Explicit solutions of regular linear discrete-time descriptor systems with constant coefficients *

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Abstract

Explicit solution formulas are presented for systems of the form $Ex^{k+1} = Ax^k + f^k$ with regular pencil $\lambda E - A$. Different results are obtained when one starts at $k = 0$ and calculates into the future (i.e., $Ex^{k+1} = Ax^k + f^k$ with $k \in \mathbb{N}$) and when one wants to get a complete solution (i.e., $Ex^{k+1} = Ax^k + f^k$ with $k \in \mathbb{Z}$).

Key words. descriptor system, strangeness index, linear discrete descriptor system, explicit solution.

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1 Introduction

Consider the square linear time-invariant continuous-time descriptor system

$$E\dot{x}(t) = Ax(t) + f(t), \quad x(t_0) = x_0, \quad t \in \mathbb{R}, \tag{1}$$

where $E, A \in \mathbb{C}^{n,n}$, $x(t) \in \mathbb{C}^n$ is the state vector, $f(t) \in \mathbb{C}^n$ is the right hand side, $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t, and $x_0 \in \mathbb{R}^n$ is an initial condition given at the point $t_0 \in \mathbb{R}$. Assuming that the pencil $\lambda E - A$ is *regular*, i.e., $\det \lambda E - A \neq 0$ for some $\lambda \in \mathbb{C}$, one can explicitly write down the unique solution of (1) with the help of the Drazin inverse, as shown in [6]. The purpose of this paper is to obtain the corresponding results for the discrete-time case, i.e., to obtain an explicit solution of the square linear time-invariant discrete-time descriptor system

$$Ex^{k+1} = Ax^k + f^k, \quad x^{k_0} = x_0, \quad k \in \mathbb{Z}, \tag{2}$$

where, again, $E, A \in \mathbb{C}^{n,n}$, $x^k \in \mathbb{C}^n$ is the state vector at $k \in \mathbb{Z}$, $f^k \in \mathbb{C}^n$ is the right hand side at $k \in \mathbb{Z}$, and $x_0 \in \mathbb{R}^n$ is an initial condition given at $k_0 \in \mathbb{Z}$. Such

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equations arise naturally by approximating $\dot{x}(t)$ via finite differences and have also been studied in other contexts, see [2, 4, 7].

We will see that the explicit solution for the continuous-time system (1) from [6] is very similar to the explicit solution of (2) in the case that one starts at a given point, let us say at $0 \in \mathbb{Z}$, and only wants to get a solution for all $k \in \mathbb{N}$. In this case, we only have to replace the integral by the sum and the exponential function by the power function. If one however wants to get a solution for all $k \in \mathbb{Z}$ the explicit solution gets more complicated and does not resemble the explicit solution from [6] so much any more. In this case we will write down the solution using a case differentiation, which represents the main result of this paper. Note that for the continuous-time case it is irrelevant whether the initial condition is fixed at the boundary or in the interior of the solution interval.

Throughout the paper we will assume that $\lambda E - A$ is a regular pencil. Any regular pencil can be reduced to the *Kronecker/Weierstraß canonical form*

$$P(\lambda E - A)Q = \lambda \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix}, \tag{3}$$

where $P, Q \in \mathbb{C}^{n,n}$ are invertible, $I_k \in \mathbb{C}^{k,k}$ is the identity matrix of dimension k, $J \in \mathbb{C}^{n_f,n_f}$ is in Jordan canonical form, $N \in \mathbb{C}^{n_\infty,n_\infty}$ is a nilpotent matrix in Jordan canonical form and $n_f, n_\infty \in \mathbb{N}$ with $n_f + n_\infty = n$. Thus, n_f is the number of finite eigenvalues and n_∞ is the number of infinite eigenvalues. The form (3) allows to determine the solution of (2) in changed coordinates (see [1]). In this paper, however, we will determine representations of the solution in the original coordinates.

To obtain a general definition of a linear discrete-time descriptor system we first introduce the discrete interval

$$\mathbb{K} := \{k \in \mathbb{Z} : k_b \leq k \leq k_f\}, k_b \in \mathbb{Z} \cup \{-\infty\}, k_f \in \mathbb{Z} \cup \{\infty\}. \tag{4}$$

Then we call

$$Ex^{k+1} = Ax^k + f^k, k \in \mathbb{K}, \tag{5}$$

where $E, A \in \mathbb{C}^{n,n}$, the pencil $\lambda E - A$ is regular, and $x^k, f^k \in \mathbb{C}^n$ a *regular linear discrete-time descriptor system with constant coefficients.* Other names for this system are *linear singular system* (e.g., [10]), *linear semi-state system*, and *linear generalized state-space system.*

Sequences are denoted by $\{x^k\}_{k=k_b}^{k_f}$ or $\{x^k\}_{k\in\mathbb{K}}$, respectively. A sequence $\{x^k\}_{k_b}^{k_f+1}$ or $\{x^k\}_{k\in\mathbb{K}\cup\{k_f+1\}}$ is called a solution of (5) if its elements fulfill all the equations.

2 The Drazin inverse

The Drazin inverse is a generalization of the inverse of a matrix to potentially singular square matrices. The properties of the Drazin inverse make it very useful for finding solutions of systems of the form (5).

Definition 1. [6] Let $E, A \in \mathbb{C}^{n,n}$, let the matrix pencil $\lambda E - A$ be regular and let the Kronecker canonical form of $\lambda E - A$ be given by (3). Then the quantity ν defined by $N^\nu = 0$, $N^{\nu-1} \neq 0$, i.e., by the index of nilpotency of N in (3), if the nilpotent block in (3) is present and by $\nu = 0$ if it is absent, is called the *index* of the matrix pencil $\lambda E - A$, and denoted by $\nu = \operatorname{ind}(\lambda E - A)$.

Definition 2. Let $E \in \mathbb{C}^{n,n}$. Further, let ν be the index of the matrix pencil $\lambda E - I_n$. Then ν is also called the *index* of E and denoted by $\operatorname{ind}(E) = \nu$.

Definition 3. Let $E \in \mathbb{C}^{n,n}$ have the index ν. A matrix $X \in \mathbb{C}^{n,n}$ satisfying

$$\begin{aligned} EX &= XE, \\ XEX &= X, \\ XE^{\nu+1} &= E^\nu, \end{aligned} \tag{6}$$

is called a *Drazin inverse* of E and denoted by E^D.

As shown in [6] the Drazin inverse of a matrix is uniquely determined. Several properties of the Drazin inverse will be used frequently in section 3, which is why we review them here.

Lemma 4. *Consider matrices $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$. Then*

$$\begin{aligned} EA^D &= A^D E, \\ E^D A &= AE^D, \\ E^D A^D &= A^D E^D, \end{aligned} \tag{7}$$

where E^D denotes the Drazin inverse of E.

Proof. See [6, Lemma 2.21]. □

Also, the following Theorem will be very useful in the next section. It represents a decomposition of a general square matrix into a part belonging to the non-zero eigenvalues and a part belonging to the zero eigenvalues.

Theorem 5. *Let $E \in \mathbb{C}^{n,n}$ with $\nu = \operatorname{ind}(E)$. There is one and only one decomposition*

$$E = \tilde{C} + \tilde{N} \tag{8}$$

with the properties

$$\tilde{C}\tilde{N} = \tilde{N}\tilde{C} = 0, \quad \tilde{N}^\nu = 0, \quad \tilde{N}^{\nu-1} \neq 0, \quad \operatorname{ind}(\tilde{C}) \leq 1. \tag{9}$$

For this decomposition the following statements hold:

$$\begin{aligned} &\tilde{C}^D\tilde{N} = \tilde{N}\tilde{C}^D = 0, \\ &E^D = \tilde{C}^D, \\ &\tilde{C}\tilde{C}^D\tilde{C} = \tilde{C}, \\ &\tilde{C}^D\tilde{C} = E^D E, \\ &\tilde{C} = EE^D E, \quad \tilde{N} = E\left(I - E^D E\right). \end{aligned} \tag{10}$$

Proof. See [6, Theorem 2.22]. □

Note, that the Drazin inverse of a matrix can easily be computed from its Jordan canonical form. To see this, assume that $E \in \mathbb{C}^{n,n}$ has the Jordan canonical form

$$E = S \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} S^{-1},$$

where $S \in \mathbb{C}^{n,n}$ is invertible, J is invertible, and N only has zero as an eigenvalue. Then, the Drazin inverse of E is given by

$$E^D = S \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1},$$

which can be shown with Definition 3 by basic computations.

3 Basic theorems

We first assume that E and A commute. This restriction will be lifted in section 4. In analogy to [6, Lemma 2.24] we obtain the following result.

Lemma 6. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ and $E = \tilde{C} + \tilde{N}$ be the decomposition (8). Let k_b and k_f be defined as in (4). Then the following assertions hold.*

1. *Let $\{x^k\}_{k=k_b}^{k_f+1}$ be a solution of (5). Set*

$$\begin{aligned} x_1^k &:= E^D E x^k, \qquad & x_2^k &:= \left(I - E^D E\right) x^k, \\ f_1^k &:= E^D E f^k, \qquad & f_2^k &:= \left(I - E^D E\right) f^k. \end{aligned} \tag{11}$$

   Then, we have

$$\begin{aligned} \tilde{C} x_1^{k+1} &= A x_1^k + f_1^k, \\ \tilde{N} x_2^{k+1} &= A x_2^k + f_2^k, \end{aligned} \tag{12}$$

   for $k = k_b, \ldots, k_f$.

2. *Let $\{x_1^k\}_{k=k_b}^{k_f+1}$ and $\{x_2^k\}_{k=k_b}^{k_f+1}$ be solutions of*

$$\begin{aligned} \tilde{C} x_1^{k+1} &= A x_1^k + E^D E f^k, \\ \tilde{N} x_2^{k+1} &= A x_2^k + (I - E^D E) f^k, \end{aligned} \tag{13}$$

   for $k = k_b, \ldots, k_f$, respectively. Then $\{x^k\}_{k=k_b}^{k_f+1}$ given by

$$x^k := E^D E x_1^k + (I - E^D E) x_2^k$$

   is a solution of (5).

3. Let $\{x_1^k\}_{k=k_b}^{k_f+1}$ *have the form*

$$x_1^k = E^D E x^k, \tag{14}$$

for some $\{x^k\}_{k=k_b}^{k_f+1}$ *and let* $\{f_1^k\}_{k=k_b}^{k_f+1}$ *have the form*

$$f_1^k = E^D E f^k,$$

for some $\{f^k\}_{k=k_b}^{k_f+1}$*. Then* $\{x_1^k\}_{k=k_b}^{k_f+1}$ *is a solution of*

$$\tilde{C} x_1^{k+1} = A x_1^k + f_1^k \tag{15}$$

if and only if $\{x_1^k\}_{k=k_b}^{k_f+1}$ *is a solution of*

$$x_1^{k+1} = E^D A x_1^k + E^D f_1^k. \tag{16}$$

Proof. From (7) and the identities (10) it follows that

$$\tilde{C}^D \tilde{C} A = E^D E A = A E^D E = A \tilde{C}^D \tilde{C} \tag{17}$$

and we know that

$$\tilde{N} x_1^k = \tilde{N} E^D E x^k = \tilde{N} \tilde{C}^D \tilde{C} x^k = 0, \tag{18}$$

$$\tilde{C} x_2^k = \left(\tilde{C} - \tilde{C} \tilde{C}^D \tilde{C} \right) x^k = 0, \tag{19}$$

$$\tilde{N} f_1^k = \tilde{N} E^D E f^k = \tilde{N} \tilde{C}^D \tilde{C} f^k = 0,$$

$$\tilde{C} f_2^k = \left(\tilde{C} - \tilde{C} \tilde{C}^D \tilde{C} \right) f^k = 0. \tag{20}$$

for all $k = k_b, \ldots, k_f + 1$. Further, from (5) we obtain

$$\left(\tilde{C} + \tilde{N} \right) \left(x_1^{k+1} + x_2^{k+1} \right) = A \left(x_1^k + x_2^k \right) + f_1^k + f_2^k. \tag{21}$$

Premultiplying (21) with $\tilde{C}^D \tilde{C}$ and using (17) as well as (9) yields

$$(\tilde{C}^D \tilde{C} \tilde{C} + \tilde{C}^D \tilde{C} \tilde{N}) \left(x_1^{k+1} + x_2^{k+1} \right) = A \tilde{C}^D \tilde{C} \left(x_1^k + x_2^k \right) + \tilde{C}^D \tilde{C} \left(f_1^k + f_2^k \right),$$

which in turn can be combined with (19) and (20) to obtain

$$\tilde{C}^D \tilde{C} \tilde{C} x_1^{k+1} = A \tilde{C}^D \tilde{C} x_1^k + f_1^k = A E^D E x_1^k + f_1^k = A x_1^k + f_1^k,$$

for all $k = k_b, \ldots, k_f$. This shows the first identity of (12), since

$$\tilde{C}^D \tilde{C} \tilde{C} = \tilde{C} \tilde{C}^D \tilde{C} = \tilde{C}.$$

Subtracting the first identity of (12) from (21) yields

$$\tilde{C} x_2^{k+1} + \tilde{N} x_1^{k+1} + \tilde{N} x_2^{k+1} = A x_2^k + f_2^k,$$

for all $k = k_b, \ldots, k_f$. By finally applying (18) and (19) this shows the second identity of (12) and thus part 1. of the assertion. To prove part 2. we apply (13) which immediately leads to

$$\begin{aligned} Ex^{k+1} &= EE^D E x_1^{k+1} + E(I - E^D E) x_2^{k+1} \\ &= \tilde{C} x_1^{k+1} + \tilde{N} x_2^{k+1} \\ &= A(x_1^k + x_2^k) + E^D E f^k + (I - E^D E) f^k \\ &= Ax^k + f^k. \end{aligned}$$

Multiplying (15) with $\tilde{C}^D = E^D$ from the left leads to

$$\tilde{C}^D \tilde{C} x_1^{k+1} = E^D A x_1^k + E^D f_1^k. \tag{22}$$

From (14) one can also obtain that

$$(I - \tilde{C}^D \tilde{C}) x_1^{k+1} = 0. \tag{23}$$

Adding (22) and (23) then immediately shows (16). Conversely, multiplying (16) by $\tilde{C}$ from the left gives

$$\begin{aligned} \tilde{C} x_1^{k+1} &= \tilde{C} \tilde{C}^D A x_1^k + \tilde{C} \tilde{C}^D f_1^k \\ &= A E^D E x_1^k + E^D E f_1^k \\ &= A x_1^k + f_1^k, \end{aligned}$$

and thus part 3. of the assertion since

$$\tilde{C} \tilde{C}^D = EE^D EE^D = E^D EE^D E = E^D E. \qquad \Box$$

Following [6, Lemma 2.25] we first consider the homogeneous case and we get the following Lemma.

Lemma 7. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$, $k_0 \in \mathbb{Z}$ and $v \in \mathbb{C}^n$. Then the following statements hold.*

1. *Let $\hat{v} = E^D E v$. Then*

$$x^k := (E^D A)^{k-k_0} \hat{v}, \quad k = k_0, k_0 + 1, \ldots \tag{24}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k = k_0, k_0 + 1, \ldots \tag{25}$$

2. *Let $\hat{v} = A^D A v$. Then*

$$x^k := (A^D E)^{k_0 - k} \hat{v}, \quad k = k_0, k_0 - 1, \ldots \tag{26}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k = k_0 - 1, k_0 - 2, \ldots \tag{27}$$

3. *Let* $\hat{v} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$. *Then*

$$x^k := \begin{cases} (E^D A)^{k-k_0}\hat{v}, & k = k_0, k_0 - 1, \ldots \\ (A^D E)^{k_0-k}\hat{v}, & k = k_0 - 1, k_0 - 2, \ldots \end{cases} \tag{28}$$

solves the homogeneous linear discrete-time descriptor system

$$Ex^{k+1} = Ax^k, \quad k \in \mathbb{Z}. \tag{29}$$

Proof.

1. With (7) we have

$$\begin{aligned} Ex^{k+1} &= E(E^D A)(E^D A)^{k-k_0} E^D E v \\ &= A(E^D A)^{k-k_0} E^D E E^D E v \\ &= A(E^D A)^{k-k_0} E^D E v \\ &= Ax^k, \end{aligned}$$

for all $k = k_0, k_0 + 1, \ldots$

2. In this case we obtain

$$\begin{aligned} Ax^k &= A(A^D E)^{k_0-k} A^D A v \\ &= A(A^D E)(A^D E)^{k_0-k-1} A^D A v \\ &= E(A^D E)^{k_0-k-1} A^D A A^D A v \\ &= E(A^D E)^{k_0-k-1} A^D A v \\ &= Ex^{k+1} \end{aligned}$$

for all $k = k_0 - 1, k_0 - 2, \ldots$

3. This follows from 1. and 2., since the definitions of x^{k_0} from 1. and 2. coincide. □

Since by (7)

$$(E^D A)^{k-k_0} E^D E v = E^D E (E^D A)^{k-k_0} v,$$

it is clear, that the solution x^k stays in the subspace $\operatorname{range}\left(E^D E\right)$ for all $k \geq k_0$. An analogous conclusion is possible for the case 2. in Lemma 7. In case 3. of Lemma 7 the solution even stays in $\operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$ all the time.

Theorem 8. *Let* $E, A \in \mathbb{C}^{n,n}$ *with* $EA = AE$ *and suppose that*

$$\operatorname{kernel}(E) \cap \operatorname{kernel}(A) = \{0\}. \tag{30}$$

Then,

$$(I - E^D E)A^D A = (I - E^D E). \tag{31}$$

Proof. See [6, Lemma 2.26]. □

According to [6, Theorem 2.27] we see the following Theorem.

Theorem 9. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (30). Also, let $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *Let $\{x^k\}_{k \geq k_0}$ be any solution of (25). Then $\{x^k\}_{k \geq k_0}$ has the form (24) for some $\hat{v} \in \operatorname{range}\left(E^D E\right)$.*
2. *Let $\{x^k\}_{k \leq k_0}$ be any solution of (27). Then $\{x^k\}_{k \leq k_0}$ has the form (26) for some $\hat{v} \in \operatorname{range}\left(A^D A\right)$.*
3. *Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (29). Then $\{x^k\}_{k \in \mathbb{Z}}$ has the form (28) for some $\hat{v} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.*

Proof. Using the decomposition (8), (10), and (7) we have

$$A\tilde{N} = AE(I - E^D E) = E(I - E^D E)A = \tilde{N}A. \tag{32}$$

Furthermore, using (31) and (10) for any $x \in \mathbb{C}^n$ one has the following series of implications.

$$\begin{aligned} A\tilde{N}x = 0 \quad &\Rightarrow \quad A^D A\tilde{N}x = 0 \\ &\Rightarrow \quad (I - E^D E)A^D A\tilde{N}x = 0 \\ &\Rightarrow \quad (I - E^D E)\tilde{N}x = 0 \\ &\Rightarrow \quad \tilde{N}x = 0. \end{aligned} \tag{33}$$

Let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (25). From Lemma 6 part 1. we get $\{x_1^k\}_{k \geq k_0}$, $\{x_2^k\}_{k \geq k_0}$ with $x^k = x_1^k + x_2^k$ which solve (12), respectively. With $\nu = \operatorname{ind}(E)$, using (9), (12), and (32) one then obtains

$$0 = \tilde{N}^\nu x_2^{k+1} = \tilde{N}^{\nu-1} A x_2^k = A\tilde{N}^{\nu-1} x_2^k,$$

for all $k \geq k_0$. From this and from (33) one can see that we also have $\tilde{N}^{\nu-1} x_2^k = 0$ for all $k \geq k_0$. Discarding the identity for $k = k_0$ then yields

$$\tilde{N}^{\nu-1} x_2^k = 0, \quad k \geq k_0 + 1.$$

Shifting the index k, i.e., replacing k by $k+1$ shows that

$$\tilde{N}^{\nu-1} x_2^{k+1} = 0, \quad k + 1 \geq k_0 + 1,$$

which is the same as

$$\tilde{N}^{\nu-1} x_2^{k+1} = 0, \quad k \geq k_0.$$

By repeating this procedure $\nu - 2$ times we finally get

$$\tilde{N} x_2^k = 0, \quad k \geq k_0.$$

Using (12) once again, this implies

$$A x_2^k = 0,$$

and thus with (11) and (31) we have

$$x_2^k = (I - E^D E)x_2^k = (I - E^D E)A^D A x_2^k = 0,$$

which means that $x^k = x_1^k$ for all $k \geq k_0$. Therefore, from Lemma 6 part 3. we know that $\{x_1^k\}$ solves

$$x_1^{k+1} = (E^D A)x_1^k,$$

for all $k \geq k_0$. Recursion of this formula shows that

$$x_1^k = (E^D A)^{k-k_0} x_1^{k_0},$$

for every $k \geq k_0$. Summing up those implications we know that for all $k \geq k_0$ we have

$$x^k = x_1^k = (E^D A)^{k-k_0} x_1^{k_0} = (E^D A)^{k-k_0} E^D E x^{k_0}, \tag{34}$$

which shows part 1. To prove part 2. let $\{x^k\}_{k \leq k_0}$ be any solution of (27). Set $l_0 := -k_0$ and $y^l := x^{-l}$ for $l \geq l_0$. By replacing $k = -l$ in (27) one obtains

$$Ex^{-l+1} = Ax^{-l}, \quad -l = -l_0 - 1, -l_0 - 2, \ldots,$$

which is equivalent to

$$Ex^{-(l-1)} = Ax^{-l}, \quad l = l_0 + 1, l_0 + 2, \ldots$$

By definition we can see that $\{y^l\}_{l \geq l_0}$ is a solution of

$$Ey^{l-1} = Ay^l, \quad l \geq l_0 + 1,$$

and also a solution of

$$Ay^{l+1} = Ey^l, \quad l \geq l_0.$$

Using the identity (34) for this reversed system means that

$$y^l = (A^D E)^{l-l_0} A^D A y^{l_0},$$

for all $l \geq l_0$. Undoing the replacements then yields

$$x^{-l} = (A^D E)^{l-l_0} A^D A x^{-l_0},$$

for all $l \geq l_0$ and thus

$$x^k = (A^D E)^{-k+k_0} A^D A x^{k_0},$$

for all $-k \geq -k_0$. Again, summing up those results shows that

$$x^k = (A^D E)^{k_0-k} A^D A x^{k_0}, \tag{35}$$

for all $k \leq k_0$. Finally, to prove part 3. let $\{x^k\}_{k \in \mathbb{Z}}$ be any solution of (29). Then from (34) we have

$$x^k = (E^D A)^{k-k_0} E^D E x^{k_0},$$

for all $k \geq k_0$ and especially for $k = k_0$ we see that

$$x^{k_0} = E^D E x^{k_0} \in \operatorname{range}\left(E^D E\right).$$

Also we know from (35) that

$$x^k = (A^D E)^{k_0 - k} A^D A x^{k_0},$$

for all $k \leq k_0$ and especially for $k = k_0$ we see that

$$x^{k_0} = A^D A x^{k_0} \in \operatorname{range}\left(A^D A\right).$$

Thus, the claim of the Theorem follows with $\hat{v} = x^{k_0}$. □

Remark 10. One may think that it is not meaningful to look at case 3. of the previous Theorem, since in most cases one starts at some time point and then calculates into the future. But as shown by the following Lemma, also those solutions (where one starts at $k_0 \in \mathbb{Z}$ and calculates a solution for $k \geq k_0$) are almost completely in the subspace $\operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.

Lemma 11. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (30). Also, let $k_0 \in \mathbb{Z}$ and let $\nu_E = \operatorname{ind}(E), \nu_A = \operatorname{ind}(A)$. Then the following statements hold.*

1. *Let $\{x^k\}_{k \geq k_0}$ be any solution of (25). Then for all $k \geq k_0 + \nu_A$ it holds that $x^k \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right)$.*
2. *Let $\{x^k\}_{k \leq k_0}$ be any solution of (27). Then for all $k \leq k_0 - \nu_E$ it holds that $x^k \in \operatorname{range}\left(E^D E\right) \cap \operatorname{range}\left(A^D A\right)$.*

Proof. Since $k \geq k_0 + \nu_A$ it follows that there exists $\hat{k} \geq 0$ such that $k = \hat{k} + k_0 + \nu_A$. From Theorem 9 using (24) and (6) we then know that for some $v \in \mathbb{C}^n$ we have

$$\begin{aligned}
A^D A x^k &= A^D A (E^D A)^{k-k_0} E^D E v \\
&= A^D A (E^D)^{k-k_0} A^{k-k_0} E^D E v \\
&= A^D A (E^D)^{k-k_0} A^{\nu_A} A^{\hat{k}} E^D E v \\
&= A^D A (E^D)^{k-k_0} A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\
&= (E^D)^{k-k_0} A^D A A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\
&= (E^D)^{k-k_0} A^D A^{\nu_A+1} A^{\hat{k}} E^D E v \\
&= (E^D A)^{k-k_0} E^D E v \\
&= x^k.
\end{aligned}$$

Also, we naturally get

$$\begin{aligned}
E^D E x^k &= E^D E (E^D A)^{k-k_0} E^D E v \\
&= (E^D A)^{k-k_0} = E^D E v = x^k,
\end{aligned} \tag{36}$$

and thus the assertion of part 1. follows. As in (36) one gets that $A^D A x^k = x^k$. Let $k = -\hat{k} + k_0 - \nu_E$ with $\hat{k} \geq 0$. Then again for some $v \in \mathbb{C}^n$ it follows that

$$\begin{aligned}
E^D E x^k &= E^D E (A^D E)^{k_0 - k} A^D A v \\
&= E^D E (A^D)^{k_0 - k} E^{k_0 - k} A^D A v \\
&= E^D E (A^D)^{k_0 - k} E^{\nu_E} E^{\hat{k}} A^D A v \\
&= E^D E (A^D)^{k_0 - k} E^D E^{\nu_E + 1} E^{\hat{k}} A^D A v \\
&= (A^D)^{k_0 - k} E^D E E^D E^{\nu_E + 1} E^{\hat{k}} A^D A v \\
&= (A^D)^{k_0 - k} E^D E^{\nu_E + 1} E^{\hat{k}} A^D A v \\
&= (A^D E)^{k_0 - k} A^D A v \\
&= x^k,
\end{aligned}$$

which proves part 2. □

Remark 12. From Lemma 11 one might presume that it is meaningful to require that the initial condition satisfies

$$x^{k_0} \in \operatorname{range}\left(A^D A\right) \cap \operatorname{range}\left(E^D E\right), \tag{37}$$

since only in this case it is possible to calculate the solution into the future (i.e., calculate x^k for $k \geq k_0$) and into the past (i.e., calculate x^k for $k \leq k_0$).
Also, only in case that (37) holds, we get something like an invertibility of the operator that calculates x^{k+1} from x^k. To understand this, imagine that a fixed x^{k_0} is given. From this we calculate a finite number of steps κ into the future. Thus, we have $x^{k_0+\kappa}$. From this state we then calculate κ steps back into the past to obtain $\tilde{x}^{k_0}$. We then have $x^{k_0} = \tilde{x}^{k_0}$ if condition (37) holds. Otherwise we cannot be sure that $x^{k_0} = \tilde{x}^{k_0}$ holds, as shown in the following example.

Example 13. Consider the homogeneous linear discrete-time descriptor system defined by

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=E} x^{k+1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{:=A} x^k,\ k \geq 0,\ x^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{38}$$

Clearly, we have $EA = AE$, $E^D = E$, $A^D = A$ and condition (30) holds. Thus, the pencil (E, A), corresponding to system (38), satisfies all assumptions of Lemma 11 which means that the iterate x^1 has to be in $\operatorname{range}\left(A^D A\right)$. Indeed,

$$Ax^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \operatorname{range}\left(A^D A\right). \tag{39}$$

Now let us calculate back one step from (39), i.e., let us consider the reversed system

$$A\tilde{x}^{l+1} = E\tilde{x}^l,\ l \leq 0,\ \tilde{x}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We see that

$$E\tilde{x}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{x}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and thus

$$\tilde{x}^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq x^0.$$

So far, we have characterized all the solutions of the homogeneous descriptor system. Thus, what we still need to characterize all solutions of (5) is one particular solution of the inhomogeneous system. On the basis of [6, Theorem 2.28] we show the following.

Theorem 14. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (30). Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}}$ with $f^k \in \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *The linear discrete-time descriptor system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \geq k_0,$$

has the particular solution $\{x_1^k + x_2^k\}_{k\geq k_0}$, where

$$\begin{aligned} x_1^k &:= \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j, \\ x_2^k &:= -(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i}, \end{aligned}$$

for $k \geq k_0$. For the construction of the iterate x^k only the f_k with $k \geq k_0$ have to be employed.

2. *The linear discrete-time descriptor system*

$$Ex^{k+1} = Ax^k + f^k, \quad k \leq k_0 - 1, \tag{40}$$

has the particular solution $\{x_1^k + x_2^k\}_{k\leq k_0}$, where

$$\begin{aligned} x_1^k &:= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1}, \\ x_2^k &:= -\sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}, \end{aligned}$$

for $k \leq k_0$. For the construction of the iterate x^k only the f_k with $k \leq k_0 - 1$ have to be employed.

Proof. Let $E = \tilde{C} + \tilde{N}$ be the decomposition (8). Then, using (7) we have the identities

$$\begin{aligned} E^D E x_1^k &= \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D E E^D f^j = x_1^k, \\ (I - E^D E) x_2^k &= -(I - E^D E)(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i} = x_2^k. \end{aligned}$$

Using (10) and (6) one can also conclude, that for all $k \geq k_0$ it follows that

$$\begin{aligned} \tilde{C} x_1^{k+1} &= \tilde{C} \sum_{j=k_0}^{k} (E^D A)^{k+1-j-1} E^D f^j \\ &= \tilde{C} \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} E^D f^j + E^D f^k \right) \\ &= \tilde{C} \left((E^D A) \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + E^D f^k \right) \\ &= A E^D E x_1^k + E^D E f^k \\ &= A x_1^k + E^D E f^k, \end{aligned}$$

and with

$$(I - E^D E) E^{\nu_E} = \begin{cases} (I - E^D E) E^D E^{\nu_E - 1} = 0, & \text{if } \nu_E \geq 1, \\ (I - E^D E) = (I - I) = 0, & \text{if } \nu_E = 0, \end{cases} \tag{41}$$

we obtain

$$\begin{aligned} \tilde{N} x_2^{k+1} &= E(I - E^D E) x_2^{k+1} \\ &= -(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^{i+1} f^{k+i+1} \\ &= -(I - E^D E) \sum_{i=0}^{\nu_E - 2} (A^D E)^{i+1} f^{k+i+1} \\ &= -(I - E^D E) A^D A \sum_{i=1}^{\nu_E - 1} (A^D E)^i f^{k+i} \\ &= -A(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i} + (I - E^D E) f^k \\ &= A x_2^k + (I - E^D E) f^k, \end{aligned}$$

by using (31). With these results and Lemma 6 part 2. one immediately gets that $\{x^k\}_{k \geq k_0}$ with

$$x^k = E^D E x_1^k + (I - E^D E) x_2^k = x_1^k + x_2^k$$

is a solution and thus part 1. of the assertion follows. To prove part 2. we perform a variable substitution. By replacing $l := -k$ and $l_0 := -k_0$ in (40) one gets the system

$$Ex^{-l+1} = Ax^{-l} + f^{-l}, \quad -l \leq -l_0 - 1,$$

which is equivalent to the system

$$Ex^{-(l-1)} = Ax^{-l} + f^{-l}, \quad l \geq l_0 + 1.$$

By further replacing $y^l := x^{-l}$ for $l \geq l_0$ one gets

$$Ey^{l-1} = Ay^l + f^{-l}, \quad l \geq l_0 + 1.$$

Shifting the index l, i.e., replacing l by $l+1$ shows that

$$Ey^l = Ay^{l+1} + f^{-l-1}, \quad l+1 \geq l_0 + 1,$$

which in turn is equivalent to

$$Ay^{l+1} = Ey^l - f^{-l-1}, \quad l \geq l_0.$$

Setting $g^l := -f^{-l-1}$ we can finally write this equation as

$$Ay^{l+1} = Ey^l + g^l, \quad l \geq l_0.$$

From the results of the first part we then get a solution of this last system as

$$y^l = \sum_{j=l_0}^{l-1} (A^D E)^{l-j-1} A^D g^j - (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D g^{l+i}.$$

Undoing the replacement $y^l = x^{-l}$ in this equations then leads to

$$x^{-l} = -\sum_{j=l_0}^{l-1} (A^D E)^{l-j-1} A^D f^{-j-1} + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{-(l+i)-1},$$

and undoing the replacement $k = -l$ finally gives us

$$\begin{aligned} x^k &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} - \sum_{j=-k_0}^{-k-1} (A^D E)^{-k-j-1} A^D f^{-j-1} \\ &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} - \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}. \quad \square \end{aligned}$$

Finding a particular solution for the case $\mathbb{K} = \mathbb{Z}$ is more complicated. Here we have to split up the system into a part belonging to the zero eigenvalues of the pencil $\lambda E - A$, a part belonging to the finite non-zero eigenvalues of $\lambda E - A$, and a part belonging to the infinite eigenvalues of $\lambda E - A$. Similar to Lemma 6 we obtain the following result.

Lemma 15. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (30). Further, let $E = \tilde{C} + \tilde{N}$ and analogously $A = \tilde{D} + \tilde{M}$ be decompositions as in (8). Let $\{x_1^k\}_{k\in\mathbb{Z}}$, $\{x_2^k\}_{k\in\mathbb{Z}}$, $\{x_3^k\}_{k\in\mathbb{Z}}$ be solutions of*

$$\begin{array}{rcl} \tilde{C}x_1^{k+1} & = & \tilde{M}x_1^k + (I - A^D A)f^k, \\ \tilde{C}x_2^{k+1} & = & \tilde{D}x_2^k + A^D A E^D E f^k, \\ \tilde{N}x_3^{k+1} & = & \tilde{D}x_3^k + (I - E^D E)f^k, \end{array}$$

respectively. Then $\{x^k\}_{k\in\mathbb{Z}}$ with

$$x^k := (I - A^D A)x_1^k + A^D A E^D E x_2^k + (I - E^D E)x_3^k,$$

is a solution of

$$Ex^{k+1} = A^k + f^k.$$

Proof. First of all, by (31) we see that

$$\begin{array}{rl} & (I - A^D A) + (I - E^D E) + A^D A E^D E \\ = & I - A^D A + I - (I - A^D A)E^D E \\ = & I - A^D A + I - (I - A^D A) = I. \end{array}$$

Furthermore, we have that

$$\begin{array}{rcl} \tilde{D}(I - A^D A) & = & AA^D A(I - A^D A) = 0, \\ \tilde{M}(A^D A E^D E) & = & A(I - A^D A)(A^D A E^D E) = 0, \\ \tilde{M}(I - E^D E) & = & A(I - A^D A)(I - E^D E) \\ & = & A\big((I - E^D E) - (I - E^D E)A^D A\big) = 0. \end{array}$$

With these identities and assumption (30) we get

$$\begin{array}{rcl} Ex^{k+1} & = & E(I - A^D A)x_1^{k+1} + EA^D A E^D E x_2^{k+1} + E(I - E^D E)x_3^{k+1} \\ & = & (I - A^D A)\tilde{C}x_1^{k+1} + A^D A E^D E\tilde{C}x_2^{k+1} + (I - E^D E)\tilde{N}x_3^{k+1} \\ & = & (I - A^D A)\tilde{M}x_1^k + (I - A^D A)f^k + \\ & & A^D A E^D E\tilde{D}x_2^k + A^D A E^D E f^k + \\ & & (I - E^D E)\tilde{D}x_3^k + (I - E^D E)f^k \\ & = & (I - A^D A)\tilde{M}x_1^k + A^D A E^D E\tilde{D}x_2^k + (I - E^D E)\tilde{D}x_3^k + f^k \\ & = & \tilde{M}(I - A^D A)x_1^k + \tilde{D}(I - A^D A)x_1^k + \\ & & \tilde{M}A^D A E^D E x_2^k + \tilde{D}A^D A E^D E x_2^k + \\ & & \tilde{M}(I - E^D E)x_3^k + \tilde{D}(I - E^D E)x_3^k + f^k \\ & = & Ax^k + f^k. \end{array}$$

Here we have used that $\tilde{D} = AA^D A$, and thus $\tilde{D}$ commutes with the matrices E and A. ∎

Using Lemma 15 we can construct a particular solution for the case $\mathbb{K} = \mathbb{Z}$, as we did in Theorem 14 for the case that $\mathbb{K} = \{k \in \mathbb{Z} : k_b \leq k\}$.

Lemma 16. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfying (30). Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then a solution $\{x^k\}_{k\in\mathbb{Z}}$ of*

$$Ex^{k+1} = Ax^k + f^k, \quad k \in \mathbb{Z},$$

is given by $x^k := x_1^k + x_2^k + x_3^k$, where

$$\begin{aligned} x_1^k &:= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1}, \\ x_2^k &:= \begin{cases} A^D A \displaystyle\sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j, & k \geq k_0, \\ -E^D E \displaystyle\sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}, & k \leq k_0, \end{cases} \\ x_3^k &:= -(I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i}, \end{aligned}$$

for $k \in \mathbb{Z}$.

Proof. Considering the decompositions $E = \tilde{C} + \tilde{N}$ and $A = \tilde{D} + \tilde{M}$ as in (8), using (7), (31), and (41) we have

$$\begin{aligned} \tilde{M} x_1^k &= A(I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\ &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^{i+1} f^{k-i-1} \\ &= (I - A^D A) \sum_{i=0}^{\nu_A - 2} (E^D A)^{i+1} f^{k-i-1} \\ &= (I - A^D A) \sum_{i=1}^{\nu_A - 1} (E^D A)^i f^{k-i} \\ &= (I - A^D A) \left(\sum_{i=0}^{\nu_A - 1} (E^D A)^i f^{k-i} - f^k \right) \\ &= -(I - A^D A) f^k + (I - A^D A) E^D E \sum_{i=0}^{\nu_A - 1} (E^D A)^i f^{k-i} \\ &= -(I - A^D A) f^k + E(I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i} \end{aligned}$$

$$\begin{aligned}
&= -(I - A^D A)f^k + Ex_1^{k+1} \\
&= -(I - A^D A)f^k + (\tilde{C} + \tilde{N})x_1^{k+1} \\
&= -(I - A^D A)f^k + \tilde{C}x_1^{k+1},
\end{aligned}$$

where the last identity holds, since x_1^k has the form $x_1^k = (I - A^D A)y_1^k$ for some y_1^k and

$$\begin{aligned}
\tilde{N}x_1^k &= E(I - E^D E)(I - A^D A)y_1^k \\
&= E\big(I - A^D A - E^D E(I - A^D A)\big)\, y_1^k = 0,
\end{aligned} \tag{42}$$

because of (31). As in Theorem 14, part 1. one obtains

$$\tilde{N}x_3^{k+1} = Ax_3^k + (I - E^D E)f^k = (\tilde{D} + \tilde{M})x_3^k + (I - E^D E)f^k.$$

Again as in (42) it follows that

$$\tilde{M}x_3^k = 0,$$

and thus

$$\tilde{N}x_3^{k+1} = \tilde{D}x_3^k + (I - E^D E)f^k.$$

Finally, for $k \geq k_0$ one has

$$\begin{aligned}
\tilde{C}x_2^{k+1} &= \tilde{C}A^D A \sum_{j=k_0}^{k} (E^D A)^{k-j} E^D f^j \\
&= \tilde{C}E^D A^D A \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} f^j + f^k \right) \\
&= EE^D EE^D A^D A \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} f^j + f^k \right) \\
&= E^D E A^D A \left(\sum_{j=k_0}^{k-1} (E^D A)^{k-j} f^j + f^k \right) \\
&= E^D E A^D A \left(E^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} f^j + f^k \right) \\
&= AA^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + A^D A E^D E f^k \\
&= AA^D AA^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + A^D A E^D E f^k \\
&= \tilde{D}A^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j + A^D A E^D E f^k \\
&= \tilde{D}x_2^k + A^D A E^D E f^k,
\end{aligned}$$

and for $k < k_0$ analogously,

$$\begin{aligned}
\tilde{D}x_2^k &= -\tilde{D}E^DE\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}A^Df^{j-1} \\
&= -AA^DAE^DEA^D\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}f^{j-1} \\
&= -AA^DE^DE\sum_{j=k+1}^{k_0}(A^DE)^{j-k-1}f^{j-1} \\
&= -AA^DE^DE\left(\sum_{j=k+2}^{k_0}(A^DE)^{j-k-1}f^{j-1}+f^k\right) \\
&= -AA^DE^DEf^k - A^DAA^DEE^DE\sum_{j=k+2}^{k_0}(A^DE)^{j-k-2}f^{j-1} \\
&= -AA^DE^DEf^k - A^DEE^DEE^DE\sum_{j=k+2}^{k_0}(A^DE)^{j-k-2}f^{j-1} \\
&= -AA^DE^DEf^k + EE^DE\left(-E^DE\sum_{j=k+2}^{k_0}(A^DE)^{j-k-2}A^Df^{j-1}\right) \\
&= -AA^DE^DEf^k + \tilde{C}x_2^{k+1}.
\end{aligned}$$

Lemma 15 then implies the assertion. □

4 Main results

In the previous section we have constructed a particular solution of the inhomogeneous problem and we have explicitly characterized all solutions of the homogeneous problem. This enables us to specify all solutions of the inhomogeneous problem.

Theorem 17. *Let $E, A \in \mathbb{C}^{n,n}$ with $EA = AE$ satisfy (30). Also, let $\nu_E = \operatorname{ind}(E)$, $\nu_A = \operatorname{ind}(A)$, $\{f^k\}_{k\in\mathbb{Z}}$ with $f^k \in \mathbb{C}^n$ and $k_0 \in \mathbb{Z}$. Then the following statements hold.*

1. *Every solution $\{x^k\}_{k\geq k_0}$ of*

$$Ex^{k+1} = Ax^k + f^k, \quad k \geq k_0 \tag{43}$$

satisfies

$$\begin{aligned}
x^k &= \left(E^DA\right)^{k-k_0}E^DEv + \sum_{j=k_0}^{k-1}\left(E^DA\right)^{k-j-1}E^Df^j \\
&\quad - \left(I - E^DE\right)\sum_{i=0}^{\nu_E-1}\left(A^DE\right)^iA^Df^{k+i},
\end{aligned}$$

for $k \geq k_0$ and for some $v \in \mathbb{C}^n$.

2. *Every solution $\{x^k\}_{k \leq k_0}$ of*

$$Ex^{k+1} = Ax^k + f^k, \quad k \leq k_0 - 1 \tag{44}$$

satisfies

$$\begin{aligned} x^k &= \left(A^D E\right)^{k_0-k} A^D A v + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\ &\quad - \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}, \end{aligned}$$

for $k \leq k_0$ and for some $v \in \mathbb{C}^n$.

3. *Every solution $\{x^k\}_{k \in \mathbb{Z}}$ of*

$$Ex^{k+1} = Ax^k + f^k, \quad k \in \mathbb{Z} \tag{45}$$

satisfies

$$\begin{aligned} x^k &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k-i-1} \\ &+ \begin{cases} (E^D A)^{k-k_0} \hat{v} + A^D A \sum_{j=k_0}^{k-1} (E^D A)^{k-j-1} E^D f^j, & k \geq k_0 \\ (A^D E)^{k_0-k} \hat{v} - E^D E \sum_{j=k+1}^{k_0} (A^D E)^{j-k-1} A^D f^{j-1}, & k \leq k_0 \end{cases} \\ &- (I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k+i}, \end{aligned} \tag{46}$$

for $k \in \mathbb{Z}$ and for some $\hat{v}$ which has the form $\hat{v} = A^D A E^D E v$, where $v \in \mathbb{C}^n$ is arbitrary.

Proof. Since the problem is linear any solution may be written as a particular solution of the inhomogeneous problem plus a solution of the homogeneous problem. Thus, we can derive the result from Theorems 9, and 14, as well as Lemmas 7, and 16. □

Since we now have an explicit representation of all solutions of (2) we can also easily specify all consistent initial conditions. We only have to look at the values of all possible solutions at k_0.

Corollary 18. *Let the assumptions of Theorem 17 hold. Consider the initial condition*

$$x^{k_0} = x_0. \tag{47}$$

Then the following statements hold.

1. *The initial value problem consisting of (43) and (47) possesses a solution if and only if there exists a $v \in \mathbb{C}^n$ with*

$$x_0 = E^D E v - \left(I - E^D E\right) \sum_{i=0}^{\nu_E - 1} \left(A^D E\right)^i A^D f^{k_0+i}.$$

If this is the case, then the solution is unique.

2. *The initial value problem consisting of (44) and (47) possesses a solution if and only if there exists a $v \in \mathbb{C}^n$ with*

$$x_0 = A^D A v + (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k_0-i-1}.$$

If this is the case, then the solution is unique.

3. *The problem consisting of (45) and (47) possesses a solution if and only if there exists a $v \in \mathbb{C}^n$ with*

$$\begin{aligned} x_0 &= (I - A^D A) \sum_{i=0}^{\nu_A - 1} (E^D A)^i E^D f^{k_0-i-1} \\ &\quad + A^D A E^D E v \\ &\quad - (I - E^D E) \sum_{i=0}^{\nu_E - 1} (A^D E)^i A^D f^{k_0+i}. \end{aligned}$$

If this is the case, then the solution is unique.

Let us now assume that E and A do not commute but are nevertheless still regular, i.e., $\det \tilde{\lambda} E - A \neq 0$ for some $\tilde{\lambda} \in \mathbb{C}$. In this situation we can use a nice trick, which is due to Campbell (see [3]).

Lemma 19. *[3] Let $E, A \in \mathbb{C}^{n,n}$ with $\lambda E - A$ regular. Let $\tilde{\lambda} \in \mathbb{C}$ be chosen such that $\tilde{\lambda} E - A$ is nonsingular. Then the matrices*

$$\tilde{E} := \left(\tilde{\lambda} E - A\right)^{-1} E, \qquad \tilde{A} := \left(\tilde{\lambda} E - A\right)^{-1} A$$

commute.

Note, that this transformation does not change the state space of the descriptor system, since the multiplication is only executed from the left. Thus, the set of solutions is not changed. Finally, we observe that condition (30) is equivalent to the regularity of the matrix pencil $\lambda E - A$.

Lemma 20. *[6, Exercise 2.8] Let $E, A \in \mathbb{C}^{n,n}$ satisfy $EA = AE$. Then $\lambda E - A$ is regular if and only if condition (30), i.e., $\operatorname{kernel}(E) \cap \operatorname{kernel}(A) = \{0\}$, holds.*

Proof. First, to show that the regularity of $\lambda E - A$ is sufficient for (30), assume to the contrary that there is an $x \neq 0$ with $x \in \operatorname{kernel}(E) \cap \operatorname{kernel}(A)$, i.e., $0 = Ex = Ax$. Then we know that $(\lambda E - A)x = 0$ for all $\lambda \in \mathbb{C}$. Since, $x \neq 0$ this means that $(\lambda E - A)$ in singular for all $\lambda \in \mathbb{C}$, which contradicts the regularity of $\lambda E - A$.
For the reverse direction, assume to the contrary that $\lambda E - A$ is singular. Since E and A commute we know that E and A can be simultaneously triangularized [5], i.e., there exists an invertible matrix P such that

$$(PEP^{-1}, PAP^{-1}) = \left(\begin{bmatrix} E_{11} & e_{12} & E_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & E_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & a_{12} & A_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \right), \tag{48}$$

where w.l.o.g. E_{11} is regular and triangular, A_{11}, A_{33}, and E_{33} are triangular, e_{12} and a_{12} are column vectors and e_{23} and a_{23} are row vectors. Note that E_{11} and A_{11} can also be square matrices of dimension 0.
Since PEP^{-1} and PAP^{-1} still commute we know that

$$\begin{aligned} \begin{bmatrix} E_{11} & e_{12} & E_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} A_{11} & a_{12} & A_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & A_{33} \end{bmatrix} &= \begin{bmatrix} E_{11}A_{11} & E_{11}a_{12} & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} = \\ \begin{bmatrix} A_{11} & a_{12} & A_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} E_{11} & e_{12} & E_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & E_{33} \end{bmatrix} &= \begin{bmatrix} A_{11}E_{11} & A_{11}e_{12} & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}. \end{aligned}$$

Now we set

$$x := P^{-1} \begin{bmatrix} -E_{11}^{-1}e_{12} \\ 1 \\ 0 \end{bmatrix} \neq 0,$$

partitioned accordingly to (48). With this we see that

$$\begin{aligned} Ex &= P^{-1}(PEP^{-1}) \begin{bmatrix} -E_{11}^{-1}e_{12} \\ 1 \\ 0 \end{bmatrix} = P^{-1} \begin{bmatrix} E_{11} & e_{12} & E_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} -E_{11}^{-1}e_{12} \\ 1 \\ 0 \end{bmatrix} = 0, \\ Ax &= P^{-1}(PAP^{-1}) \begin{bmatrix} -E_{11}^{-1}e_{12} \\ 1 \\ 0 \end{bmatrix} = P^{-1} \begin{bmatrix} A_{11} & a_{12} & A_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} -E_{11}^{-1}e_{12} \\ 1 \\ 0 \end{bmatrix} = 0, \end{aligned}$$

where the last identity holds since E_{11} and A_{11} commute and $E_{11}a_{12} = A_{11}e_{12}$. Thus, we found a $x \neq 0$ in $\operatorname{kernel}(E) \cap \operatorname{kernel}(A)$ which contradicts the assumption. □

Lemma 19 and Lemma 20 show, that the assumptions of Theorem 17 and Corollary 18 can essentially be reduced to the regularity of the matrix pencil $\lambda E - A$. To obtain those even more general statements one would have to perform the replacements

$$E \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} E, \qquad A \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} A, \qquad f \leftarrow \left(\tilde{\lambda}E - A\right)^{-1} f,$$

in Theorem 17 and Corollary 18, with $\tilde{\lambda}$ chosen as in Lemma 19.

5 Conclusion

In this text we concentrated on regular systems, i.e., on systems of the form (5) where the matrix pencil $\lambda E - A$ is regular. For such systems we have presented the explicit solution with the help of the Drazin inverse. In contrast to the continuous-time case one has to distinguish between four different cases for such systems. The first case is where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \geq k_0$. The second case is where one has an initial condition given at point $k_0 \in \mathbb{Z}$ and only wants to get a solution for indices $k \leq k_0$. These first two cases are closely related, since the first case can be transferred into the second one by a variable substitution.

The third case is really different from the first two cases. Here also an initial condition is given at some point $k_0 \in \mathbb{Z}$ but one is looking for a solution for indices $k \geq k_0$ as well as for indices $k \leq k_0$. This puts stronger restrictions on the initial condition, i.e., the set of consistent initial conditions in the third case is smaller than in the first or second case.

The fourth case has not been examined in this paper. It consists of the case, where one only wants to get a solution on a finite interval, i.e., a solution for all $k \in \mathbb{K} = \{k \in \mathbb{Z} : k_b \leq k \leq k_f\}, k_b \in \mathbb{Z}, k_f \in \mathbb{Z}$. This case is more complicated, as boundary value conditions have to be introduced on both ends of the interval to fix a unique solution. This case has been studied in [8, 9] but these results seem questionable against the background of our, since the additional final condition only has an influence on the last but $\operatorname{ind}(A)$ final elements of the solution sequence.

The case differentiation at (46) does not seem optimal and there may be nicer notations available, e.g., one could define an operator that realizes the case differentiation. Perhaps even nicer notations are possible that do not involve a case differentiation at all.

The results in this paper could in principle be used to actually compute solutions of systems of the form (5) but another method employing the singular value decomposition seems to be better suited for this purpose, see [1, Chapter 5].

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