. Lemma 3.1, we can construct a solution representation $\mathcal{P}$ with $\mathcal{P}(\mathcal{I}_0 \times \mathcal{U}(x_0^*) = (\mathcal{I}_0 \times \mathcal{U}(x_0^*) \cap \mathbb{S}$.
Let $\hat{x} \in C^1([t_0, \hat{t}), \mathbb{S}(\cdot))$ be a curve in $\mathbb{S}(\cdot)$ with $\hat{x}(t_0) = x_0$. Along $\hat{x}$, then $G(t, \mathcal{P}(t, \hat{x}(t))) = 0$ and considering the total time derivative, this implies that $(G_t + G_x \dot{\mathcal{P}})(t, \hat{x}(t)) = 0$ on $[t_0, \hat{t})$. Hence,

$$-(G_x^+ G_t)(t, \hat{x}(t)) = (G_x^+ G_x \dot{\mathcal{P}})(t, \hat{x}(t))$$

on $[t_0, \hat{t})$, and noting that $\dot{\mathbb{S}}_{(t,x)} = -(G_x^+ G_t)(t, x)$, cp. Lemma 2.6, we have proved the assertion. Similarly, we get that $(G_x \mathcal{P}_x)(t, x) = 0$ on $(\mathcal{I}_0 \times \mathcal{U}(x_0^*)) \cap \mathbb{S}$. As $\ker(G_x(t, x)) = \mathrm{T}_x \mathbb{S}(t)$ with $\dim(\ker(G_x(t, x))) = d$, cp. Lemma 2.6, this proves that $\mathcal{P}_x(t, x) \in \mathrm{T}_x \mathbb{S}(t)$. Hence, if $\operatorname{rank}(\mathcal{P}_x) = d$, then $\operatorname{span}(\mathcal{P}_x(t, x)) = \mathrm{T}_x \mathbb{S}(t)$. ☐

Via its time derivative, a characterizing function $\mathcal{P}$ yields a representation of the change in time of $\mathbb{S}$ and, if $\operatorname{rank}(\mathcal{P}_x(t, x)) = \dim(\mathbb{S})$, then also of the tangent space $\mathrm{T}_x \mathbb{S}(t)$. As the representation of $\dot{\mathbb{S}}_{(t,x)}$ is derived using the identity $G(t, \mathcal{P}(t, x)) = 0$ on $\mathbb{S}$, which is valid for every combination of defining function $G$ and characterizing function $\mathcal{P}$, the representation $\dot{\mathbb{S}}_{(t,x)} = (G_x^+ G_x \mathcal{P}_t)(t, x)$ holds for every combination of defining function $G$ and characterizing function $\mathcal{P}$.
For a time-varying, affine linear subspace $\mathbb{L}_v$, the characterizing function $\mathcal{P}_v$ is an affine projection onto $\mathbb{L}_v$, cp. Corollary 3.3.
Using Lemma 3.2 and Lemma 3.3, we can characterize the intersection $\mathbb{S}_+ := \mathbb{S} \cap \mathbb{R} \times \mathbb{R}^n$ of an embedded submanifold $\mathbb{S}$ with the nonnegative orthant.

## 5 Examples

Example 5.1 illustrates that for a linear subspace $\mathbb{L}$, the characterizing function agrees with the orthogonal projection onto $\mathbb{L}$.

**Example 5.1.** *Consider the time-varying subspace $\mathbb{L}$ defined on the interval $\mathcal{I} = (-1, \infty)$, whose $t$-sections are given by $\mathbb{L}(t) = \operatorname{range}(P(t))$, where*

$$P = \begin{bmatrix} \frac{t+1}{t+2} & \frac{\sqrt{t+1}}{t+2} & 0 \\ \frac{\sqrt{t+1}}{t+2} & \frac{1}{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P^c = \begin{bmatrix} \frac{1}{t+2} & -\frac{\sqrt{t+1}}{t+2} & 0 \\ -\frac{\sqrt{t+1}}{t+2} & \frac{t+1}{t+2} & 0 \\ 0 & 0 & t+2 \end{bmatrix}, \tag{40}$$

*are pointwise complementary orthogonal projections onto $\mathbb{L}$ and $\mathbb{L}^\perp$, respectively. The projection $P$ can be diagonalized with respect to the matrix*

$$T = \begin{bmatrix} \frac{\sqrt{t+1}}{\sqrt{t+2}} & -\frac{1}{\sqrt{t+2}} & 0 \\ \frac{1}{\sqrt{t+2}} & \frac{\sqrt{t+1}}{\sqrt{t+2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Then, $P$ is specifying for $\mathbb{L}$ with locally defining function $[0, I_2]T^T$. Considering the homogeneous space, the parametric description induced by $P$ vanishes identically, i.e., $g(t, x) = 0$. The characterizing function $\mathcal{P}$ induced by $P, g$ trivially agrees with $P$.*

Example 5.2 and 5.3 illustrate how a time-varying, embedded submanifold is characterized by a locally defining function and how the characterizing function and parameterization obtained by projection are computed for a time-varying embedded submanifold.

**Example 5.2.** *Consider the level set* $\mathbb{S} := G^{-1}(0)$*, where*

$$G(t, x) = x_1^2 + x_2^2 - x_3 e^{-\alpha t},$$

*for* $t \in \mathcal{I} = (0, \infty)$*,* $x_1, x_2, x_3 \in \mathbb{R}$ *and* $\alpha > 0$*. Noting that* $G \in C^\infty(\mathcal{I} \times \mathbb{R}^3, \mathbb{R})$ *and, on* $\mathcal{I}$*,*

$$\operatorname{rank}\big(DG(t,x)\big) = \operatorname{rank}\left(\begin{bmatrix}\alpha e^{-\alpha t} x_3 & 2x_1 & 2x_2 & -e^{-\alpha t}\end{bmatrix}\right) = 1,$$

*we have verified that* $\mathbb{S}$ *is a time-varying, embedded* $C^\infty$*-submanifold, cp. Lemma 2.6.*
*Now, we construct a parametric description and characterizing function* $\mathcal{P}_\sigma$ *that is induced by the projections*

$$P_\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_\sigma^\perp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

*The partial derivatives of* $G$ *are given by*

$$G_x(t,x) = \begin{bmatrix} 2x_1 & 2x_2 & -e^{-\alpha t} \end{bmatrix}, \quad G_t(t,x) = \alpha e^{-\alpha t} x_3,$$

*implying that* $(G_x(t,x)P_\sigma^\perp)^+ = [0, 0, -e^{-\alpha t}]^T$ *On* $\mathcal{I}$*, then* $G_x$ *and* $P_\sigma$ *satisfy*

$$(G_x(t,x)P_\sigma^\perp)^+(G_x(t,x)P_\sigma^\perp) = P_\sigma^\perp, \quad (G_x(t,x)P_\sigma^\perp)^+(G_x(t,x)P_\sigma^\perp) = 1,$$

*i.e.,* $P_\sigma$ *is a specifying projection for* $\mathbb{S}$ *with defining function* $G$*. Partitioning the variables into* $x_P = [x_1, x_2, 0]^T$*,* $x_P = [0, 0, x_3]^T$*, the parametric description induced by* $P_\sigma$*,* $G$ *is given by*

$$g_{\sigma,3}(t, x_P) = \begin{bmatrix} 0 & 0 & e^{\alpha t}(x_{P,1}^2 + x_{P,2}^2) \end{bmatrix}^T.$$

*The characterizing function induced by* $P_\sigma$ *and* $g_\sigma$ *is given by*

$$\mathcal{P}_\sigma(t,x) = \begin{bmatrix} x_1 & x_2 & e^{\alpha t}(x_1^2 + x_2^2) \end{bmatrix}^T. \tag{41}$$

**Example 5.3.** *Consider the level set* $\mathbb{S} := G^{-1}(0)$*, where*

$$G(t,x) = \tfrac{\sqrt{t+1}}{t+2}\big(\sqrt{t+1}x_1 + x_2\big)x_3 - v(t)$$

*is defined on* $\mathcal{I} \times \mathbb{R}^3$ *with* $\mathcal{I} = (-1, \infty)$ *and* $v \in C^1(\mathcal{I} \to \mathbb{R})$*. Noting that* $G \in C^1(\mathcal{I} \times \mathbb{R}^3, \mathbb{R})$ *and* $\operatorname{rank}(DG(t,x)) = 1$*, where*

$$DG(t,x) = \left[\tfrac{d}{dt}\big[\tfrac{\sqrt{t+1}}{t+2}\big(\sqrt{t+1}x_1 + x_2\big)x_3\big] - \dot v(t) \quad \tfrac{t+1}{t+2}x_3 \quad \tfrac{\sqrt{t+1}}{t+2}x_3 \quad \tfrac{\sqrt{t+1}}{t+2}\big(\sqrt{t+1}x_1 + x_2\big)\right],$$

*we have verified that* $\mathbb{S}$ *is a time-varying, embedded* $C^1$*-submanifold, cp. Lemma 2.6.*
*We construct a parametric description and characterizing function* $\mathcal{P}_\sigma$ *induced by the projections* $P, P^c$ *given in (40). Noting that* $(G_xP)^+(G_xP) = P$*, we find that* $P^c$ *is specifying for* $\mathbb{S}$ *with defining function* $G$*. The parameterization induced by* $G$ *and* $P^c$ *is given by*

$$g(t, x_{P^c}) = \begin{bmatrix} \frac{v_1}{x_{P^c 3}}(t) & \frac{v_1(t)}{x_{P^c 3}(t)\sqrt{t+1}} & 0 \end{bmatrix}^T$$

*and the characterizing function by*

$$\mathcal{P}(t,x) = \begin{bmatrix} \frac{x_1-\sqrt{t+1}x_2}{t+2} + \frac{v_1}{x_3}(t) \\ -\frac{\sqrt{t+1}(x_1-\sqrt{t+1}x_2)}{t+2} + \frac{v_1(t)}{x_3(t)\sqrt{t+1}} \\ x_3 \end{bmatrix}.$$

*Noting that $P_1^c(t)g(t,x_{P^c}) = 0$, we verify the projection properties of the characterizing function given in Theorem 4.1.*

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